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ELEMENTS

OF

QUATERNIONS.

BY THE LATE

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EDITED BY HIS SON,

WILLIAM EDWIN HAMILTON, A. B. T. C. D., C. E.

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TO THE

RIGHT HONORABLE WILLIAM EARL OF ROSSE,

CHANCELLOR OF THE UNIVERSITY OF DUBLIN,

This Volume

IS, BY PERMISSION, DEDICATED,

BY

THE EDITOR.

M577200

In my late father's Will no instructions were left as to the publication of his Writings, nor specially as to that of the "ELEMENTS OF QUATERNIONS," which, but for his late fatal illness, would have been before now, in all their completeness, in the hands of the Public.

My brother, the Rev. A. H. Hamilton, who was named Executor, being too much engaged in his clerical duties to undertake the publication, deputed this task to me.

It was then for me to consider how I could best fulfil my triple duty in this matter—First, and chiefly, to the dead; secondly, to the present public; and, thirdly, to succeeding generations. I came to the conclusion that my duty was to publish the work as I found it, adding merely proof sheets, partially corrected by my late father and from which I removed a few typographical errors, and editing only in the literal sense of giving forth.

Shortly before my father's death, I had several conversations with him on the subject of the "ELEMENTS." In these he spoke of anticipated applications of Quaternions to Electricity, and to all questions in which the idea of Polarity is involved—applications which he never in his own lifetime expected to be able fully to develope, bows to be reserved for the hands of another Ulysses. He also discussed a good deal the nature of his own forthcoming Preface; and I may intimate, that after dealing with its more important topics, he intended to advert to the great labour which the writing of the "ELEMENTS" had cost him—labour both mental and mechanical; as, besides a mass of subsidiary and unprinted calculations, he wrote out all the manuscript, and corrected the proof sheets, without assistance.

And here I must gratefully acknowledge the generous act of the Board of Trinity College, Dublin, in relieving us of the remaining pecuniary liability, and thus incurring the main expense, of the publication of this volume. The announcement of their intention to do so, gratifying as it was, surprised me the less, when I remembered that they had, after the publication of my father's former book, "Lectures on Quaternions," defrayed its entire cost; an extension of their liberality beyond what was recorded by him at the end of his Preface to the "Lectures," which doubtless he would have acknowledged, had he lived to complete the Preface of the "ELEMENTS."

He intended also, I know, to express his sense of the care bestowed upon the typographical correctness of this volume by Mr. M. H. Gill of the University Press, and upon the delineation of the figures by the Engraver, Mr. Oldham.

I annex the commencement of a Preface, left in manuscript by my father, and which he might possibly have modified or rewritten. Believing that I have thus best fulfilled my part as trustee of the unpublished "ELEMENTS," I now place them in the hands of the scientific public.

WILLIAM EDWIN HAMILTON.

January 1st, 1866.

PREFACE.*

[1.] THE volume now submitted to the public is founded on the same principles as the "LECTURES,"(1) which were published on the same subject about ten years ago: but the plan adopted is entirely new, and the present work can in no sense be considered as a second edition of that former one. The *Table of Contents*, by collecting into one view the headings of the various Chapters and Sections, may suffice to give, to readers already acquainted with the subject, a notion of the course pursued: but it seems proper to offer here a few introductory remarks, especially as regards the method of exposition, which it has been thought convenient on this occasion to adopt.

[2.] The present treatise is divided into Three Books, each designed to develope one guiding conception or view, and to illustrate it by a sufficient but not excessive number of examples or applications. The First Book relates to the *Conception of a Vector*, considered as a *directed right line*, in space of three dimensions. The Second Book introduces a *First Conception of a Quaternion*, considered as *the Quotient of two such Vectors*. And the Third Book treats of *Products and Powers of Vectors*, regarded as constituting a *Second Principal Form* of the *Conception of Quaternions in Geometry*.

* * * * * *

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* This fragment, by the Author, was found in one of his manuscript books by the Editor.

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This short First Chapter should be read with care by a beginner; any misconception of the meaning of the word "Vector" being fatal to progress in the Quaternions. The Chapter contains explanations also of the connected, but not all equally important, words or phrases, "revector," "provector," "transvector," "actual and null vectors," "opposite and successive vectors," "origin and term of a vector," "equal and unequal vectors," "addition and subtraction of vectors," "multiples and fractions of vectors," &c.; with the notation B – A, for the Vector (or directed right line) AB: and a deduction of the result, essential but *not peculiar*‡ to quaternions, that (what is here called) the *vector-sum*, of two co-initial sides of a parallelogram, is the intermediate and co-initial *diagonal*. The term "Scalar" is also introduced, in connexion with *coefficients of vectors*.

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* This *Chapter* may be referred to, as I. i.; the next as I. ii.; the first Chapter of the Second Book, as II. i.; and similarly for the rest.

† This *Section* may be referred to, as I. i. 1; the next, as I. i. 2; the sixth Section of the second Chapter of the Third Book, as III. ii. 6; and so on.

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BOOK II.

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CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING QUOTIENTS OF VECTORS, 103–239

Very little, if any, of this Chapter II. i., should be omitted, even in a first perusal; since it contains the most essential conceptions and notations of the Calculus of Quaternions, at least so far as *quotients* of vectors are concerned, with numerous geometrical illustrations. Still there are a few investigations respecting circumscribed cones, imaginary intersections, and ellipsoids, in the thirteenth Section, which a student *may* pass over, and which will be indicated in the proper place in this Table.

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It is shown, by consideration of an *angle on a desk*, or inclined plane, that the *complex relation* of one vector to another, in *length* and

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in *direction*, involves generally a system of *four numerical elements*. Many other motives, leading to the adoption of the *name*, "Quaternion," for the subject of the present Calculus, from its fundamental connexion with the *number* "Four," are found to present themselves in the course of the work.

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In the five foregoing Sections it is shown, among other things, that the *plane* of a quaternion is generally an *essential element* of its constitution, so that *diplanar quaternions* are *unequal;* but that the *square* of *every right radial* (or *right versor*) is equal to *negative unity*, *whatever* its *plane* may be. The *Symbol* $\sqrt{-1}$ admits then of a *real interpretation*, in this as in several other systems; but *when* thus treated as *real*, it is in the present Calculus *too vague* to be useful: on which account it is found convenient to *retain the old signification* of that symbol, as denoting the (uninterpreted) *Imaginary of Algebra*, or what may here be called the *scalar imaginary*, in investigations respecting *non-real intersections*, or *non-real contacts*, in geometry.

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The student ought to make himself *familiar* with these laws, which are all included in the Fundamental Formula,

$$i^2 = j^2 = k^2 = ijk = -1. \qquad \text{(A)}$$

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In fact, a QUATERNION may be *symbolically defined* to be a *Quadrinomial Expression* of the form,

$$q = w + ix + jy + kz, \qquad \text{(B)}$$

in which w, x, y, z are *four scalars*, or ordinary algebraic quantities, while i, j, k are *three new symbols*, obeying the *laws* contained in the formula (A), and therefore *not subject* to all the usual rules of *algebra*: since we have, for instance,

$$ij = +k, \quad \text{but} \quad ji = -k; \quad \text{and} \quad i^2j^2k^2 = -(ijk)^2.$$

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CHAPTER II.

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CHAPTER I.

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The first six Sections of this Chapter ought to be read, even in a first perusal of the work.

SECTION 1.—On a First Method of Interpreting a Product of Two Vectors as a Quaternion, ... 301–303

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This *first interpretation* treats the *product* $\beta . \alpha$, as equal to the *quotient* $\beta : \alpha^{-1}$; where α^{-1} (or Rα) is the previously defined *Reciprocal* (II. ii. 7) of the vector α, namely a *second vector*, which has an *inverse length*, and an *opposite direction*. *Multiplication of Vectors* is thus proved to be (like that of Quaternions) a *Distributive*, but *not* generally a *Commutative Operation*. The *Square of a Vector* is shown to be always a *Negative Scalar*, namely the *negative* of the *square* of the *tensor* of that vector, or of the *number* which expresses its *length*; and some geometrical applications of this fertile principle, to *spheres*, &c., are given. The *Index* of the *Right Part* of a *Product of Two Co-initial Vectors*, OA, OB, is proved to be a right line, *perpendicular to the Plane of the Triangle* OAB, and representing by its *length* the *Double Area* of that triangle; while the *Rotation round this Index, from the Multiplier to the Multiplicand*, is *positive*. This *right part*, or *vector part*, V$\alpha\beta$, of the product *vanishes*, when the *factors* are *parallel* (to one common line); and the *scalar* part, S$\alpha\beta$, when they are *rectangular*.

SECTION 3.—On a Second Method of arriving at the same Interpretation, of a Binary Product of Vectors, . . . 308–310

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In this *second interpretation*, which is found to agree in all its results with the first, but is better adapted to an extension of the theory, as in the following Sections, to *ternary products* of vectors, a *product of two vectors* is treated as the product of the two *right quaternions*, of which those vectors are the *indices* (II. i. 5). It is shown that, on the same plan, the *Sum of a Scalar and a Vector is a Quaternion.*

SECTION 6.—On the Interpretation of a Product of Three or more Vectors as a Quaternion, ... 316–330

This interpretation is effected by the substitution, as in recent Sections, of *Right Quaternions* for *Vectors*, without change of *order* of the *factors*. *Multiplication of Vectors*, like that of Quaternions, is thus proved to be an *Associative Operation.* A vector, generally, is reduced to the *Standard Trinomial Form*,

$$\rho = ix + jy + kz; \qquad \text{(C)}$$

in which i, j, k are the peculiar symbols already considered (II. i. 10), but are regarded *now* as denoting *Three Rectangular Vector-Units*, while the *three scalars* x, y, z are simply *rectangular co-ordinates;* from the known theory of which last, illustrations of results are derived. The *Scalar of the Product of Three coinitial Vectors*, OA, OB, OC, is found to represent, with a *sign* depending on the *direction of a rotation*, the *Volume of the Parallelepiped* under those three lines; so that it *vanishes* when they are *complanar. Constructions* are given also for *products of successive sides of triangles*, and other *closed polygons*, inscribed in *circles*, or in *spheres;* for example, a *characteristic property of the circle* is contained in the theorem, that the product of the *four* successive sides of an *inscribed quadrilateral* is a *scalar:* and an equally *characteristic* (but less obvious) *property of the sphere* is included in this other theorem, that the product of the *five* successive sides of an *inscribed gauche pentagon* is equal to a *tangential vector*, drawn from the point at which the pentagon *begins* (or *ends*). Some general *Formulæ* of *Transformation of Vector Expressions* are given, with which a student ought to render himself *very familiar*, as they are of continual occurrence in the *practice* of this Calculus; especially the four formulæ (pp. 316, 317):

$$\mathrm{V}.\gamma\mathrm{V}\beta\alpha = \alpha\mathrm{S}\beta\gamma - \beta\mathrm{S}\gamma\alpha; \qquad \text{(D)}$$

$$\mathrm{V}\gamma\beta\alpha = \alpha\mathrm{S}\beta\gamma - \beta\mathrm{S}\gamma\alpha + \gamma\mathrm{S}\alpha\beta; \qquad \text{(E)}$$

$$\rho\mathrm{S}\alpha\beta\gamma = \alpha\mathrm{S}\beta\gamma\rho + \beta\mathrm{S}\gamma\alpha\rho + \gamma\mathrm{S}\alpha\beta\rho; \qquad \text{(F)}$$

$$\rho\mathrm{S}\alpha\beta\gamma = \mathrm{V}\beta\gamma\mathrm{S}\alpha\rho + \mathrm{V}\gamma\alpha\mathrm{S}\beta\rho + \mathrm{V}\alpha\beta\mathrm{S}\gamma\rho; \qquad \text{(G)}$$

in which $\alpha, \beta, \gamma, \rho$ are *any four vectors*, while S and V are signs of the operations of taking separately the *scalar* and *vector parts* of a quaternion. On the whole, this Section (III. i. 6) must be considered to be (as regards the present exposition) an important one; and if it have been read with care, after a perusal of the portions previously indicated, no difficulty will be experienced in passing to any subsequent applications of Quaternions, in the present or any other work.

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SECTION 7.—On the Fourth Proportional to Three Diplanar Vectors, ... 331–349

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SECTION 10.—On the Interpretation of a Power of a Vector as a Quaternion, ... 364–384

It may be well to read this Section (III. i. 10), especially for the *Exponential Connexions* which it establishes, between *Quaternions* and *Spherical Trigonometry*, or rather *Polygonometry*, by a species of *extension of Moivre's theorem*, from the *plane* to *space*, or to the *sphere*. For example, there is given (in p. 381) an *equation of six terms*, which holds good for *every spherical pentagon*, and is deduced in this way from an *extended exponential formula*. The calculations in the sub-articles to Art. 312 (pp. 375–379) may however be passed over; and perhaps Art. 315, with its sub-articles (pp. 383, 384). But Art. 314, and its sub-articles, pp. 381–383, should be read, on account of the *exponential forms* which they contain, of equations of the *circle*, *ellipse*, *logarithmic spirals* (circular and elliptic), *helix*, and *screw surface.*

SECTION 11.—On Powers and Logarithms of Diplanar Quaternions; with some Additional Formulæ, ... 384–390

It may suffice to read Art. 316, and its first eleven sub-articles, pp. 384–386. In this Section, the adopted *Logarithm*, lq, of a Quaternion q, is the *simplest root*, q', of the transcendental equation,

$$1 + q' + \frac{q'^2}{2} + \frac{q'^3}{2.3} + \&c. = q;$$

and its expression is found to be,

$$\mathrm{l}q = \mathrm{lT}q + \angle q \, . \mathrm{UV}q, \qquad \text{(H)}$$

in which T and U are the signs of *tensor* and *versor*, while $\angle q$ is the *angle* of q, supposed usually to be between 0 and π. Such *logarithms* are found to be often *useful* in this Calculus, although they do not *generally* possess the elementary property, that the *sum* of the logarithms of two quaternions is equal to the logarithm of their *product*: this apparent paradox, or at least *deviation from ordinary algebraic rules*, arising necessarily from the corresponding property of *quaternion multiplication*, which has been already seen to be *not generally* a *commutative operation* ($q'q''$ *not* $= q''q'$, unless q' and q'' be *complanar*). And *here*, perhaps, a student might consider his *first perusal* of this work as *closed.*[*]

CHAPTER II.

ON DIFFERENTIALS AND DEVELOPMENTS OF FUNCTIONS OF QUATERNIONS; AND ON SOME APPLICATIONS OF QUATERNIONS TO GEOMETRICAL AND PHYSICAL QUESTIONS, ... 391–495

It has been already said, that this Chapter may be omitted in a first perusal of the work.

SECTION 1.—On the Definition of Simultaneous Differentials, ... 391–393

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[*] If he should choose to proceed to the *Differential Calculus of Quaternions* in the next Chapter (III. ii.), and to the *Geometrical* and other *Applications* in the third Chapter (III. iii.) of the present Book, it might be useful to read at this stage the last Section (I. iii. 7) of the First Book, which treats of *Differentials of Vectors* (pp. 98-102); and perhaps the omitted parts of the Section II. i. 13, namely Articles 213-220, with their subarticles (pp. 214-233), which relate, among other things, to a *Construction of the Ellipsoid*, suggested by the present Calculus. But the writer will now abstain from making any further suggestions of this kind, after having indicated as above what appeared to him a *minimum* course of study, amounting to rather less than 200 pages (or parts of pages) of this Volume, which will be recapitulated for the convenience of the student at the end of the present Table.

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SECTION 2.—Elementary Illustrations of the Definition, from Algebra and Geometry, ... 394–398

In the view here adopted (comp. I. iii. 7), *differentials* are *not necessarily*, nor even *generally*, *small*. But it is shown at a later stage (Art. 401, pp. 626–630), that the principles of this Calculus *allow* us, whenever any advantage may be thereby gained, to treat differentials as *infinitesimals*; and so to *abridge calculation*, at least in many applications.

SECTION 3.—On some general Consequences of the Definition, ... 398–409

Partial differentials and *derivatives* are introduced; and differentials of *functions of functions*.

SECTION 4.—Examples of Quaternion Differentiation, . . . 409–419

One of the most important *rules* is, to differentiate the *factors* of a quaternion *product*, *in situ*; thus (by p. 405),

$$\mathrm{d}\,.\,qq' = \mathrm{d}q\,.\,q' + q\,.\,\mathrm{d}q'. \qquad \text{(I)}$$

The formula (p. 399), $\mathrm{d}\,.\,q^{-1} = -q^{-1}\mathrm{d}q\,.\,q^{-1}$, (J)

for the differential of the *reciprocal* of a quaternion (or vector), is also very often useful; and so are the equations (p. 413),

$$\frac{\mathrm{dT}q}{\mathrm{T}q} = \mathrm{S}\frac{\mathrm{d}q}{q}; \qquad \frac{\mathrm{dU}q}{\mathrm{U}q} = \mathrm{V}\frac{\mathrm{d}q}{q}; \qquad \text{(K)}$$

and (p. 411), $\mathrm{d}\,.\,a^t = \frac{\pi}{2}\,a^{t+1}\mathrm{d}t$; (L)

q being any quaternion, and a any constant vector-unit, while t is a variable scalar. It is important to *remember* (comp. III. i. 11), that we have *not* in *quaternions* the *usual* equation,

$$\mathrm{dl}q = \frac{\mathrm{d}q}{q},$$

unless q and $\mathrm{d}q$ be *complanar*; and therefore that we have *not generally*,

$$\mathrm{dl}\rho = \frac{\mathrm{d}\rho}{\rho},$$

if ρ be a *variable vector*; although we *have*, in this Calculus, the scarcely less simple equation, which is useful in questions respecting *orbital motion*,

$$\mathrm{dl}\frac{\rho}{a} = \frac{\mathrm{d}\rho}{\rho}, \qquad \text{(M)}$$

if a be any constant vector, and if the *plane* of a and ρ be *given* (or constant).

SECTION 5.—On Successive Differentials and Developments, of Functions of Quaternions, ... 420–435

In this Section principles are established (pp. 423–426), respecting quaternion *functions* which *vanish together*; and a form of development (pp. 427, 428) is assigned, *analogous** *to Taylor's Series*, and like it capable of being concisely expressed by the *symbolical equation*, $1 + \Delta = \epsilon^d$ (p. 432). As an example of partial and successive differentiation, the expression (pp. 432, 433),

$$\rho = rk^t j^s k j^{-s} k^{-t},$$

which may represent *any vector*, is operated on; and an application is made, by means of *definite integration* (pp. 434, 435), to deduce the known area and volume of a sphere, or of portions thereof; together with the theorem, that the *vector sum* of the *directed elements* of a *spheric segment* is *zero*: each *element of surface* being represented by an *inward normal*, proportional to the elementary area, and corresponding in hydrostatics to the *pressure of a fluid* on that element.

SECTION 6.—On the Differentiation of Implicit Functions of Quaternions; and on the General Inversion of a Linear Function, of a Vector or a Quaternion: with some connected Investigations, ... 435–495

In this Section it is shown, among other things, that a *Linear and Vector Symbol*, ϕ, of *Operation on a Vector*, ρ, satisfies (p. 443) a *Symbolic and Cubic Equation*, of the form,

$$0 = m - m'\phi + m''\phi^2 - \phi^3; \qquad \text{(N)}$$

whence

$$m\phi^{-1} = m' - m''\phi + \phi^2 = \psi, \qquad \text{(N')}$$

= *another symbol of linear operation*, which it is shown how to deduce otherwise from ϕ, as well as the three scalar constants, m, m', m''. The connected *algebraical cubic* (pp. 460, 461),

$$M = m + m'c + m''c^2 + c^3 = 0, \qquad \text{(O)}$$

is found to have important applications; and it is proved† (pp. 460, 462) that if $S\lambda\phi\rho = S\rho\phi\lambda$, independently of λ and ρ, in which case the function ϕ is said to be *self-conjugate*, then this last cubic has *three real roots*, c_1, c_2, c_3; while, in the same case, the *vector equation*,

$$V\rho\phi\rho = 0, \qquad \text{(P)}$$

is satisfied by a system of *Three Real and Rectangular Directions*: namely (compare pp. 468, 469, and the Section III. iii. 7), those of the *axes* of a (biconcyclic) *system* of *surfaces* of the *second order*, represented by the *scalar equation*,

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* At a later stage (Art. 375, pp. 509, 510), a *new Enunciation of Taylor's Theorem* is given, with a *new proof*, but still in a *form* adapted to quaternions.

† A simplified proof, of some of the chief results for this important *case of self-conjugation*, is given at a later stage, in the few first subarticles to Art. 415 (pp. 698, 699).

$$S\rho\phi\rho = C\rho^2 + C', \text{ in which } C \text{ and } C' \text{ are constants.} \qquad (Q)$$

Cases are discussed; and *general forms* (called cyclic, rectangular, focal, bifocal, &c., from their chief geometrical uses) are assigned, for the vector and scalar functions $\phi\rho$ and $S\rho\phi\rho$: one useful pair of such (*cyclic*) forms being, with real and constant values of g, λ, μ,

$$\phi\rho = g\rho + V\lambda\rho\mu, \qquad S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho. \qquad (R)$$

And finally it is shown (pp. 491, 492) that if fq be a *linear and quaternion function of a quaternion*, q, then the *Symbol of Operation*, f, satisfies a certain *Symbolic and Biquadratic Equation*, analogous to the *cubic* equation in ϕ, and capable of similar applications.

CHAPTER III.

ON SOME ADDITIONAL APPLICATIONS OF QUATERNIONS, WITH SOME CONCLUDING REMARKS, . . . 495 to the end.

This Chapter, like the one preceding it, may be omitted in a first perusal of the Volume, as has indeed been already remarked.

SECTION 1.—Remarks Introductory to this Concluding Chapter, ... 495–496

SECTION 2.—On Tangents and Normal Planes to Curves in Space, ... 496–501

SECTION 3.—On Normals and Tangent Planes to Surfaces, 501–510

SECTION 4.—On Osculating Planes, and Absolute Normals, to Curves of Double Curvature, ... 511–515

SECTION 5.—On Geodetic Lines, and Families of Surfaces, 515–531

In these Sections, $d\rho$ usually denotes a *tangent to a curve*, and ν a *normal to a surface*. Some of the theorems or constructions may perhaps be new; for instance, those connected with the *cone of parallels* (pp. 498, 513, &c.) to the *tangents to a curve of double curvature*; and possibly the theorem (p. 525), respecting *reciprocal curves in space*: at least, the deductions here given of these results may serve as exemplifications of the Calculus employed. In treating of *Families of Surfaces* by quaternions, a sort of *analogue* (pp. 529, 530) to the formation and integration of *Partial Differential Equations* presents itself; as indeed it had done, on a similar occasion, in the *Lectures* (p. 574).

SECTION 6.—On Osculating Circles and Spheres, to Curves in Space; with some connected Constructions, . . . 531–630

The analysis, however condensed, of this long Section (III. iii. 6), cannot conveniently be performed otherwise than under the heads of the respective *Articles* (389–401) which compose it: each Article

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being followed by several subarticles, which form with it a sort of *Series.*[*]

ARTICLE 389.—*Osculating Circle defined*, as the *limit* of a circle, which *touches a given curve* (plane or of double curvature) at a *given point* P, and *cuts* the curve at a *near* point Q (see Fig. 77, p. 511). Deduction and interpretation of general expressions for the vector κ of the *centre* K of the circle so defined. The *reciprocal of the radius* KP being called the *vector of curvature*, we have *generally*,

$$\textit{Vector of Curvature} = (\rho-\kappa)^{-1} = \frac{\mathrm{dUd}\rho}{\mathrm{Td}\rho} = \frac{1}{\mathrm{d}\rho}\,\mathrm{V}\,\frac{\mathrm{d}^2\rho}{\mathrm{d}\rho} = \&\text{c.}; \qquad \text{(S)}$$

and if the *arc* (s) of the curve be made the independent variable, then

$$\text{Vector of Curvature} = \rho'' = \mathrm{D}_s^{\,2}\rho = \frac{\mathrm{d}^2\rho}{\mathrm{d}s^2}. \qquad \text{(S')}$$

Examples: curvatures of helix, ellipse, hyperbola, logarithmic spiral; locus of centres of curvature of helix, plane evolute of plane ellipse, 531–535

ARTICLE 390.—Abridged general calculations; return from (S') to (S), ... 535, 536

ARTICLE 391.—Centre determined by three scalar equations; *Polar Axis, Polar Developable*, ... 537

ARTICLE 392.—*Vector Equation* of osculating circle, ... 538, 539

ARTICLE 393.—*Intersection* (or intersections) of a circle with a *plane curve* to which it osculates; example, hyperbola, ... 539–541

ARTICLE 394.—Intersection (or intersections) of a *spherical curve* with a *small circle* osculating thereto; example, *spherical conic; constructions* for the spherical *centre* (or *pole*) of the circle osculating to such a curve, and for the point of *intersection* above mentioned, . . 541–549

ARTICLE 395.—*Osculating Sphere*, to a curve of double curvature, defined as the *limit* of a sphere, which *contains the osculating circle* to the curve at a *given point* P, and *cuts* the same curve at a *near* point Q (comp. Art. 389). The *centre* S, of the sphere so found, is (as usual) the point in which the *polar axis* (Art. 391) *touches the cusp-edge* of the *polar developable*. Other general construction for the same centre (p. 551, comp. p. 573). General expressions for the *vector*, σ = OS, and for the *radius*, $R = \overline{\text{SP}}$; R^{-1} is the *spherical curvature* (comp. Art. 397). *Condition of Sphericity* ($S = 1$), and *Coefficient of Non-sphericity* ($S - 1$), for a curve in space. When this last coefficient is *positive* (as it is for the helix), the *curve* lies *outside* the *sphere*, at least in the neighbourhood of the point of osculation, ... 549–553

ARTICLE 396.—Notations $\tau, \tau', \ldots$ for $\mathrm{D}_s\rho$, $\mathrm{D}_s^{\,2}\rho$, &c.; properties of a curve depending on the *square* (s^2) of its *arc*, measured from a given point P; τ = *unit-tangent*, τ' = *vector of curvature*, $r^{-1} = \mathrm{T}\tau'$ = *curvature* (or *first* curvature, comp. Art. 397), $\nu = \tau\tau'$ = *binormal*; the

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[*] A *Table of initial Pages* of all the *Articles* will be elsewhere given, which will much facilitate *reference*.

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three planes, respectively perpendicular to τ, τ', ν, are the *normal plane*, the *rectifying plane*, and the *osculating plane;* general theory of *emanant lines and planes, vector of rotation, axis of displacement, osculating screw surface; condition of developability* of *surface of emanants*, 554–559

ARTICLE 397.—Properties depending on the *cube* (s^3) of the *arc; Radius* r (denoted here, for distinction, by a roman letter), and *Vector* $\mathrm{r}^{-1}\tau$, of *Second Curvature; this radius* r may be *either* positive or *negative* (whereas the radius r of *first* curvature is always treated as *positive*), and its reciprocal r^{-1} may be thus expressed (pp. 563, 559),

$$\textit{Second Curvature}^* = \mathrm{r}^{-1} = \mathrm{S}\,\frac{d^3\rho}{\mathrm{V}d\rho d^2\rho},\ (\mathrm{T}),\ \text{or},\ \mathrm{r}^{-1} = \mathrm{S}\,\frac{\tau''}{\tau\tau'}, \qquad (\mathrm{T}')$$

the independent variable being the *arc* in (T′), while it is *arbitrary* in (T): but quaternions supply a vast variety of *other expressions* for this important scalar (see, for instance, the *Table* in pp. 574, 575). We have also (by p. 560, comp. Arts. 389, 395, 396),

$$\textit{Vector of Spherical Curvature} = \mathrm{SP}^{-1} = (\rho - \sigma)^{-1} = \&\text{c.}, \qquad (\mathrm{U})$$

= *projection of vector* (τ') of (*simple* or *first*) *curvature, on radius* (R) *of osculating sphere:* and if p and P denote the *linear* and *angular elevations*, of the centre (S) of this *sphere* above the osculating *plane*, then (by same page 560),

$$p = r \tan P = R \sin P = r'\mathrm{r} = \mathrm{r}\mathrm{D}_s r. \qquad (\mathrm{U}')$$

Again (pp. 560, 561), if we write (comp. Art. 396),

$$\lambda = \mathrm{V}\,\frac{\tau''}{\tau'} = \mathrm{r}^{-1}\tau + \tau\tau' = \textit{Vector of Second Curvature plus Binormal}, \ (\mathrm{V})$$

this line λ may be called the *Rectifying Vector;* and if H denote the *inclination* (considered first by Lancret), of this *rectifying line* (λ) to the *tangent* (τ) to the curve, then

$$\tan H = r'^{-1} \tan P = r^{-1}\mathrm{r}. \qquad (\mathrm{V}')$$

Known right cone with rectifying line for its *axis*, and with H for its *semiangle*, which *osculates* at P to the *developable locus of tangents* to the curve (or by p. 568 to the *cone of parallels* already mentioned); *new right cone*, with a *new semiangle*, C, connected with H by the relation (p. 562),

$$\tan C = \frac{3}{4} \tan H, \qquad (\mathrm{V}'')$$

which osculates to the *cone of chords*, drawn from the given point P

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* In this Article, or Series, 397, and indeed also in 396 and 398, several references are given to a very interesting Memoir by M. de Saint-Venant, "*Sur les lignes courbes non planes:*" in which, however, that able writer *objects* to such known phrases as *second curvature, torsion*, &c., and proposes in their stead a new name "*cambrure*," which it has not been thought necessary here to adopt. (*Journal de l'Ecole Polytechnique*, Cahier xxx)

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to other points Q of the given curve. *Other* osculating *cones, cylinders, helix*, and *parabola;* this last being (pp. 562, 566) the parabola which *osculates to the projection of the curve, on its own osculating plane. Deviation* of *curve*, at any near point Q, from the osculating *circle* at P, decomposed (p. 566) into *two rectangular deviations*, from osculating *helix* and *parabola*. Additional formulæ (p. 576), for the general theory of *emanants* (Art. 396); case of *normally* emanant *lines*, or of *tangentially* emanant *planes*. General *auxiliary spherical curve* (pp. 576–578, comp. p. 515); new proof of the second expression (V') for tan H, and of the theorem that if this *ratio* of *curvatures* be constant, the proposed curve is a *geodetic* on a *cylinder:* new proof that if *each* curvature $(r^{-1}, \mathrm{r}^{-1})$ be *constant*, the cylinder is *right*, and therefore the curve a *helix*, ... 559–578

ARTICLE 398.—Properties of a curve in space, depending on the *fourth* and *fifth powers* (s^4, s^5) of its *arc* (s), ... 578–612

This *Series* 398 is so much longer than any other in the Volume, and is supposed to contain so much original matter, that it seems necessary here to subdivide the analysis under several separate heads, lettered as (*a*), (*b*), (*c*), &c.

(*a*). Neglecting s^5, we may write (p. 578, comp. Art. 396),

$$\mathrm{OP}_s = \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'' + \tfrac{1}{24}s^4\tau''' ; \qquad \text{(W)}$$

or (comp. p. 587), $\rho_s = \rho + x_s\tau + y_s r\tau' + z_s r\nu$, (W')

with expressions (p. 588) for the *coefficients* (or *co-ordinates*) x_s, y_s, z_s, in terms of r', r', r'', r, r', and s. If s^5 be taken into account, it becomes necessary to add to the expression (W) the term, $\tfrac{1}{120}s^5\tau^{\mathrm{IV}}$; with corresponding additions to the scalar coefficients in (W'), introducing r''' and r'': the laws for forming which additional terms, and for extending them to higher powers of the arc, are assigned in a subsequent Series (399, pp. 612, 617).

(*b*). Analogous expressions for τ''', ν'', κ'', λ', σ', and p', R', P', H', to serve in questions in which s^5 is neglected, are assigned (in p. 579); τ'' ν', κ', λ, σ, and p, R, P, H, having been previously expressed (in Series 397); while τ^{IV}, ν''', κ''', λ'', σ'', &c. enter into investigations which take account of s^5: the arc s being treated as the independent variable in all *these* derivations.

(*c*). One of the chief results of the present Series (398), is the introduction (p. 581, &c.) of a *new auxiliary angle*, J, *analogous* in several respects to the *known angle* H (397), but belonging to a *higher order* of theorems, respecting *curves in space:* because the *new* angle J depends on the *fourth* (and lower) powers of the arc s, while Lancret's angle H depends only on s^3 (including s^1 and s^2). In fact, while tan H is represented by the expressions (V'), whereof one is r'^{-1} tan P, tan J admits (with many transformations) of the following analogous expression (p. 581),

$$\tan J = R'^{-1} \tan P ; \qquad \text{(X)}$$

where R' depends* by (*b*) on s^4, while r' and P depend (397) on no higher power than s^3.

(*d*). To give a more distinct *geometrical meaning* to this new angle J, than can be easily gathered from such a formula as (X), respecting which it may be observed, in passing, that J is in general more simply defined by expressions for its *cotangent* (pp. 581, 588), than for its tangent, we are to conceive that, at each point P of any proposed curve of double curvature, there is drawn a *tangent plane* to the *sphere*, which *osculates* (395) to the curve at that point; and that then the *envelope* of all these *planes* is determined, which envelope (for reasons afterwards more fully explained) is called here (p. 581) the "*Circumscribed Developable*:" being a surface *analogous* to the "*Rectifying Developable*" of Lancret, but belonging (*c*) to a *higher order* of questions. And then, as the *known* angle H denotes (397) the *inclination*, suitably measured, of the *rectifying line* (λ), which is a *generatrix* of the *rectifying* developable, to the *tangent* (τ) to the *curve*; so the *new* angle J represents the inclination of a *generating line* (ϕ), of what has just been called the *circumscribed* developable, to the *same* tangent (τ), measured likewise in a defined direction (p. 581), but in the tangent plane to the sphere. It may be noted as another analogy (p. 582), that while H is a *right angle* for a *plane curve*, so J is *right* when the curve is *spherical*. For the *helix* (p. 585), the angles H and J are *equal*; and the rectifying and circumscribed *developables coincide*, with each other and with the *right cylinder*, on which the helix is a geodetic line.

(*e*). If the recent line ϕ be measured from the given point P, in a suitable direction (as contrasted with the opposite), and with a suitable length, it becomes what may be called (comp. 396) the *Vector of Rotation of the Tangent Plane* (*d*) *to the Osculating Sphere*; and then it satisfies, among others, the equations (pp. 579, 581, comp. (V)),

$$\phi = \mathrm{V}\frac{\nu''}{\nu'}, \quad \mathrm{T}\phi = R^{-1}\operatorname{cosec} J; \tag{X'}$$

this last being an expression for the *velocity of rotation* of the *plane* just mentioned, or of its *normal*, namely the spherical *radius* R, if the *given curve* be conceived to be described by a point moving with a *con-*

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[*] In other words, the calculation of r' and P introduces no differentials higher than the *third order*; but that of R' requires the *fourth order* of differentials. In the language of modern geometry, the *former* can be determined by the consideration of *four consecutive points of the curve*, or by that of *two consecutive osculating circles*; but the *latter* requires the consideration of *two* consecutive *osculating spheres*, and therefore of *five* consecutive *points* of the curve (supposed to be one of double curvature). *Other* investigations, in the present and immediately following Series (398, 399), especially those connected with what we shall shortly call the *Osculating Twisted Cubic*, will be found to involve the consideration of *six* consecutive points of a curve.

stant velocity, assumed = 1. And if we denote by V the point in which the *given radius* R or PS is *nearest* to a *consecutive radius* of the same kind, or to the radius of a *consecutive osculating sphere*, then this point V *divides the line* PS *internally*, into *segments* which may (ultimately) be thus expressed (pp. 580, 581),

$$\overline{\mathrm{PV}} = R \sin^2 J, \quad \overline{\mathrm{VS}} = R \cos^2 J. \qquad (\mathrm{X}'')$$

But these and other connected results, depending on s^4, have their known *analogues* (with H for J, and r for R), in that earlier theory (e) which introduces only s^3 (besides s^1 and s^2): and they are all *included* in the general theory of *emanant* lines and planes (396, 397), of which some new geometrical illustrations (pp. 582–584) are here given.

(f). New auxiliary scalar n $(= p^{-1}RR' = \cot J \sec P = \&\mathrm{c}.)$, = *velocity of centre* S of *osculating sphere*, if the velocity of the point P of the *given curve* be taken as *unity* (e); n vanishes with R', $\cot J$, and (comp. 395) the coefficient $S - 1$ $(= nr\mathrm{r}^{-1})$ of non-sphericity, for the case of a spherical curve (p. 584). Arcs, first and second curvatures, and rectifying planes and lines, of the cusp-edges of the polar and rectifying* developables; these can *all* be expressed without going beyond s^5, and *some* without using any higher power than s^4, or differentials of the orders corresponding; $r_1 = n\mathrm{r}$, and $\mathrm{r}_1 = nr$, are the *scalar radii* of first and second curvature of the *former* cusp-edge, r_1 being *positive* when *that* curve turns its *concavity* at S towards the *given* curve at P: determination of the point R, in which the *latter* cusp-edge is touched by the rectifying line λ to the original curve (pp. 584–587).

(g). Equation with one arbitrary constant (p. 587), of a *cone of the second order*, which has its *vertex* at the given point P, and has *contact* of the *third order* (or *four-side contact*) with the *cone of chords* (397) from that point; equation (p. 590) of a *cylinder* of the second order, which has an arbitrary line PB from P as one *side*, and has contact of the *fourth order* (or *five-point contact*) with the *curve* at P; the constant above mentioned can be so determined, that the right line PB shall be a *side* of the *cone also*, and therefore a *part* of the *intersection* of *cone* and *cylinder;* and then the *remaining* or *curvilinear part*, of the *complete intersection* of those two surfaces of the second order, is (by known principles) a *gauche curve of the third order*, or what is briefly called[*] a *Twisted Cubic*: and *this last curve*, in virtue of its construction above described, and whatever the assumed direction of the auxiliary line PE may be, has *contact of the fourth order* (or *five-point contact*) *with the given curve* of double curvature at P (pp. 587–590, comp. pp. 568, 572).

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* The *rectifying plane*, of the cusp-edge of the rectifying developable, is the plane of λ and r', of which the formula LIV'. in p. 587 is the equation; and the *rectifying line* RH, of the same cusp-edge, *intersects the absolute normal* PK to the given curve, or the *radius* (r) of first curvature, in the point H in which that radius is *nearest* (e) to a *consecutive radius* of the same kind. But this last theorem, which is here deduced by quaternions, had been previously arrived at by M. de Saint-Venant (comp. the Note to p. xv.), through an entirely different analysis, confirmed by geometrical considerations.

(h). Determination (p. 590) of the *constant* in the equation of the *cone* (g), so that *this* cone may have *contact* of the *fourth order* (or *five-side contact*) with the *cone of chords* from P; the cone thus found may be called the *Osculating Oblique Cone* (comp. 397), of the second order, to that cone of chords; and the coefficients of its equation involve only r, r, r', r', r'', r'', but *not* r''', although this last derivative is of no higher order than r'', since each depends only on s^3 (and lower powers), or introduces only *fifth differentials*. Again, the *cylinder* (g) will have contact of the *fifth order* (or *six-point contact*) with the given *curve* at P, if the line PE, which is by construction a *side* of that cylinder, and has hitherto had an *arbitrary direction*, be *now* obliged to be a side of a certain *cubic cone*, of which the equation (p. 590) involves as constants not only $r\mathrm{r}r'\mathrm{r}'r''\mathrm{r}''$, like that of the *osculating cone* just determined, but *also* r'''. The *two cones* last mentioned have the *tangent* (τ) to the given curve for a *common side*,[†] but they have also *three other* common sides, whereof *one* at least is *real*, since they are assigned by a *cubic equation* (same p. 590); and by taking *this side* for the line PE in (g), there results a *new cylinder* of the *second order*, which *cuts the osculating oblique cone*, *partly* in that *right line* PE itself, and *partly* in a *gauche curve* of the *third order*, which it is proposed to call an *Osculating Twisted Cubic* (comp. again (g)), because it has *contact of the fifth order* (or *six-point contact*) *with the given curve* at P (pp. 590, 591).

(i). *In general*, and independently of any question of *osculation*, a *Twisted Cubic* (g), if passing through the *origin* O, may be represented by any one of the *vector equations* (pp. 592, 593),

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[*] By Dr. Salmon, in his excellent Treatise on *Analytic Geometry of Three Dimensions* (Dublin, 1862), which is several times cited in the Notes to this final Chapter (III. iii.) of these *Elements*. The gauche curves, above mentioned, have been studied with much success, of late years, by M. Chasles, Sig. Cremona, and other geometers: but their existence, and some of their leading properties, appear to have been first perceived and published by Prof. Möbius (see his *Barycentric Calculus*, Leipzig, 1827, pp. 114–122, especially p. 117).

[†] *This side*, however, *counts as three* (p. 614), in the system of the *six lines of intersection* (real or imaginary) of these *two cones*, which have a *common vertex* P, and are respectively of the *second* and *third orders* (or degrees). Additional light will be thrown on this whole subject, in the following Series (399); in which also it will be shown that there is only *one* osculating twisted cubic, at a given point, to a given curve of double curvature; and that *this cubic curve can be determined, without resolving any cubic* or other *equation*.

$$\mathrm{V}\alpha\rho + \mathrm{V}\rho\phi\rho = 0, \quad (\mathrm{Y}); \quad \text{or} \quad (\phi + c)\,\rho = \alpha, \quad (\mathrm{Y}')$$

or $\rho = (\phi + c)^{-1}\alpha$, (Y''); or $\mathrm{V}\alpha\rho + \rho\mathrm{V}\gamma\rho + \mathrm{V}\rho\mathrm{V}\lambda\rho\mu = 0$, (Y''') in which $\alpha, \gamma, \lambda, \mu$ are *real* and *constant vectors*, but c is a *variable scalar*; while $\phi\rho$ denotes (comp. the Section III. ii. 6, or pp. xii., xiii.) a *linear and vector function*, which is *here* generally *not self-conjugate*, of the *variable vector* ρ of the *cubic curve*. The *number* of the *scalar constants*, in the form (Y'''), or in any other form of the equation, is found to be *ten* (p. 593), with the foregoing supposition that the curve passes *through the origin*, a restriction which it is easy to remove. The curve (Y) is *cut*, as it ought to be, in *three points* (real or imaginary), by an *arbitrary secant plane*; and its three *asymptotes* (real or imaginary) have the *directions* of the *three vector roots* β (see again the last cited Section) of the equation (same p. 593),

$$\mathrm{V}\beta\phi\beta = 0: \qquad (\mathrm{Z})$$

so that by (P), p. xii., *these three asymptotes compose a real and rectangular system*, for the *case of self-conjugation* of the function ϕ in (Y).

(*j*). *Deviation* of a near point P_s of the given curve, from the *sphere* (395) which *osculates* at the given point P; this deviation (by p. 593, comp. pp. 553, 584) is

$$\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}} = \frac{r_1 s^4}{24r\mathrm{T}^2 R} = \frac{R' s^4}{24r\mathrm{T}p} = \frac{n s^4}{24r\mathrm{T}R} = \&\text{c.}; \qquad (\mathrm{A}_1)$$

it is ultimately equal (p. 595) to the *quarter* of the deviation (397) of the same near point P_s from the *osculating circle* at P, multiplied by the *sine* of the *small angle* SPS_s, which the *small arc* SS_s of the *locus of* the *spheric centre* S (or of the *cusp-edge* of the polar developable) *subtends* at the same point P; and it has an *outward* or an *inward* direction, according as this *last arc* is *concave* or *convex* (*f*) at S, towards the *given curve* at P (pp. 585, 595). It is also ultimately equal (p. 596) to the deviation $\overline{\mathrm{PS}_s} - \overline{\mathrm{P}_s\mathrm{S}_s}$, of the *given point* P from the *near sphere*, which *osculates* at the *near point* P_s; and likewise (p. 597) to the *component*, in the direction of SP, of the deviation of that near point from the osculating *circle* at P, measured in a direction parallel to the normal plane at that point, if this *last* deviation be *now* expressed to the accuracy of the *fourth order*: whereas it has *hitherto* been considered sufficient to develope this *deviation from the osculating circle* (397) as far as the *third order* (or third dimension of s); and therefore to treat it as having a direction, *tangential to the osculating* sphere (comp. pp. 566, 594).

(*k*). The deviation (A_1) is also equal to the *third part* (p. 598) of the *deviation* of the *near point* P_s from the *given circle* (which osculates at P), if measured *in the near normal plane* (at P_s), and *decomposed* in the direction of the *radius* R_s of the *near sphere*; or to the *third part* (with direction preserved) of the deviation of the *new near point* in which the given circle is *cut* by the near plane, *from the near sphere*: or finally to the *third part* (as before, and still with an unchanged direction) of the **deviation from the** *given sphere*, of that *other near point* C, in which the *near circle* (osculating at P$_s$) is cut by the *given normal plane* (at P), and which is found to satisfy the equation,

$$\overline{\mathrm{SC}} = 3\overline{\mathrm{SP}_s} - 2\overline{\mathrm{SP}}. \tag{B$_1$}$$

Geometrical connexions (p. 599) between these various results (*j*) (*k*), illustrated by a diagram (Fig. 83).

(*l*). The *Surface*, which is the *Locus of the Osculating Circle* to a given curve in space, may be represented rigorously by the *vector expression* (p. 600),

$$\omega_{s,u} = \rho_s + r_s\tau_s \sin u + r_s^2\tau_s' \operatorname{vers} u; \tag{C$_1$}$$

in which s and u are two independent scalar variables, whereof s is (as before) the *arc* PP$_s$ *of the given curve*, but is *not now treated as small*: and u is the (small or large) *angle subtended at the centre* K$_s$ of the circle, by the *arc of that circle*, measured from its *point of osculation* P$_s$. But the *same superficial locus* (comp. 392) may be represented also by the *vector equation* (p. 611), involving *apparently* only one scalar variable (s),

$$\mathrm{V}\frac{2\tau_s}{\omega - \rho_s} + \nu_s = 0, \tag{D$_1$}$$

in which $\nu_s = \tau_s\tau_s'$, and $\omega = \omega_{s,u} =$ the vector of an arbitrary point of the surface. The general method (p. 501), of the Section III. iii. 3, shows that the *normal* to this *surface* (C$_1$), at any proposed point thereof, has the direction of $\omega_{s,u} - \sigma_s$; that is (p. 600), the direction of the *radius* of the *sphere*, which contains the *circle* through that point, and has the *same point of osculation* P$_s$ to the given curve. The *locus of the osculating circle* is therefore found, by this little calculation with quaternions, to be at the same time the *Envelope of the Osculating Sphere*, as was to be expected from geometrical considerations (comp. the Note to p. 600).

(*m*). The *curvilinear locus of the point* C in (*k*) is *one branch* of the *section of the surface* (*l*), made by the normal plane to the given curve at P; and if D be the projection of C on the tangent at P to this new curve, which tangent PD has a direction *perpendicular* to the radius PS or R of the osculating sphere at P (see again Fig. 83, in p. 599), while the ordinate DC is *parallel* to that radius, then (attending only to principal terms, pp. 598, 599) we have the expressions,

$$\mathrm{PD} = \frac{Rs^3}{6r^2\mathrm{r}}\mathrm{U}\tau(\sigma - \rho), \quad \mathrm{DC} = \frac{-ns^4}{8r\mathrm{r}R}\mathrm{U}(\sigma \acute{-} \rho), \tag{E$_1$}$$

and therefore ultimately (p. 600),

$$\frac{\mathrm{DC}^3}{\mathrm{PD}^4} = \frac{81}{32}\cdot\frac{n^3r^5\mathrm{r}(\sigma - \rho)}{R^8} = \text{const.}; \tag{F$_1$}$$

from which it follows that P is a *singular point* of the *section* here considered, but *not a cusp* of that section, although the *curvature* at P is *infinite*: the *ordinate* DC varying ultimately as the *power* with exponent $\frac{4}{3}$ of the *abscissa* PD. *Contrast* (pp. 600, 601), of this section, with that of the developable *Locus of Tangents*, made by the same normal plane at P to the given curve; the vectors analogous to PD and DC are in *this* case nearly equal to $-\frac{1}{2}s^2\tau'$ and $-\frac{1}{3}s^3\mathrm{r}^{-1}\nu$; so that the latter varies ultimately as the power $\frac{3}{2}$ of the former, and the point P *is* (as it is known to be) a *cusp* of this last section.

(*n*). A given *Curve* of double curvature is therefore *generally* a *Singular Line* (p. 601), although *not a cusp-edge*, upon that *Surface* (*l*), which is at once the *Locus* of its osculating *Circle*, and the *Envelope* of its osculating *Sphere*: and the *new developable surface* (*d*), as being *circumscribed to this superficial locus* (or *envelope*), so as to *touch it along this singular line* (p. 612), may naturally be called, as above, the *Circumscribed Developable* (p. 581).

(*o*). Additional light may be thrown on this whole theory of the *singular line* (*n*), by considering (pp. 601–611) a problem which was discussed by Monge, in two distinct Sections (xxii. xxvi.) of his well-known *Analyse* (comp. the Notes to pp. 602, 603, 609, 610 of these *Elements*); namely, to determine the *envelope of a sphere with varying radius R*, whereof the *centre* S traverses a *given curve in space*; or briefly, to find the *Envelope of a Sphere with One varying Parameter* (comp. p. 624): especially for the *Case of Coincidence* (p. 603, &c.), of what are *usually two distinct branches* (p. 602) of a certain *Characteristic Curve* (or *arête de rebroussement*), namely the *curvilinear envelope* (real or imaginary) of all the *circles*, along which the *superficial envelope* of the *spheres* is *touched* by those spheres themselves.

(*p*). Quaternion forms (pp. 603, 604) of the *condition of coincidence* (*o*); one of these can be at once translated into Monge's *equation of condition* (p. 603), or into an equation slightly more general, as leaving the independent variable arbitrary; but a simpler and more easily interpretable form is the following (p. 604),

$$r_1\,\mathrm{d}r = \pm R\,\mathrm{d}R, \qquad (\mathrm{G}_1)$$

in which *r* is the radius of the *circle of contact*, of a sphere with its envelope (*o*), while r_1 is the radius of (first) *curvature* of the *curve* (S), which is the *locus* of the *centre* S of the sphere.

(*q*). The *singular line* into which the *two branches* of the curvilinear envelope are *fused*, when this *condition* is satisfied, is in general an *orthogonal trajectory* (p. 607) to the *osculating planes* of the *curve* (S); *that* curve, which is *now* the *given* one, is therefore (comp. 391, 395) the *cusp-edge* (p. 607) of the *polar developable*, corresponding to the singular line just mentioned, or to what may be called the *curve* (P), which was *formerly* the given curve. In this way there arise many verifications of formulæ (pp. 607, 608); for example, the equation (G_1) is easily shown to be consistent with the results of (*f*).

(*r*). With the geometrical hints thus gained from interpretation of quaternion results, there is now no difficulty in assigning the *Complete and General Integral* of the *Equation of Condition* (*p*), which was presented by Monge under the form (comp. p. 603) of a *non-linear differential equation of the second order*, involving *three variables*

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(s, ψ, ϖ) considered as *functions of a fourth* (a), namely the *co-ordinates* of the *centre* of the *sphere*, regarded as *varying with the radius*, but which does not appear to have been either *integrated* or *interpreted* by that illustrious analyst. The general integral *here found* presents itself at first in a *quaternion form* (p. 609), but is easily *translated* (p. 610) into the usual language of analysis. A *less general integral* is also assigned, and its geometrical signification exhibited, as answering to a case for which the *singular line* lately considered reduces itself to a *singular point* (pp. 610, 611).

(*s*). Among the *verifications* (*q*) of this whole theory, it is shown (pp. 608, 609) that although, when the *two branches* (*o*) of the general *curvilinear envelope* of the *circles* of the system are *real* and *distinct*, *each branch* is a *cusp-edge* (or *arête de rebroussement*, as Monge perceived it to be), upon the *superficial envelope* of the *spheres*, yet in the *case of fusion* (*p*) this *cuspidal character* is *lost* (as was likewise seen by Monge[*]): and that *then* a *section of the surface*, made by a *normal plane* to the *singular line*, has precisely the *form* (*m*), expressed by the equation (F_1). In short, the result is in many ways confirmed, by calculation and by geometry, that when the *condition of coincidence* (*p*) is satisfied, the *Surface* is, as in (*n*), at once the *Envelope of the osculating Sphere* and the *Locus of the osculating Circle*, to that *Singular Line* on itself, into which by (*q*) the *two branches* (*o*) of its *general cusp-edge* are *fused*.

(*t*). Other applications of preceding formulæ might be given; for instance, the formula for κ'' enables us to assign general expressions (p. 611) for the *centre* and *radius* of the *circle*, which *osculates* at X to the *locus of the centre of the osculating circle*, to a given *curve in space*: with an elementary verification, for the case of the *plane evolute of the plane evolute of a plane curve*. But it is time to conclude this long analysis, which however could scarcely have been much abridged, of the results of Series 398, and to pass to a more brief account of the investigations in the following Series.

ARTICLE 399.—Additional general investigations, respecting that *gauche curve* of the *third order* (or degree), which has been above called an *Osculating Twisted Cubic* (398, (*h*)), to any proposed curve of double curvature; with applications to the case, where the given curve is a *helix*, ... 612–621

(*a*). In general (p. 614), the *tangent* PT to the given curve is a *nodal side* of the *cubic cone* 398, (*h*); *one tangent plane* to that cone (C_3), along that side, being the *osculating plane* (*P*) to the curve, and therefore touching also, along the same side, the *osculating oblique cone* (C_2) of the *second order*, to the *cone of chords* (397) from P; while the *other* tangent plane to the cubic cone (C_3) *crosses* that *first* plane (*P*), or the *quadric cone* (C_2), at an angle of which the trigonometric *cotan-*

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[*] Compare the first Note to p. 609 of these *Elements*.

gent ($\frac{1}{2}r'$) is equal to *half the differential of the radius* (r) *of second curvature, divided by the differential of the arc* (s). And the *three common sides*, PE, PE′, PE″, of these *two cones*, which remain when the *tangent* PT is *excluded*, and of which *one* at least must be *real*, are the *parallels* through the given point P to the *three asymptotes* (398, (*i*)) to the gauche curve sought; being also *sides* of *three quadric cylinders*, say (L_2), (L'_2), (L''_2), which contain those asymptotes as *other sides* (or generating lines): and of which *each contains the twisted cubic* sought, and is *cut in it* by the *quadric cone* (C_2).

(*b*). On applying this *First Method* to the case of a given *helix*, it is found (p. 614) that the *general cubic cone* (C_3) breaks up into the system of a *new quadric cone*, (C_2'), and a *new plane* (P'); which latter is the *rectifying plane* (396) of the helix, or the *tangent plane* at P to the *right cylinder*, whereon that given curve is traced. The two quadric cones, (C_2) and (C_2'), *touch each other* and the plane (P') along the tangent PT, and have *no other real common side:* whence *two* of the sought *asymptotes*, and *two* of the corresponding *cylinders* (*a*), are in this case *imaginary*, although they can still be used in *calculation* (pp. 614, 615, 617). But the *plane* (P') *cuts* the *cone* (C_2), not only in the tangent PT, but also in a *second real side* PE, to which the *real asymptote* is *parallel* (*a*); and which is at the same time a *side* of a *real quadric cylinder* (L_2), which has that asymptote for *another side* (p. 617), and *contains the twisted cubic :* this gauche curve being thus the *curvilinear part* (p. 615) of the *intersection* of the *real cone* (C_2), with the *real cylinder* (L_2).

(*c*). Transformations and verifications of this result; *fractional expressions* (p. 616), for the co-ordinates of the twisted cubic; expression (p. 615) for the *deviation* of the *helix* from that *osculating curve*, which deviation is directed *inwards*, and is of the *sixth order :* the *least* distance, between the *tangent* PT and the *real asymptote*, is a right line PB, which is *cut internally* (p. 617) by the *axis* of the *right cylinder* (*b*), in a point A such that PA is to AB as *three* to *seven*.

(*d*). The *First Method* (*a*), which had been established in the preceding Series (398), *succeeds* then for the case of the *helix*, with a facility which arises chiefly from the circumstance (*b*), that for *this case* the *general cubic cone* (C_3) breaks up into *two* separate loci, whereof *one* is a *plane* (P'). But *usually* the foregoing method requires, as in 398, (*h*)), the solution of a *cubic equation :* an inconvenience which is completely avoided, by the employment of a *Second General Method*, as follows.

(*e*). This *Second Method* consists in taking, for a *second locus* of the *gauche osculatrix* sought, a certain *Cubic Surface* (S_3), of which every point is the *vertex** of a quadric *cone*, having *six-point con-*

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* It is known that the *locus of the vertex* of a *quadric cone*, which passes through *six given points of space*, A, B, C, D, E, F, whereof no four are in one

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tact with the given curve at P: so that this *new surface* is cut by the *plane at infinity*, in the *same cubic curve* as the *cubic cone* (C_3). It is found (p. 620) to be a *Ruled Surface*, with the tangent PT for a *Singular Line*; and when this *right line* is *set aside*, the *remaining* (that is, the *curvilinear*) *part* of the intersection of the *two loci*, (C_2) and (S_3), is the *Osculating Twisted Cubic* sought: which *gauche osculatrix* is thus *completely* and *generally determined*, without any such difficulty or *apparent variety*, as might be supposed to attend the solution of a *cubic equation* (*d*), and with new verifications for the case of the *helix* (p. 621).

Article 400.—On Involutes and Evolutes in Space, ... 621–626

(*a*). The usual points of Monge's theory are deduced from the two fundamental quaternion equations (p. 621),

$$\mathrm{S}(\sigma-\rho)\rho'=0, \quad \mathrm{V}(\sigma-\rho)\sigma'=0, \qquad (\mathrm{H}_1)$$

in which ρ and σ are corresponding vectors of involute and evolute; together with a theorem of Prof. De Morgan (p. 622), respecting the case when the involute is a spherical curve.

(*b*). An *involute in space* is generally the *only real part* (p. 624) of the *envelope* of a certain variable *sphere* (comp. 398), which has its *centre* on the *evolute*, while its *radius* R is the variable *intercept* between the two curves: but because we have *here* the relation (p. 622, comp. p. 602),

$$R'^2+\sigma'^2=0, \qquad (\mathrm{H}_1')$$

the *circles of contact* (398, (*o*)) reduce themselves each to a *point* (or rather to a *pair of imaginary right lines*, intersecting in a real point), and the preceding theory (398), of envelopes of spheres with one varying parameter, undergoes important modifications in its results, the conditions of the application being different. In particular, the *involute* is indeed, as the equations (H_1) express, an *orthogonal trajectory* to the *tangents* of the *evolute*; but *not* to the *osculating planes*

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plane, is generally a *Surface*, say (S_4), of the *Fourth Degree*: in fact, it is *cut* by the plane of the triangle ABC in a system of *four right lines*, whereof three are the sides of that triangle, and the fourth is the intersection of the two planes, ABC and DEF. If then we investigate the intersection of this surface (S_4) with the *quadric cone*, (A.BCDEF), or say (C_2), which has A for vertex, and passes through the five other given points, we might expect to find (in *some sense*) a curve of the *eighth degree*. But when we set aside the *five right lines*, AB, AC, AD, AE, AF, which are *common* to the two surfaces here considered, we find that the (remaining or) *curvilinear part* of the *complete intersection* is reduced to a *curve of the third degree*, which is precisely the *twisted cubic through the six given points*. In applying this general (and perhaps new) method, to the problem of the *osculating* twisted cubic to a *curve*, the osculating *plane* to that curve may be *excluded*, as foreign to the question: and then the *quartic surface* (S_4) is reduced to the *cubic surface* (S_3), above described.

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of that curve, as the *singular line* (398, (q)) of the former envelope *was*, to those of the curve which was the *locus* of the *centres* of the *spheres* before considered, when a certain *condition of coincidence* (or of *fusion*, 398, (p)) was satisfied.

(c). *Curvature of hodograph of evolute* (p. 625); if P, P_1, P_2, .. and S, S_1, S_2, .. be *corresponding points* of involute and evolute, and if we draw right lines ST_1, ST_2, .. in the directions of S_1P_1, S_2P_2, .. and with a common length $= \overline{SP}$, the *spherical curve* PT_1T_2 .. will have *contact of the second order* at P, *with the involute* PP_1P_2 .. (pp. 625, 626).

Article 401.—Calculations abridged, by the treatment of *quaternion differentials* (which have hitherto been *finite*, comp. p. xi.) as *infinitesimals*;* new deductions of *osculating plane*, *circle*, and *sphere*, with the *vector equation* (392) of the circle; and of the first and second *curvatures* of a *curve in space*, ... 626–630

Section 7.—On Surfaces of the Second Order; and on Curvatures of Surfaces, ... 630–706

Article 402.—References to some equations of *Surfaces*, in earlier parts of the Volume, ... 630, 631

Article 403.—Quaternion equations of the *Sphere* ($\rho^2 = -1$, &c.), 631–633

In some of these equations, the notation N for *norm* is employed (comp. the Section II. i. 6).

Article 404.—Quaternion equations of the *Ellipsoid*, ... 633–635

One of the simplest of these forms is (pp. 307, 635) the equation,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad (I_1)$$

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* Although, for the sake of brevity, and even of clearness, some *phrases* have been used in the foregoing analysis of the Series 398 and 399, such as *four-side* or *five-side contact* between *cones*, and *five-point* or *six-point* contact between *curves*, or between a *curve* and a *surface*, which are borrowed from the doctrine of *consecutive points* and *lines*, and therefore from that of *infinitesimals*; with a few other expressions of modern geometry, such as the *plane at infinity*, &c.; yet the *reasonings* in the *text* of these *Elements* have all been rigorously *reduced*, so far, or are all obviously reducible, to the fundamental conception of *Limits*: compare the *definitions* of the *osculating circle* and *sphere*, assigned in Articles 389, 395. The object of Art. 401 is to make it visible how, without abandoning such *ultimate reference to limits*, it is possible to *abridge calculation*, in several cases, by treating (at this stage) the *differential symbols*, $d\rho$, $d^2\rho$, &c., *as if they represented infinitely small differences*, $\Delta\rho$, $\Delta^2\rho$, &c.; without taking the trouble to *write* these latter symbols *first*, as denoting *finite differences*, in the *rigorous statement* of a problem, of which statement it is not always easy to assign the proper form, for the case of points, &c., at *finite distances*: and *then* having the additional trouble of *reducing* the complex expressions so found to *simpler forms*, in which *differentials* shall finally appear. In short, it is shown that in *Quaternions*, as in other parts of Analysis, the *rigour of limits* can be combined with the *facility of infinitesimals*.

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in which ι and κ are real and constant vectors, in the directions of the cyclic normals. This *form* (I_1) is intimately connected with, and indeed served to suggest, that *Construction of the Ellipsoid* (II. i. 13), by means of a *Diacentric Sphere* and a *Point* (p. 227, comp. Fig. 53, p. 226), which was among the *earliest geometrical results* of the Quaternions. The *three semiaxes*, a, b, c, are expressed (comp. p. 230) in terms of ι, κ as follows:

$$a = T\iota + T\kappa; \quad b = \frac{\kappa^2 - \iota^2}{T(\iota - \kappa)}; \quad c = T\iota - T\kappa; \qquad (I_1')$$

whence

$$ab^{-1}c = T(\iota - \kappa). \qquad (I_1'')$$

ARTICLE 405.—*General Central Surface* of the *Second Order* (or central quadric), $S\rho\phi\rho = f\rho = 1$, ... 636–638

ARTICLE 406.—*General Cone* of the *Second Order* (or quadric cone), $S\rho\phi\rho = f\rho = 0$, ... 638–643

ARTICLE 407.—*Bifocal Form* of the equation of a *central* but *non-conical surface* of the second order: with some quaternion formulæ, relating to *Confocal Surfaces*, ... 643–653

(*a*). The *bifocal form* here adopted (comp. the Section III. ii. 6) is the equation,

$$Cf\rho = (S\alpha\rho)^2 - 2eS\alpha\rho S\alpha'\rho + (S\alpha'\rho)^2 + (1 - e^2)\rho^2 = C, \qquad (J_1)$$

in which,

$$C = (e^2 - 1)(e + S\alpha\alpha')l^2. \qquad (J_1')$$

α, α' are two (real) *focal unit-lines*, common to the whole *system* of confocals; the (real and positive) scalar l is also *constant* for that system: but the scalar e *varies*, in passing from surface to surface, and may be regarded as a *parameter*, of which the value serves to *distinguish* one confocal, say (e), from another (pp. 643, 644).

(*b*). The *squares* (p. 644) of the three *scalar semiaxes* (real or imaginary), arranged in algebraically descending order, are,

$$a^2 = (e + 1)l^2, \quad b^2 = (e + S\alpha\alpha')l^2, \quad c^2 = (e - 1)l^2; \qquad (K_1)$$

whence

$$l^2 = \frac{a^2 - c^2}{2}, \quad e = \frac{a^2 + c^2}{a^2 - c^2}; \qquad (L_1)$$

and the three *vector semiaxes* corresponding are,

$$aU(\alpha + \alpha'), \quad bUV\alpha\alpha', \quad cU(\alpha - \alpha'). \qquad (M_1)$$

(*c*). *Rectangular*, *unifocal*, and *cyclic forms* (pp. 644, 648, 650), of the scalar function $f\rho$, to each of which corresponds a form of the vector function $\phi\rho$; deduction, by a new analysis, of several known theorems* (pp. 644, 645, 648, 652, 653) respecting *confocal surfaces*, and their *focal conics;* the lines α, α' are *asymptotes* to the *focal hyperbola* (p. 647), whatever the *species* of the surface may be: references (in Notes to pp. 648, 649) to the *Lectures*,* for the *focal ellipse* of the *Ellipsoid*, and for several different *generations* of this last surface.

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* For example, it is proved by quaternions (pp. 652, 653), that the *focal lines* of the *focal cone*, which has any proposed point P for vertex, and rests on the focal hyperbola, are *generating lines* of the *single-sheeted hyperboloid* (of the given confocal system), which passes through that point: and an *extension* of this result, to the focal lines of *any cone circumscribed to a confocal*, is deduced by a similar analysis, in a subsequent Series (408, p. 656). But such known theorems respecting confocals can only be alluded to, in these Contents.

(*d*). *General Exponential Transformation* (p. 651), of the equation of *any central quadric;*

$$\rho = x\alpha + y\mathrm{V}\alpha^t\beta, \quad (\mathrm{N}_1), \quad \text{with } x^2 f\alpha + y^2 f\mathrm{UV}\alpha\alpha' = 1, \qquad (\mathrm{N}_1')$$

and

$$\beta = \frac{(\alpha' - e\alpha)\,\mathrm{UV}\alpha\alpha'}{e + \mathrm{S}\alpha\alpha'}; \qquad (\mathrm{N}_1'')$$

this *auxiliary vector* β is *constant*, for any *one* confocal (e); the *exponent*, t, in (N_1), is an *arbitrary* or *variable scalar;* and the *coefficients*, x and y, are *two other* scalar variables, which are however *connected* with each other by the relation (N_1').

(*e*). If any *fixed value* be assigned to t, the equation (N_1) then represents the *section* made by a plane through α (p. 651), which section is an *ellipse* if the surface be an *ellipsoid*, but an *hyperbola* for *either hyperboloid;* and the *cutting plane* makes with the *focal plane* of α, α', or with the plane of the focal hyperbola, an angle $= \frac{1}{2}t\pi$.

(*f*). If, on the other hand, we allow t to *vary*, but assign to x and y any *constant values* consistent with (N_1'), the equation (N_1) then represents an *ellipse* (p. 651), whatever the *species* of the *surface* may be; x represents the *distance of its centre* from the *centre* o of the *surface*, measured along the *focal line* α; y is the *radius* of a *right cylinder*, with α for its *axis*, of which the ellipse is a *section*, or the radius of a *circle* in a plane perpendicular to α, into which that ellipse can be orthogonally *projected:* and the angle $\frac{1}{2}t\pi$ is now the *excentric anomaly*. *Such elliptic sections* of a central quadric may be otherwise obtained from the *unifocal form* (*c*) of the equation of the surface; they are, in some points of view, almost as interesting as the known *circular sections:* and it is proposed (p. 649) to call them *Centro-Focal Ellipses.*

(*g*). And it is obvious that, by interchanging the *two focal lines* α, α' in (*d*), a *Second Exponential Transformation* is obtained, with a *Second System* of *centro-focal ellipses*, whereof the proposed *surface* is the *locus*, as well as of the *first system* (*f*), but which have their *centres* on the line α', and are projected into *circles*, on a plane perpendicular to this *latter line* (p. 649).

(*h*). *Equation of Confocals* (p. 652),

$$\mathrm{V}\nu_{,}\phi\nu_{,} = \mathrm{V}\nu\phi_{,}\nu. \qquad (\mathrm{O}_1)$$

Article 408.—On Circumscribed Quadric Cones; and on the Umbilics of a central quadric, ... 653-663

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* *Lectures on Quaternions* (by the present author), Dublin, Hodges and Smith, 1853.

(*a*). Equations (p. 653) of *Conjugate Points*, and of *Conjugate Directions*, with respect to the surface $f\rho = 1$,

$$f(\rho, \rho') = 1, \quad (P_1), \text{ and } f(\rho, \rho') = 0; \qquad (P_1')$$

Condition of Contact, of the same surface with the *right line* PP',

$$(f(\rho, \rho') - 1)^2 = (f\rho - 1)\,(f\rho' - 1); \qquad (Q_1)$$

this latter is also a form of the equation of the *Cone*, with vertex at P', which is *circumscribed* to the same quadric ($f\rho = 1$).

(*b*). The condition (Q_1) may also be thus transformed (p. 654),

$$F\mathrm{V}\rho\rho' = a^2b^2c^2 f(\rho - \rho'), \qquad (Q_1')$$

F being a scalar function, connected with *f* by certain relations of *reciprocity* (comp. p. 483); and a simple *geometrical interpretation* may be assigned, for this last equation.

(*c*). The *Reciprocal Cone*, or *Cone of Normals* σ at P' to the circumscribed cone (Q_1) or (Q_1'), may be represented (p. 655) by the very simple equation,

$$F(\sigma : \mathrm{S}\rho'\sigma) = 1; \qquad (Q_1'')$$

which likewise admits of an extremely simple interpretation.

(*d*). A given *right line* (p. 656) is *touched* by *two confocals*, and other known results are easy consequences of the present analysis; for example (pp. 658, 659), the cone circumscribed to any surface of the system, from any point of *either* of the *two real focal curves*, is a *cone of revolution* (real or imaginary): but a similar conclusion holds good, when the *vertex* is on the *third* (or *imaginary*) *focal*, and even more generally (p. 663), when that vertex is *any point* of the (known and imaginary) *developable envelope* of the *confocal system*.

(*e*). A central quadric has in general *Twelve Umbilics* (p. 659), whereof only *four* (at most) can be *real*, and which are its *intersections* with the *three focal curves*: and these *twelve points* are ranged, *three by three*, on *eight imaginary right lines* (p. 662), which *intersect the circle at infinity*, and which it is proposed to call the *Eight Umbilicar Generatrices* of the surface.

(*f*). These (imaginary) *umbilicar generatrices* of a quadric are found to possess several interesting properties, especially in relation to the *lines of curvature*: and their *locus*, for a *confocal system*, is a *developable surface* (p. 663), namely the known *envelope* (*d*) of that system.

ARTICLE 409.—Geodetic Lines on Central Surfaces of the Second Order, ... 664–667

(*a*). One form of the *general* differential *equation of geodetics* on an *arbitrary surface* being, by III. iii. 5 (p. 515),

$$\mathrm{V}\nu d^2\rho = 0, \qquad (R_1), \quad \text{if} \quad \mathrm{T}d\rho = \text{const.}, \qquad (R_1')$$

this is shown (p. 664) to conduct, for central quadrics, to the first integral,

$$P^{-2}D^{-2} = \mathrm{T}\nu^2 f\,\mathrm{U}d\rho = h = \text{const.}; \qquad (S_1)$$

where *P* is the *perpendicular from the centre* O on the *tangent plane*, and D is the (real or imaginary) *semidiameter* of the surface, which is *parallel to the tangent* ($d\rho$) to the curve. The known equation of Joachimstal, $P.D = \text{const.}$, is therefore proved anew; this last *constant*, however, being by no means necessarily *real*, if the surface be *not* an *ellipsoid*.

(b). Deduction (p. 665) of a theorem of M. Chasles, that the *tangents to a geodetic*, on any one central quadric (e), *touch* also a *common confocal* ($e_{\prime}$); and of an integral (p. 666) of the form,

$$e_1 \sin^2 v_1 + e_2 \cos^2 v_1 = e_{\prime} = \text{const.}, \qquad (\text{S}_1{}')$$

which agrees with one of M. Liouville.

(c). Without the restriction ($\text{R}_1{}'$), the differential of the scalar h in (S_1) may be thus decomposed into factors (p. 666),

$$dh = d.P^{-2}D^{-2} = 2S\nu d\nu d\rho^{-1}.S\nu d\rho^{-1}d^2\rho; \qquad (\text{S}_1{}'')$$

but, by the lately cited Section (III. iii. 5, p. 515), the differential equation of the *second order*,

$$S\nu d\rho d^2\rho = 0, \qquad (\text{R}_1{}'')$$

with an arbitrary scalar variable, represents the *geodetic lines* on *any surface:* the theorem (a) is therefore in this way reproduced.

(d). But we see, at the same time, by ($\text{S}_1{}''$), that the quantity h, or $P.D = h^{-\frac{1}{2}}$, is *constant*, *not only* for the *geodetics* on a central quadric, but *also* for a certain *other set* of curves, determined by the differential equation of the *first order*, $S\nu d\nu d\rho = 0$, which will be seen, in the next Series, to represent the *lines of curvature*.

Article 410.—On Lines of Curvature generally; and in particular on such lines, for the case of a Central Quadric, ... 667-674

(a). The differential equation (comp. 409, (d)),

$$S\nu d\nu d\rho = 0, \qquad (\text{T}_1)$$

represents (p. 667) the *Lines of Curvature*, upon an *arbitrary surface;* because it is a *limiting form* of this other equation,

$$S\nu\Delta\nu\Delta\rho = 0, \qquad (\text{T}_1{}')$$

which is the *condition of intersection* (or of parallelism), of the *normals* drawn at the extremities of the two vectors ρ and $\rho + \Delta\rho$.

(b). The normal vector ν, in the equation (T_1), may be multiplied (pp. 673, 700) by any constant or variable scalar n, without any real change in that equation; but in this whole theory, of the treatment of *Curvatures of Surfaces* by Quaternions, it is advantageous to consider the expression $S\nu d\rho$ as denoting the *exact differential* of *some scalar function* of ρ; for then (by pp. 486, 487) we shall have an equation of the form,

$$d\nu = \phi d\rho = \text{a } \textit{self-conjugate function} \text{ of } d\rho, \qquad (\text{U}_1)$$

which usually involves ρ also. For instance, we may write generally (p. 669, comp. (R), p. xiii),

$$d\nu = g d\rho + V\lambda d\rho\mu; \qquad (\text{U}_1{}')$$

the scalar g, and the vectors λ, μ being *real*, and being *generally*[*] functions of ρ, but not involving $d\rho$.

(*c*). This being understood, the *two*[†] *directions* of the *tangent* $d\rho$, which satisfy at once the general equation (T_1) of the lines of curvature, and the differential equation $S\nu d\rho = 0$ of the surface, are easily found to be represented by the two vector expressions (p. 669),

$$UV\nu\lambda \pm UV\nu\mu; \qquad (T_1'')$$

they are therefore generally *rectangular* to each other, as they have long been known to be.

(*d*). The *surface itself* remaining still quite *arbitrary*, it is found useful to introduce the conception of an *Auxiliary Surface of the Second Order* (p. 670), of which the variable vector is $\rho + \rho'$, and the equation is,

$$S\rho'\phi\rho' = g\rho'^2 + S\lambda\rho'\mu\rho' = 1, \qquad (U_1'')$$

or more generally = const.; and it is proposed to call *this* surface, of which the *centre* is at the given point P, the *Index Surface*, partly because its *diametral section*, made by the *tangent plane* to the *given* surface at P, is a certain *Index Curve* (p. 668), which may be considered to coincide with the known "*indicatrice*" of Dupin.

(*e*). The expressions (T_1'') show (p. 670), that whatever the *given* surface may be, the *tangents* to the lines of curvature *bisect the angles* formed by the *traces* of the two *cyclic planes* of the *Index Surface* (*d*), on the *tangent plane* to the given surface; these two tangents have also (as was seen by Dupin) the directions of the *axes* of the *Index Curve* (p. 668); and they are distinguished (as he likewise saw) from all *other* tangents to the given surface, at the given point P, by the condition that *each* is *perpendicular to its own conjugate*, with respect to that indicating curve: the *equation* of such *conjugation*, of two tangents τ and τ', being in the present notation (see again p. 668),

$$S\tau\phi\tau' = 0, \quad \text{or } S\tau'\phi\tau = 0. \qquad (U_1''')$$

(*f*). New proof (p. 669) of another theorem of Dupin, namely that *if a developable be circumscribed* to *any surface*, along *any curve* thereon, its *generating lines* are everywhere *conjugate*, as tangents to the surface, to the corresponding *tangents to the curve*.

(*g*). Case of a central quadric; new proof (p. 671) of still another theorem of Dupin, namely that the *curve of orthogonal intersection* (p. 645), of two *confocal surfaces*, is a *line of curvature* on *each*.

(*h*). The system of the *eight umbilicar generatrices* (408, (*e*)), of a central quadric, is the *imaginary envelope* of the *lines of curvature* on that surface (p. 671); and *each* such *generatrix* is *itself* an *imaginary*

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[*] For the case of a *central quadric*, g, λ, μ are *constants*.

[†] *Generally two*; but in some cases *more*. It will soon be seen, that *three lines of curvature* pass through an *umbilic* of a *quadric*.

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line of curvature thereon: so that *through each* of the *twelve umbilics* (see again 408, (*e*)) there pass *three lines of curvature* (comp. p. 677), whereof however only *one*, at most, can be real: namely *two generatrices*, and a *principal section* of the surface. These last results, which are perhaps new, will be illustrated, and otherwise proved, in the following Series (411).

ARTICLE 411.—Additional illustrations and confirmations of the foregoing theory, for the case of a *Central** *Quadric*; and especially of the theorem respecting the *Three Lines of Curvature through an Umbilic*, whereof *two* are always *imaginary* and *rectilinear*, ... 674–679

(*a*). The general equation of condition (T_1'), or $S\nu\Delta\nu\Delta\rho = 0$, for the intersection of two finitely distant normals, may be easily transformed for the case of a quadric, so as to express (p. 675), that *when the normals* at P and P′ *intersect* (or are parallel), the *chord* PP′ is *perpendicular to its own polar.*

(*b*). Under the same conditions, if the point P be given, the *locus* of the *chord* PP′ is *usually* (p. 676) a *quadric cone*, say (*C*); and therefore the locus of the *point* P′ is usually a *quartic curve*, with P for a *double point*, whereat *two branches* of the *curve cut each other* at *right angles*, and *touch* the *two lines* of curvature.

(*c*). If the point P be one of a *principal section* of the given surface, but *not an umbilic*, the *cone* (*C*) breaks up into a *pair of planes*, whereof *one*, say (*P*), is the plane of the section, and the *other*, (*P*′), is perpendicular thereto, and is *not tangential* to the surface; and thus the *quartic* (*b*) breaks up into a *pair of conics* through P, whereof one is the principal section itself, and the other is perpendicular to it.

(*d*). But *if* the given point P be an *umbilic*, the *second plane* (*P*′) becomes a *tangent plane* to the surface; and the *second conic* (*c*) breaks up, at the same time, into a *pair of imaginary*† *right lines*, namely the *two umbilicar generatrices* through P (pp. 676, 678, 679).

(*e*). It follows that the *normal* PN at a *real umbilic* P (of an ellipsoid, or a double-sheeted hyperboloid) is *not intersected by any other real normal*, except those which are in the *same principal section*; but that this real normal PN *is* intersected, in an *imaginary sense*, by *all the normals* P′N′, which are drawn at points P′ of *either* of the two *imaginary generatrices* through the real *umbilic* P; so that *each* of these imaginary *right lines* is seen anew to be a *line** *of curvature*, on the surface (comp. 410, (h)), because all the normals P′N′, at points of this line are situated in one *common* (*imaginary*) *normal plane* (p. 676): and as before, there are thus *three lines* of curvature *through an umbilic*.

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* Many, indeed most, of the results apply, without modification, to the case of the *Paraboloids*; and the rest can easily be adapted to this latter case, by the consideration of infinitely distant points. We shall therefore often, for conciseness, omit the term *central*, and simply speak of *quadrics*, or surfaces of the *second order*.

† It is well known that the *single-sheeted hyperboloid*, which (alone of central quadrics) has *real generating lines*, has at the same time *no real umbilics* (comp. pp. 661, 662).

(f). These geometrical results are in various ways deducible from calculation with quaternions; for example, a form of the equation of the lines of curvature on a quadric is seen (p. 677) to become an *identity* at an umbilic ($\nu \parallel \lambda$): while the *differential* of that equation breaks up into two *factors*, whereof *one* represents the tangent to the *principal section*, while the *other* ($S\lambda d^2\rho = 0$) assigns the directions of the *two generatrices*.

(g). The equation of the *cone* (C), which has already presented itself as a certain *locus of chords* (b), admits of many quaternion transformations; for instance (see p. 675), it may be written thus,

$$\frac{S\alpha\rho\Delta\rho}{S\alpha\Delta\rho} + \frac{S\alpha'\rho\Delta\rho}{S\alpha'\Delta\rho} = 0, \qquad (V_1)$$

ρ being the vector of the vertex P, and $\rho + \Delta\rho$ that of any other point P′ of the cone; while α, α' are still, as in 407, (a), two real *focal lines*, of which the *lengths* are *here arbitrary*, but of which the *directions* are *constant*, as before, for a whole *confocal system*.

(h). This cone (C), or (V_1), is also the *locus* (p. 678) of a system

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* It might be natural to suppose, from the known general theory (410, (c)) of the *two* rectangular directions, that *each* such generatrix PP′ is *crossed perpendicularly*, at every one of its *non-umbilicar points* P′, by a *second* (and *distinct*, although *imaginary*) *line of curvature*. But it is an almost equally well known and *received result* of modern geometry, paradoxical as it must at first appear, that *when a right line is directed to the circle at infinity*, as (by 408, (e)) the generatrices in question are, then *this imaginary line is everywhere perpendicular to itself*. Compare the Notes to pages 459, 672. Quaternions are not at all responsible for the *introduction* of this principle into geometry, but they *recognise* and *employ* it, under the following very simple form: that *if a non-evanescent vector be directed to the circle at infinity*, it is *an imaginary value of the symbol* $0^{\frac{1}{2}}$ (comp. pp. 300, 459, 662, 671, 672); and conversely, that *when this last symbol represents a vector which is not null*, the vector thus denoted is *an imaginary line*, which *cuts that circle*. It may be noted here, that such is the case with the *reciprocal polar* of *every chord* of a *quadric*, connecting *any two umbilics* which are *not in one principal plane*; and that thus the *quadratic equation* (XXI., in p. 669) from which the *two directions* (410, (c)) can *usually* be derived, becomes an *identity* for *every umbilic*, real *or* imaginary: as it ought to do, for consistency with the foregoing theory of the *three lines* through that umbilic. And as an additional illustration of the *coincidence* of directions of the lines of curvature at any *non-umbilicar point* P′ of an umbilicar generatrix, it may be added that the *cone of chords* (C), in 411, (b), is found to *touch the quadric along that generatrix*, when its *vertex* is at any *such* point P′.

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of *three rectangular lines;* and if it be cut by any plane perpendicular to a side, and not passing through the vertex, the *section* is an *equilateral hyperbola.*

(*i*). The same cone (C) has, for *three* of its *sides* PP′, the *normals* (p. 677) to the *three confocals* (p. 644) of a given system which pass through its vertex P; and therefore also, by 410, (g), the *tangents* to the *three lines of curvature* through that point, which are the *intersections* of those three confocals.

(*j*). And because its equation (V_1) does not involve the constant l, of 407, (a), (b), we arrive at the following theorem (p. 678):—*If indefinitely many quadrics, with a common centre* O, *have their asymptotic cones biconfocal, and pass through a common point* P, *their normals at that point have a quadric cone* (C) *for their locus.*

ARTICLE 412.—On Centres of Curvature of Surfaces, ... 679–689

(*a*). If σ be the vector of the *centre* S of *curvature* of a normal section of an *arbitrary surface,* which touches one of the two lines of curvature thereon, at any given point P, we have the two fundamental equations (p. 679),

$$\sigma = \rho + R\mathrm{U}\nu, \quad (\mathrm{W}_1), \quad \text{and} \quad R^{-1}\mathrm{d}\rho + \mathrm{dU}\nu = 0; \qquad (\mathrm{W}_1{}')$$

whence

$$\mathrm{V}\mathrm{d}\rho\mathrm{dU}\nu = 0, \quad (\mathrm{W}_1{}''), \quad \text{and} \quad \frac{\mathrm{T}\nu}{R} + \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = 0; \qquad (\mathrm{W}_1{}''')$$

the equation ($W_1{}''$) being a *new form* of the *general* differential equation of the *lines of curvature.*

(*b*). Deduction (pp. 680, 681, &c.) of some known theorems from these equations; and of some which introduce the new and general conception of the *Index Surface* (410, (d)), as well as that of the known *Index Curve.*

(*c*). Introducing the auxiliary scalar (p. 682),

$$r = \frac{\mathrm{T}\nu}{R} = -\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = -\mathrm{S}\tau^{-1}\phi\tau, \qquad (\mathrm{X}_1)$$

in which τ ($\parallel \mathrm{d}\rho$) is a tangent to a line of curvature, while $\mathrm{d}\nu = \phi\mathrm{d}\rho$, as in ($U_1$), the two values of r, which answer to the two rectangular directions ($T_1{}''$) in 410, (c), are given (p. 680) by the expression,

$$r = -g - \mathrm{T}\lambda\mu \,.\, \cos\left(\angle\frac{\nu}{\lambda} \mp \angle\frac{\nu}{\mu}\right), \qquad (\mathrm{X}'_1)$$

in which g, λ, μ are, for any given point P, the constants in the equation ($U_1{}''$) of the *index surface;* the *difference* of the *two curvatures* R^{-1} therefore *vanishes at an umbilic* of the *given surface,* whatever the *form* of that surface may be: that is, at a point, where $\nu \parallel \lambda$ or $\parallel \mu$, and where consequently the *index curve* is *a circle.*

(*d*). At any *other* point P of the given surface, which is as yet entirely *arbitrary,* the values of r may be thus expressed (p. 681),

$$r_1 = a_1{}^{-2}, \; r_2 = a_2{}^{-2}, \qquad (\mathrm{X}_1{}'')$$

a_1, a_2 being the *scalar semiaxes* (real or imaginary) of the *index curve* (*defined,* comp. 410, (d), by the equations $\mathrm{S}\rho'\phi\rho' = 1$, $\mathrm{S}\nu\rho' = 0$).

(*e*). The *quadratic equation*, of which r_1 and r_2, or the *inverse squares* of the two last *semiaxes*, are the *roots*, may be written (p. 683) under the *symbolical form*,

$$S\nu^{-1}(\phi + r)^{-1}\nu = 0; \qquad (Y_1)$$

which may be developed (same page) into this other form,

$$r^2 + rS\nu^{-1}\chi\nu + S\nu^{-1}\psi\nu = 0, \qquad (Y'_1)$$

the linear and vector functions, ψ and χ, being derived from the function ϕ, on the plan of the Section III. ii. 6 (pp. 440, 443).

(*f*). Hence, *generally*, the *product* of the *two curvatures* of a *surface* is expressed (same p. 683) by the formula,

$$R_1^{-1}R_2^{-1} = r_1 r_2 T\nu^{-2} = -S\frac{1}{\nu}\psi\frac{1}{\nu}; \qquad (Z_1)$$

which will be found useful in the following series (418), in connexion with the theory of the *Measure of Curvature*.

(*g*). The given surface being still quite general, if we write (p. 686),

$$\tau = Ud\rho, \quad \tau' = U(\nu d\rho), \ (A_2), \text{ and therefore } \tau\tau' = U\nu, \qquad (A'_2)$$

so that τ and τ' are *unit tangents* to the lines of curvature, it is easily proved that

$$d\tau' = \tau S\tau' d\tau, \ (B_2), \quad \text{or that} \quad V\tau d\tau' = 0; \qquad (B'_2)$$

this *general parallelism* of $d\tau'$ to τ being geometrically explained, by observing that a line of curvature *on any surface* is, at the same time, a line of curvature *on the developable normal surface*, which *rests upon that line*, and to which τ' or $\nu\tau$ is *normal*, if τ be *tangential* to the line.

(*h*). If the *vector of curvature* (389) of a line of curvature be *projected on the normal* ν to the given surface, the projection (p. 686) is the vector of curvature of the *normal section* of that surface, which has the same tangent τ; but this result, and an analogous one (same page) for the developable normal surface (*g*), are virtually included in Meusnier's theorem, which will be proved by quaternions in Series 414.

(*i*). The vector σ of a centre S of curvature of the given surface, answering to a given point P thereon, may (by (W_1) and (X_1)) be expressed by the equation,

$$\sigma = \rho + r^{-1}\nu; \qquad (C_2)$$

which may be regarded also as a *general form* of the *Vector Equation* of the *Surface of Centres*, or of the *locus* of the centre S: the variable vector ρ of the point P of the *given* surface being supposed (p. 501) to be expressed as a vector function of two independent and scalar variables, whereof therefore ν, r, and σ become also functions, although the two last involve an ambiguous *sign*, on account of the *Two Sheets* of the surface of centres.

(*j*). The *normal* at S, to what may be called the *First Sheet*, has the direction of the *tangent* τ to what may (on the same plan) be called the *First Line of Curvature* at P; and the vector υ of the point *corresponding* to s, on the corresponding *sheet* of the *Reciprocal* (comp. pp. 507, 508) *of the Surface of Centres*, has (by p. 684) the expression,

$$\upsilon = \tau (S\rho\tau)^{-1}; \qquad (D_2)$$

which may also be considered (comp. (i)) to be a form of the *Vector Equation* of that *Reciprocal Surface*.

(k). The vector υ satisfies generally (by same page) the *equations of reciprocity*,

$$S\upsilon\sigma = S\sigma\upsilon = 1, \quad S\upsilon\delta\sigma = 0, \quad S\sigma\delta\upsilon = 0, \qquad (D_2')$$

$\delta\sigma$, $\delta\upsilon$ denoting any infinitesimal variations of the vectors σ and υ, consistent with the equations of the surface of centres and its reciprocal, or any *linear* and *vector elements* of those two surfaces, at two corresponding points; we have also the relations (pp. 684, 685),

$$S\rho\upsilon = 1, \quad S\nu\upsilon = 0, \quad S\nu\upsilon\phi\upsilon = 0. \qquad (D_2'')$$

(l). The equation $S\upsilon(\omega - \rho) = 0$, or more simply,

$$S\upsilon\omega = 1, \qquad (E_2)$$

in which ω is a variable vector, represents (p. 684) the *normal plane* to the *first line* (j) of curvature at P; or the *tangent plane* at s to the *first sheet* of the surface of centres: or finally, the tangent plane to that *developable normal surface* (g), which *rests* upon the *second line* of curvature, and *touches the first sheet* along a certain *curve*, whereof we shall shortly meet with an example. And if υ be regarded, comp. (i), as a vector function of *two* scalar variables, the *envelope of the variable plane* (E_2) is a *sheet of the surface of centres*; or rather, on account of the ambiguous *sign* (i), it is that surface of centres *itself*: while, in like manner, the *reciprocal surface* (j) is the *envelope* of this *other plane*,

$$S\sigma\omega = 1. \qquad (E_2')$$

(m). The equations (W_1), (W_1') give (comp. the Note to p. 684),

$$d\sigma = dR.U\nu; \qquad (F_2)$$

combining which with (C_2), we see that the equations (H_1) of p. xxv. are satisfied, when the derived vectors ρ' and σ' are changed to the corresponding differentials, $d\rho$ and $d\sigma$. The known theorem (of Monge), that each *Line of Curvature* is generally an *involute*, with the corresponding *Curve of Centres* for one of its *evolutes* (400), is therefore in this way reproduced: and the connected theorem (also of Monge), that *this evolute* is a *geodetic on its own sheet* of the surface of centres, follows easily from what precedes.

(n). In the foregoing paragraphs of this analysis, the *given* surface has throughout been *arbitrary*, or *general*, as stated in (d) and (g). But if we now consider specially the *case* of a *central quadric*, several less general but interesting results arise, whereof *many*, but perhaps *not all*, are known; and of which some may be mentioned here.

(*o*). Supposing, then, that not only $d\nu = \phi d\rho$, but also $\nu = \phi\rho$, and $S\rho\nu = f\rho = 1$, the *Index Surface* (410, (*d*)) becomes simply (p. 670) the *given* surface, with its *centre transported* from O to P; whence many simplifications follow.

(*p*). For example, the *semiaxes* a_1, a_2 of the *index curve* are now equal (p. 681) to the semiaxes of the *diametral section* of the given surface, made by a plane parallel to the tangent plane; and $T\nu$ is, as in 409, the *reciprocal* P^{-1} of the *perpendicular*, from the centre on this latter plane; whence (by (X_1) and X_1'')) these known expressions for the two* curvatures result:

$$R_1^{-1} = P\mathrm{a}_1^{-2}; \quad R_2^{-1} = P\mathrm{a}_2^{-2}. \qquad (G_2)$$

(*q*). Hence, by (*e*), if a *new* surface be *derived* from a *given* central quadric (of *any species*), as the *locus of the extremities of normals*, erected at the centre, *to the planes of diametral sections* of the given surface, *each such normal* (when real) having the *length* of *one* of the *semiaxes* of that *section*, the *equation of this new surface*† admits (p. 683) of being written thus:

$$S\rho\,(\phi - \rho^{-2})^{-1}\rho = 0. \qquad (H_2)$$

(*r*). Under the conditions (*o*), the expression (C_2) for σ gives (p. 684) the two converse forms,

$$\sigma = r^{-1}(\phi + r)\,\rho, \quad (I_2), \qquad \rho = r\,(\phi + r)^{-1}\,\sigma; \qquad (I_2')$$

whence (pp. 684, 689),

$$\nu = r\,(\phi + r)^{-1}\,\phi\sigma, \quad (J_2), \qquad \sigma = (\phi^{-1} + r^{-1})\,\nu; \qquad (J_2')$$

and therefore (p. 689), by (*d*), (*p*), and by the theory (407) of confocal surfaces,

$$\sigma_1 = \phi_2^{-1}\nu = \phi_2^{-1}\phi\rho, \qquad (K_2)$$

if ϕ_2 be formed from ϕ by changing the semiaxes abc to $a_2b_2c_2$; it being understood that the given quadric (abc) is cut by the two confocals ($a_1b_1c_1$) and ($a_2b_2c_2$), in the *first* and *second lines* of curvature through the given point P: and that σ_1 is here the vector of that *first centre* S of curvature, which answers to the *first line* (comp. (*j*)). Of course, on the same plan, we have the analogous expression,

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* Throughout the present Series 412, we attend only (comp. (*a*)) to the curvatures of the *two normal sections* of a surface, which have the directions of the *two lines* of curvature: these being in fact what are always regarded as the *two principal curvatures* (or simply as *the* two curvatures) of the *surface*. But, in a shortly subsequent Series (414), the more general case will be considered, of the curvature of *any section*, normal or oblique.

† When the *given* surface is an *ellipsoid*, the *derived* surface is the celebrated *Wave Surface* of Fresnel: which thus has (H_2) for a *symbolical form* of its equation. When the given surface is an *hyperboloid*, and a *semiaxis* of a section is *imaginary*, the (scalar and now positive) *square*, of the (imaginary) *normal* erected, is still to be made equal to the *square* of that semiaxis.

$$\sigma_2 = \phi_1^{-1}\nu = \phi_1^{-1}\phi\rho, \qquad (\mathrm{K}_2')$$

for the vector of the *second centre*.

(*s*). These expressions for σ_1, σ_2 include (p. 689) a theorem of Dr. Salmon, namely that the *centres of curvature* of a given quadric at a given point are the *poles of the tangent plane*, with respect to the two confocals through that point; and *either* of them may be regarded, by admission of an ambiguous *sign* (comp. (*i*)), as a *new Vector Form*[*] of the *Equation of the Surface of Centres*, for the *case* (*o*) of a given *central quadric*.

(*t*). In connexion with the same expressions for σ_1, σ_2, it may be observed that if r_1, r_2 be the corresponding values of the auxiliary scalar r in (*c*), and if τ, τ' still denote the unit tangents (*g*) to the first and second lines of curvature, while abc, $a_1b_1c_1$, and $a_2b_2c_2$ retain their recent significations (*r*), then (comp. pp. 686, 687, see also p. 652),

$$r_1 = f\tau = f\mathrm{U}d\rho = (a^2 - a_2^2)^{-1} = \&c., \qquad (\mathrm{L}_2)$$

and

$$r_2 = f\tau' = f\mathrm{U}\nu d\rho = (a^2 - a_1^2)^{-1} = \&c.; \qquad (\mathrm{L}_2')$$

this association of r_1 and σ_1 with a_2, &c., and of r_2 and σ_2 with a_1, &c., arising from the circumstance that the *tangents* τ and τ' have respectively the directions of the *normals* ν_2 and ν_1, to the two confocal surfaces, $(a_2b_2c_2)$ and $(a_1b_1c_1)$.

(*u*). By the properties of such surfaces, the scalar here called r_2 is therefore *constant*, in the whole extent of a *first line* of curvature; and the same *constancy* of r_2, or the equation,

$$d f\mathrm{U}\nu d\rho = 0, \qquad (\mathrm{M}_2)$$

may in various ways be proved by quaternions (p. 687).

(*v*). Writing simply r and r' for r_1 and r_2, so that r' is constant, but r variable, for a *first line* of curvature, while conversely r is constant and r' variable for a *second line*, it is found (pp. 684, 685, 586), that the *scalar equation* of the *surface of centres* (*i*) may be regarded as the result of the elimination of r^{-1} between the *two* equations,

$$1 = \mathrm{S}.\sigma(1 + r^{-1}\phi)^{-2}\phi\sigma, \quad (\mathrm{N}_2), \quad \text{and} \quad 0 = \mathrm{S}.\sigma(1 + r^{-1}\phi)^{-3}\phi^2\sigma; \quad (\mathrm{N}_2')$$

whereof the latter is the *derivative* of the former, with respect to the scalar r^{-1}. It follows (comp. p. 688), that the *First Sheet* of the *Surface of Centres is touched by an Auxiliary Quadric* (N_2), *along a Quartic Curve* (N_2) (N_2'), which curve is the *Locus* of the *Centres of First Curvature*, for all the points of a *Line of Second Curvature;* the *same sheet* being also touched (see again p. 688), along the *same curve*, by the *developable normal surface* (*l*), which *rests* on the *same second line:* with permission to interchange the words, *first* and *second*, throughout the whole of this enunciation.

(*w*). The given surface being still a central quadric (*o*), the vectors ρ, σ, ν can be expressed as functions of v (comp. (*j*) (*k*) (*l*)), and conversely the latter can be expressed as a function of any one of the former; we have, for example, the reciprocal equations (p. 685),

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[*] Dr. Salmon's result, that this surface of centres is of the *twelfth degree*, may be easily deduced from this form.

$$\sigma = (1 + r^{-1}\phi)^2 \phi^{-1}\upsilon, \quad (\mathrm{O}_2), \quad \text{and} \quad \upsilon = (1 + r^{-1}\phi)^{-2} \phi\sigma; \qquad (\mathrm{O}_2')$$

from which last the formula (N_2) may be obtained anew, by observing (k) that $\mathrm{S}\sigma\upsilon = 1$. Hence also, by ($r$), we can infer the expressions,*

$$\rho = (\phi^{-1} + r^{-1})\,\upsilon = \phi_2{}^{-1}\,\upsilon, \ (\mathrm{P}_2), \text{ and } \upsilon = \phi_2\,\rho = \nu_2; \qquad (\mathrm{P}_2')$$

and in fact it is easy to see otherwise (comp. p. 645), that $\nu_2 \parallel \tau \parallel \upsilon$, and $\mathrm{S}\rho\nu_2 = 1 = \mathrm{S}\rho\upsilon$, whence $\nu_2 = \upsilon$ as before.

(x). More fully, the *two sheets* of the *reciprocal* (j) of the surface of centres may have their separate *vector equations* written thus,

$$\upsilon_1 = \phi_2\,\rho = \nu_2, \quad \upsilon_2 = \phi_1\rho = \nu_1; \qquad (\mathrm{P}_2'')$$

and the *scalar equation*† of this *reciprocal surface* itself, considered as including *both* sheets, may (by page 685) be thus written, the functions f and F being related as in 408, (b),

$$\upsilon^4 = (F\upsilon - 1)\,f\upsilon, \qquad (\mathrm{Q}_2)$$

with several equivalent forms; one way of obtaining this equation being the elimination of r between the two following (same p. 685):

$$F\upsilon + r^{-1}\,\upsilon^2 = 1, \ (\mathrm{Q}_2'); \ f\upsilon + r\upsilon^2 = 0. \qquad (\mathrm{Q}_2'')$$

(y). The two last equations may also be written thus, for the *first sheet* of the reciprocal surface,

$$F_2\,\upsilon_1 = 1, \ (\mathrm{R}_2), \text{ and } f\mathrm{U}\upsilon_1 = r, \qquad (\mathrm{R}_2')$$

in which (comp. pp. 685, 689),

$$F_2\upsilon = \mathrm{S}\,\upsilon\phi_2{}^{-1}\upsilon = \mathrm{S}\upsilon\,(\phi^{-1} + r^{-1})\,\upsilon; \qquad (\mathrm{R}_2'')$$

and accordingly (comp. pp. 483, 645), we have $F_2\nu_2 = F\nu = 1$, and $f\mathrm{U}\nu_2 = f\tau = r$.

(z). For a *line of second curvature* on the given surface, the scalar r is *constant*, as before; and then the two equations (Q_2'), (Q_2''), or (R_2), (R'_2), represent jointly (comp. the slightly different enunciation in p. 688) a certain *quartic curve*, in which the *quadric reciprocal* (R_2), of the *second confocal* $(a_2\,b_2\,c_2)$, *intersects the first sheet* (y) of the *Reciprocal Surface* (Q_2); this quartic curve, being at the same time the intersection of the *quadric surface* (Q_2') or (R_2), with the *quadric cone* (Q_2'') or (R_2'), which is *biconcyclic* with the *given quadric*, $f\rho = 1$.

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[*] The equation $\upsilon = \nu_2$, = the normal to the confocal $(a_2\,b_2\,c_2)$ at P, is not actually given in the text of Series 412; but it is easily deduced, as above, from the formulæ and methods of that Series.

[†] The equation (Q_2) is one of the *fourth degree;* and, when expanded by co-ordinates, it agrees perfectly with that which was first assigned by Dr. Booth (see a Note to p. 685), for the *Tangential Equation* of the *Surface of Centres* of a quadric, or for the Cartesian equation of the *Reciprocal Surface.*

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ARTICLE 413.—On the Measure of Curvature of a Surface, . . 689–693.

The object of this short Series 413 is the deduction by quaternions, somewhat more briefly and perhaps more clearly than in the *Lectures*, of the principal results of Gauss (comp. Note to p. 690), respecting the *Measure of Curvature of a Surface*, and questions therewith connected.

(*a*). Let P, P$_1$, P$_2$ be *any three near points* on a given but *arbitrary surface*, and R, R$_1$, R$_2$ the three *corresponding points* (near to *each other*) on the *unit sphere*, which are determined by the *parallelism* of the *radii* OR, OR$_1$, OR$_2$ to the *normals* PN, P$_1$N$_1$, P$_2$ N$_2$; then the areas of the *two small triangles* thus formed will bear to each other the *ultimate ratio* p. 690),

$$\text{lim.}\ \frac{\Delta \mathrm{RR_1R_2}}{\Delta \mathrm{PP_1P_2}} = \frac{\mathrm{V.dU}\nu\delta\mathrm{U}\nu}{\mathrm{Vd}\rho\delta\rho} = -\mathrm{S}\frac{1}{\nu}\psi\frac{1}{\nu}; \qquad (\mathrm{S}_2)$$

whence, with Gauss's *definition* of the *measure of curvature*, as the *ultimate ratio of corresponding areas* on surface and sphere, we have, by the formula (Z$_1$) in 412, (*f*), his *fundamental theorem*,

$$\textit{Measure of Curvature} = R_1^{-1}R_2^{-1}, \qquad (\mathrm{S}_2')$$

= *Product of the two Principal Curvatures of Sections.*

(*b*). If the vector ρ of the surface be considered as a function of two scalar variables, *t* and *u*, and if derivations with respect to these be denoted by upper and lower accents, this general transformation results (p. 691),

$$\textit{Measure of Curvature} = \mathrm{S}\frac{\rho''}{\nu}\,\mathrm{S}\frac{\rho_{,,}}{\nu} - \left(\mathrm{S}\frac{\rho'_{,}}{\nu}\right)^2, \qquad (\mathrm{T}_2)$$

in which $\nu = \mathrm{V}\rho'\rho_{,}$; (T_2')

with a verification for the notation *pqrst* of Monge.

(*c*). The square of a linear element d*s*, of the given but arbitrary surface, may be expressed (p. 691) as follows:

$$\mathrm{d}s^2 = (\mathrm{Td}\rho^2 =)\, e\mathrm{d}t^2 + 2f\mathrm{d}t\mathrm{d}u + g\mathrm{d}u^2; \qquad (\mathrm{U}_2)$$

and with the recent use (*b*) of accents, the *measure* (T$_2$) is proved (same page) to be an explicit function of the ten scalars,

$$e, f, g;\quad e', f', g';\quad e_{,}, f_{,}, g_{,};\quad \text{and}\quad e_{,,} - 2f'_{,} + g''; \qquad (\mathrm{U}_2')$$

the form of this function (p. 692) agreeing, in all its details, with the corresponding expression assigned by Gauss.*

(*d*). Hence follow at once (p. 692) two of the most important results of that great mathematician on this subject; namely, that *every Deformation of a Surface*, consistent with the conception of it as an *infinitely thin* and *flexible* but *inextensible solid*, leaves *unaltered*,

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* References are given, in Notes to pp. 690, &c. of the present Series 413, to the pages of Gauss's beautiful Memoir, "*Disquisitiones generales circa Superficies Curvas*," as reprinted in the Additions to Liouville's Monge.

Ist, the *Measure of Curvature at any Point*, and IInd, the *Total Curvature of any Area*: this last being the *area* of the *corresponding portion* (s) of the *unit-sphere*.

(e). By a suitable choice of t and u, as certain *geodetic co-ordinates*, the expression (U_2) may be reduced (p. 692) to the following,

$$ds^2 = dt^2 + n^2 du^2 ; \qquad (U_2'')$$

where t is the *length* of a geodetic arc AP, from a fixed point A to a variable point P of the surface, and u is the *angle* BAP which this variable arc makes with a fixed geodetic AB: so that in the immediate neighbourhood of A, we have $n = t$, and $n' = D_t n = 1$.

(*f*). The general expression (*c*) for the *measure of curvature* takes thus the very simple form (p. 692),

$$R_1^{-1} R_2^{-1} = -n^{-1} n'' = -n^{-1} D_t^2 n ; \qquad (V_2)$$

and we have (comp. (*d*)) the equation (p. 693),

$$\textit{Total Curvature of Area } \mathrm{APQ} = \Delta u - \int n' du ; \qquad (V_2')$$

this *area* being bounded by *two geodetics*, AP and AQ, which make with each other an angle $= \Delta u$, and by an *arc* PQ of an *arbitrary curve* on the given surface, for which t, and therefore n', may be conceived to be a given function of u.

(*g*). If this arc PQ be *itself* a *geodetic*, and if we denote by v the variable angle which it makes at P with AP prolonged, so that $\tan v = n du : dt$, it is found that $dv = -n' du$; and thus the equation (V_2') conducts (p. 693) to another very remarkable and general theorem of Gauss, for an *arbitrary surface*, which may be thus expressed,

$$\textit{Total Curvature of a Geodetic Triangle } \mathrm{ABC} = \mathrm{A} + \mathrm{B} + \mathrm{C} - \pi, \qquad (V_2'')$$

= what may be called the *Spheroidal Excess* of that triangle, the *total area* (4π) of the *unit-sphere* being represented by *eight right angles*: with *extensions* to *Geodetic Polygons*, and *modifications* for the case of what may on the same plan be called the *Spheroidal Defect*, when the *two curvatures* of the surface are *oppositely directed*.

ARTICLE 414.—On Curvatures of Sections (Normal and Oblique) of Surfaces; and on Geodetic Curvatures, ... 694–698

(*a*). The curvatures considered in the two preceding Series having been those of the *principal normal sections* of a surface, the present Series 414 treats briefly the more general case, where the section is made by an *arbitrary plane*, such as the *osculating plane* at P to an *arbitrary curve* upon the surface.

(*b*). The *vector of curvature* (389) of any such curve or section being $(\rho - \kappa)^{-1} = D_s^2 \rho$, its *normal* and *tangential components* are found to be (p. 694),

$$(\rho - \sigma)^{-1} = \nu^{-1} S \frac{d\nu}{d\rho} = (\rho - \sigma_1)^{-1} \cos^2 v + (\rho - \sigma_2)^{-1} \sin^2 v, \qquad (W_2)$$

and

$$(\rho - \xi)^{-1} = \nu^{-1} d\rho^{-1} S \nu d\rho^{-1} d^2 \rho ; \qquad (W_2')$$

the former component being the *Vector of Normal Curvature* of the

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Surface, for the direction of the *tangent* to the curve: and the latter being the *Vector of Geodetic Curvature* of the same *Curve* (or section).

(*c*). In the foregoing expressions, σ and ξ are the vectors of the points S and X, in which the *axis* of the *osculating circle* to the *curve* intersects respectively the *normal* and the *tangent plane* to the *surface* (p. 694); S is also the *centre* of the *sphere*, which *osculates to the surface* in the direction $d\rho$ of the tangent; σ_1, σ_2 are the vectors of the *two centres* S_1, S_2, *of curvature of the surface*, considered in Series 412, which are at the same time the centres of the *two osculating spheres*, of which the curvatures are (algebraically) the *greatest* and *least*: and v is the *angle* at which the curve here considered crosses the *first line of curvature*.

(*d*). The equation (W_2) contains a theorem of Euler, under the form (p. 695),

$$R^{-1} = R_1^{-1}\cos^2 v + R_2^{-1}\sin^2 v; \qquad (W_2'')$$

it contains also Meusnier's theorem (same page), under the form (comp. 412, (*h*)) that the *vector of normal curvature* (*b*) of a *surface*, for any given direction, is the *projection on the normal* ν, of the *vector of oblique curvature*, whatever the *inclination* of the plane of the section to the tangent plane may be.

(*e*). The expression (W_2'), for the *vector of geodetic curvature*, admits (p. 697) of various transformations, with corresponding expressions for the *radius* $T(\rho - \xi)$ of geodetic curvature, which is also the *radius of plane curvature* of the *developed curve*, when the developable circumscribed to the given surface along the given curve is *unfolded* into a plane: and when this radius is *constant*, so that the developed curve is a *circle*, or part of one, it is proposed (p. 698) to call the given curve a *Didonia* (as in the *Lectures*), from its possession of a certain *isoperimetrical property*, which was first considered by M. Delaunay, and is represented in quaternions by the formula (p. 697),

$$\delta\int S(U\nu.d\rho\delta\rho) + c\delta\int Td\rho = 0; \qquad (X_2)$$

or

$$\sigma^{-1}d\rho = V(U\nu.dUd\rho), \qquad (X'_2)$$

by the rules of what may be called the *Calculus of Variations in Quaternions*: *c* being a constant, which represents generally (p. 698) the *radius* of the developed circle, and becomes *infinite* for *geodetic lines*, which are thus included as a *case of Didonias*.

ARTICLE 415.—Supplementary Remarks, ... 698–706

(*a*). Simplified proof (referred to in a Note to p. xii), of the general existence of a system of *three real and rectangular directions*, which satisfy the vector equation $V\rho\phi\rho = 0$, (P), when ϕ is a linear, vector, and *self-conjugate* function; and of a system of *three real roots* of the cubic equation $M = 0$ (p. xii), under the same condition (pp. 698–700).

(*b*). It may happen (p. 701) that the *differential equation*,

$$S\nu d\rho = 0, \qquad (Y_2)$$

is *integrable*, or represents a *system of surfaces*, *without* the *expression* Sνdρ being an *exact differential*, as it was in 410, (*b*). In this case, there exists some scalar *factor*, *n*, such that S$n\nu$dρ *is* the exact differential of a scalar function of ρ, without the assumption that this vector ρ is *itself* a *function* of a *scalar variable*, *t*; and then if we write (pp. 701, 702, comp. p. xxx),

$$\mathrm{d}\nu = \phi\mathrm{d}\rho, \quad \mathrm{d}.n\nu = \varphi\mathrm{d}\rho, \qquad (\mathrm{Y}_2{}')$$

this *new vector function* φ will be *self-conjugate*, although the function ϕ is *not* such *now*, as it *was* in the equation (U_1).

(*c*). In this manner it is found (p. 702), that the *Condition** *of Integrability* of the equation (Y_2) is expressed by the very simple formula,

$$\mathrm{S}\gamma\nu = 0; \qquad (\mathrm{Y}_2{}'')$$

in which γ is a *vector function* of ρ, *not* generally *linear*, and deduced from ϕ on the plan of the Section III. ii. 6 (p. 442), by the relation,

$$\phi\mathrm{d}\rho - \phi'\mathrm{d}\rho = 2\mathrm{V}\gamma\mathrm{d}\rho; \qquad (\mathrm{Y}_2{}''')$$

ϕ' being the *conjugate* of ϕ, but *not here equal* to it.

(*d*). Connexions (pp. 702, 703) of the *Mixed Transformations* in the last cited Section, with the known *Modular and Umbilicar Generations* of a surface of the second order.

(*e*). The equation (p. 704),

$$\mathrm{T}(\rho - \mathrm{V}.\beta\mathrm{V}\gamma\alpha) = \mathrm{T}(\alpha - \mathrm{V}.\gamma\mathrm{V}\beta\rho), \qquad (\mathrm{Z}_2)$$

in which α, β, γ are *any three vector constants*, represents a *central quadric*, and appears to offer a *new mode of generation*† of such a surface, on which there is not room to enter, at this late stage of the work.

(*f*). The vector of the *centre* of the quadric, represented by the equation $f\rho - 2\mathrm{S}\epsilon\rho =$ const., with $f\rho = \mathrm{S}\rho\phi\rho$, is generally $\kappa = \phi^{-1}\epsilon = m^{-1}\psi\epsilon$ (p. 704); case of *paraboloids*, and of *cylinders*.

(*g*). The equation (p. 705),

$$\mathrm{S}q\rho q'\rho q''\rho + \mathrm{S}\rho\phi\rho + \mathrm{S}\gamma\rho + C = 0, \qquad (\mathrm{Z}_2{}')$$

represents the *general surface* of the *third degree*, or briefly the *General Cubic Surface*; C being a constant scalar, γ a constant vector, and q, q', q'' three constant quaternions, while $\phi\rho$ is here again a linear, vector, and self-conjugate function of ρ.

(*h*). The *General Cubic Cone*, with its vertex at the origin, is thus represented in quaternions by the monomial equation (same page),

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* It is shown, in a Note to p. 702, that this *monomial equation* (Y''_2) becomes, when expanded, the known *equation of six terms*, which expresses the condition of integrability of the differential equation $p\mathrm{d}x + q\mathrm{d}y + r\mathrm{d}z = 0$.

† In a Note to p. 649 (already mentioned in p. xxviii), the reader will find references to the *Lectures*, for several different *generations of the ellipsoid*, derived from quaternion forms of its equation.

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$$S q\rho q'\rho q''\rho = 0. \qquad (Z_2'')$$

(*i*). *Screw Surface*, *Screw Sections* (p. 705); *Skew Centre* of *Skew Arch*, with illustration by a diagram (Fig. 85, p. 706).

Section 8.—On a few Specimens of Physical Applications of Quaternions, with some Concluding Remarks, 707 to the end.

Article 416.—On the Statics of a Rigid Body, ... 707–709

(*a*). *Equation of Equilibrium*,

$$V\gamma\Sigma\beta = \Sigma V\alpha\beta; \qquad (A_3)$$

each α is a *vector of application*; β the corresponding *vector of applied force*; γ an *arbitrary* vector: and this *one* quaternion formula (A_3) is equivalent to the system of the *six* usual scalar equations ($X = 0$, $Y = 0$, $Z = 0$, $L = 0$, $M = 0$, $N = 0$).

(*b*.) When $S(\Sigma\beta . \Sigma V\alpha\beta) = 0$, (B_3), but *not* $\Sigma\beta = 0$, (C_3) the applied forces have an *unique resultant* $= \Sigma\beta$, which acts along the line whereof (A_3) is then the equation, with γ for its variable vector.

(*c*). When the condition (C_3) *is* satisfied, the forces compound themselves generally into *one couple*, of which the *axis* $= \Sigma V\alpha\beta$, whatever may be the position of the assumed origin o of vectors.

(*d*). When $\Sigma V\alpha\beta = 0$, (D_3), with or without (C_3), the forces have no tendency to turn the body round *that* point o; and when the equation (A_3) holds good, as in (*a*), for an *arbitrary* vector γ, the forces do not tend to produce a rotation* round *any* point c, so that they completely *balance* each other, as before, and *both* the conditions (C_3) and (D_3) are satisfied.

(*e*). In the general case, when *neither* (C_3) nor (D_3) is satisfied, if q be an *auxiliary quaternion*, such that

$$q\Sigma\beta = \Sigma V\alpha\beta, \qquad (E_3)$$

then Vq is the *vector perpendicular* from the origin, on the *central axis* of the system; and if $c = Sq$, then $c\Sigma\beta$ represents, both in quantity and in direction, the *axis of the central couple*.

(*f*). If Q be *another* auxiliary quaternion, such that

$$Q\Sigma\beta = \Sigma\alpha\beta, \qquad (F_3)$$

with $T\Sigma\beta > 0$, then $SQ = c =$ *central moment* divided by *total force*;

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* It is easy to prove that the *moment* of the *force* β, acting at the end of the vector α from o, and estimated with respect to any unit-line ι from the same origin, or the *energy* with which the force so acting tends to cause the body to *turn* round that line ι, regarded as a *fixed axis*, is represented by the scalar, $-S\iota\alpha\beta$, or $S\iota^{-1}\alpha\beta$; so that when the condition (D_3) is satisfied, the applied forces have no tendency to produce rotation round *any axis through the origin*: which origin becomes an *arbitrary point* c, when the *equation of equilibrium* (A_3) holds good.

and VQ is the vector γ of a point C *upon the central axis* which does not vary with the origin O, and which there are reasons for considering as the *Central Point* of the system, or as the *general centre of applied forces*: in fact, for the *case of parallelism*, this point C coincides with what is usually called the centre of parallel forces.

(*g*). Conceptions of the *Total Moment* $\Sigma\alpha\beta$, regarded as being *generally* a *quaternion*; and of the *Total Tension*, $-\Sigma\alpha\beta$, considered as a *scalar* to which that quaternion with its sign changed reduces itself for the case of *equilibrium* (*a*), and of which the value is in that case *independent of the origin* of vectors.

(*h*). *Principle of Virtual Velocities*,

$$\Sigma S\beta\delta\alpha = 0, \qquad (G_3)$$

ARTICLE 417.—On the Dynamics of a Rigid Body, ... 709-713

(*a*). *General Equation of Dynamics*,

$$\Sigma m S\,(D_t^2\alpha - \xi)\,\delta\alpha = 0\,; \qquad (H_3)$$

the vector ξ representing the accelerating force, or $m\xi$ the moving force, acting on a particle m of which the vector at the time t is α; and $\delta\alpha$ being any infinitesimal variation of this last vector, geometrically compatible with the connexions between the parts of the system, which need not *here* be a rigid one.

(*b*). For the case of a *free system*, we may change each $\delta\alpha$ to $\epsilon + V\iota\alpha$, ϵ and ι being *any two* infinitesimal vectors, which do not change in passing from one particle m to another; and thus the general equation (H_3) furnishes *two* general vector equations, namely,

$$\Sigma m\,(D_t^2\alpha - \xi) = 0, \quad (I_3), \qquad \text{and} \qquad \Sigma m V\alpha\,(D_t^2\alpha - \xi) = 0\,; \quad (J_3)$$

which contain respectively the law of the *motion of the centre of gravity*, and the law of *description of areas*.

(*c*). If a *body* be supposed to be *rigid*, and to have a *fixed point* O, then only the equation (J_3) need be retained; and we may write,

$$D_t\alpha = V\iota\alpha, \qquad (K_3)$$

ι being *here* a *finite* vector, namely the *Vector Axis of Instantaneous Rotation*: its *versor* $U\iota$ denoting the *direction* of that axis, and its *tensor* $T\iota$ representing the *angular velocity* of the body about it, at the time t.

(*d*). When the forces vanish, or balance each other, or compound themselves into a single force acting at the fixed point, as for the case of a heavy body turning freely about its centre of gravity, then

$$\Sigma m V\alpha\xi = 0, \quad (L_3)\,; \quad \text{and if we write,} \quad \phi\iota = \Sigma m\alpha V\alpha\iota, \qquad (M_3)$$

so that ϕ again denotes a linear, vector, and self-conjugate function, we shall have the equations,

$$\phi D_t\iota + V\iota\phi\iota = 0, \quad (N_3)\,; \qquad \phi\iota + \gamma = 0, \quad (O_3)\,; \qquad S\iota\phi\iota = h^2\,; \quad (P_3)$$

whence $\quad S\iota\gamma + h^2 = 0, \quad (Q_3), \quad$ and $\quad \phi D_t\iota = V\iota\gamma\,; \quad (R_3)$

the vector γ being what we may call the *Constant of Areas*, and the scalar h^2 being the *Constant of Living Force*.

(*e*). One of Poinsot's representations of the *motion of a body*, under the circumstances last supposed, is thus reproduced under the form, that the *Ellipsoid of Living Force* (P_3), with its *centre* at the *fixed point* O, *rolls without gliding* on the *fixed plane* (Q_3), which is *parallel to the Plane of Areas* $(S\iota\gamma = 0)$; the variable *semidiameter of contact*, ι, being the *vector-axis* (*c*) of *instantaneous rotation* of the body.

(*f*). The *Moment of Inertia*, with respect to *any axis* ι through O, is equal to the *living force* (h^2) divided by the *square* $(T\iota^2)$ of the *semidiameter of the ellipsoid* (P_3), which has the direction of that axis; and hence may be derived, with the help of the first *general construction* of an ellipsoid, suggested by quaternions, a simple geometrical representation (p. 711) of the *square-root* of the moment of inertia of a body, with respect to *any axis* AD passing through a given point A, as a certain *right line* $\overline{\text{BD}}$, if $\overline{\text{CD}} = \overline{\text{CA}}$, with the help of two other points B and C, which are likewise fixed in the body, but may be chosen in more ways than one.

(*g*). A *cone* of the second degree,

$$S\iota\nu = 0, \quad (S_3), \quad \text{with } \nu = \gamma^2\phi\iota - h^2\phi^2\iota, \qquad (T_3)$$

is *fixed* in the *body*, but *rolls* in *space* on that *other cone*, which is the *locus* of the instantaneous *axis* ι; and thus a *second representation*, proposed by Poinsot, is found for the *motion of the body*, as the *rolling of one cone on another*.

(*h*). Some of Mac Cullagh's results, respecting the motion here considered, are obtained with equal ease by the same quaternion analysis; for example, the line γ, although *fixed in space*, describes *in the body* an easily assigned *cone of the second degree* (p. 712), which cuts the *reciprocal ellipsoid*,

$$S\gamma\phi^{-1}\gamma = h^2, \qquad (U_3)$$

in a certain *sphero-conic:* and the *cone of normals* to the last mentioned cone (or the locus of the line $\iota + h^2\gamma^{-1}$) *rolls on the plane of areas* $(S\iota\gamma = 0)$.

(*i*). The *Three (Principal) Axes of Inertia* of the body, for the given point O, have the *directions* (p. 712) of the *three rectangular and vector roots* (comp. (P), p. xii., and the paragraph 415, (*a*), p. xlii.) of the equation

$$V\iota\phi\iota = 0, \quad (V_3), \quad \text{because, for each, } D_t\iota = 0; \qquad (V_3')$$

and if A, B, C denote the three *Principal Moments* of inertia corresponding, then the *Symbolical Cubic* in ϕ (comp. the formula (N) in page xii.) may be thus written,

$$(\phi + A)\ (\phi + B)\ (\phi + C) = 0. \qquad (W_3)$$

(*j*). Passage (p. 713), from moments referred to axes passing through a given point O, to those which correspond to respectively parallel axes, through any other point Ω of the body.

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ARTICLE 418.—On the motions of a System of Bodies, considered as free particles $m, m', ..$ which attract each other according to the law of the Inverse Square 713-717

(*a*). Equation of motion of the system,

$$\Sigma m \mathrm{S} \mathrm{D}_t^2 \alpha \delta \alpha + \delta P = 0, \quad (\mathrm{X}_3), \quad \text{if } P = \Sigma m m' \mathrm{T} (\alpha - \alpha')^{-1}; \quad (\mathrm{Y}_3)$$

α is the vector, at the time t, of the mass or particle m; P is the *potential* (or *force-function*); and the infinitesimal variations $\delta\alpha$ are arbitrary.

(*b*). Extension of the *notation of derivatives*,

$$\delta P = \Sigma \mathrm{S} (\mathrm{D}_\alpha P . \delta \alpha). \qquad (\mathrm{Z}_3)$$

(*c*). The differential equations of motion of the separate masses $m, ..$ become thus,

$$m \mathrm{D}_t^2 \alpha + \mathrm{D}_\alpha P = 0, \ldots; \qquad (\mathrm{A}_4)$$

and the laws of the centre of gravity, of areas, and of living force, are obtained under the forms,

$$\Sigma m \mathrm{D}_t \alpha = \beta, \quad (\mathrm{B}_4); \quad \Sigma m \mathrm{V} \alpha \mathrm{D}_t \alpha = \gamma; \qquad (\mathrm{C}_4)$$

and $$T = -\tfrac{1}{2} \Sigma m (\mathrm{D}_t \alpha)^2 = P + H; \qquad (\mathrm{D}_4)$$

β, γ being two vector constants, and H a scalar constant.

(*d*). Writing,

$$F = \int_0^t (P + T)\, \mathrm{d}t, \quad (\mathrm{E}_4), \quad \text{and } V = \int_0^t 2T \mathrm{d}t = F + tH, \quad (\mathrm{F}_4)$$

F may be called the *Principal* Function*, and V the *Characteristic Function*, of the motion of the system; *each* depending on the *final vectors* of *position*, $\alpha, \alpha', ..$ and on the *initial vectors*, $\alpha_0, \alpha'_0, ..$; but F depending *also* (explicitly) on the *time*, t, while V (= the *Action*) depends instead on the *constant* H of *living force*, in addition to those final and initial vectors: the *masses* $m, m', ..$ being supposed to be known, or constant.

(*e*). We are led thus to equations of the forms,

$$m \mathrm{D}_t \alpha + \mathrm{D}_\alpha F = 0, \ldots \quad (\mathrm{G}_4); \quad -m \mathrm{D}_0 \alpha + \mathrm{D}_{\alpha_0} F = 0, \ldots \quad (\mathrm{H}_4);$$

$$(\mathrm{D}_t F) = -H, \qquad (\mathrm{I}_4)$$

whereof the system (G_4) contains what may be called the *Intermediate Integrals*, while the system (H_4) contains the *Final Integrals*, of the differential *Equations of Motion* (A_4).

(*f*). In like manner we find equations of the forms,

$$\mathrm{D}_\alpha V = -m \mathrm{D}_t \alpha, \ldots \quad (\mathrm{J}_4); \quad \mathrm{D}_{\alpha_0} V = m \mathrm{D}_0 \alpha, \ldots \quad (\mathrm{K}_4); \quad \mathrm{D}_H V = t; \quad (\mathrm{L}_4)$$

the *intermediate integrals* (*e*) being here the result of the elimination

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* References are given to two Essays by the present writer, "*On a General Method in Dynamics*," in the *Philosophical Transactions* for 1834 and 1835, in which the *Action* (V), and a certain *other* function (S), which is here denoted by F, were called, as above, the *Characteristic* and *Principal Functions*. But the *analysis* here used, as being founded on the *Calculus of Quaternions*, is altogether unlike the analysis which was employed in those former Essays.

Pages.

of H, between the system (J_4) and the equation (L_4); and the *final integrals*, of the same system of differential equations (A_4), being now (theoretically) obtained, by eliminating the same constant H between (K_4) and (L_4).

(*g*). The functions F and V are obliged to satisfy certain *Partial Differential Equations in Quaternions*, of which those relative to the final vectors α, α', .. are the following,

$$(\mathrm{D}_t F) - \tfrac{1}{2}\Sigma m^{-1}(\mathrm{D}_\alpha F)^2 = P, \ (\mathrm{M}_4); \quad \tfrac{1}{2}\Sigma m^{-1}(\mathrm{D}_\alpha V)^2 + P + H = 0; \ (\mathrm{N}_4)$$

and they are subject to certain geometrical conditions, from which can be deduced, in a new way, and as new verifications, the law of motion of the centre of gravity, and the law of description of areas.

(*h*). General approximate expressions (p. 717) for the functions F and V, and for their derivatives H and t, for the case of a *short motion* of the system.

ARTICLE 419.—On the Relative Motion of a Binary System; and on the Law of the Circular Hodograph, ... 717-733

(*a*). The vector of one body from the other being α, and the distance being r ($= \mathrm{T}\alpha$), while the sum of the masses is M, the differential equation of the relative motion is, with the law of the inverse square,

$$\mathrm{D}^2\alpha = M\alpha^{-1}r^{-1}; \qquad (\mathrm{O}_4)$$

D being here used as a characteristic of derivation, with respect to the time t.

(*b*). As a first integral, which holds good also for any *other law* of *central force*, we have

$$\mathrm{V}\alpha\mathrm{D}\alpha = \beta = \text{a constant vector}; \qquad (\mathrm{P}_4)$$

which *includes* the *two* usual laws, of the *constant plane* ($\perp \beta$), and of the *constant areal velocity* $\left(\dfrac{c}{2} = \tfrac{1}{2}\mathrm{T}\beta\right)$.

(*c*). Writing $\tau = \mathrm{D}\alpha$ = *vector of relative velocity*, and conceiving this new vector τ to be drawn from that one of the two bodies which is here selected for the origin o, the locus of the extremities of the vector τ is (by earlier definitions) the *Hodograph of the Relative Motion;* and this hodograph is proved to be, for the *Law of the Inverse Square*, a *Circle*.

(*d*). In fact, it is shown (p. 720), that for any *law* of central force, the *radius of curvature* of the hodograph is equal to the force, multiplied into the square of the distance, and divided by the doubled areal velocity; or by the *constant parallelogram* c, under the vectors (α and τ) of *position* and *velocity*, or of the *orbit* and the *hodograph*.

(*e*). It follows then, conversely, that the law of the inverse square is the *only law* which renders the hodograph generally a *circle;* so that the law of nature may be characterized, as the *Law of the Circular Hodograph:* from which latter law, however, it is easy to deduce the *form* of the *Orbit*, as a *conic section* with a *focus* at o.

(*f*). If the *semiparameter* of this *orbit* be denoted, as usual, by p, and if h be the *radius* of the *hodograph*, then (p. 719),

$$h = Mc^{-1} = cp^{-1} = (Mp^{-1})^{\frac{1}{2}}. \qquad (Q_4)$$

(*g*). The orbital *excentricity* e is also the hodographic excentricity, in the sense that eh is the distance of the centre H of the hodograph, from the point O which is here treated as the centre of force.

(*h*). The orbit is an *ellipse*, when the point O is *interior* to the hodographic circle ($e < 1$); it is a *parabola*, when O is *on the circumference* of that circle ($e = 1$); and it is an *hyperbola*, when O is an *exterior point* ($e > 1$). And in all these cases, if we write

$$a = p(1 - e^2)^{-1} = ch^{-1}(1 - e^2)^{-1}, \qquad (R_4)$$

the constant a will have its usual signification, relatively to the orbit.

(*i*). The quantity Mr^{-1} being here called the *Potential*, and denoted by P, geometrical *constructions* for this quantity P are assigned, with the help of the hodograph (p. 723); and for the *harmonic mean*, $2M(r + r')^{-1}$, between the *two potentials*, P and P', which answer to the extremities T, T′ of any proposed *chord* of that circle: all which constructions are illustrated by a new diagram (Fig. 86).

(*j*). If U be the *pole* of the chord TT′; M, M′ the points in which the line OU cuts the circle; L the middle point, and N the pole, of the *new chord* MM′, *one* secant from which last pole is thus the line NTT′; U′ the intersection of *this* secant with the chord MM′, or the *harmonic conjugate* of the point U, with respect to the same chord; and $\mathrm{NT_{,}T'_{,}}$ any *near secant* from N, while $\mathrm{U_{,}}$ (on the line OU) is the pole of the *near chord* $\mathrm{T_{,}T'_{,}}$: then the *two small arcs*, $\mathrm{T_{,}T}$ and $\mathrm{T'T'_{,}}$, of the hodograph, intercepted *between these two secants*, are proved to be ultimately *proportional* to the *two potentials*, P and P'; or to the *two ordinates* TV, T′V′, namely the perpendiculars let fall from T and T′, on what may here be called the *hodographic axis* LN. Also, the *harmonic mean* between these two ordinates is obviously (by the construction) the line U′L; while UT, UT′, and $\mathrm{U_{,}T_{,}}$, $\mathrm{U_{,}T'_{,}}$ are *four tangents* to the hodograph, so that *this* circle is *cut orthogonally*, in the *two pairs of points*, T, T′ and $\mathrm{T_{,}}$, $\mathrm{T'_{,}}$, by *two other circles*, which have the two near points U, $\mathrm{U_{,}}$ for their *centres* (pp. 724, 725).

(*k*). In general, for *any motion of a point* (absolute or relative, in one plane or in space, for example, in the motion of the centre of the *moon* about that of the *earth*, under the perturbations produced by the attractions of the sun and planets), with α for the *variable vector* (418) of *position* of the point, the *time* dt which corresponds to any *vector-element* $dD\alpha$ of the *hodograph*, or what may be called the time of *hodographically describing* that element, is the *quotient* obtained by *dividing* the same *element of the hodograph*, by the *vector of acceleration* $D^2\alpha$ in the *orbit*; because we may write *generally* (p. 724),

$$dt = \frac{dD\alpha}{D^2\alpha}, \quad \text{or} \quad dt = \frac{TdD\alpha}{TD^2\alpha}, \quad \text{if} \quad dt > 0. \qquad (S_4)$$

(*l*). For the law of the *inverse square* (comp. (*a*) and (*i*)), the measure of the *force* is,

$$\mathrm{TD}^2\alpha = Mr^{-2} = M^{-1}P^2; \qquad (\mathrm{T}_4)$$

the *times* dt, dt', of hodographically describing the small circular *arcs* T,T and T'T,' of the hodograph, being found by multiplying the lengths (*j*) of those two arcs by the mass, and dividing each product by the square of the potential corresponding, are therefore *inversely* as those *two potentials*, P, P', or *directly* as the *distances*, r, r', in the *orbit :* so that we have the proportion,

$$\mathrm{d}t : \mathrm{d}t' : \mathrm{d}t + \mathrm{d}t' = r : r' : r + r'. \qquad (\mathrm{U}_4)$$

(*m*). If we suppose that the *mass*, M, and the *five points* O, L, M, U, U, upon the chord MM' are *given*, or *constant*, but that the *radius*, h, of the *hodograph*, or the position of the *centre* H on the hodographic *axis* LN, is *altered*, it is found in this way (p. 725) that although the *two elements of time*, dt, dt', *separately vary*, yet their *sum* remains *unchanged :* from which it follows, that even if the *two circular arcs*, T,T, T'T,', be *not small*, but still intercepted (*j*) between *two secants* from the *pole* N of the *fixed chord* MM', the *sum* (say, $\Delta t + \Delta t'$) of the *two times* is *independent of the radius*, h.

(*n*). And hence may be deduced (p. 726), by supposing *one* secant to become a *tangent*, this *Theorem of Hodographic Isochronism*, which was communicated without demonstration, several years ago, to the Royal Irish Academy,* and has since been treated as a subject of investigation by several able writers:

If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.

(*o*). This *common time* can easily be expressed (p. 726), under the form of the definite integral,

$$\textit{Time of } \mathrm{TMT}' = \frac{2M}{g^3}\int_0^w \frac{\mathrm{d}w}{(1 - e'\cos w)^2}; \qquad (\mathrm{V}_4)$$

$2g$ being the length of the *fixed chord* MM'; e' the quotient LO : LM, which reduces itself to -1 when O is at M', that is for the case of a *parabolic orbit ;* e' lying *between* ± 1 for an *ellipse*, and *outside* those limits for an *hyperbola*, but being, in all these cases, *constant ;* while w is a certain auxiliary angle, of which the sine $= \overline{\mathrm{UT}} : \overline{\mathrm{UL}}$ (p. 727), or $= s(r + r')^{-1}$, if s denote the length $\overline{\mathrm{PP}}'$ of the *chord of the orbit, corresponding* to the *chord* TT' of the *hodograph ;* and w varies from 0 to π, when the whole *periodic time* $2\pi n^{-1}$ for a *closed orbit* is to be computed: with the verification, that the integral (V_4) gives, in this last case,

$$M = a^3n^2, \text{ as usual.} \qquad (\mathrm{W}_4)$$

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* See the *Proceedings* of the 16th of March, 1847. It is understood that the *common centre* O of *force* is occupied by a *common mass*, M.

(*p*). By examining the general composition of the definite integral (V_4), or by more purely geometrical considerations, which are illustrated by Fig. 87, it is found that, with the law of the inverse square, the *time* t of describing an *arc* PP′ of the *orbit* (closed or unclosed) is a *function* (p. 729) of the *three ratios*,

$$\frac{a^3}{M}, \quad \frac{r+r'}{a}, \quad \frac{s}{r+r'}; \qquad (X_4)$$

and therefore simply a function of the chord (s, or $\overline{\text{PP}'}$) of the orbit, and of the *sum of the distances* ($r + r'$, or $\overline{\text{OP}} + \overline{\text{OP}'}$) when M and a are given: which is a form of the *Theorem of Lambert*.

(*q*). The same important theorem may be otherwise deduced, through a quite different analysis, by an employment of *partial derivatives*, and of *partial differential equations in quaternions*, which is analogous to that used in a recent investigation (418), respecting the motions of an *attracting system* of *any number of bodies*, m, m', &c.

(*r*). Writing now (comp. p. xlvii) the following expression for the *relative living force*, or for the *mass* ($M = m + m'$), multiplied into the *square* of the *relative velocity* (TDα),

$$2T = -M\mathrm{D}\alpha^2 = 2(P + H) = M(2r^{-1} - a^{-1}); \qquad (Y_4)$$

introducing the *two new integrals* (p. 729),

$$F = \int_0^t (P + T)\,\mathrm{d}t, \quad (Z_4), \quad \text{and} \quad V = \int_0^t 2T\mathrm{d}t = F + tH, \qquad (A_5)$$

which have thus (comp. (E_4) and (F_4)) the *same forms* as before, but with *different* (although *analogous*) *significations*, and may still be called the *Principal* and *Characteristic Functions* of the motion; and denoting by α, α' (instead of α_0, α) the *initial* and *final vectors* of *position*, or of the *orbit*, while r, r' are the two *distances*, and τ, τ' the two corresponding *vectors of velocity*, or of the *hodograph*: it is found that when M is given, F may be treated as a function of α, α', t, or of r, r', s, t, and V as a function of α, α', a, or of r, r', s, and H; and that their *partial derivatives*, in the first view of these two functions, are (p. 729),

$$\mathrm{D}_\alpha F = \mathrm{D}_\alpha V = \tau, \quad (B_5); \qquad \mathrm{D}_{\alpha'} F = \mathrm{D}_{\alpha'} V = -\tau'; \qquad (C_5)$$

$$(\mathrm{D}_t)\, F = -H, \quad (D_5); \quad \text{and} \quad \mathrm{D}_H V = \frac{2a^2}{M}\mathrm{D}_a V = t; \qquad (E_5)$$

while, in the second view of the same functions, they satisfy the *two partial differential equations* (p. 730),

$$\mathrm{D}_r F = \mathrm{D}_r' F, \quad (F_5), \quad \text{and} \quad \mathrm{D}_r V = \mathrm{D}_r' V; \qquad (G_5)$$

along with *two other* equations of the same kind, but of the second degree, for *each* of the functions here considered, which are analogous to those mentioned in p. xlviii.

(*s*). The equations (F_5) (G_5) express, that the *two distances*, r and r', enter into each of the two functions only by their *sum*; so that, if M be still treated as given, F may be regarded as a function of the *three* quantities, $r+r'$, s, and t; while V, and therefore also t by (E_5), is found in like manner to be a function of the *three* scalars, $r+r'$, s, and a: which last result respecting the *time* agrees with (p), and furnishes a *new proof of Lambert's Theorem*.

(t). The *three* partial differential equations (r) in V conduct, by merely *algebraical* combinations, to expressions for the three partial derivatives, $D_r V$, $D_{r'} V\ (= D_r V)$, and $D_s V$; and thus, with the help of (E_5), to *two new definite integrals*[*] (p. 731), which express respectively the *Action* and the *Time*, in the *relative motion of a binary system* here considered, namely, the two following:

$$V = \int_{-s}^{s} \left(\frac{M}{r+r'+s} - \frac{M}{4a} \right)^{\frac{1}{2}} ds; \qquad (H_5)$$

$$t = \tfrac{1}{2} \int_{-s}^{s} \left(\frac{4M}{r+r'+s} - \frac{M}{a} \right)^{-\frac{1}{2}} ds; \qquad (I_5)$$

whereof the latter is not to be extended, without modification, beyond the limits within which the radical is finite.

ARTICLE 420.—On the determination of the Distance of a Comet, or new Planet, from the Earth, ... 733, 734

(a). The masses of earth and comet being neglected, and the mass of the sun being denoted by M, let r and w denote the distances of earth and comet from sun, and z their distance from each other, while α is the heliocentric vector of the earth ($T\alpha = r$), known by the theory of the sun, and ρ is the unit-vector, determined by observation, which is directed from the earth to the comet. Then it is easily proved by quaternions, that we have the equation (p. 734),

$$\frac{S\rho D\rho D^2\rho}{S\rho D\rho U\alpha} = \frac{r}{z}\left(\frac{M}{r^3} - \frac{M}{w^3} \right), \qquad (J_5)$$

with
$$w^2 = r^2 + z^2 - 2zS\alpha\rho; \qquad (K_5)$$

eliminating w between these two formulæ, clearing of fractions, and dividing by z, we are therefore conducted in this way to an algebraical *equation of the seventh degree*, whereof *one root* is the *sought distance*, z.

(b). The final equation, thus obtained, differs only by its notation, and by the facility of its deduction, from that assigned for the same purpose in the *Mécanique Céleste*; and the *rule* of Laplace there given, for determining, by inspection of a celestial *globe*, *which* of the two bodies (earth and comet) is the *nearer to the sun*, results at sight from the formula (J_5).

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[*] References are given to the *First Essay*, &c., by the present writer (comp. the Note to p. xlvii.), in which were assigned integrals, substantially equivalent to (H_5) and (I_5), but deduced by a quite different analysis. It has recently been remarked to him, by his friend Professor Tait of Edinburgh, that while the *area* described, with Newton's Law, about the *full focus* of an orbit, has long been known to be proportional to the *time* corresponding, so the *area about the empty focus* represents (or is proportional to) the *action*.

ARTICLE 421.—On the Development of the Disturbing Force of the Sun on the Moon; or of one Planet on another, which is nearer than itself to the Sun, ... 734-736

(a). Let α, σ be the geocentric vectors of moon and sun; $r (= T\alpha)$, and $s (= T\sigma)$, their geocentric distances; M the sum of the masses of earth and moon; S the mass of the sun; and D (as in recent Series) the mark of derivation with respect to the time: then the *differential equation of the disturbed motion of the moon about the earth* is,

$$D^2\alpha = M\phi\alpha + \eta, \quad (L_5), \quad \text{if} \quad \phi\alpha = \phi(\alpha) = \alpha^{-1}T\alpha^{-1}, \qquad (M_5)$$

and $\eta = \textit{Vector of Disturbing Force} = S(\phi\sigma - \phi(\sigma - \alpha))$; $\quad (N_5)$

ϕ denoting here a *vector function*, but *not* a *linear* one.

(b). If we neglect η, the equation (L_5) reduces itself to the form $D^2\alpha = M\phi\alpha$; which contains (comp. (O_4)) the laws of *undisturbed elliptic motion*.

(c). If we develope the disturbing vector η, according to ascending powers of the quotient $r : s$, of the distances of moon and sun from the earth, we obtain an *infinite series* of terms, each representing a *finite group* of *partial disturbing forces*, which may be thus denoted,

$$\eta = \eta_1 + \eta_2 + \eta_3 + \&c.; \qquad (O_5)$$

$$\eta_1 = \eta_{1,1} + \eta_{1,2}, \quad \eta_2 = \eta_{2,1} + \eta_{2,2} + \eta_{2,3}, \ \&c.; \qquad (P_5)$$

these partial forces *increasing* in *number*, but *diminishing* in *intensity*, in the passage from any one group to the following; and being connected with each other, *within* any such group, by simple *numerical ratios* and *angular relations*.

(d). For example, the *two forces* $\eta_{1,1}$, $\eta_{1,2}$ of the *first group* are, *rigorously*, proportional to the numbers 1 and 3; the *three* forces $\eta_{2,1}$, $\eta_{2,2}$, $\eta_{2,3}$ of the *second* group are as the numbers 1, 2, 5; and the *four* forces of the *third* group are proportional to 5, 9, 15, 35: while the separate *intensities* of the *first forces*, in these *three first groups*, have the expressions,

$$T\eta_{1,1} = \frac{Sr}{2s^3}; \qquad T\eta_{2,1} = \frac{3Sr^2}{8s^4}; \qquad T\eta_{3,1} = \frac{5Sr^3}{16s^5}. \qquad (Q_5)$$

(e). All these *partial forces* are conceived to *act at the moon*; but their *directions* may be represented by the respectively *parallel unit-lines* $U\eta_{1,1}$, &c., drawn *from the earth*, and terminating on a *great circle* of the *celestial sphere* (supposed here to have its radius equal to unity), which passes *through the geocentric* (or *apparent*) *places*, ☉ and ☽, of the sun and moon *in the heavens*.

(f). Denoting then the *geocentric elongation* ☉☽ of *moon* from *sun* (in the *plane of the three bodies*) by $+\theta$; and by ☉$_1$, ☉$_2$, and ☽$_1$, ☽$_2$, ☽$_3$, what may be called *two fictitious suns*, and *three fictitious moons*, of which the corresponding elongations from ☉, in the *same great* *circle*, are $+2\theta$, -2θ, and $-\theta$, $+3\theta$, -3θ, as illustrated by Fig. 88 (p. 735); it is found that the *directions* of the *two forces* of the *first group* are represented by the *two radii* of this *unit-circle*, which terminate in ☽ and ☽$_1$; those of the *three* forces of the *second* group, by the three radii to ☉$_1$, ☉, and ☉$_2$; and those of the *four* forces of the third group, by the radii to ☽$_2$, ☽, ☽$_1$, and ☽$_3$; with facilities for *extending* all these results (with the requisite modifications), to the *fourth* and *subsequent groups*, by the *same quaternion analysis*.

(*g*). And it is important to observe, that *no supposition* is here made respecting any *smallness* of *excentricities* or *inclinations* (p. 736); so that *all the formulæ apply*, with the necessary changes of *geocentric* to *heliocentric* vectors, &c., to the *perturbations of the motion of a comet about the sun*, produced by the *attraction of a planet*, which is (at the time) *more distant* than the comet from the sun.

Article 422.—On Fresnel's Wave, ... 736–756

(*a*). If ρ and μ be two corresponding vectors, of *ray-velocity* and *wave-slowness*, or briefly *Ray* and *Index*, in a biaxal crystal, the velocity of light in a vacuum being unity; and if $\delta\rho$ and $\delta\mu$ be any infinitesimal *variations* of these two vectors, consistent with the equations (supposed to be as yet unknown), of the *Wave* (or *wave-surface*), and its *reciprocal*, the *Index-Surface* (or *surface of wave-slowness*): we have then first the fundamental *Equations of Reciprocity* (comp. p. 417),

$$S\mu\rho = -1, \quad (R_5); \qquad S\mu\delta\rho = 0, \quad (S_5); \qquad S\rho\delta\mu = 0, \quad (T_5)$$

which are independent of any hypothesis respecting the *vibrations* of the *ether*.

(*b*). If $\delta\rho$ be next regarded as a *displacement* (or vibration), *tangential* to the *wave*, and if $\delta\epsilon$ denote the *elastic force* resulting, there exists then, on Fresnel's principles, a *relation* between these two small vectors; which relation may (with our notations) be expressed by either of the two following equations,

$$\delta\epsilon = \phi^{-1}\delta\rho, \quad (U_5), \quad \text{or} \quad \delta\rho = \phi\delta\epsilon; \qquad (V_5)$$

the function ϕ being of that linear, vector, and *self-conjugate* kind, which has been frequently employed in these Elements.

(*c*). The fundamental connexion, between the functional symbol ϕ, and the optical constants *abc* of the crystal, is expressed (p. 741, comp. the formula (W_3) in p. xlvi) by the *symbolic and cubic equation*,

$$(\phi + a^{-2})(\phi + b^{-2})(\phi + c^{-2}) = 0; \qquad (W_5)$$

of which an extensive use is made in the present Series.

(*d*). The *normal component*, $\mu^{-1}S\mu\delta\epsilon$, of the elastic force $\delta\epsilon$, is *ineffective* in Fresnel's theory, on account of the supposed *incompressibility of the ether*; and the *tangential component*, $\phi^{-1}\delta\rho - \mu^{-1}S\mu\delta\epsilon$, is (in the same theory, and with present notations) to be equated to

$\mu^{-2}\delta\rho$, for the propagation of a *rectilinear vibration* (p. 737); we obtain then thus, for such a vibration or *tangential displacement*, $\delta\rho$, the expression,

$$\delta\rho = (\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}S\mu\delta\epsilon; \qquad (X_5)$$

and therefore by (S_5) the equation,

$$0 = S\mu^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}, \qquad (Y_5)$$

which is a *Symbolical Form* of the scalar *Equation of the Index-Surface*, and may be thus transformed,

$$1 = S\mu(\mu^2 - \phi)^{-1}\mu. \qquad (Z^5)$$

(*e*). The *Wave-Surface*, as being the *reciprocal* (*a*) of the *index-surface* (*d*), is easily found (p. 738) to be represented by this *other Symbolical Equation*,

$$0 = S\rho^{-1}(\phi - \rho^{-2})^{-1}\rho^{-1}; \qquad (A_6)$$

or

$$1 = S\rho(\rho^2 - \phi^{-1})^{-1}\rho. \qquad (B_6)$$

(*f*). In such transitions, from one of these reciprocal surfaces to the other, it is found convenient to introduce *two auxiliary vectors*, υ and $\omega(=\phi\upsilon)$, namely the lines ou and ow of Fig. 89; *both* drawn from the *common centre* o of the two surfaces; but υ terminating (p. 738) on the *tangent plane* to the *wave*, and being *parallel* to the direction of the *elastic force* $\delta\epsilon$; whereas ω terminates (p. 739) on the tangent plane to the *index-surface*, and is parallel to the *displacement* $\delta\rho$.

(*g*). Besides the relation,

$$\omega = \phi\upsilon, \quad \text{or} \quad \upsilon = \phi^{-1}\omega, \qquad (C_6)$$

connecting the two new vectors (*f*) with *each other*, they are connected with ρ and μ by the equations (pp. 738, 739),

$$S\mu\upsilon = -1, \quad (D_6); \qquad S\rho\upsilon = 0; \qquad (E_6)$$
$$S\rho\omega = -1, \quad (F_6); \qquad S\mu\omega = 0; \qquad (G_6)$$

and generally (p. 739), the following *Rule of the Interchanges* holds good: *In any formula involving* ρ, μ, υ, ω, and ϕ, or some of them, it is permitted to *exchange* ρ with μ, υ with ω, and ϕ with ϕ^{-1}; *provided* that we at the same time interchange $\delta\rho$ with $\delta\epsilon$, but *not generally** $\delta\mu$ with $\delta\rho$, when these variations, or any of them occur.

(*h*). We have also the relations (pp. 739, 740),

$$-\rho^{-1} = \upsilon^{-1}V\upsilon\mu = \mu + \upsilon^{-1}; \qquad (H_6)$$
$$-\mu^{-1} = \omega^{-1}V\omega\rho = \rho + \omega^{-1}; \qquad (I_6)$$

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* This apparent *exception* arises (pp. 739, 740) from the circumstance, that $\delta\rho$ and $\delta\epsilon$ have their *directions generally fixed*, in this whole investigation (although subject to a *common reversal* by ±), when ρ and μ are given; whereas $\delta\mu$ continues to be used, as in (*a*), to denote *any infinitesimal vector, tangential to the index-surface* at the end of μ.

with others easily deduced, which may all be illustrated by the above-cited Fig. 89.

(*i*). Among such deductions, the following equations (p. 740) may be mentioned,

$$(\mathrm{V}\upsilon\phi\upsilon)^2 + \mathrm{S}\upsilon\phi\upsilon = 0, \quad (\mathrm{J}_6); \qquad (\mathrm{V}\omega\phi^{-1}\omega)^2 + \mathrm{S}\omega\phi^{-1}\omega = 0; \quad (\mathrm{K}_6)$$

which show that the *Locus* of *each* of the *two Auxiliary Points*, U and W, wherein the two vectors υ and ω terminate (*f*), is a *Surface of the Fourth Degree*, or briefly, a *Quartic Surface*; of which *two loci* the *constructions* may be connected (as stated in p. 741) with those of the *two reciprocal ellipsoids*,

$$\mathrm{S}\rho\phi\rho = 1, \quad (\mathrm{L}_6), \quad \text{and} \quad \mathrm{S}\rho\phi^{-1}\rho = 1; \qquad (\mathrm{M}_6)$$

ρ denoting, for each, an *arbitrary semidiameter*.

(*j*). It is, however, a much more interesting use of these *two ellipsoids*, of which (by (W_5), &c.) the *scalar semiaxes* are a, b, c for the *first*, and a^{-1}, b^{-1}, c^{-1} for the *second*, to observe that they may be employed (pp. 738, 739) for the *Constructions* of the *Wave* and the *Index-Surface*, respectively, by a very simple rule, which (at least for the *first* of these two reciprocal surfaces (a)) was assigned by Fresnel himself.

(*k*). In fact, on comparing the *symbolical form* (A_6) of the equation of the *Wave*, with the form (H_2) in p. xxxvii, or with the equation 412, XLI., in p. 683, we derive at once *Fresnel's Construction*: namely, that if the *ellipsoid* (*abc*) be *cut*, by an arbitrary *plane* through its centre, and if *perpendiculars* to that plane be erected at that central point, which shall have the *lengths* of the *semiaxes* of the *section*, then *the locus of the extremities, of the perpendiculars so erected, will be the sought Wave-Surface.*

(*l*). A precisely similar construction applies, to the derivation of the *Index-Surface* from the ellipsoid $(a^{-1}b^{-1}c^{-1})$: and thus the *two auxiliary surfaces*, (L_6) and (M_6), may be briefly called the *Generating Ellipsoid*, and the *Reciprocal Ellipsoid*.

(*m*). The cubic (W_5) in ϕ enables us easily to express (p. 741) the *inverse function* $(\phi + e)^{-1}$, where e is any scalar; and thus, by changing e to $-\rho^{-2}$, &c., *new forms* of the equation (A_6) of the *wave* are obtained, whereof one is,

$$0 = (\phi^{-1}\rho)^2 + (\rho^2 + a^2 + b^2 + c^2)\,\mathrm{S}\rho\phi^{-1}\rho - a^2b^2c^2; \qquad (\mathrm{N}_6)$$

with an analogous equation in μ (comp. the *rule* in (g)), to represent the *index-surface*: so that each of these two surfaces is of the *fourth degree*, as indeed is otherwise known.

(*n*). If *either* $\mathrm{S}\rho\phi^{-1}\rho$ or ρ^2 be treated as *constant* in (N_6), the *degree* of that equation is depressed from the *fourth* to the *second*; and therefore the *Wave* is *cut*, by *each* of the *two concentric quadrics*,

$$\mathrm{S}\rho\phi^{-1}\rho = h^4, \quad (\mathrm{O}_6), \qquad \rho^2 + r^2 = 0, \qquad (\mathrm{P}_6)$$

in a (real or imaginary) *curve of the fourth degree*: of which *two quar-*

tic curves, answering to all scalar values of the constants h and r, the wave is the *common locus*.

(*o*). The *new ellipsoid* (O_6) is *similar* to the ellipsoid (M_6), and *similarly placed*, while the *sphere* (P_6) has r for *radius;* and *every quartic* of the *second system* (n) is a *sphero-conic*, because it is, by the equation (A_6) of the wave, the *intersection* of that *sphere* (P_6) with the *concentric* and *quadric cone*,

$$0 = S\rho\,(\phi + r^{-2})^{-1}\rho\,; \tag{Q_6}$$

or, by (B_6), with this *other concentric quadric*,[*]

$$-1 = S\rho\,(\phi^{-1} + r^{2})^{-1}\rho, \tag{R_6}$$

whereof the *conjugate* (obtained by changing -1 to $+1$ in the last equation) has

$$a^2 - r^2,\ b^2 - r^2,\ c^2 - r^2, \tag{S_6}$$

for the squares of its scalar semiaxes, and is therefore *confocal with the generating ellipsoid* (L_6).

(*p*). For *any point* P of the *wave*, or at the end of *any ray* ρ, the *tangents* to the *two curves* (n) have the directions of ω and $\mu\omega$; so that *these two quartics cross each other at right angles*, and *each* is a *common orthogonal* in *all* the curves of the *other system*.

(*q*). But the *vibration* $\delta\rho$ is easily proved to be parallel to ω; hence the curves of the *first system* (n) are *Lines of Vibration of the Wave:* and the curves of the *second system* are the *Orthogonal Trajectories*[†] to those *Lines.*

(*r*). In general, the *vibration* $\delta\rho$ has (on Fresnel's principles) the direction of the *projection* of the *ray* ρ on the *tangent plane* to the *wave;* and the *elastic force* $\delta\epsilon$ has in like manner the direction of the *projection* of the *index-vector* μ on the tangent plane to the *index-surface:* so that the *ray* is thus *perpendicular* to the *elastic force*

ARTICLE 423.—Mac Cullagh's Theorem of the Polar Plane, . . 757–762

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[*] For *real curves* of the *second system* (n), this *new quadric* (R_6) is an *hyperboloid*, with *one sheet* or with *two*, according as the constant r lies between a and b, or between b and c; and, of course, the *conjugate hyperboloid* (o) has *two sheets* or *one*, in the same two cases respectively.

[†] In a different theory of light (comp. the next Series, 423), these *sphero-conics* on the wave are *themselves* the *lines of vibration.*

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* This *Table* was mentioned in the Note to p. xiv. of the *Contents*, as one likely to facilitate *reference*. In fact, the references in the text of the *Elements* are almost entirely to *Articles* (with their sub-articles), and not to *pages*.

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NOTE.—It appears by these Tables that the Author intended to have completed the work by the addition of Seven Articles, and Two Figures.—ED.

ELEMENTS OF QUATERNIONS.

BOOK I.

ON VECTORS, CONSIDERED WITHOUT REFERENCE TO ANGLES, OR TO ROTATIONS.

CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING VECTORS.

SECTION 1.—*On the Conception of a Vector; and on Equality of Vectors.*

ART. 1.—A right line AB, considered as having not only *length*, but also *direction*, is said to be a VECTOR. Its initial point A is said to be its *origin;* and its final point B is said to be its *term*. A vector AB is conceived to be (or to construct) the *difference* of its two extreme points; or, more fully, to be the result of the *subtraction* of its own origin from its own term; and, in conformity with this *conception*, it is also denoted by the *symbol* B − A: a notation which will be found to be extensively useful, on account of the analogies which it serves to express between geometrical and algebraical operations. When the extreme points A and B are *distinct*, the vector AB or B − A is said to be an *actual* (or an *effective*) vector; but when (as a limit) those two points are conceived to *coincide*, the vector AA or A − A, which then results, is said to be *null*. *Opposite* vectors, such as AB and BA, or B − A and A − B, are sometimes called vector and *revector*. *Successive* vectors, such as AB and BC, or B − A and C − B, are occasionally said to be vector and *provector:* the line AC, or C − A, which is

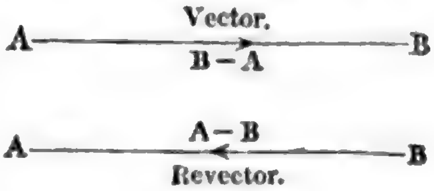

Fig. 1.

drawn from the origin A of the first to the term C of the second, being then said to be the *transvector*. At a later stage, we shall have to consider *vector-arcs* and *vector-angles*; but at present, our only *vectors* are (as above) *right lines*.

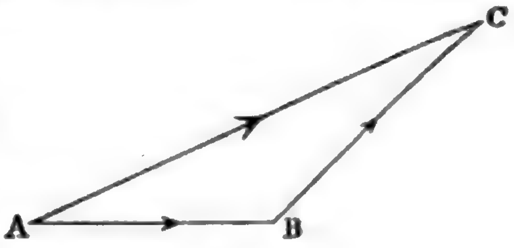

Fig. 2.

2. Two vectors are said to be EQUAL to each other, or the *equation* $AB = CD$, or $B - A = D - C$, is said to hold good, when (and only when) the origin and term of the one can be brought to *coincide* respectively with the corresponding points of the other, by *transports* (or by *translations*) *without rotation*. It follows that *all null vectors are equal*, and may therefore be denoted by a *common symbol*, such as that used for *zero*; so that we may write,

$$A - A = B - B = \&c. = 0;$$

but that two *actual* vectors, AB and CD, are *not* (in the present *full sense*) *equal* to each other, unless they have not merely *equal lengths*, but also *similar directions*. If then they do not happen to be *parts* of one *common line*, they must be *opposite sides* of a *parallelogram*, ABDC; the two lines AD, BC becoming thus the two *diagonals* of such a figure, and consequently *bisecting* each other, in some point E. Conversely, if the two equations,

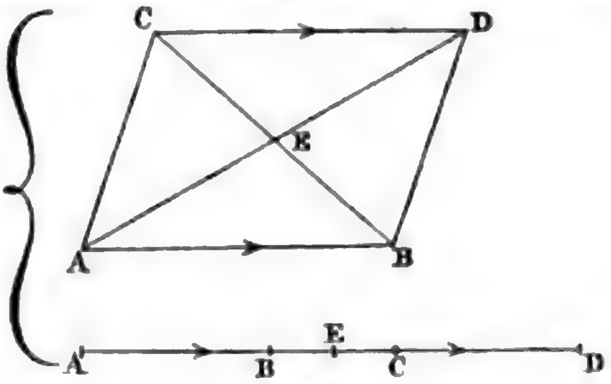

Fig. 3.

$$D - E = E - A, \quad \text{and} \quad C - E = E - B,$$

are satisfied, so that the two lines AD and BC are *commedial*, or have a *common middle point* E, then *even* if they be parts of *one* right line, the equation $D - C = B - A$ is satisfied. *Two radii*, AB, AC, of any *one circle* (or *sphere*), can *never* be *equal vectors*; because their *directions differ*.

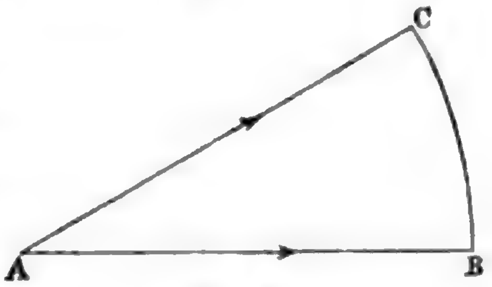

Fig. 4.

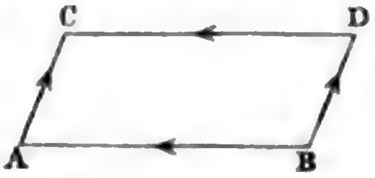

Fig. 5.

3. An *equation* between *vectors*, considered as an *equidifference* of *points*, admits of *inversion* and *alternation*; or in symbols, if

$$\mathrm{D} - \mathrm{C} = \mathrm{B} - \mathrm{A},$$

then

$$\mathrm{C} - \mathrm{D} = \mathrm{A} - \mathrm{B},$$

and

$$\mathrm{D} - \mathrm{B} = \mathrm{C} - \mathrm{A}.$$

Two vectors, CD and EF, which are equal to the *same third* vector, AB, are also equal to *each other;* and these *three* equal vectors are, in general, the three parallel *edges* of a *prism*.

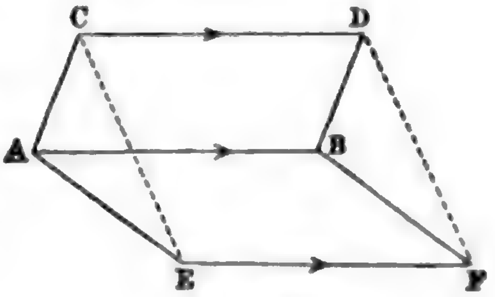

Fig. 6.

SECTION 2.—*On Differences and Sums of Vectors taken two by two.*

4. In order to be able to write, as in algebra,

$$(\mathrm{C}' - \mathrm{A}') - (\mathrm{B} - \mathrm{A}) = \mathrm{C} - \mathrm{B}, \text{ if } \mathrm{C}' - \mathrm{A}' = \mathrm{C} - \mathrm{A},$$

we next define, that when a first vector AB is *subtracted* from a second vector AC which is *co-initial* with it, or from a third vector A′C′ which is *equal* to that second vector, the *remainder* is that fourth vector BC, which is drawn from the term B of the first to the term C of the second vector: so that if a vector be subtracted from a *transvector* (Art. 1), the remainder is the *provector* corresponding. It is evident that this *geometrical subtraction* of *vectors* answers to a *decomposition* of *vections* (or of *motions*); and that, by such a decomposition of a *null* vection into two *opposite* vections, we have the formula,

$$0 - (\mathrm{B} - \mathrm{A}) = (\mathrm{A} - \mathrm{A}) - (\mathrm{B} - \mathrm{A}) = \mathrm{A} - \mathrm{B};$$

so that, if an *actual* vector AB be subtracted from a *null* vector AA, the remainder is the *revector* BA. If then we agree to *abridge*, generally, an expression of the form $0 - a$ to the shorter form, $-a$, we may write briefly, $-\mathrm{AB} = \mathrm{BA}$; a and $-a$ being thus symbols of *opposite* vectors, while a and $-(-a)$ are, for the same reason, symbols of one *common* vector: so that we may write, as in algebra, the *identity*,

$$-(-a) = a.$$

5. Aiming still at agreement with algebra, and adopting on that account the *formula of relation* between the *two signs*, + and −,

$$(b-a)+a=b,$$

in which we shall say as usual that $b-a$ is *added* to a, and that their *sum* is b, while relatively to it they may be jointly called *summands*, we shall have the two following consequences:

I. If a *vector*, AB or $\mathrm{B}-\mathrm{A}$, be *added* to its own *origin* A, the *sum* is its *term* B (Art. 1); and

II. If a *provector* BC be added to a vector AB, the sum is the *transvector* AC; or in symbols,

$$\text{I.}\,.\,(\mathrm{B}-\mathrm{A})+\mathrm{A}=\mathrm{B};\ \text{and II.}\,.\,(\mathrm{C}-\mathrm{B})+(\mathrm{B}-\mathrm{A})=\mathrm{C}-\mathrm{A}.$$

In fact, the first equation is an *immediate* consequence of the general formula which, as above, *connects* the *signs* + and −, when combined with the *conception* (Art. 1) of a *vector* as a *difference* of *two points*; and the second is a result of the same formula, combined with the *definition* of the geometrical *subtraction* of *one* such vector from *another*, which was assigned in Art. 4, and according to which we have (as in algebra) for *any three points*, A, B, C, the *identity*,

$$(\mathrm{C}-\mathrm{A})-(\mathrm{B}-\mathrm{A})=\mathrm{C}-\mathrm{B}.$$

It is clear that this *geometrical addition* of *successive vectors* corresponds (comp. Art. 4) to a *composition* of *successive vections*, or *motions*; and that the *sum* of two *opposite* vectors (or of vector and *revector*) is a *null line*; so that

$$\mathrm{BA}+\mathrm{AB}=0,\ \text{or}\ (\mathrm{A}-\mathrm{B})+(\mathrm{B}-\mathrm{A})=0.$$

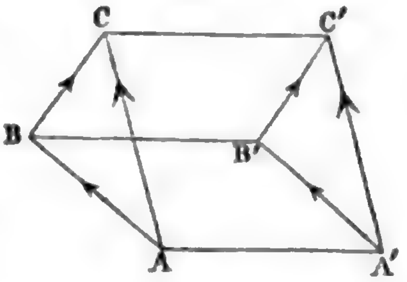

Fig. 7.

It follows also that the *sums* of *equal pairs* of successive vectors are *equal*; or more fully that

$$\text{if}\ \mathrm{B}'-\mathrm{A}'=\mathrm{B}-\mathrm{A},\ \text{and}\ \mathrm{C}'-\mathrm{B}'=\mathrm{C}-\mathrm{B},\ \text{then}\ \mathrm{C}'-\mathrm{A}'=\mathrm{C}-\mathrm{A};$$

the *two triangles*, ABC and A′B′C′, being in general the two *opposite faces* of a *prism* (comp. Art. 3).

6. Again, in order to have, as in algebra,

$$(C' - B') + (B - A) = C - A, \text{ if } C' - B' = C - B,$$

we shall define that if there be two successive vectors, AB, BC, and if a third vector B′C′ be *equal* to the second, but *not successive* to the first, the *sum* obtained by adding the third to the first is that fourth vector, AC, which is drawn from the origin A of the first to the term C of the second. It follows that the sum of any *two co-initial sides*, AB, AC, of any *parallelogram* ABDC, is the intermediate and co-initial *diagonal* AD; or, in symbols,

$$(C - A) + (B - A) = D - A, \text{ if } D - C = B - A;$$

because we have then (by 3) $C - A = D - B$.

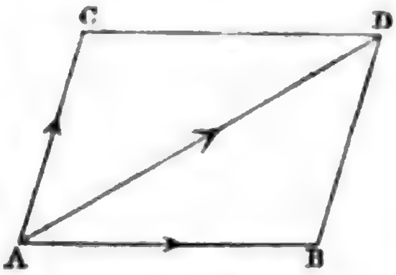

Fig. 8.

7. The *sum* of any *two* given vectors has thus a *value* which is independent of their *order*; or, in symbols, $\alpha + \beta = \beta + \alpha$. If *equal* vectors be added to *equal* vectors, the *sums* are equal vectors, even if the *summands* be not given as *successive* (comp. 5); and if a *null* vector be added to an *actual* vector, the sum is that actual vector; or, in symbols, $0 + \alpha = \alpha$. If then we agree to *abridge* generally (comp. 4) the expression $0 + \alpha$ to $+\alpha$, and if α still denote a *vector*, then $+\alpha$, and $+(+\alpha)$, &c., are *other* symbols for the *same* vector; and we have, as in algebra, the identities,

$$-(-\alpha) = +\alpha, \quad +(-\alpha) = -(+\alpha) = -\alpha, \quad (+\alpha) + (-\alpha) = 0, \text{ \&c.}$$

SECTION 3.—*On Sums of three or more Vectors.*

8. The *sum* of *three* given vectors, α, β, γ, is next defined to be that fourth vector,

$$\delta = \gamma + (\beta + \alpha), \quad \text{or briefly,} \quad \delta = \gamma + \beta + \alpha,$$

which is obtained by adding the third to the sum of the first and second; and in like manner the sum of *any number* of vectors is formed by adding the *last* to the sum of all that precede it: also, for any four vectors, α, β, γ, δ, the sum $\delta + (\gamma + \beta + \alpha)$ is denoted simply by $\delta + \gamma + \beta + \alpha$, without parentheses, and so on for any number of summands.

9. The sum of any number of *successive* vectors, AB, BC, CD, is thus the line AD, which is drawn from the origin A of the first, to the term D of the last; and because, when there are *three* such vectors, we can draw (as in Fig. 9) the two *diagonals* AC, BD of the (plane or gauche) quadrilateral ABCD, and may then at pleasure regard AD, either as the sum of AB, BD, or as the sum of AC, CD, we are allowed to establish the following general *formula of association*, for the case of *any three summand lines*, α, β, γ:

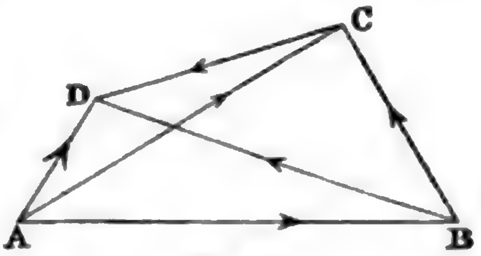

Fig. 9.

$$(\gamma + \beta) + \alpha = \gamma + (\beta + \alpha) = \gamma + \beta + \alpha;$$

by combining which with the *formula of commutation* (Art. 7), namely, with the equation,

$$\alpha + \beta = \beta + \alpha,$$

which had been previously established for the case of any *two* such summands, it is easy to conclude that the *Addition of Vectors* is always both an *Associative* and a *Commutative Operation.* In other words, the *sum* of *any number* of *given vectors* has a *value* which is independent of their *order*, and of the mode of *grouping* them; so that if the *lengths* and *directions* of the summands be *preserved*, the length and direction of the *sum* will *also* remain unchanged: except that this last *direction* may be regarded as *indeterminate*, when the *length* of the *sum-line* happens to *vanish*, as in the case which we are about to consider.

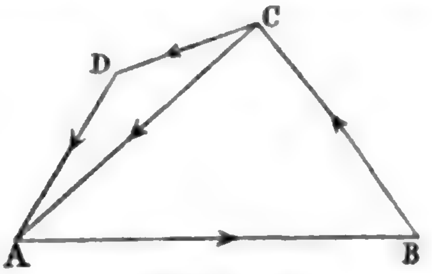

Fig. 10.

10. When any n summand-lines, AB, BC, CA, or AB, BC, CD, DA, &c., arranged in any one order, are the n *successive sides* of a *triangle* ABC, or of a *quadrilateral* ABCD, or of *any other closed polygon*, their *sum* is a *null line*, AA; and conversely,

when the sum of any given system of n vectors is thus equal to *zero*, they may be made (*in any order, by transports without rotation*) the n successive *sides* of a *closed* polygon (plane or gauche). Hence, if there be *given* any such polygon (P), suppose a *pentagon* ABCDE, it is possible to construct another closed polygon (P′), such as $A'B'C'D'E'$, with an *arbitrary initial point* A', but with the *same number* of sides, $A'B'$, .. $E'A'$, which *new sides* shall be *equal* (as vectors) to the *old* sides AB, .. EA, taken in any *arbitrary order*. For example, if we draw *four* successive vectors, as follows,

$$A'B' = CD, \quad B'C' = AB, \quad C'D' = EA, \quad D'E' = BC,$$

and then *complete* the *new pentagon* by drawing the line $E'A'$, this *closing side* of the *second* figure (P′) will be *equal* to the *remaining side* DE of the *first* figure (P).

and [illegible] may be made on [illegible]

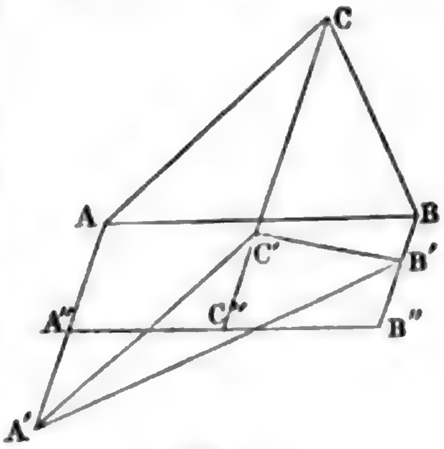

Fig. 11.

11. Since a *closed* figure ABC .. is *still* a closed one, when all its points are *projected* on any assumed *plane*, by any system of parallel ordinates (although the *area* of the projected figure $A'B'C'$... may happen to *vanish*), it follows that if the *sum* of any number of given vectors α, β, γ, .. be *zero*, and if we project them all *on any one plane* by *parallel lines* drawn from their extremities, the *sum* of the *projected vectors* α', β', γ', .. will *likewise* be *null*; so that these *latter* vectors, like the *former*, *can* be so *placed* as to *become* the *successive sides* of a *closed polygon*, even if they be not *already* such. (In Fig. 11, $A''B''C''$ is considered as *such* a polygon, namely, as a *triangle with evanescent area*; and we have the equation,

$$A''B'' + B''C'' + C''A'' = 0,$$

as well as

$$A'B' + B'C' + C'A' = 0, \text{ and } AB + BC + CA = 0.)$$

SECTION 4.—*On Coefficients of Vectors.*

12. The *simple* or *single* vector, a, is also denoted by $1a$, or by $1 \,.\, a$, or by $(+1)\,a$; and in like manner, the *double* vector, $a + a$, is denoted by $2a$, or $2 \,.\, a$, or $(+2)\,a$, &c.; the *rule* being, that for any algebraical integer, m, regarded as a *coefficient* by which the vector a is *multiplied*, we have always,

$$1a + ma = (1 + m)\,a\,;$$

the symbol $1 + m$ being here interpreted as in algebra. Thus, $0a = 0$, the zero on the one side denoting a *null coefficient*, and the zero on the other side denoting a *null vector;* because by the rule,

$$1a + 0a = (1 + 0)\,a = 1a = a, \text{ and } \therefore 0a = a - a = 0.$$

Again, because $(1)\,a + (-1)\,a = (1 - 1)\,a = 0a = 0$, we have $(-1)a = 0 - a = -a = -(1a)$; in like manner, since $(1)a + (-2)\,a = (1 - 2)\,a = (-1)\,a = -a$, we infer that $(-2)\,a = -a - a = -(2a)$; and generally, $(-m)\,a = -(ma)$, whatever whole number m may be: so that we may, without danger of confusion, *omit the parentheses* in these last symbols, and write simply, $-1a$, $-2a$, $-ma$.

13. It follows that *whatever two whole numbers* (positive or negative, or null) may be represented by m and n, and *whatever two vectors* may be denoted by a and β, we have always, as in algebra, the formulæ,

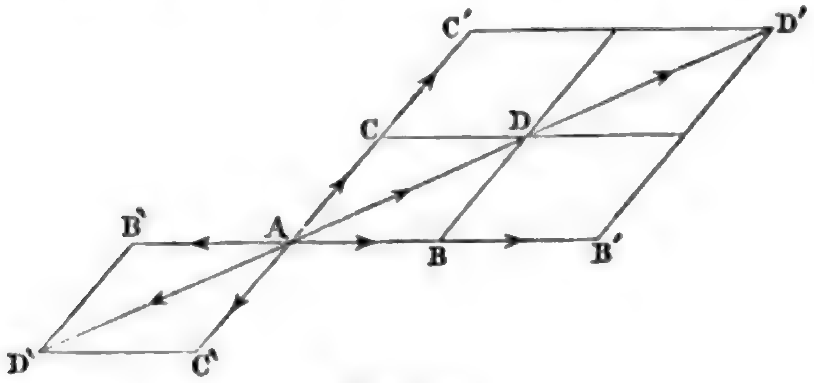

Fig. 12.

$$na \pm ma = (n \pm m)\,a, \qquad n\,(ma) = (nm)\,a = nma,$$

and (compare Fig. 12),

$$m\,(\beta \pm a) = m\beta \pm ma\,;$$

so that the *multiplication of vectors by coefficients* is a *doubly distributive operation*, at least if the *multipliers* be *whole numbers;* a restriction which, however, will soon be removed.

14. If $m\alpha = \beta$, the coefficient m being still *whole*, the vector β is said to be a *multiple* of α; and conversely (at least if the integer m be different from *zero*), the vector α is said to be a *sub-multiple* of β. A *multiple of a sub-multiple* of a vector is said to be a *fraction* of that vector; thus, if $\beta = m\alpha$, and $\gamma = n\alpha$, then γ is a fraction of β, which is denoted as follows, $\gamma = \frac{n}{m}\beta$; also β is said to be *multiplied* by the fractional coefficient $\frac{n}{m}$, and γ is said to be the *product* of this multiplication. It follows that if x and y be *any two fractions* (positive or negative or null, whole numbers being included), and if α and β be *any two vectors*, then

$$y\alpha \pm x\alpha = (y \pm x)\alpha, \quad y(x\alpha) = (yx)\alpha = yx\alpha, \quad x(\beta \pm \alpha) = x\beta \pm x\alpha;$$

results which include those of Art. 13, and may be extended to the case where x and y are *incommensurable coefficients*, considered as *limits* of *fractional* ones.

15. For any actual vector α, and for any coefficient x, of any of the foregoing kinds, the *product* $x\alpha$, interpreted as above, represents always a *vector* β, which has the *same direction* as the *multiplicand-line* α, if $x > 0$, but has the *opposite* direction if $x < 0$, becoming *null* if $x = 0$. Conversely, if α and β be *any two actual vectors*, with directions *either* similar *or* opposite, in *each* of which two cases we shall say that they are *parallel vectors*, and shall write $\beta \parallel \alpha$ (because *both* are then *parallel*, in the *usual* sense of the word, to *one common line*), we can always find, or conceive as found, a *coefficient* $x \gtrless 0$, which shall satisfy the equation $\beta = x\alpha$; or, as we shall also write it, $\beta = \alpha x$; and the positive or negative *number* x, so found, will bear to ± 1 the same *ratio*, as that which the *length* of the line β bears to the length of α.

16. Hence it is natural to say that this *coefficient* x is the *quotient* which results, from the *division of the vector* β, *by the parallel vector* α; and to write, accordingly,

$$x = \beta \div \alpha, \quad \text{or } x = \beta : \alpha, \quad \text{or } x = \frac{\beta}{\alpha};$$

so that we shall have, identically, as in algebra, at least if the *divisor-line* α be an *actual* vector, and if the *dividend-line* β be *parallel* thereto, the equations,

$$(\beta : \alpha) \cdot \alpha = \frac{\beta}{\alpha}\alpha = \beta, \quad \text{and} \quad x\alpha : \alpha = \frac{x\alpha}{\alpha} = x;$$

which will afterwards be *extended*, by *definition*, to the case of *non-parallel* vectors. We may write also, under the same conditions, $\alpha = \frac{\beta}{x}$, and may say that the *vector* α is the quotient of the *division* of the other vector β *by the number* x; so that we shall have these other identities,

$$\frac{\beta}{x} \cdot x = (\alpha x =) \beta, \quad \text{and} \quad \frac{\alpha x}{x} = \alpha.$$

17. The positive or negative *quotient*, $x = \frac{\beta}{\alpha}$, which is thus obtained by the *division* of one of two *parallel vectors* by another, including *zero* as a *limit*, may also be called a SCALAR; because it can always be found, and in a certain sense *constructed*, by the *comparison of positions upon one common scale* (or *axis*); or can be put under the form,

$$x = \frac{\text{C} - \text{A}}{\text{B} - \text{A}} = \frac{\text{AC}}{\text{AB}},$$

where the *three points*, A, B, C, are *collinear* (as in the figure annexed). Such *scalars* are, therefore, simply the REALS (or *real quantities*) of *Algebra*; but, in combination with the *not less real* VECTORS above considered, they form *one* of the *main elements* of the *System*, or *Calculus*, to

Fig. 13.

which the present work relates. In fact it will be shown, at a later stage, that there is an important sense in which we can conceive a scalar to be *added* to a vector; and that the *sum* so obtained, or the combination,

"*Scalar plus Vector,*"

is a QUATERNION.

◆

CHAPTER II.

APPLICATIONS TO POINTS AND LINES IN A GIVEN PLANE.

SECTION 1.—*On Linear Equations connecting two Co-initial Vectors.*

18. WHEN several vectors, OA, OB, . . are all drawn from one common point O, that point is said to be the *Origin of the System;* and each particular vector, such as OA, is said to be *the vector of its own term*, A. In the present and future sections we shall always suppose, if the contrary be not expressed, that all the vectors α, β, . . which we may have occasion to consider, are thus drawn from one *common origin*. But if it be desired to *change* that origin O, without changing the *term-points* A, . . we shall only have to *subtract*, from *each* of their *old* vectors α, . . one *common vector* ω, namely, the *old vector* OO′ of the *new origin* O′; since the *remainders*, $\alpha - \omega$, $\beta - \omega$, . . will be the *new vectors* α', β', . . of the *old points* A, B, . . . For example, we shall have

$$\alpha' = \mathrm{O'A} = \mathrm{A} - \mathrm{O'} = (\mathrm{A} - \mathrm{O}) - (\mathrm{O'} - \mathrm{O}) = \mathrm{OA} - \mathrm{OO'} = \alpha - \omega.$$

19. If *two* vectors α, β, or OA, OB, be thus drawn from a given origin O, and if their *directions* be either similar *or* opposite, so that the *three points*, O, A, B, are situated *on one right line* (as in the figure

Fig. 14.

annexed), then (by 16, 17) their *quotient* $\frac{\beta}{\alpha}$ is some positive or negative *scalar*, such as x; and conversely, the equation $\beta = x\alpha$, interpreted with this reference to an *origin*, expresses the *condition of collinearity*, of the points O, A, B; the particular *values*, $x = 0$, $x = 1$, corresponding to the particular *positions*, O and A, of the *variable point* B, whereof the *indefinite right line* OA is the *locus*.

20. The *linear equation*, connecting the *two* vectors α and β, acquires a more symmetric *form*, when we write it thus:

$$a\alpha + b\beta = 0;$$

where a and b are *two* scalars, of which however only the *ratio* is important. The *condition of coincidence*, of the two points A and B, answering above to $x = 1$, is now $\frac{-a}{b} = 1$; or, more symmetrically,

$$a + b = 0.$$

Accordingly, when $a = -b$, the linear equation becomes

$$b(\beta - \alpha) = 0, \quad \text{or} \quad \beta - \alpha = 0,$$

since we do not suppose that *both* the coefficients vanish; and the equation $\beta = \alpha$, or $\mathrm{OB} = \mathrm{OA}$, requires that the *point* B should *coincide* with the point A: a case which may also be conveniently expressed by the formula,

$$\mathrm{B} = \mathrm{A};$$

coincident points being thus treated (in *notation* at least) as *equal*. In general, the linear equation gives,

$$a \,.\, \mathrm{OA} + b \,.\, \mathrm{OB} = 0, \quad \text{and therefore} \quad a : b = \mathrm{BO} : \mathrm{OA}.$$

SECTION 2.—*On Linear Equations between three co-initial Vectors.*

21. If two (actual and co-initial) vectors, α, β, be *not* connected by *any* equation of the form $a\alpha + b\beta = 0$, with *any two* scalar coefficients a and b whatever, their *directions* can *neither* be similar *nor* opposite to each other; they therefore *determine*

a plane AOB, in which the (now actual) vector, represented by the sum $a\alpha + b\beta$, is situated. For if, for the sake of symmetry, we denote this sum by the symbol $-c\gamma$, where c is some *third scalar*, and $\gamma = \text{OC}$ is some *third vector*, so that the *three* co-initial vectors, α, β, γ, are connected by the *linear equation*,

$$a\alpha + b\beta + c\gamma = 0;$$

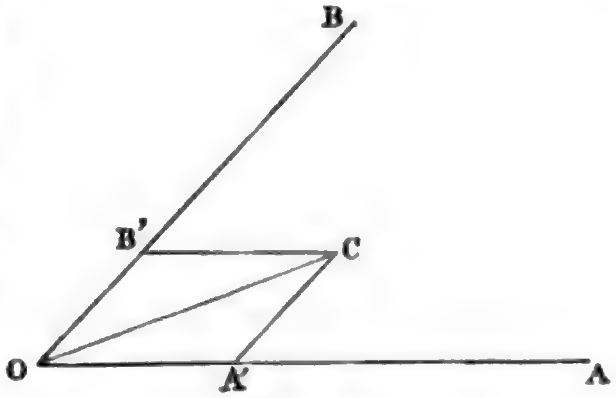

Fig. 15.

and if we make

$$\text{OA}' = \frac{-a\alpha}{c}, \qquad \text{OB}' = \frac{-b\beta}{c};$$

then the two auxiliary points, A′ and B′, will be situated (by 19) on the two indefinite right lines, OA, OB, respectively: and we shall have the equation,

$$\text{OC} = \text{OA}' + \text{OB}',$$

so that the figure A′OB′C is (by 6) a parallelogram, and consequently plane.

22. Conversely, if C be *any point in the plane* AOB, we can draw from it the *ordinates*, CA′ and CB′, to the lines OA and OB, and can determine the ratios of the three scalars, a, b, c, so as to satisfy the two equations,

$$\frac{a}{c} = -\frac{\text{OA}'}{\text{OA}}, \qquad \frac{b}{c} = -\frac{\text{OB}'}{\text{OB}};$$

after which we shall have the recent expressions for OA′, OB′, with the relation $\text{OC} = \text{OA}' + \text{OB}'$ as before; and shall thus be brought back to the linear equation $a\alpha + b\beta + c\gamma = 0$, which equation may therefore be said to express the *condition of complanarity* of the *four points*, O, A, B, C. And if we write it under the form,

$$x\alpha + y\beta + z\gamma = 0,$$

and consider the vectors α and β as *given*, but γ as a *variable vector*, while x, y, z *are variable scalars*, the *locus* of the *variable point* C will then be the *given plane*, OAB.

23. It may happen that the point C is situated *on the right line* AB, which is here considered as a *given* one. In that case (comp. Art. 17, Fig. 13), the quotient $\frac{\text{AC}}{\text{AB}}$ must be equal to some scalar, suppose t; so that we shall have an equation of the form,

$$\frac{\gamma - \alpha}{\beta - \alpha} = t, \quad \text{or} \quad \gamma = \alpha + t(\beta - \alpha), \quad \text{or} \quad (1 - t)\alpha + t\beta - \gamma = 0;$$

by comparing which last form with the linear equation of Art. 21, we see that the *condition of collinearity* of the three points A, B, C, in the given plane OAB, is expressed by the formula,

$$a + b + c = 0.$$

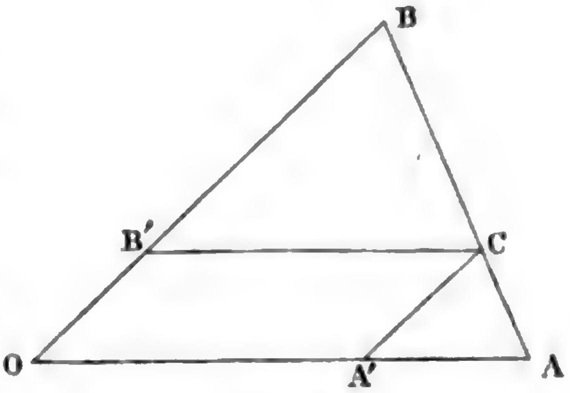

Fig. 16.

This condition may also be thus written,

$$1 = \frac{-a}{c} + \frac{-b}{c}, \quad \text{or} \quad \frac{\text{OA}'}{\text{OA}} + \frac{\text{OB}'}{\text{OB}} = 1;$$

and under this last form it expresses a *geometrical relation*, which is otherwise known to exist.

24. When we have thus the *two* equations,

$$a\alpha + b\beta + c\gamma = 0, \quad \text{and} \quad a + b + c = 0,$$

so that the three co-initial vectors α, β, γ terminate on one right line, and may on that account be said to be *termino-collinear*, if we eliminate, successively and separately, each of the three scalars a, b, c, we are conducted to these three other equations, expressing certain *ratios of segments* :

$$b(\beta - \alpha) + c(\gamma - \alpha) = 0, \qquad c(\gamma - \beta) + a(\alpha - \beta) = 0,$$
$$a(\alpha - \gamma) + b(\beta - \gamma) = 0;$$

or

$$0 = b.\text{AB} + c.\text{AC} = c.\text{BC} + a.\text{BA} = a.\text{CA} + b.\text{CB}.$$

Hence follows this *proportion*, between *coefficients* and *segments*,

$$a : b : c = \text{BC} : \text{CA} : \text{AB}.$$

We might also have observed that the proposed equations give,

$$a = \frac{b\beta + c\gamma}{b + c}, \qquad \beta = \frac{c\gamma + a a}{c + a}, \qquad \gamma = \frac{a a + b\beta}{a + b};$$

whence

$$\frac{\text{AC}}{\text{AB}} = \frac{\gamma - a}{\beta - a} = \frac{b}{a + b} = -\frac{b}{c}, \text{ \&c.}$$

25. If we still treat a and β as *given*, but regard γ and $\frac{y}{x}$ as *variable*, the equation

$$\gamma = \frac{xa + y\beta}{x + y}$$

will express that the *variable point* C is situated *somewhere* on the *indefinite right line* AB, or that it has this *line* for its *locus:* while it *divides* the *finite* line AB into *segments*, of which the *variable quotient* is,

$$\frac{\text{AC}}{\text{CB}} = \frac{y}{x}.$$

Let C′ be *another point* on the *same line*, and let its vector be,

$$\gamma' = \frac{x'a + y'\beta}{x' + y'};$$

then, in like manner, we shall have this *other ratio of segments*,

$$\frac{\text{AC}'}{\text{C}'\text{B}} = \frac{y'}{x'}.$$

If, then, we agree to employ, generally, *for any group of four collinear points*, the *notation*,

$$(\text{ABCD}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DA}} = \frac{\text{AB}}{\text{BC}} : \frac{\text{AD}}{\text{DC}};$$

so that this *symbol*,

$$(\text{ABCD}),$$

may be said to denote the *anharmonic function*, or *anharmonic quotient*, or simply the *anharmonic of the group*, A, B, C, D: we shall have, in the present case, the equation,

$$(\text{ACBC}') = \frac{\text{AC}}{\text{CB}} : \frac{\text{AC}'}{\text{C}'\text{B}} = \frac{yx'}{xy'}.$$

26. When the *anharmonic quotient* becomes equal to *negative unity*, the *group* becomes (as is well known) *harmonic*. If then we have the two equations,

$$\gamma = \frac{x\alpha + y\beta}{x + y}, \qquad \gamma' = \frac{x\alpha - y\beta}{x - y},$$

the two points C and C′ are *harmonically conjugate* to each other, with respect to the *two given points*, A and B; and when they *vary together*, in consequence of the variation of the value of $\frac{y}{x}$, they form (in a well-known sense), on the indefinite right line AB, *divisions in involution*; the *double points* (or *foci*) of this involution, namely, the points of which each is *its own* conjugate, being the points A and B themselves. As a verification, if we denote by μ the vector of the middle point M of the given interval AB, so that

$$\beta - \mu = \mu - \alpha, \text{ or } \mu = \tfrac{1}{2}(\alpha + \beta),$$

we easily find that

$$\frac{\gamma - \mu}{\beta - \mu} = \frac{y - x}{y + x} = \frac{\beta - \mu}{\gamma' - \mu}, \qquad \text{or} \quad \frac{\mathrm{MC}}{\mathrm{MB}} = \frac{\mathrm{MB}}{\mathrm{MC'}};$$

so that the *rectangle* under the *distances* MC, MC′, of the two *variable* but *conjugate points*, C, C′, from the *centre* M of the involution, is equal to the *constant square* of *half* the *interval* between the two *double points*, A, B.

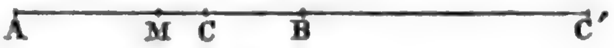

Fig. 17.

More generally, if we write

$$\gamma = \frac{x\alpha + y\beta}{x + y}, \qquad \gamma' = \frac{lx\alpha + my\beta}{lx + my},$$

where the anharmonic quotient $\frac{l}{m} = \frac{yx'}{xy'}$ is *any constant scalar*, then in another known and modern[*] phraseology, the points C and C′ will form, on the indefinite line AB, *two homographic divisions*, of which A and B are still the *double points*. More generally still, if we establish the two equations,

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[*] See the *Géométrie Supérieure* of M. Chasles, p. 107. (Paris, 1852.)

$$\gamma = \frac{x\alpha + y\beta}{x + y}, \quad \text{and} \quad \gamma' = \frac{lx\alpha' + my\beta'}{lx + my},$$

$\frac{l}{m}$ being still constant, but $\frac{y}{x}$ variable, while $\alpha' = \mathrm{OA}'$, $\beta' = \mathrm{OB}'$, and $\gamma' = \mathrm{OC}'$, the *two given lines*, AB and A′B′, are then *homographically divided*, by the *two variable points*, C and C′, not now supposed to move along one *common* line.

27. When the linear equation $a\alpha + b\beta + c\gamma = 0$ subsists, *without* the relation $a + b + c = 0$ between its coefficients, then the three co-initial vectors α, β, γ are still *complanar*, but they no longer terminate *on one right line*; their *term-points* A, B, C being now the corners of a *triangle*.

In this more general case, we may propose to find the vectors α', β', γ' of the three points,

$$\mathrm{A}' = \mathrm{OA} \cdot \mathrm{BC}, \quad \mathrm{B}' = \mathrm{OB} \cdot \mathrm{CA}, \quad \mathrm{C}' = \mathrm{OC} \cdot \mathrm{AB};$$

that is to say, of the points in which the lines drawn from the origin O to the three *corners* of the triangle *intersect* the three respectively opposite *sides*. The three *collineations* OAA′, &c., give (by 19) three expressions of the forms,

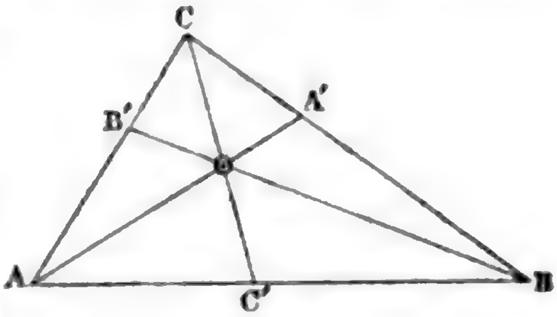

Fig. 18.

$$\alpha' = x\alpha, \qquad \beta' = y\beta, \qquad \gamma' = z\gamma,$$

where x, y, z are three scalars, which it is required to determine by means of the three *other* collineations, A′BC, &c., with the help of relations derived from the principle of Art. 23. Substituting therefore for α its value $x^{-1}\alpha'$, in the *given* linear equation, and equating to zero the sum of the coefficients of the *new* linear equation which results, namely,

$$x^{-1}a\alpha' + b\beta + c\gamma;$$

and eliminating similarly β, γ, each in its turn, from the original equation; we find the values,

$$x = \frac{-a}{b + c}, \qquad y = \frac{-b}{c + a}, \qquad z = \frac{-c}{a + b};$$

whence the sought vectors are expressed in either of the two following ways:

$$\text{I.}\ \ldots\ \alpha' = \frac{-a\alpha}{b+c}, \qquad \beta' = \frac{-b\beta}{c+a}, \qquad \gamma' = \frac{-c\gamma}{a+b};$$

or

$$\text{II.}\ \ldots\ \alpha' = \frac{b\beta + c\gamma}{b+c}, \qquad \beta' = \frac{c\gamma + a\alpha}{c+a}, \qquad \gamma' = \frac{a\alpha + b\beta}{a+b}.$$

In fact we see, by *one* of these expressions for α', that A′ is on the line OA; and by the *other* expression for the same vector α', that the same point A′ is on the line BC. As another verification, we may observe that the last expressions for α', β', γ', coincide with those which were found in Art. 24, for α, β, γ themselves, on the particular supposition that the three points A, B, C were collinear.

28. We may next propose to determine the *ratios* of the *segments* of the *sides* of the triangle ABC, made by the points A′, B′, C′. For this purpose, we may write the last equations for α', β', γ' under the form,

$$0 = b(\alpha' - \beta) - c(\gamma - \alpha') = c(\beta' - \gamma) - a(\alpha - \beta') = a(\gamma' - \alpha) - b(\beta - \gamma');$$

and we see that they then give the required ratios, as follows:

$$\frac{\text{BA}'}{\text{A}'\text{C}} = \frac{c}{b}, \qquad \frac{\text{CB}'}{\text{B}'\text{A}} = \frac{a}{c}, \qquad \frac{\text{AC}'}{\text{C}'\text{B}} = \frac{b}{a};$$

whence we obtain at once the known *equation of six segments*,

$$\frac{\text{BA}'}{\text{A}'\text{C}} \cdot \frac{\text{CB}'}{\text{B}'\text{A}} \cdot \frac{\text{AC}'}{\text{C}'\text{B}} = 1,$$

as the *condition of concurrence* of the three right lines AA′, BB′, CC′, in a *common point*, such as O. It is easy also to infer, from the same ratios of segments, the following *proportion* of *coefficients* and *areas*,

$$a : b : c = \text{OBC} : \text{OCA} : \text{OAB},$$

in which we must, in general, attend to algebraic *signs*; a *triangle* being conceived to *pass* (*through zero*) from *positive* to *negative*, or *vice versâ*, as *compared* with any *given* triangle in its own plane, when (in the course of any continuous change) its *vertex crosses* its *base*. It may be observed that *with this convention* (which is, in fact, a *necessary* one, for the establishment of *general formulæ*) we have, for *any three points*, the equation

$$\text{ABC} + \text{BAC} = 0,$$

exactly as we had (in Art. 5) for any *two* points, the equation

$$\text{AB} + \text{BA} = 0.$$

More fully, we have, on this plan, the formulæ,

$$\text{ABC} = -\,\text{BAC} = \text{BCA} = -\,\text{CBA} = \text{CAB} = -\,\text{ACB};$$

and *any two complanar triangles*, ABC, A′B′C′, bear *to each other* a *positive* or a *negative ratio*, according as the two *rotations*, which may be conceived to be denoted by the same *symbols* ABC, A′B′C′, are *similarly* or *oppositely directed*.

29. If A′ and B′ *bisect* respectively the sides BC and CA, then

$$a = b = c,$$

and C′ bisects AB; whence the known theorem follows, that *the three bisectors of the sides of a triangle concur*, in a point which is often called the *centre of gravity*, but which we prefer to call the *mean point* of the triangle, and which is here the *origin* O. At the same time, the first expressions in Art. 27 for α', β', γ' become,

$$\alpha' = -\frac{\alpha}{2}, \qquad \beta' = -\frac{\beta}{2}, \qquad \gamma' = -\frac{\gamma}{2};$$

whence this other known theorem results, that *the three bisectors trisect each other*.

30. The linear equation between α, β, γ reduces itself, in the case last considered, to the form,

$$\alpha + \beta + \gamma = 0, \quad \text{or} \quad \text{OA} + \text{OB} + \text{OC} = 0;$$

the three vectors α, β, γ, or OA, OB, OC, are therefore, in this case, adapted (by Art. 10) to become the *successive sides* of a *triangle*, by *transports without rotation* ; and accordingly, if we complete (as in Fig. 19) the parallelogram AOBD, the triangle OAD will have the property in question. It follows (by 11) that if we *project* the *four points* O, A, B, C, by any system of *parallel ordinates*, into four *other* points, $\mathrm{O}_{,}$, $\mathrm{A}_{,}$, $\mathrm{B}_{,}$, $\mathrm{C}_{,}$, on any assumed *plane*, the *sum* of the three *projected vectors*, $\alpha_{,}$, $\beta_{,}$, $\gamma_{,}$, or $\mathrm{O_{,}A_{,}}$, &c., will be *null;* so that we shall have the *new linear equation*,

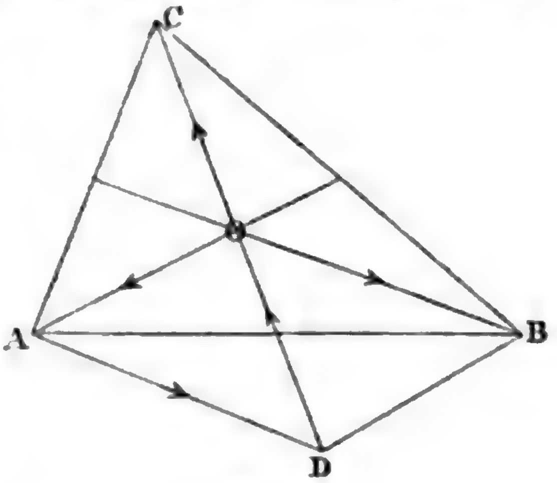

Fig. 19.

$$\alpha_{,}+\beta_{,}+\gamma_{,}=0,$$

or,

$$\mathrm{O_{,}A_{,}}+\mathrm{O_{,}B_{,}}+\mathrm{O_{,}C_{,}}=0\,;$$

and in fact it is evident (see Fig. 20) that the *projected mean point* $\mathrm{O}_{,}$ will be the *mean point of the projected triangle*, $\mathrm{A}_{,}$, $\mathrm{B}_{,}$, $\mathrm{C}_{,}$. We shall have also the equation,

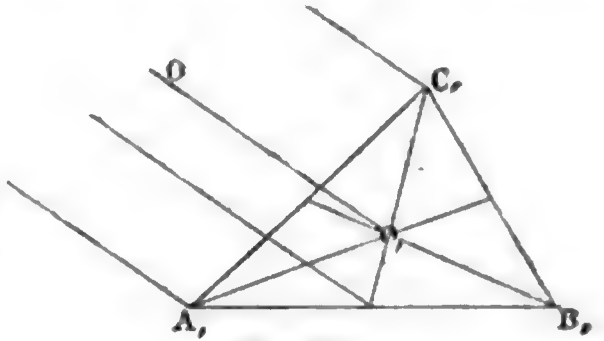

Fig. 20.

$$(\alpha_{,}-\alpha)+(\beta_{,}-\beta)+(\gamma_{,}-\gamma)=0\,;$$

where

$$\alpha_{,}-\alpha=\mathrm{O_{,}A_{,}}-\mathrm{OA}=(\mathrm{O_{,}A}+\mathrm{AA_{,}})-(\mathrm{OO_{,}}+\mathrm{O_{,}A})=\mathrm{AA_{,}}-\mathrm{OO_{,}}\,;$$

hence

$$\mathrm{OO_{,}}=\tfrac{1}{3}(\mathrm{AA_{,}}+\mathrm{BB_{,}}+\mathrm{CC_{,}}),$$

or the *ordinate of the mean point of a triangle* is the *mean* of the *ordinates of the three corners*.

SECTION 3.—*On Plane Geometrical Nets.*

31. Resuming the more general case of Art. 27, in which the coefficients a, b, c are supposed to be *unequal*, we may next inquire, in what points A'', B'', C'' do the lines $\mathrm{B'C'}$, $\mathrm{C'A'}$, $\mathrm{A'B'}$ meet respectively the sides BC, CA, AB, of the triangle; or may seek to assign the vectors α'', β'', γ'' of the points of intersection (comp. 27),

$$\text{A}'' = \text{B}'\text{C}' \cdot \text{BC}, \qquad \text{B}'' = \text{C}'\text{A}' \cdot \text{CA}, \qquad \text{C}'' = \text{A}'\text{B}' \cdot \text{AB}.$$

The first expressions in Art. 27 for β', γ', give the equations,

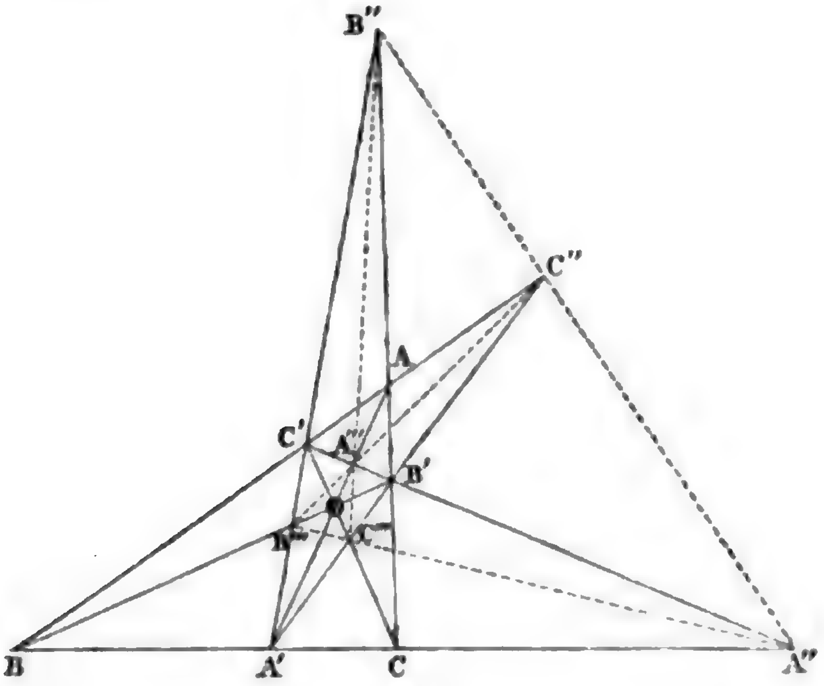

Fig. 21.

$$(c+a)\beta' + b\beta = 0, \qquad (a+b)\gamma' + c\gamma = 0;$$

whence

$$\frac{b\beta - c\gamma}{b-c} = \frac{(a+b)\gamma' - (c+a)\beta'}{(a+b)-(c+a)};$$

but (by 25) one member is the vector of a point on BC, and the other of a point on B′C′; each therefore is a value for the vector α'' of A″, and similarly for β'' and γ''. We may therefore write,

$$\alpha'' = \frac{b\beta - c\gamma}{b-c}, \qquad \beta'' = \frac{c\gamma - a\alpha}{c-a}, \qquad \gamma'' = \frac{a\alpha - b\beta}{a-b};$$

and by comparing these expressions with the second set of values of α', β', γ' in Art. 27, we see (by 26) that the points A″, B″, C″ are, respectively, the *harmonic conjugates* (as they are indeed known to be) of the points A′, B′, C′, with respect to the three pairs of points, B, C; C, A; A, B; so that, in the notation of Art. 25, we have the equations,

$$(\text{BA}'\text{CA}'') = (\text{CB}'\text{AB}'') = (\text{AC}'\text{BC}'') = -1.$$

And because the expressions for α'', β'', γ'' conduct to the following linear equation between those three vectors,

$$(b - c)\alpha'' + (c - a)\beta'' + (a - b)\gamma'' = 0,$$

with the relation

$$(b - c) + (c - a) + (a - b) = 0$$

between its coefficients, we arrive (by 23) at this other known theorem, that *the three points* A'', B'', C'' *are collinear*, as indicated by one of the *dotted lines* in the recent Fig. 21.

32. The line $\mathrm{A''B'C'}$ may represent *any rectilinear transversal*, cutting the sides of a triangle ABC; and because we have

$$\frac{\mathrm{BA''}}{\mathrm{A''C}} = \frac{\alpha'' - \beta}{\gamma - \alpha''} = -\frac{c}{b},$$

while $\frac{\mathrm{CB'}}{\mathrm{B'A}} = \frac{a}{c}$, and $\frac{\mathrm{AC'}}{\mathrm{C'B}} = \frac{b}{a}$, as before, we arrive at this *other equation of six segments*, for any triangle cut by a right line (comp. 28),

$$\frac{\mathrm{BA''}}{\mathrm{A''C}} \cdot \frac{\mathrm{CB'}}{\mathrm{B'A}} \cdot \frac{\mathrm{AC'}}{\mathrm{C'B}} = -1;$$

which again agrees with known results.

33. Eliminating β and γ between either set of expressions (27) for β' and γ', with the help of the given linear equation, we arrive at this other equation, connecting the three vectors α, β', γ':

$$0 = -a\alpha + (c + a)\beta' + (a + b)\gamma'.$$

Treating this on the same plan as the given equation between α, β, γ, we find that if (as in Fig. 21) we make,

$$\mathrm{A}''' = \mathrm{OA} \cdot \mathrm{B'C'}, \qquad \mathrm{B}''' = \mathrm{OB} \cdot \mathrm{C'A'}, \qquad \mathrm{C}''' = \mathrm{OC} \cdot \mathrm{A'B'},$$

the vectors of these three new points of intersection may be expressed in either of the two following ways, whereof the first is shorter, but the second is, for some purposes (comp. 34, 36) more convenient:

$$\text{I.} \ldots \alpha''' = \frac{a\alpha}{2a + b + c}, \qquad \beta''' = \frac{b\beta}{2b + c + a}, \qquad \gamma''' = \frac{c\gamma}{2c + a + b};$$

or

$$\text{II.} \ldots \alpha''' = \frac{2a\alpha + b\beta + c\gamma}{2a + b + c}, \qquad \beta''' = \frac{2b\beta + c\gamma + a\alpha}{2b + c + a},$$

$$\gamma''' = \frac{2c\gamma + a\alpha + b\beta}{2c + a + b}.$$

And the three equations, of which the following is one,

$$(b - c)\alpha'' - (2b + c + a)\beta''' + (2c + a + b)\gamma''' = 0,$$

with the relations between their coefficients which are evident on inspection, show (by 23) that we have the three additional *collineations*, A''B'''C''', B''C'''A''', C''A'''B''', as indicated by three of the dotted lines in the figure. Also, because we have the two expressions,

$$\alpha''' = \frac{(a + b)\gamma' + (c + a)\beta'}{(a + b) + (c + a)}, \qquad \alpha'' = \frac{(a + b)\gamma' - (c + a)\beta'}{(a + b) - (c + a)},$$

we see (by 26) that the two points A'', A''' are *harmonically conjugate* with respect to B' and C'; and similarly for the two other pairs of points, B'', B''', and C'', C''', compared with C', A', and with A', B': so that, in a notation already employed (25, 31), we may write,

$$(\text{B}'\text{A}'''\text{C}'\text{A}'') = (\text{C}'\text{B}'''\text{A}'\text{B}'') = (\text{A}'\text{C}'''\text{B}'\text{C}'') = -1.$$

34. If we *begin*, as above, with any *four complanar points*, O, A, B, C, of which no three are collinear, we can (as in Fig. 18), by what may be called a *First Construction*, derive from them six lines, connecting them two by two, and intersecting each other in three new points, A', B', C'; and then by a *Second Construction* (represented in Fig. 21), we may connect these by three new lines, which will give, by their intersections with the former lines, six new points, A'', . . C'''. We might proceed to connect these with each other, and with the given points, by sixteen new lines, or lines of a *Third Construction*, namely, the four dotted lines of Fig. 21, and twelve other lines, whereof three should be drawn from each of the four given points: and these would be found to determine eighty-four new points of intersection, of which some may be seen, although they are not marked, in the figure.

But *however far* these processes of *linear construction* may be continued, so as to form what has been called* a *plane*

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* By Prof. A. F. Möbius, in page 274 of his *Barycentric Calculus* (der barycentrische Calcul, Leipzig, 1827).

geometrical net, the *vectors* of the points thus determined have all one *common property:* namely, that each can be represented by an expression of the form,

$$\rho = \frac{xa\alpha + yb\beta + zc\gamma}{xa + yb + zc};$$

where the *coefficients* x, y, z are some *whole numbers*. In fact we see (by 27, 31, 33) that such expressions can be assigned for the *nine* derived vectors, $\alpha', \ldots \gamma'''$, which alone have been hitherto considered; and it is not difficult to perceive, from the nature of the calculations employed, that a similar result must hold good, for every vector subsequently deduced. But this and other connected results will become more completely evident, and their *geometrical signification* will be better understood, after a somewhat closer consideration of *anharmonic quotients*, and the introduction of a certain system of *anharmonic co-ordinates*, for points and lines in one plane, to which we shall next proceed: reserving, for a subsequent Chapter, any applications of the same theory to *space*.

SECTION 4.—*On Anharmonic Co-ordinates and Equations of Points and Lines in one Plane.*

35. If we compare the last equations of Art. 33 with the corresponding equations of Art. 31, we see that the *harmonic group* BA′CA″, on the side BC of the triangle ABC in Fig. 21, has been simply *reflected* into another such group, B′A‴C′A″, on the line B′C′, by a *harmonic pencil* of four rays, all passing through the point O; and similarly for the other groups. More generally, let OA, OB, OC, OD, or briefly O . ABCD, be *any* pencil, with the point O for *vertex;* and let the *new ray* OD be cut, as in Fig. 22, by the three sides of the triangle ABC, in the three points A_1, B_1, C_1; let also

$$\mathrm{OA}_1 = \alpha_1 = \frac{yb\beta + zc\gamma}{yb + zc},$$

so that (by 25) we shall have the anharmonic quotients,

$$(\mathrm{BA'CA_1}) = \frac{y}{z}, \qquad (\mathrm{CA'BA_1}) = \frac{z}{y};$$

and let us seek to express the two other vectors of intersection, β_1 and γ_1, with a view to determining the anharmonic ratios of the groups on the two other sides. The given equation (27),

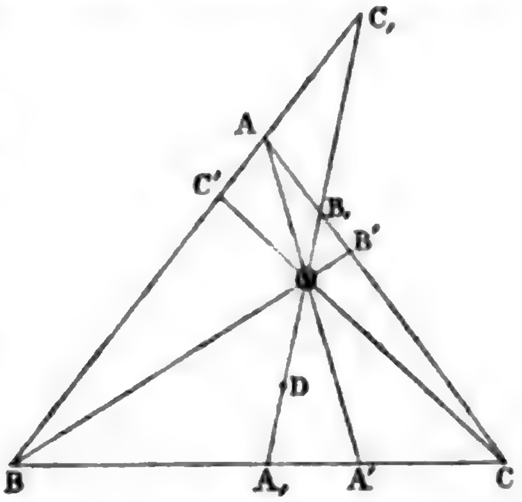

Fig. 22.

$$a\alpha + b\beta + c\gamma = 0,$$

shows us at once that these two vectors are,

$$\text{OB}_1 = \beta_1 = \frac{(y-z)c\gamma + ya\alpha}{(y-z)c + ya};$$

$$\text{OC}_1 = \gamma_1 = \frac{(z-y)b\beta + za\alpha}{(z-y)b + za};$$

whence we derive (by 25) these two other anharmonics,

$$(\text{CB}'\text{AB}_1) = \frac{y-z}{y}; \qquad (\text{BC}'\text{AC}_1) = \frac{z-y}{z};$$

so that we have the relations,

$$(\text{CB}'\text{AB}_1) + (\text{CA}'\text{BA}_1) = (\text{BC}'\text{AC}_1) + (\text{BA}'\text{CA}_1) = 1.$$

But in general, for any four collinear points A, B, C, D, it is not difficult to prove that

$$\frac{\text{AB}}{\text{BC}} \cdot \text{CD} + \frac{\text{AC}}{\text{CB}} \cdot \text{BD} = \text{DA};$$

whence by the definition (25) of the signification of the symbol (ABCD), the following identity is derived,

$$(\text{ABCD}) + (\text{ACBD}) = 1.$$

Comparing this, then, with the recently found relations, we have, for Fig. 22, the following anharmonic equations:

$$(\text{CAB}'\text{B}_1) = (\text{CA}'\text{BA}_1) = \frac{z}{y};$$

$$(\text{BAC}'\text{C}_1) = (\text{BA}'\text{CA}_1) = \frac{y}{z};$$

and we see that (as was to be expected from known principles) the anharmonic of the *group* does not change, when we pass from one side of the triangle, considered as a *transversal* of the pencil, to another such side, or transversal. We may therefore speak (as usual) of such an anharmonic of a *group*, as being at the same time the *Anharmonic of a Pencil;* and, with attention to the *order of the rays*, and to the definition (25), may *denote* the two last anharmonics by the two following reciprocal expressions:

$$(\mathrm{O\,.\,CABD}) = \frac{z}{y}; \qquad (\mathrm{O\,.\,BACD}) = \frac{y}{z};$$

with other resulting values, when the *order* of the rays is changed; it being understood that

$$(\mathrm{O\,.\,CABD}) = (\mathrm{C'A'B'D'}),$$

if the rays OC, OA, OB, OD be cut, in the points C', A', B', D', by any one right line.

36. The expression (34),

$$\rho = \frac{xa\alpha + yb\beta + zc\gamma}{xa + yb + zc},$$

may represent the vector of *any point* P *in the given plane*, by a suitable choice of the *coefficients* x, y, x, or simply of their *ratios*. For since (by 22) the three complanar vectors PA, PB, PC must be connected by some linear equation, of the form

$$a'\,.\,\mathrm{PA} + b'\,.\,\mathrm{PB} + c'\,.\,\mathrm{PC} = 0,$$

or

$$a'(\alpha - \rho) + b'(\beta - \rho) + c'(\gamma - \rho) = 0,$$

which gives

$$\rho = \frac{a'\alpha + b'\beta + c'\gamma}{a' + b' + c'},$$

we have only to write

$$\frac{a'}{a} = x, \qquad \frac{b'}{b} = y, \qquad \frac{c'}{c} = z,$$

and the proposed expression for ρ will be obtained. Hence it is easy to infer, on principles already explained, that if we write (compare the annexed Fig. 23),

$$\text{P}_1 = \text{PA}\cdot\text{BC}, \qquad \text{P}_2 = \text{PB}\cdot\text{CA}, \qquad \text{P}_3 = \text{PC}\cdot\text{AB},$$

we shall have, with the same coefficients xyz, the following expressions for the vectors OP_1, OP_2, OP_3, or ρ_1, ρ_2, ρ_3, of these three points of intersection, P_1, P_2, P_3:

$$\rho_1 = \frac{yb\beta + zc\gamma}{yb + zc}, \qquad \rho_2 = \frac{zc\gamma + xa\alpha}{zc + xa},$$

$$\rho_3 = \frac{xa\alpha + yb\beta}{xa + yb};$$

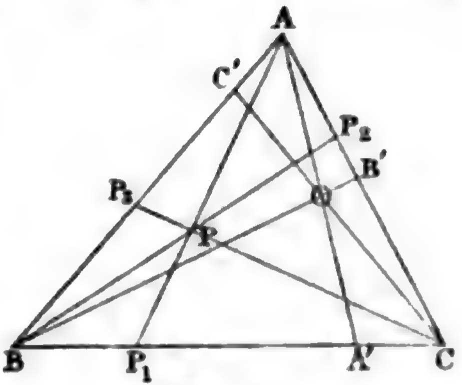

Fig. 28.

which give at once the following anharmonics of pencils, or of groups,

$$(\text{A}\,.\,\text{BOCP}) = (\text{BA}'\text{CP}_1) = \frac{y}{z};$$

$$(\text{B}\,.\,\text{COAP}) = (\text{CB}'\text{AP}_2) = \frac{z}{x};$$

$$(\text{C}\,.\,\text{AOBP}) = (\text{AC}'\text{BP}_3) = \frac{x}{y};$$

whereof we see that the product is unity. *Any two of these three pencils* suffice to *determine the position of the point* P, when the *triangle* ABC, and the *origin* O are given; and therefore it appears that the *three coefficients x, y, z*, or *any scalars proportional to them*, of which the *quotients* thus represent the *anharmonics* of those pencils, may be conveniently called the ANHARMONIC CO-ORDINATES *of that point*, P, *with respect to the given triangle and origin:* while the point P itself may be *denoted* by the *Symbol*,

$$\text{P} = (x, y, z).$$

With this notation, the thirteen points of Fig. 21 come to be thus symbolized:

$\text{A} = (1, 0, 0)$, $\text{B} = (0, 1, 0)$, $\text{C} = (0, 0, 1)$, $\text{O} = (1, 1, 1)$;
$\text{A}' = (0, 1, 1)$, $\text{B}' = (1, 0, 1)$, $\text{C}' = (1, 1, 0)$;
$\text{A}'' = (0, 1, -1)$, $\text{B}'' = (-1, 0, 1)$, $\text{C}'' = (1, -1, 0)$;
$\text{A}''' = (2, 1, 1)$, $\text{B}''' = (1, 2, 1)$, $\text{C}''' = (1, 1, 2)$.

37. If P_1 and P_2 be any two points in the given plane,

$$P_1 = (x_1, y_1, z_1), \qquad P_2 = (x_2, y_2, z_2),$$

and if t and u be any two scalar coefficients, then the following *third point*,

$$P = (tx_1 + ux_2,\ ty_1 + uy_2,\ tz_1 + uz_2),$$

is *collinear* with the two former points, or (in other words) is situated *on the right line* P_1P_2. For, if we make

$$x = tx_1 + ux_2, \qquad y = ty_1 + uy_2, \qquad z = tz_1 + uz_2,$$

and

$$\rho_1 = \frac{x_1 a\alpha + \,.\,.}{x_1 a + \,.\,.}, \qquad \rho_2 = \frac{x_2 a\alpha + \,.\,.}{x_2 a + \,.\,.}, \qquad \rho = \frac{xa\alpha + \,.\,.}{xa + \,.\,.},$$

these *vectors* of the three points P_1P_2P are connected by the *linear equation*,

$$t(x_1a + \,.\,.)\rho_1 + u(x_2a + \,.\,.)\rho_2 - (xa + \,.\,.)\rho = 0;$$

in which (comp. 23), the *sum* of the *coefficients* is *zero*. Conversely, the point P cannot be collinear with P_1, P_2, unless its co-ordinates admit of being thus expressed in terms of theirs. It follows that if a *variable point* P be obliged to *move along a given right line* P_1P_2, or if it have such a *line* (in the given plane) for its *locus*, its *co-ordinates* xyz must satisfy a *homogeneous equation of the first degree, with constant coefficients*; which, in the known notation of determinants, may be thus written,

$$0 = \begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix};$$

or, more fully,

$$0 = x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(x_1y_2 - y_1x_2);$$

or briefly,

$$0 = lx + my + nz,$$

where l, m, n are *three constant scalars*, whereof the *quotients* determine the *position* of the right *line* Λ, which is thus the *locus* of the point P. It is natural to call the *equation*, which thus *connects* the *co-ordinates* of the *point* P, the *Anharmonic Equation of the Line* Λ; and we shall find it convenient also to speak of the *coefficients* l, m, n, in that equation, as being the *Anharmonic Co-ordinates of that Line:* which line may also be *denoted* by the *Symbol*,

$$\Lambda = [l, m, n].$$

38. For example, the three sides BC, CA, AB of the given triangle have thus for their *equations*,

$$x = 0, \qquad y = 0, \qquad z = 0,$$

and for their *symbols*,

$$[1, 0, 0], \qquad [0, 1, 0], \qquad [0, 0, 1].$$

The three additional lines OA, OB, OC, of Fig. 18, have, in like manner, for their equations and symbols,

$$y - z = 0, \qquad z - x = 0, \qquad x - y = 0,$$
$$[0, 1, -1], \qquad [-1, 0, 1], \qquad [1, -1, 0].$$

The lines B′C′A″, C′A′B″, A′B′C″, of Fig. 21, are

$$y + z - x = 0, \qquad z + x - y = 0, \qquad x + y - z = 0,$$

or

$$[-1, 1, 1], \qquad [1, -1, 1], \qquad [1, 1, -1];$$

the lines A″B‴C‴, B″C‴A‴, C″A‴B‴, of the same figure, are in like manner represented by the equations and symbols,

$$y + z - 3x = 0, \qquad z + x - 3y = 0, \qquad x + y - 3z = 0,$$
$$[-3, 1, 1], \qquad [1, -3, 1], \qquad [1, 1, -3];$$

and the line A″B″C″ is

$$x + y + z = 0, \quad \text{or} \quad [1, 1, 1].$$

Finally, we may remark that on the same plan, the equation and the symbol of what is often called the *line at infinity*, or of the *locus of all the infinitely distant points in the given plane*, are respectively,

$$ax + by + cz = 0, \quad \text{and} \quad [a, b, c];$$

because the *linear function*, $ax + by + cz$, of the *co-ordinates* z, y, z of a point P in the plane, is the *denominator* of the expression (34, 36) for the *vector* ρ of that point: so that the *point* P is at an infinite distance from the origin O, when, and only when, this linear function *vanishes*.

39. These *anharmonic co-ordinates of a line*, although above interpreted (37) with reference to the *equation* of that line, considered as connecting the co-ordinates of a variable *point* thereof, are capable of receiving an independent geometrical interpretation. For the three points L, M, N, in which the line Λ, or $[l, m, n]$, or $lx + my + nz = 0$, intersects the three sides BC, CA, AB of the given triangle ABC, or the three given lines $x = 0$, $y = 0$, $z = 0$ (38), may evidently (on the plan of 36) be thus denoted:

$$\mathrm{L} = (0, n, -m); \qquad \mathrm{M} = (-n, 0, l); \qquad \mathrm{N} = (m, -l, 0).$$

But we had also (by 36),

$$\mathrm{A}'' = (0, 1, -1); \qquad \mathrm{B}'' = (-1, 0, 1); \qquad \mathrm{C}'' = (1, -1, 0);$$

whence it is easy to infer, on the principles of recent articles, that

$$\frac{n}{m} = (\mathrm{BA''CL}); \qquad \frac{l}{n} = (\mathrm{CB''AM}); \qquad \frac{m}{l} = (\mathrm{AC''BN});$$

with the resulting relation,

$$(\mathrm{BA''CL}) \, . \, (\mathrm{CB''AM}) \, . \, (\mathrm{AC''BN}) = 1.$$

40. Conversely, this last equation is easily proved, with the help of the known and general relation between *segments* (32), applied to *any two transversals*, A″B″C″ and LMN, of any triangle ABC. In fact, we have thus the two equations,

$$\frac{\mathrm{BA''}}{\mathrm{A''C}} \cdot \frac{\mathrm{CB''}}{\mathrm{B''A}} \cdot \frac{\mathrm{AC''}}{\mathrm{C''B}} = -1, \qquad \frac{\mathrm{BL}}{\mathrm{LC}} \cdot \frac{\mathrm{CM}}{\mathrm{MA}} \cdot \frac{\mathrm{AN}}{\mathrm{NB}} = -1;$$

on dividing the former of which by the latter, the last formula of the last article results. We might therefore in this way have been led, *without* any consideration of a *variable point* P, to introduce *three auxiliary scalars, l, m, n, defined* as having their *quotients* $\frac{n}{m}, \frac{l}{n}, \frac{m}{n}$ equal respectively, as in 39, to the three anharmonics of groups,

$$(\text{BA}''\text{CL}), \qquad (\text{CB}''\text{AM}), \qquad (\text{AC}''\text{BN});$$

and then it would have been evident that these three scalars, *l, m, n* (or any others proportional thereto), are sufficient to *determine the position of the right line* Λ, or LMN, considered as a *transversal* of the given triangle ABC: so that they might naturally have been called, on this account, as above, *the anharmonic co-ordinates* of that *line.* But although the anharmonic co-ordinates of a point and of a line may thus be *independently defined,* yet the *geometrical utility* of such definitions will be found to depend mainly on their *combination:* or on the formula $lx + my + nz = 0$ of 37, which may at pleasure be considered as expressing, either that the *variable point* (x, y, z) is *situated somewhere* upon the *given right line* $[l, m, n]$; or else that the *variable line* $[l, m, n]$ *passes, in some direction, through the given point* (x, y, z).

41. If Λ_1 and Λ_2 be any *two* right lines in the given plane,

$$\Lambda_1 = [l_1, m_1, n_1], \qquad \Lambda_2 = [l_2, m_2, n_2],$$

then any *third* right line Λ in the same plane, which passes *through the intersection* $\Lambda_1 \cdot \Lambda_2$, or (in other words) which *concurs* with them (at a finite or infinite distance), may be represented (comp. 37) by a symbol of the form,

$$\Lambda = [tl_1 + ul_2,\ tm_1 + um_2,\ tn_1 + un_2],$$

where *t* and *u* are scalar coefficients. Or, what comes to the same thing, if *l, m, n* be the anharmonic co-ordinates of the line Λ, then (comp. again 37), the equation

$$0 = l\,(m_1 n_2 - n_1 m_2) + \&\text{c.} = \begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix},$$

must be satisfied; because, if (X, Y, Z) be the supposed point *common* to the three lines, the three equations

$$lX + mY + nZ = 0, \quad l_1 X + m_1 Y + n_1 Z = 0, \quad l_2 X + m_2 Y + n_2 Z = 0,$$

must co-exist. Conversely, this coexistence will be possible and the three lines will have a common point (which may be infinitely distant), if the recent *condition of concurrence* be satisfied. For example, because $[a, b, c]$ has been seen (in 38) to be the symbol of the line at infinity (at least if we still retain the same significations of the scalars a, b, c as in articles 27, &c.), it follows that

$$\Lambda = [l, m, n], \quad \text{and} \quad \Lambda' = [l + ua, \; m + ub, \; n + uc],$$

are symbols of two *parallel lines*; because they *concur at infinity*. In general, all problems respecting intersections of right lines, collineations of points, &c., in the given plane, when treated by this *anharmonic method*, conduct to easy *eliminations* between *linear equations* (of the *scalar* kind), on which we need not here delay: the *mechanism* of such *calculations* being for the most part the *same* as in the known method of *trilinear co-ordinates*: although (as we have seen) the *geometrical interpretations* are altogether *different*.

SECTION 5.—*On Plane Geometrical Nets, resumed.*

42. If we now *resume*, for a moment, the consideration of those plane geometrical *nets*, which were mentioned in Art. 34; and agree to call those points and lines, in the given plane, *rational points* and *rational lines*, respectively, which have their *anharmonic co-ordinates equal* (or *proportional*) to *whole numbers*; because then the *anharmonic quotients*, which were discussed in the last Section, are *rational*; but to say that a point or line is *irrational*, or that it is *irrationally related* to the given system of *four initial points* O, A, B, C, when its anharmonic co-ordinates are *not* thus *all* equal (or proportional) to *integers*; it is clear that *whatever four points* we may assume as *initial*, and *however far* the construction of the net may be carried, the *net-points* and *net-lines* which result will *all* be *rational*, in the sense just now defined. In fact, we *begin* with such; and the subsequent *eliminations* (41) can never after-

wards conduct to any, that are of the contrary kind: the right line which *connects* two rational points being always a rational line; and the point of *intersection* of two rational lines being necessarily a rational point. The assertion made in Art. 34 is therefore fully justified.

43. Conversely, *every rational point* of the given plane, with respect to the four assumed initial points OABC, *is a point of the net* which those four points determine. To prove this, it is evidently sufficient to show that every rational point $A_1 = (0, y, z)$, on any one side BC of the given triangle ABC, can be so constructed. Making, as in Fig. 22,

$$B_1 = OA_1 \cdot CA, \quad \text{and} \quad C_1 = OA_1 \cdot AB,$$

we have (by 35, 36) the expressions,

$$B_1 = (y, 0, y - z), \qquad C_1 = (z, z - y, 0);$$

from which it is easy to infer (by 36, 37), that

$$C'B_1 \cdot BC = (0, y, z - y), \qquad B'C_1 \cdot BC = (0, y - z, z);$$

and thus we can reduce the linear construction of the rational point $(0, y, z)$, in which the two *whole* numbers y and z may be supposed to be *prime* to each other, to depend on that of the point $(0, 1, 1)$, which has already been constructed as A′. It follows that although *no irrational point* Q of the plane *can be a net-point*, yet *every such point* can be *indefinitely approached* to, by continuing the linear construction; so that it can be *included within* a *quadrilateral interstice* $P_1P_2P_3P_4$, or even within a *triangular interstice* $P_1P_2P_3$, which interstice of the net can be made *as small* as we may desire. Analogous remarks apply to *irrational lines* in the plane, which can *never coincide* with *net-lines*, but may always be indefinitely *approximated* to by such.

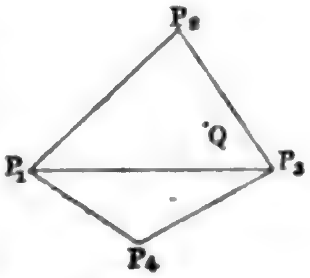

Fig. 24.

44. If P, P_1, P_2 be any three *collinear* points of the net, so that the formulæ of 37 apply, and if P′ be any *fourth* net-point (x', y', z') upon the same line, then writing

$$x_1a + y_1b + z_1c = v_1, \qquad x_2a + y_2b + z_2c = v_2,$$

we shall have two expressions of the forms,

$$\rho = \frac{tv_1\rho_1 + uv_2\rho_2}{tv_1 + uv_2}, \qquad \rho' = \frac{t'v_1\rho_1 + u'v_2\rho_2}{t'v_1 + u'v_2},$$

in which the coefficients $tut'u'$ are *rational*, because the co-ordinates xyz, &c., are such, whatever the constants abc may be. We have therefore (by 25) the following rational expression for the anharmonic of this *net-group:*

$$(\mathrm{P}_1\mathrm{P}\mathrm{P}_2\mathrm{P}') = \frac{ut'}{tu'} = \frac{(yx_1 - xy_1)\,(y'x_2 - x'y_2)}{(xy_2 - yx_2)\,(x'y_1 - y'x_1)};$$

and similarly for every other group of the same kind. Hence *every group* of four collinear net-points, and consequently also *every pencil* of four concurrent net-lines, has a *rational value* for its *anharmonic function;* which value depends *only* on the *processes* of *linear construction* employed, in arriving at that group or pencil, and is quite *independent* of the *configuration* or *arrangement* of the *four initial points:* because the *three initial constants*, *a*, *b*, *c*, *disappear* from the expression which results. It was thus that, in Fig. 21, the *nine pencils*, which had the nine derived points $\mathrm{A}' \,..\, \mathrm{C}'''$ for their vertices, were all *harmonic* pencils, in whatever manner the four points O, A, B, C might be arranged. In general, it may be said that plane geometrical nets are all *homographic figures*;[*] and conversely, in any two such plane *figures*, *corresponding points* may be considered as either *coinciding*, or at least (by 43) as indefinitely approaching to coincidence, with *similarly constructed points* of two plane *nets*: that is, with points of which (in their respective systems) the *anharmonic co-ordinates* (36) are *equal integers*.

45. Without entering here[†] on any *general* theory of *transformation* of anharmonic co-ordinates, we may already see that if we select *any four net-points* O_1, A_1, B_1, C_1, of which no three are collinear, *every other point* P of the same net is *rationally related* (42) *to these;* because (by 44) the three new anhar-

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[*] Compare the *Géométrie Supérieure* of M. Chasles, p. 362.

[†] See Note A, on *Anharmonic Co-ordinates*.

monics of pencils, $(A_1 \,.\, B_1O_1C_1P) = \frac{y_1}{z_1}$, &c., are rational: and therefore (comp. 36) the *new co-ordinates* x_1, y_1, z_1 of the point P, as well its *old* co-ordinates xyz, are equal or proportional to *whole numbers*. It follows (by 43) that *every point* P of the net can be *linearly constructed*, if *any four* such points be *given* (no three being collinear, as above); or, in other words, that the *whole net* can be *reconstructed*,* if *any one* of its *quadrilaterals* (such as the *interstice* in Fig. 24) be *known*. As an example, we may suppose that the four points $OA'B'C'$ in Fig. 21 are given, and that it is required to *recover* from them the three points ABC, which had previously been among the *data* of the construction. For this purpose, it is only necessary to determine first the three auxiliary points A''', B''', C''', as the intersections $OA' \cdot B'C'$, &c.; and next the three other auxiliary points A'', B'', C'', as $B'C' \cdot B'''C'''$, &c.: after which the formulæ, $A = B'B'' \cdot C'C''$, &c., will enable us to return, as required, to the points A, B, C, as intersections of known right lines.

SECTION 6.—*On Anharmonic Equations, and Vector Expressions, for Curves in a given Plane.*

46. When, in the expressions 34 or 36 for a variable *vector* $\rho = OP$, the three variable *scalars* (or anharmonic co-ordinates) x, y, z are *connected* by any given algebraic *equation*, such as

$$f_p(x, y, z) = 0,$$

supposed to be rational and integral, and homogeneous of the pth *degree*, then the *locus* of the *term* P (Art. 1) of that vector is a *plane curve* of the pth *order;* because (comp. 37) it is *cut* in *p points* (distinct or coincident, and real or imaginary), by any *given right line*, $lx + my + nz = 0$, in the given plane.

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* This theorem (45) of the possible *reconstruction of a plane net*, from any one of its *quadrilaterals*, and the theorem (43) respecting the possibility of indefinitely *approaching* by *net-lines* to the points above called *irrational* (42), without ever *reaching* such points by any processes of *linear construction* of the kind here considered, have been taken, as regards their substance (although investigated by a totally different analysis), from that highly original treatise of MÖBIUS, which was referred to in a former note (p. 23). Compare Note B, upon the *Barycentric Calculus;* and the remarks in the following Chapter, upon *nets in space*.

For example, if we write

$$\rho = \frac{t^2 a\alpha + u^2 b\beta + v^2 c\gamma}{t^2 a + u^2 b + v^2 c},$$

where t, u, v are three new variable scalars, of which we shall suppose that the sum is zero, then, by eliminating these between the four equations,

$$x = t^2, \quad y = u^2, \quad z = v^2, \quad t + u + v = 0,$$

we are conducted to the following equation of the *second* degree,

$$0 = f_p = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy;$$

so that here $p = 2$, and the locus of P is a *conic section*. In fact, it is the conic which *touches* the *sides* of the *given triangle* ABC, at the points above called A′, B′, C′; for if we seek its *intersections* with the side BC, by making $x = 0$ (38), we obtain a *quadratic* with *equal* roots, namely, $(y - z)^2 = 0$; which shows that there is *contact* with this side at the point (0, 1, 1), or A′ (36): and similarly for the two other sides.

47. If the point O, in which the three right lines AA′, BB′, CC′ *concur*, be (as in Fig. 18, &c.) *interior* to the triangle ABC, the sides of that triangle are then all cut *internally*, by the points A′, B′, C′ of contact with the conic; so that in this case (by 28) the ratios of the constants a, b, c are all *positive*, and the *denominator* of the recent expression (46) for ρ cannot *vanish*, for any *real* values of the variable scalars t, u, v; and consequently no *such* values can render *infinite* that *vector* ρ. The *conic* is therefore generally in this case, as in Fig. 25, an *inscribed ellipse;* which becomes however the *inscribed circle*, when

$$a^{-1} : b^{-1} : c^{-1} = \mathrm{s} - \mathrm{a} : \mathrm{s} - \mathrm{b} : \mathrm{s} - \mathrm{c};$$

a, b, c denoting here the lengths of the sides of the triangle, and s being their semi-sum.

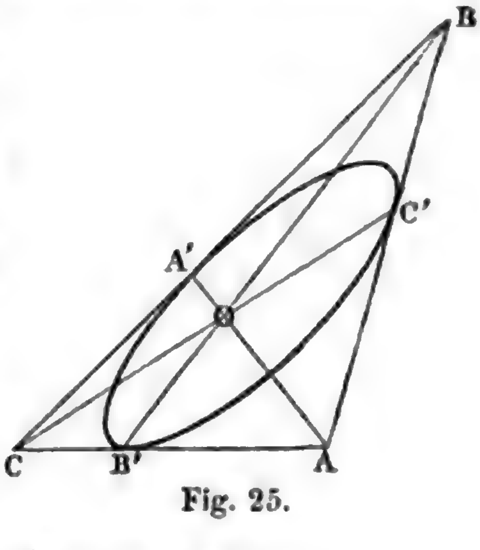

Fig. 25.

48. But if the point of concourse O be *exterior* to the *triangle of tangents* ABC, so that *two* of its sides are cut *externally*, then *two* of the three ratios of *segments* (28) are *negative;* and therefore *one* of the three *constants* a, b, c may be treated as < 0, but each of the two others as > 0. Thus if we suppose that

$$b > 0, \qquad c > 0, \qquad a < 0, \qquad a + b > 0, \qquad a + c > 0,$$

A′ will be a point on the side B *itself*, but the points B′, C′, O will be on the lines AC, AB, AA′ *prolonged*, as in Fig. 26; and then the conic A′B′C′ will be an *ellipse* (including the case of a *circle*), or a *parabola*, or an *hyperbola*, according as the *roots* of the *quadratic*,

$$(a + c)\,t^2 + 2ctu + (b + c)\,u^2 = 0,$$

obtained by equating the denominator (46) of the vector ρ to zero, are either, Ist, *imaginary;* or IInd, *real* and *equal*; or IIIrd, *real* and *unequal:* that is, according as we have

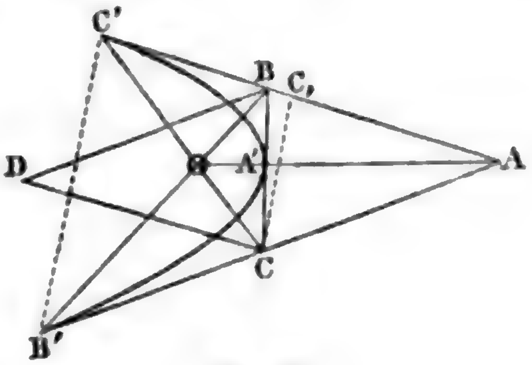

Fig. 26.

$$bc + ca + ab > 0, \quad \text{or} \; = 0, \quad \text{or} \; < 0;$$

or (because the product abc is here negative), according as

$$a^{-1} + b^{-1} + c^{-1} < 0, \quad \text{or} \; = 0, \quad \text{or} \; > 0.$$

For example, if the conic be what is often called the *exscribed circle*, the known ratios of segments give the proportion,

$$a^{-1} : b^{-1} : c^{-1} = -\mathrm{s} : \mathrm{s} - \mathrm{c} : \mathrm{s} - \mathrm{b};$$

and

$$-\mathrm{s} + \mathrm{s} - \mathrm{c} + \mathrm{s} - \mathrm{b} < 0.$$

49. More generally, if C, be (as in Fig. 26) a point upon the side AB, or on that side prolonged, such that CC, is parallel to the chord B′C′, then

$$\mathrm{C_{\prime}C'} : \mathrm{AC'} = \mathrm{CB'} : \mathrm{AB'} = -a : c, \quad \text{and} \quad \mathrm{AB} : \mathrm{AC'} = a + b : b;$$

writing then the *condition* (48) *of ellipticity* (or circularity) under the form, $\frac{-a}{c} < \frac{a+b}{b}$, we see that the conic is an ellipse, parabola, or hyperbola, according as CC′ < or = or > AB; the arrangement being *still*, in other respects, that which is represented in Fig. 26. Or, to express the same thing more symmetrically, if we complete the parallelogram CABD, then according as the point D falls, Ist, *beyond the chord* B′C′, with respect to the point A; or IInd, *on that chord;* or IIIrd, *within the triangle* AB′C′, the general arrangement of the same Figure being retained, the curve is *elliptic*, or *parabolic*, or *hyperbolic*. In that *other* arrangement or configuration, which answers to the system of inequalities, $b > 0$, $c > 0$, $a + b + c < 0$, the point A′ is still upon the side BC *itself*, but O is on the line A′A prolonged *through* A; and *then* the inequality,

$$a(b + c) + bc < -(b^2 + bc + c^2) < 0,$$

shows that the conic is *necessarily* an hyperbola; whereof it is easily seen that *one branch* is touched by the side BC at A′, while the *other* branch is touched in B′ and C′, by the sides CA and BA prolonged through A. The curve is also hyperbolic, if either $a + b$ or $a + c$ be negative, while b and c are positive as before.

50. When the quadratic (48) has its roots real and unequal, so that the conic is an *hyperbola*, then the *directions* of the *asymptotes* may be found, by substituting those roots, or the values of t, u, v which correspond to them (or any scalars *proportional* thereto), in the *numerator* of the expression (46) for ρ; and similarly we can find the direction of the *axis* of the *parabola*, for the case when the roots are real but equal: for we shall thus obtain the directions, or direction, in which a right line OP must be drawn from O, so as to *meet the conic at infinity*. And the same *conditions* as before, for distinguishing the *species* of the conic, may be otherwise obtained by combining the *anharmonic equation*, $f = 0$ (46), of that conic, with the corresponding equation $ax + by + cz = 0$ (38) of the *line at infinity;* so as to inquire (on known principles of modern geometry) whether that *line* meets that *curve* in *two*

imaginary points, or *touches* it, or *cuts* it, in points which (although *infinitely distant*) are here to be considered as *real*.

51. In general, if $f(x, y, z) = 0$ be the anharmonic equation (46) of *any plane curve*, considered as the *locus* of a variable *point* P; and if the *differential** of this equation be thus denoted,

$$0 = \mathrm{d}f(x, y, z) = X\mathrm{d}x + Y\mathrm{d}y + Z\mathrm{d}z;$$

then because, by the supposed *homogeneity* (46) of the function f, we have the relation

$$Xx + Yy + Zz = 0,$$

we shall have also this other but analogous relation,

$$Xx' + Yy' + Zz' = 0,$$

if

$$x' - x : y' - y : z' - z = \mathrm{d}x : \mathrm{d}y : \mathrm{d}z;$$

that is (by the principles of Art. 37), if $\text{P}' = (x', y', z')$ be *any point upon the tangent to the curve*, drawn at the point $\text{P} = (x, y, z)$, and regarded as the *limit of a secant*. The *symbol* (37) of this *tangent* at P may therefore be thus written,

$$[X, Y, Z], \quad \text{or} \quad [\mathrm{D}_x f, \mathrm{D}_y f, \mathrm{D}_z f];$$

where D_x, D_y, D_z are known *characteristics of partial derivation*.

52. For example, when f has the form assigned in 46, as answering to the conic lately considered, we have $\mathrm{D}_x f = 2(x - y - z)$, &c.; whence the tangent at any point (x, y, z) of this curve may be denoted by the symbol,

$$[x - y - z, \qquad y - z - x, \qquad z - x - y];$$

in which, as usual, the co-ordinates of the line may be replaced by any others proportional to them. Thus at the point A′, or (by 36) at (0, 1, 1), which is evidently (by the form of f) a point upon the curve, the tangent is the line $[-2, 0, 0]$, or $[1, 0, 0]$; that is (by 38), the side BC of the given triangle, as

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* In the theory of *quaternions*, as distinguished from (although including) that of *vectors*, it will be found necessary to introduce a *new definition of differentials*, on account of the *non-commutative* property of *quaternion-multiplication*: but, for the present, the *usual* significations of the signs d and D are sufficient.

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was otherwise found before (46). And in general it is easy to see that the recent symbol denotes the right line, which is (in a well known sense) the *polar* of the point (x, y, z), with respect to the same given conic; or that the line $[X', Y', Z']$ is the polar of the point (x', y', z'): because the equation

$$Xx' + Yy' + Zz' = 0,$$

which for a *conic* may be written as $X'x + Y'y + Z'z = 0$, expresses (by 51) the condition requisite, in order that a point (x, y, z) of the curve* should belong to a tangent which passes through the point (x', y', z'). Conversely, the *point* (x, y, z) is (in the same well-known sense) the *pole* of the *line* $[X, Y, Z]$; so that the *centre* of the conic, which is (by known principles) the *pole of the line at infinity* (38), is the point which satisfies the conditions $a^{-1}X = b^{-1}Y = c^{-1}Z$; it is therefore, for the present conic, the point $\text{K} = (b + c,\ c + a,\ a + b)$, of which the *vector* OK is easily reduced, by the help of the linear equation, $a\alpha + b\beta + c\gamma = 0$ (27), to the form,

$$\kappa = -\frac{a^2\alpha + b^2\beta + c^2\gamma}{2(bc + ca + ab)};$$

with the verification that the *denominator vanishes*, by 48, when the conic is a *parabola*. In the more general case, when this denominator is different from zero, it can be shown that *every chord* of the curve, which is drawn *through the extremity* K of the vector κ, is *bisected* at that point K: which point would therefore in this way be seen again to be the *centre*.

53. Instead of the *inscribed conic* (46), which has been the subject of recent articles, we may, as another example, consider that *exscribed* (or *circumscribed*) conic, which passes through the three corners A, B, C of the given triangle, and touches there the lines AA″, BB″, CC″ of Fig. 21. The anharmonic equation of this new conic is easily seen to be,

$$yz + zx + xy = 0;$$

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* If the curve $f = 0$ were of a degree *higher* than the *second*, then the two equations above written would represent what are called the *first polar*, and the *last* or the *line-polar*, of the point (x', y', z'), with respect to the given curve.

the vector of a variable point P of the curve may therefore be expressed as follows,

$$\rho = \frac{t^{-1}a\alpha + u^{-1}b\beta + v^{-1}c\gamma}{t^{-1}a + u^{-1}b + v^{-1}c},$$

with the condition $t + u + v = 0$, as before. The vector of its centre K′ is found to be,

$$\kappa' = \frac{2(a^2\alpha + b^2\beta + c^2\gamma)}{a^2 + b^2 + c^2 - 2bc - 2ca - 2ab};$$

and it is an ellipse, a parabola, or an hyperbola, according as the denominator of this last expression is negative, or null, or positive. And because these two recent *vectors*, κ, κ', bear a *scalar ratio* to each other, it follows (by 19) that *the three points* O, K, K′ *are collinear;* or in other words, that the *line of centres* KK′, of the *two conics* here considered, *passes through the point of concourse* O *of the three lines* AA′, BB′, CC′. More generally, if L be the *pole* of *any given right line* $\Lambda = [l, m, n]$ (37), with respect to the *inscribed* conic (46), and if L′ be the pole of the *same line* Λ with respect to the *exscribed* conic of the present article, it can be shown that the *vectors* OL, OL′, or λ, λ', of these *two poles* are of the forms,

$$\lambda = k\,(la\alpha + mb\beta + nc\gamma), \qquad \lambda' = k'\,(la\alpha + mb\beta + nc\gamma),$$

where k and k' are *scalars;* the three points O, L, L′ are therefore ranged *on one right line.*

54. As an example of a *vector-expression* for a curve of an order *higher than the second*, the following may be taken:

$$\mathrm{OP} = \rho = \frac{t^3a\alpha + u^3b\beta + v^3c\gamma}{t^3a + u^3b + v^3c};$$

with $t + u + v = 0$, as before. Making $x = t^3$, $y = u^3$, $z = v^3$, we find here by elimination of t, u, v the *anharmonic equation,*

$$(x + y + z)^3 - 27xyz = 0;$$

the locus of the point P is therefore, in this example, a *curve of the third order*, or briefly a *cubic curve.* The *mechanism* (41) of calculations with *anharmonic co-ordinates* is so much the same as that of the known *trilinear method*, that it may suffice to remark briefly here that the *sides* of the given triangle ABC are the *three* (real) *tangents of inflexion*; the *points of inflexion* being those which are marked as A″, B″, C″ in Fig. 21; and the *origin* of vectors O being a *conjugate point*.* If $a = b = c$, in which case (by 29) this origin O becomes (as in Fig. 19) the *mean point* of the triangle, the *chord of inflexion* A″B″C″ is then the *line at infinity*, and the curve takes the form represented in Fig. 27; having *three infinite branches*, inscribed within the angles vertically opposite to those of the given triangle ABC, of which the sides are the *three asymptotes*.

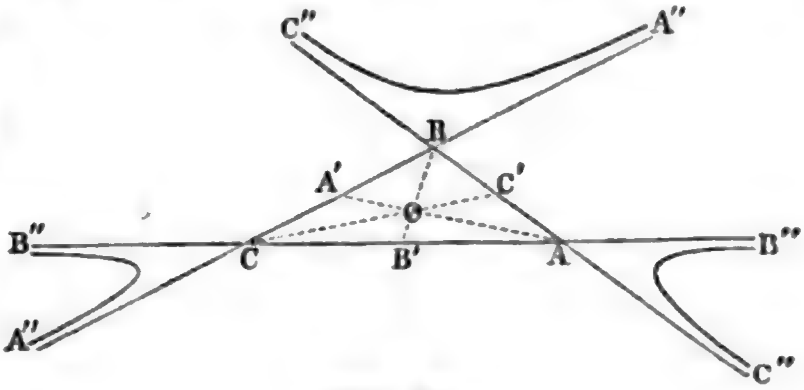

Fig. 27.

55. It would be improper to enter here into any details of discussion of such cubic curves, for which the reader will naturally turn to other works.† But it may be remarked, in passing, that because the *general cubic* may be represented, on the present plan, by combining the general expression of Art. 34 or 36 for the vector ρ, with the scalar equation

$$s^3 = 27kxyz, \quad \text{where} \quad s = x + y + z;$$

k denoting an arbitrary constant, which becomes equal to unity, when the origin is (as in 54) a conjugate point; it follows that if $P = (x, y, z)$ and $P' = (x', y', z')$ be *any two points of the curve*, and if we make $s' = x' + y' + z'$, we shall have the relation,

$$xyzs'^3 = x'y'z's^3, \quad \text{or} \quad \frac{xs'}{sx'} \cdot \frac{ys'}{sy'} \cdot \frac{zs'}{sz'} = 1:$$

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* Answering to the values $t = 1$, $u = \theta$, $v = \theta^2$, where θ is one of the imaginary cube-roots of unity; which values of t, u, v give $x = y = z$, and $\rho = 0$.

† Especially the excellent Treatise on *Higher Plane Curves*, by the Rev. George Salmon, F. T. C. D., &c. Dublin, 1852.

in which it is not difficult to prove that

$$\frac{xs'}{sx'} = (\text{A}''.\,\text{PBP}'\text{B}''); \quad \frac{ys'}{sy'} = (\text{B}''.\,\text{PCP}'\text{C}''); \quad \frac{zs'}{sz'} = (\text{C}''.\,\text{PAP}'\text{A}'');$$

the notation (35) of *anharmonics of pencils* being retained. We obtain therefore thus the following *Theorem*:—"*If the sides of any given plane* triangle* ABC *be cut* (as in Fig. 21) *by any given rectilinear transversal* A″B″C″, *and if any two points* P and P′ *in its plane be such as to satisfy the anharmonic relation*

$$(\text{A}''.\,\text{PBP}'\text{B}'')\,.\,(\text{B}''.\,\text{PCP}'\text{C}'')\,.\,(\text{C}''.\,\text{PAP}'\text{A}'') = 1,$$

then these two points P, P′ *are on one common cubic curve, which has the three collinear points* A″, B″, C″ *for its three real points of inflexion, and has the sides* BC, CA, AB *of the triangle for its three tangents at those points*;" a result which seems to offer a *new geometrical generation for curves of the third order*.

56. Whatever the *order* of a plane curve may be, or whatever may be the *degree* p of the *function* f in 46, we saw in 51 that the *tangent* to the curve at any point $\text{P} = (x, y, z)$ is the right line

$$\Lambda = [l, m, n], \quad \text{if} \quad l = \text{D}_x f, \quad m = \text{D}_y f, \quad n = \text{D}_z f;$$

expressions which, by the supposed *homogeneity* of f, give the relation, $lx + my + nz = 0$, and therefore enable us to establish the system of the *two* following differential equations,

$$l\text{d}x + m\text{d}y + n\text{d}z = 0, \qquad x\text{d}l + y\text{d}m + z\text{d}n = 0.$$

If then, by elimination of the ratios of x, y, z, we arrive at a *new homogeneous equation* of the form,

$$0 = \text{F}(\text{D}_x f,\ \text{D}_y f,\ \text{D}_z f),$$

as one that is true for all values of x, y, z which render the function $f = 0$ (although it may require to be cleared of *factors*, introduced by this *elimination*), we shall have the equation

$$\text{F}(l, m, n) = 0,$$

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* This Theorem may be extended, with scarcely any modification, from *plane* to *spherical curves*, of the *third order*.

as a *condition* that must be *satisfied by the tangent* Λ to the curve, *in all the positions which can be assumed by that right line.* And, by *comparing* the two differential equations,

$$\mathrm{dF}(l, m, n) = 0, \qquad x\mathrm{d}l + y\mathrm{d}m + z\mathrm{d}n = 0,$$

we see that we may write the *proportion*,

$x : y : z = \mathrm{D}_l\mathrm{F} : \mathrm{D}_m\mathrm{F} : \mathrm{D}_n\mathrm{F}$, and the *symbol* $\mathrm{P} = (\mathrm{D}_l\mathrm{F}, \mathrm{D}_m\mathrm{F}, \mathrm{D}_n\mathrm{F})$,

if (x, y, z) be, as above, the *point of contact* P of the *variable line* $[l, m, n]$, in any one of its positions, *with the curve* which is its *envelope*. Hence we can pass (or return) *from the tangential equation* $\mathrm{F} = 0$, of a curve *considered as the envelope of a right line* Λ, *to the local equation* $f = 0$, of the same curve considered (as in 46) as the *locus of a point* P: since, if we obtain, by elimination of the ratios of l, m, n, an equation of the form

$$0 = f(\mathrm{D}_l\mathrm{F},\ \mathrm{D}_m\mathrm{F},\ \mathrm{D}_n\mathrm{F}),$$

(cleared, if it be necessary, of foreign factors) as a consequence of the homogeneous equation $\mathrm{F} = 0$, we have only to *substitute* for these *partial derivatives*, $\mathrm{D}_l\mathrm{F}$, &c., the *anharmonic co-ordinates* x, y, z, to which they are proportional. And when the functions f and F are not only *homogeneous* (as we shall *always* suppose them to be), but also *rational* and *integral* (which it is sometimes convenient *not* to assume them as being), then, while the *degree* of the function f, or of the *local equation*, marks (as before) the *order* of the curve, the degree of the *other homogeneous function* F, or of the *tangential equation* $\mathrm{F} = 0$, is easily seen to denote, in this *anharmonic method* (as, from the analogy of *other* and older methods, it might have been expected to do), the *class* of the curve to which that equation belongs: or the *number of tangents* (distinct or coincident, and real or imaginary), which can be drawn *to that curve, from an arbitrary point* in its plane.

57. As an example (comp. 52), if we eliminate x, y, z between the equations,

$$l = x - y - z, \quad m = y - z - x, \quad n = z - x - y, \quad lx + my + nz = 0,$$

where l, m, n are the co-ordinates of the tangent to the *inscribed* *conic* of Art. 46, we are conducted to the following tangential equation of that conic, or *curve of the second class*,

$$\text{F}(l, m, n) = mn + nl + lm = 0\,;$$

with the verification that the sides [1, 0, 0], &c. (38), of the triangle ABC are among the lines which satisfy this equation. Conversely, if this *tangential equation* were *given*, we might (by 56) *derive* from it expressions for the *co-ordinates of contact* x, y, z, as follows:

$$x = \text{D}_l\text{F} = m + n, \qquad y = n + l, \qquad z = l + m\,;$$

with the verification that the side [1, 0, 0] touches the conic, considered now as an *envelope*, in the point (0, 1, 1), or A′, as before: and then, by eliminating l, m, n, we should be brought back to the *local equation*, $f = 0$, of 46. In like manner, from the local equation $f = yz + zx + xy = 0$ of the *exscribed conic* (53), we can derive by differentiation the *tangential co-ordinates*,[*]

$$l = \text{D}_x f = y + z, \qquad m = z + x, \qquad n = x + y,$$

and so obtain by elimination the tangential equation, namely,

$$\text{F}(l, m, n) = l^2 + m^2 + n^2 - 2mn - 2nl - 2lm = 0\,;$$

from which we could in turn *deduce* the *local* equation. And (comp. 40), the very simple formula

$$lx + my + nz = 0,$$

which we have so often had occasion to employ, as *connecting two sets* of anharmonic co-ordinates, may not only be considered (as in 37) as the *local equation of a given right line* Λ, *along* which a *point* P *moves*, but also as the *tangential equation of a given point, round which a right line turns:* according as we suppose the set l, m, n, or the set x, y, z, to be *given*. Thus, while the right *line* A″B″C″, or [1, 1, 1], of Fig. 21, was represented in 38 by the *equation* $x + y + z = 0$, the *point* O of the same figure, or the point (1, 1, 1), may be represented by the *analogous equation*,

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[*] This *name* of "*tangential co-ordinates*" appears to have been first introduced by Dr. Booth in a Tract published in 1840, to which the author of the present Elements cannot now more particularly refer: but the *system* of Dr. Booth was entirely different from his own. See the reference in Salmon's *Higher Plane Curves*, note to page 16.

$$l + m + n = 0;$$

because the *co-ordinates* l, m, n of *every line*, which passes *through this point* O, must satisfy this equation of the first degree, as may be seen exemplified, in the same Art. 38, by the lines OA, OB, OC.

58. To give an instance or two of the use of forms, which, although *homogeneous*, are yet not *rational* and *integral* (56), we may write the local equation of the *inscribed* conic (46) as follows:

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 0;$$

and then (suppressing the common numerical factor $\frac{1}{2}$), the partial derivatives are

$$l = x^{-\frac{1}{2}}, \qquad m = y^{-\frac{1}{2}}, \qquad n = z^{-\frac{1}{2}};$$

so that a form of the tangential equation for this conic is,

$$l^{-1} + m^{-1} + n^{-1} = 0;$$

which evidently, when cleared of fractions, agrees with the first form of the last Article: with the verification (48), that $a^{-1} + b^{-1} + c^{-1} = 0$ when the curve is a *parabola;* that is, when it is *touched* (50) by the *line at infinity* (38). For the *exscribed* conic (53), we may write the *local* equation thus,

$$x^{-1} + y^{-1} + z^{-1} = 0;$$

whence it is allowed to write also,

$$l = x^{-2}, \qquad m = y^{-2}, \qquad n = z^{-2},$$

and

$$l^{\frac{1}{2}} + m^{\frac{1}{2}} + n^{\frac{1}{2}} = 0;$$

a form of the tangential equation which, when cleared of radicals, agrees again with 57. And it is evident that we could return, with equal ease, from these tangential to these local equations.

59. For the *cubic curve* with a *conjugate point* (54), the local equation may be thus written,*

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* Compare Salmon's *Higher Plane Curves*, page 172.

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0\,;$$

we may therefore assume for its tangential co-ordinates the expressions,

$$l = x^{-\frac{2}{3}}, \qquad m = y^{-\frac{2}{3}}, \qquad n = z^{-\frac{2}{3}}\,;$$

and a form of its tangential equation is thus found to be,

$$l^{-\frac{1}{2}} + m^{-\frac{1}{2}} + n^{-\frac{1}{2}} = 0.$$

Conversely, if this tangential form were *given*, we might return to the local equation, by making

$$x = l^{-\frac{3}{2}}, \qquad y = m^{-\frac{3}{2}}, \qquad z = n^{-\frac{3}{2}},$$

which would give $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, as before. The tangential equation just now found becomes, when it is cleared of *radicals*,

$$0 = l^{-2} + m^{-2} + n^{-2} - 2m^{-1}n^{-1} - 2n^{-1}l^{-1} - 2l^{-1}m^{-1}\,;$$

or, when it is also cleared of *fractions*,

$$0 = \mathrm{F} = m^2n^2 + n^2l^2 + l^2m^2 - 2nl^2m - 2lm^2n - 2mn^2l\,;$$

of which the *biquadratic* form shows (by 56) that *this cubic* is a *curve of the fourth class*, as indeed it is known to be. The *inflexional* character (54) of the points A″, B″, C″ upon this curve is here recognised by the circumstance, that when we make $m - n = 0$, in order to find the four tangents from $\mathrm{A}'' = (0, 1, -1)$ (36), the resulting biquadratic, $0 = m^4 - 4lm^3$, has *three equal roots;* so that the line $[1, 0, 0]$, or the side BC, *counts as three*, and is therefore a *tangent of inflexion:* the *fourth tangent* from A″ being the line $[1, 4, 4]$, which touches the cubic at the point $(-8, 1, 1)$.

60. In general, the two equations (56),

$$n\mathrm{D}_x f - l\mathrm{D}_z f = 0, \qquad n\mathrm{D}_y f - m\mathrm{D}_z f = 0,$$

may be considered as expressing that the homogeneous equation,

$$f(nx, ny, -lx - my) = 0,$$

which is obtained by eliminating z with the help of the relation $lx + my + nz = 0$, from $f(x, y, z) = 0$, and which we may denote by $\phi(x, y) = 0$, has *two equal roots* $x : y$, if l, m, n be still the co-ordinates of a *tangent* to the curve f; an *equality* which obviously corresponds to the *coincidence* of *two intersections* of that line with that curve. Conversely, if we seek by the usual methods the *condition of equality* of two roots $x : y$ of the homogeneous equation of the pth degree,

$$0 = \phi(x, y) = f(nx, ny, -lx - my),$$

by eliminating the ratio $x : y$ between the two *derived* homogeneous equations, $0 = \mathrm{D}_x\phi$, $0 = \mathrm{D}_y\phi$, we shall in general be conducted to a result of the *dimension* $2p(p-1)$ in l, m, n, and of the *form*,

$$0 = n^{p(p-1)}\,\mathrm{F}(l, m, n);$$

and so, by the rejection of the *foreign factor* $n^{p(p-1)}$, introduced by this *elimination*,[*] we shall obtain the *tangential equation* $\mathrm{F} = 0$, which will be in general of the *degree* $p(p-1)$; such being generally the known *class* (56) of the curve of which the *order* (46) is denoted by p: with (of course) a similar mode of passing, reciprocally, from a tangential to a local equation.

61. As an example, when the function f has the *cubic form* assigned in 54, we are thus led to investigate the condition for the existence of two equal roots in the *cubic equation*,

$$0 = \phi(x, y) = \{(n - l)\,x + (m - l)\,y\}^3 + 27n^2xy(lx + my),$$

by eliminating $x : y$ between *two derived* and *quadratic equations;* and the result presents itself, in the first instance, as of the *twelfth dimension* in the tangential co-ordinates l, m, n; but it is found to be *divisible by* n^6, and when this division is effected, it is reduced to the *sixth degree*, thus appearing to imply that the curve is of the *sixth class*, as in fact the *general cubic* is well known to be. A *further reduction* is however possible in the present case, on account of the *conjugate point* O (54), which introduces (comp. 57) the *quadratic factor*,

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[*] Compare the method employed in Salmon's *Higher Plane Curves*, page 98, to find the equation of the *reciprocal* of a given curve, with respect to the imaginary conic, $x^2 + y^2 + z^2 = 0$. In general, if the function F be deduced from f as above, then $\mathrm{F}(xyz) = 0$, and $f(xyz) = 0$ are equations of *two reciprocal curves*.

$$(l + m + n)^2 = 0\,;$$

and when *this factor also* is *set aside*, the tangential equation is found to be *reduced to the biquadratic form*[*] already assigned in 59; the *algebraic division*, last performed, corresponding to the known *geometric depression* of a *cubic curve* with a *double point*, from the *sixth* to the *fourth class*. But it is time to close this Section on *Plane Curves*; and to proceed, as in the next Chapter we propose to do, to the consideration and comparison of *vectors of points in space*.

◆

CHAPTER III.

APPLICATIONS OF VECTORS TO SPACE.

SECTION 1.—*On Linear Equations between Vectors not Coplanar.*

62. When three given and actual vectors OA, OB, OC, or α, β, γ, are *not* contained in any common plane, and when the three scalars a, b, c do not *all* vanish, then (by 21, 22) the expression $a\alpha + b\beta + c\gamma$ cannot become equal to *zero*; it must therefore represent *some actual vector* (1), which we may, for the sake of symmetry, denote by the symbol $-d\delta$: where the *new* (actual) *vector* δ, or OD, is not contained in any one

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[*] If we multiply that form $\text{F} = 0$ (59) by z^2, and then change nz to $-lx - my$, we obtain a biquadratic equation in $l : m$, namely,

$$0 = \psi(l, m) = (l - m)^2 (lx + my)^2 + 2lm (l + m) (lx + my) z + l^2m^2z^2\,;$$

and if we then eliminate $l : m$ between the two derived cubics, $0 = \text{D}_l\psi$, $0 = \text{D}_m\psi$, we are conducted to the following equation of the twelfth degree, $0 = x^3y^3z^3 f(x, y, z)$, where f has the same cubic form as in 54. We are therefore thus brought *back* (comp. 59) from the *tangential* to the *local* equation of the cubic curve (54); complicated, however, as we see, with the *factor* $x^3y^3z^3$, which corresponds to the system of the three real tangents of inflexion to that curve, each tangent being taken three times. The reason why we have not here been obliged to reject *also* the foreign factor, z^{12}, as by the general theory (60) we might have expected to be, is that we multiplied the biquadratic function F *only* by z^2, and *not* by z^4.

of the three given and distinct planes, BOC, COA, AOB, unless some one, at least, of the three given coefficients a, b, c, vanishes; and where the *new scalar*, d, is either greater or less than zero. We shall thus have a *linear equation between four vectors,*

$$a\alpha + b\beta + c\gamma + d\delta = 0;$$

which will give

$$\delta = \frac{-a\alpha}{d} + \frac{-b\beta}{d} + \frac{-c\gamma}{d}, \quad \text{or} \quad \mathrm{OD} = \mathrm{OA}' + \mathrm{OB}' + \mathrm{OC}';$$

where OA', OB', OC', or $\frac{-a\alpha}{d}$, $\frac{-b\beta}{d}$, $\frac{-c\gamma}{d}$, are the vectors of the three points A′, B′, C′, into which the point D is *projected, on the three given lines* OA, OB, OC, by planes drawn parallel to the three given planes, BOC, &c.; so that they are the three *co-initial edges* of a *parallelepiped,* whereof the *sum,* OD or δ, is the *internal* and *co-initial diagonal* (comp. 6). Or we may project D *on the three planes,* by lines DA″, DB″, DC″ parallel to the three given lines, and then shall have $\mathrm{OA}'' = \mathrm{OB}' + \mathrm{OC}' = \frac{b\beta + c\gamma}{-d}$, &c., and

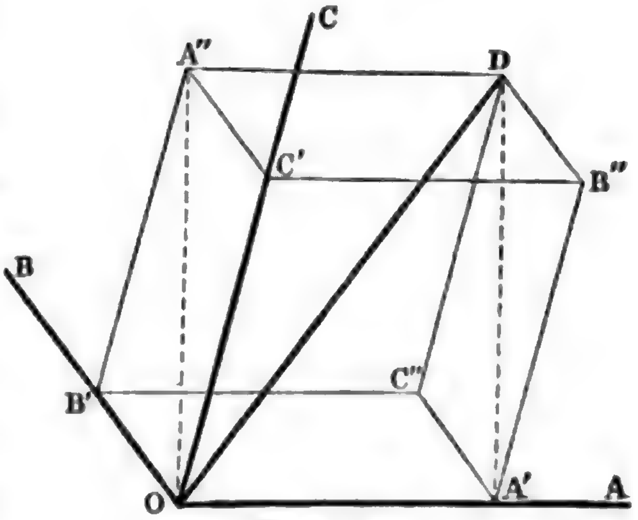

Fig. 28.

$$\delta = \mathrm{OD} = \mathrm{OA}' + \mathrm{OA}'' = \mathrm{OB}' + \mathrm{OB}'' = \mathrm{OC}' + \mathrm{OC}''.$$

And it is evident that this construction will apply to *any fifth point* D *of space,* if the *four* points OABC be still supposed to be *given,* and *not complanar:* but that some at least of the *three ratios* of the *four scalars* a, b, c, d (which last letter is not *here* used as a mark of *differentiation*) will *vary* with the *position* of the point D, or with the *value* of its vector δ. For example, we shall have $a = 0$, if D be situated in the plane BOC; and similarly for the two other given planes through O.

63. We may inquire (comp. 23), *what relation* between these scalar *coefficients* must exist, in order that the point D

may be situated *in the fourth given plane* ABC; or what is the *condition of complanarity* of the *four points*, A, B, C, D. Since the *three vectors* DA, DB, DC are now supposed to be *complanar*, they must (by 22) be connected by a *linear equation*, of the form

$$a(\alpha-\delta)+b(\beta-\delta)+c(\gamma-\delta)=0;$$

comparing which with the recent and more general form (62), we see that the required *condition* is,

$$a+b+c+d=0.$$

This equation may be written (comp. again 23) as

$$\frac{-a}{d}+\frac{-b}{d}+\frac{-c}{d}=1, \quad \text{or} \quad \frac{\mathrm{OA}'}{\mathrm{OA}}+\frac{\mathrm{OB}'}{\mathrm{OB}}+\frac{\mathrm{OC}'}{\mathrm{OC}}=1;$$

and, under this last form, it expresses a known *geometrical property* of a *plane* ABCD, referred to three *co-ordinate axes* OA, OB, OC, which are drawn from any common *origin* O, and *terminate* upon the plane. We have also, in this *case of complanarity* (comp. 28), the following *proportion* of *coefficients* and *areas*:

$$a:b:c:-d=\mathrm{DBC}:\mathrm{DCA}:\mathrm{DAB}:\mathrm{ABC};$$

or, more symmetrically, with attention to *signs* of areas,

$$a:b:c:d=\mathrm{BCD}:-\mathrm{CDA}:\mathrm{DAB}:-\mathrm{ABC};$$

where Fig. 18 may serve for illustration, if we conceive O in that Figure to be replaced by D.

64. When we have thus at once the *two* equations,

$$a\alpha+b\beta+c\gamma+d\delta=0, \quad \text{and} \quad a+b+c+d=0,$$

so that the *four co-initial vectors* α, β, γ, δ *terminate* (as above) *on one common plane*, and may therefore be said (comp. 24) to be *termino-complanar*, it is evident that the *two right lines*, DA and BC, which connect *two pairs* of the *four complanar points*, must *intersect* each other in some point A′ of the plane, at a finite or infinite distance. And there i no difficulty in perceiving, on the plan of 31, that the *vectors* of the *three* *points* A′, B′, C′ of intersection, which thus result, are the following:

$$\left\{\begin{array}{ll} \text{for } \mathrm{A}' = \mathrm{BC}\cdot\mathrm{DA}, & \alpha' = \dfrac{b\beta + c\gamma}{b + c} = \dfrac{a\alpha + d\delta}{a + d}; \\ \text{for } \mathrm{B}' = \mathrm{CA}\cdot\mathrm{DB}, & \beta' = \dfrac{c\gamma + a\alpha}{c + a} = \dfrac{b\beta + d\delta}{b + d}; \\ \text{for } \mathrm{C}' = \mathrm{AB}\cdot\mathrm{DC}, & \gamma' = \dfrac{a\alpha + b\beta}{a + b} = \dfrac{c\gamma + d\delta}{c + d}; \end{array}\right.$$

expressions which are *independent* of the *position* of the arbitrary *origin* O, and which accordingly coincide with the corresponding expressions in 27, when we place that origin in the point D, or make $\delta = 0$. Indeed, these last results hold good (comp. 31), even when the *four vectors* α, β, γ, δ, or the *five points* O, A, B, C, D, are *all* complanar. For, although there *then* exist *two* linear equations between those four vectors, which may in general be written thus,

$$a'\alpha + b'\beta + c'\gamma + d'\delta = 0, \qquad a''\alpha + b''\beta + c''\gamma + d''\delta = 0,$$

without the relations, $a' + \&c. = 0$, $a'' + \&c. = 0$, between the coefficients, yet if we form from these *another* linear equation, of the form,

$$(a'' + ta')\alpha + (b'' + tb')\beta + (c'' + tc')\gamma + (d'' + td')\delta = 0,$$

and determine t by the condition,

$$t = -\frac{a'' + b'' + c'' + d''}{a' + b' + c' + d'},$$

we shall only have to make $a = a'' + ta'$, &c., and the two equations written at the commencement of the present article will then both be satisfied; and will conduct to the expressions assigned above, for the three vectors of intersection: which *vectors* may thus be found, without its being *necessary* to employ those processes of *scalar elimination*, which were treated of in the foregoing Chapter.

As an *Example*, let the two given equations be (comp. 27, 38),

$$a\alpha + b\beta + c\gamma = 0, \qquad (2a + b + c)\alpha''' - a\alpha = 0;$$

and let it be required to determine the vectors of the intersections of the three pairs of lines BC, AA‴; CA, BA‴; and AB, CA‴. Forming the combination,

$$(2a+b+c)\alpha''' - a\alpha + t(a\alpha + b\beta + c\gamma) = 0,$$

and determining t by the condition,

$$(2a+b+c) - a + t(a+b+c) = 0,$$

which gives $t = -1$, we have for the three sought vectors the expressions,

$$\frac{b\beta + c\gamma}{b+c}, \quad \frac{c\gamma + 2a\alpha}{c+2a}, \quad \frac{2a\alpha + b\beta}{2a+b};$$

whereof the first $= \alpha'$, by 27. Accordingly, in Fig. 21, the line AA‴ intersects BC in the point A′; and although the two *other* points of intersection here considered, which belong to what has been called (in 34) a *Third Construction*, are not marked in that Figure, yet their *anharmonic symbols* (36), namely, (2, 0, 1) and (2, 1, 0), might have been otherwise found by combining the equations $y = 0$ and $x = 2z$ for the two lines CA, BA‴; and by combining $z = 0$, $x = 2y$ for the remaining pair of lines.

65. In the more general case, when the *four given points* A, B, C, D, are *not* in any *common plane*, let E be *any fifth given point* of space, not situated on any one of the *four faces* of the *given pyramid* ABCD, nor on any such face prolonged; and let its vector $\text{OE} = \varepsilon$. Then the *four co-initial vectors*, EA, EB, EC, ED, whereof (by supposition) no three are complanar, and which do not terminate upon one plane, must be (by 62) connected by some equation of the form,

$$a.\text{EA} + b.\text{EB} + c.\text{EC} + d.\text{ED} = 0;$$

where the *four scalars*, a, b, c, d, and their *sum*, which we shall denote by $-e$, are *all different from zero*. Hence, because $\text{EA} = \alpha - \varepsilon$, &c., we may establish the following *linear equation between five co-initial vectors*, $\alpha, \beta, \gamma, \delta, \varepsilon$, whereof *no four are termino-complanar* (64),

$$a\alpha + b\beta + c\gamma + d\delta + e\varepsilon = 0;$$

with the *relation*, $a + b + c + d + e = 0$, between the *five scalars* a, b, c, d, e, whereof no one now separately vanishes. Hence also, $\varepsilon = (a\alpha + b\beta + c\gamma + d\delta) : (a + b + c + d)$, &c.

66. Under these conditions, if we write

$$\text{D}_1 = \text{DE} \cdot \text{ABC}, \quad \text{and} \quad \text{OD}_1 = \delta_1,$$

that is, if we denote by δ_1 the vector of the point D_1 in which the right line DE intersects the plane ABC, we shall have

$$\delta_1 = \frac{a\alpha + b\beta + c\gamma}{a+b+c} = \frac{d\delta + e\epsilon}{d+e}.$$

In fact, these two expressions are *equivalent*, or represent one *common* vector, in virtue of the given equations; but the first shows (by 63) that this vector δ_1 terminates on the *plane* ABC, and the second shows (by 25) that it terminates on the *line* DE; its extremity D_1 must therefore be, as required, the *intersection* of this line with that plane. We have therefore the two equations,

$$\text{I.}\ldots a(\alpha - \delta_1) + b(\beta - \delta_1) + c(\gamma - \delta_1) = 0;$$
$$\text{II.}\ldots d(\delta - \delta_1) + e(\epsilon - \delta_1) = 0;$$

whence (by 28 and 24) follow the two proportions,

$$\text{I}'.\ldots a : b : c = D_1BC : D_1CA : D_1AB;$$
$$\text{II}'.\ldots d : e = ED_1 : D_1D;$$

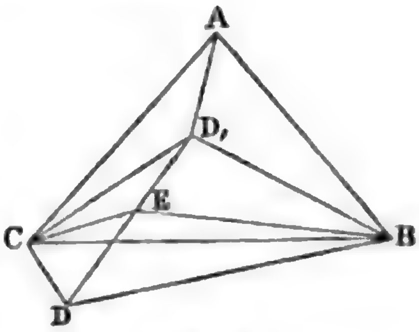

Fig. 29.

the arrangement of the points, in the annexed Fig. 29, answering to the case where all the four coefficients *a*, *b*, *c*, *d* are *positive* (or have one *common sign*), and when therefore the remaining coefficient *e* is *negative* (or has the *opposite* sign).

67. For the three *complanar triangles*, in the first proportion, we may substitute any three *pyramidal volumes*, which *rest* upon those triangles as their *bases*, and which have one *common vertex*, such as D or E; and because the collineation DED_1 gives $DD_1BC - ED_1BC = DEBC$, &c., we may write this other proportion,

$$\text{I}''.\ldots a : b : c = DEBC : DECA : DEAB.$$

Again, the same collineation gives

$$ED_1 : DD_1 = EABC : DABC;$$

we have therefore, by II′., the proportion,

$$\text{II}''.\ldots d : -e = EABC : DABC.$$

But

$$DEBC + DECA + DEAB + EABC = DABC,$$

and

$$a + b + c + d = -e;$$

we may therefore establish the following *fuller* formula of proportion, between *coefficients* and *volumes*:

III. . . $a : b : c : d : -e = \text{DEBC} : \text{DECA} : \text{DEAB} : \text{EABC} : \text{DABC}$;

the *ratios* of all these five *pyramids* to each other being considered as *positive*, for the particular *arrangement* of the *points* which is represented in the recent figure.

68. The *formula* III. may however be regarded as perfectly *general*, if we agree to say that a *pyramidal volume changes sign*, or rather that it *changes its algebraical character*, as *positive* or *negative*, in *comparison* with a *given* pyramid, and with a given *arrangement* of points, *in passing through zero* (comp. 28); namely *when*, in the course of any continuous change, *any one of its vertices crosses the corresponding base*. With this *convention** we shall have, generally,

$$\text{DABC} = -\text{ADBC} = \text{ABDC} = -\text{ABCD}, \quad \text{DEBC} = \text{BCDE}, \quad \text{DECA} = \text{CDEA};$$

the proportion III. may therefore be expressed in the following more *symmetric*, but equally *general form*:

III′. . . $a : b : c : d : e = \text{BCDE} : \text{CDEA} : \text{DEAB} : \text{EABC} : \text{ABCD}$;

the *sum* of *these five pyramids* being *always* equal to *zero*, when *signs* (as above) are attended to.

69. We saw (in 24) that the two equations,

$$a\alpha + b\beta + c\gamma = 0, \qquad a + b + c = 0,$$

gave the proportion of *segments*,

$$a : b : c = \text{BC} : \text{CA} : \text{AB},$$

whatever might be the position of the *origin* O. In like manner we saw (in 63) that the two other equations,

---

* Among the consequences of this convention respecting *signs* of *volumes*, which has already been adopted by some modern geometers, and which indeed is *necessary* (comp. 28) for the establishment of *general formulæ*, one is that *any two pyramids*, ABCD, A′B′C′D′, bear to each other a *positive* or a *negative ratio*, according as the two *rotations*, BCD and B′C′D′, supposed to be *seen* respectively from the points A and A′, have *similar* or *opposite directions*, as right-handed or left-handed.

$$a\alpha + b\beta + c\gamma + d\delta = 0, \qquad a + b + c + d = 0,$$

gave the proportion of *areas*,

$$a : b : c : d = \mathrm{BCD} : - \mathrm{CDA} : \mathrm{DAB} : - \mathrm{ABC};$$

where again the origin is arbitrary. And we have just deduced (in 68) a corresponding proportion of *volumes*, from the two analogous equations (65),

$$a\alpha + b\beta + c\gamma + d\delta + e\epsilon = 0, \qquad a + b + c + d + e = 0,$$

with an equally arbitrary origin. If then we conceive these *segments*, *areas*, and *volumes* to be replaced by the *scalars* to which they are thus *proportional*, we may establish the *three general formulæ :*

I. $\mathrm{OA}.\mathrm{BC} + \mathrm{OB}.\mathrm{CA} + \mathrm{OC}.\mathrm{AB} = 0$;
II. $\mathrm{OA}.\mathrm{BCD} - \mathrm{OB}.\mathrm{CDA} + \mathrm{OC}.\mathrm{DAB} - \mathrm{OD}.\mathrm{ABC} = 0$;
III. $\mathrm{OA}.\mathrm{BCDE} + \mathrm{OB}.\mathrm{CDEA} + \mathrm{OC}.\mathrm{DEAB} + \mathrm{OD}.\mathrm{EABC} + \mathrm{OE}.\mathrm{ABCD} = 0$;

where in I., A, B, C are *any three collinear points ;*
in II., A, B, C, D are *any four complanar points ;*
and in III., A, B, C, D, E are *any five points of space ;*

while O is, in *each* of the three formulæ, an entirely *arbitrary point.* It must, however, be remembered, that the *additions* and *subtractions* are supposed to be performed according to the *rules of vectors*, as stated in the First Chapter of the present Book; the segments, or areas, or volumes, which the equations indicate, being treated as *coefficients* of those vectors. We might still further *abridge* the *notations*, while retaining the *meaning* of these formulæ, by *omitting the symbol* of the *arbitrary origin* O; and by thus writing,*

I′. $\mathrm{A}.\mathrm{BC} + \mathrm{B}.\mathrm{CA} + \mathrm{C}.\mathrm{AB} = 0,$

for any three collinear points; with corresponding formulæ II′. and III′., for any four complanar points, and for any five points of space.

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* We should thus have some of the *notations* of the *Barycentric Calculus* (see Note B), but employed here with different *interpretations.*

SECTION 2.— *On Quinary Symbols for Points and Planes in Space.*

70. The equations of Art. 65 being still supposed to hold good, the vector ρ of any point P of space may, in indefinitely many ways, be expressed (comp. 36) under the form:

$$\text{I.} \ldots \text{OP} = \rho = \frac{xa\alpha + yb\beta + zc\gamma + wd\delta + ve\epsilon}{xa + yb + zc + wd + ve};$$

in which the *ratios of the differences* of the *five coefficients*, *xyzwv*, *determine the position of the point.* In fact, because the four points ABCD are not in any common plane, there necessarily exists (comp. 65) a determined *linear relation* between the *four vectors* drawn to them from the point P, which may be written thus,

$$x'a \,.\, \text{PA} + y'b \,.\, \text{PB} + z'c \,.\, \text{PC} + w'd \,.\, \text{PD} = 0,$$

giving the expression,

$$\text{II.} \ldots \rho = \frac{x'a\alpha + y'b\beta + z'c\gamma + w'd\delta}{x'a + y'b + z'c + w'd},$$

in which the *ratios* of the *four scalars* $x'y'z'w'$, depend upon, and conversely determine, the *position* of P; writing, then,

$$x = tx' + v, \qquad y = ty' + v, \qquad z = tz' + v, \qquad w = tw' + v,$$

where t and v are *two* new and *arbitrary* scalars, and remembering that $a\alpha + \ldots + e\epsilon = 0$, and $a + \ldots + e = 0$ (65), we are conducted to the form for ρ, assigned above.

71. When the *vector* ρ is thus expressed, the *point* P may be denoted by the *Quinary Symbol* (x, y, z, w, v); and we may write the *equation*,

$$\text{P} = (x, y, z, w, v).$$

But we see that the *same point* P may *also* be denoted by this *other symbol*, of the same kind, (x', y', z', w', v'), provided that the following *proportion* between *differences of coefficients* (70) holds good:

$$x' - v' : y' - v' : z' - v' : w' - v' = x - v : y - v : z - v : w - v.$$

Under this condition, we shall therefore write the following *formula of congruence*,

$$(x', y', z', w', v') \equiv (x, y, z, w, v),$$

to express that these *two quinary symbols*, although *not identical in composition*, have yet the *same geometrical signification*, or denote one *common point.* And we shall reserve the symbolic *equation*,

$$(x', y', z', w', v') = (x, y, z, w, v),$$

to express that the *five coefficients*, $x' \ldots v'$, of the one symbol, are *separately equal* to the *corresponding* coefficients of the other, $x' = x, \,..\, v' = v$.

72. Writing also, generally,

$$(tx,\ ty,\ tz,\ tw,\ tv) = t\,(x,\ y,\ z,\ w,\ v),$$
$$(x' + x, \,..\, v' + v) = (x', \,..\, v') + (x, \,..\, v), \ \&c.,$$

and abridging the particular symbol* (1, 1, 1, 1, 1) to (U), while (Q), (Q'), .. may briefly denote the quinary symbols $(x, \,..\, v)$, $(x', \,..\, v')$, .. we may thus establish the *congruence* (71),

$$(Q') \equiv (Q), \text{ if } (Q) = t\,(Q') + u\,(U);$$

in which t and u are arbitrary coefficients. For example,

$$(0, 0, 0, 0, 1) \equiv (1, 1, 1, 1, 0), \quad \text{and} \quad (0, 0, 0, 1, 1) \equiv (1, 1, 1, 0, 0);$$

each symbol of the first pair denoting (65) the given point E; and each symbol of the second pair denoting (66) the derived point D_1. When the coefficients are *so simple* as in these last expressions, we may occasionally *omit the commas*, and thus write, still more briefly,

$$(00001) \equiv (11110); \qquad (00011) \equiv (11100).$$

73. If *three vectors*, ρ, ρ', ρ'', expressed each under the *first form* (70), be *termino-collinear* (24) and if we denote their denominators, $xa + ..$, $x'a + ..$, $x''a + ..$, by m, m', m'', they must then (23) be connected by a *linear equation*, with a *null sum* of coefficients, which may be written thus:

$$tm\rho + t'm'\rho' + t''m''\rho'' = 0; \qquad tm + t'm' + t''m'' + 0.$$

We have, therefore, the two *equations of condition*,

$$t\,(xa\alpha + .. + ve\epsilon) + t'\,(x'a\alpha + .. + v'e\epsilon) + t''\,(x''a\alpha + .. + v''e\epsilon) = 0;$$
$$t\,(xa + .. + ve) + t'\,(x'a + .. + v'e) + t''\,(x''a + .. + v''e) = 0;$$

where t, t', t'' are three new scalars, while the five vectors $\alpha \,..\, \epsilon$, and the five scalars $a \,..\, e$, are subject only to the two equations (65): but these equations of condition are satisfied by supposing that

$$tx + t'x' + t''x'' = \,..\, = tv + t'v' + t''v'' = -u,$$

where u is some new scalar, and they cannot be satisfied otherwise. Hence the *condition of collinearity* of the *three points* P, P′, P″, in which the three vectors ρ, ρ', ρ'' terminate, and of which the quinary symbols are (Q), (Q'), (Q''), may briefly be expressed by the equation,

---

* *This quinary symbol* (U) denotes *no determined point*, since it corresponds (by 70, 71) to the *indeterminate vector* $\rho = \frac{0}{0}$; but it admits of useful *combinations* with *other* quinary symbols, as above.

$$t(Q) + t'(Q') + t''(Q'') = -u(U);$$

so that *if any four scalars, t, t', t'', u, can be found, which satisfy this last symbolic equation,* then, but not in any other case, *those three points* PP′P″ *are ranged on one right line.* For example, the three points D, E, D_1, which are denoted (72) by the quinary symbols, (00010), (00001), (11100), are *collinear;* because the *sum* of these three *symbols* is (U). And if we have the equation,

$$(Q'') = t(Q) + t'(Q') + u(U),$$

where t, t', u are *any three scalars,* then (Q'') is a symbol for a point P″, *on the right line* PP′. For example, the symbol $(0, 0, 0, t, t')$ may denote *any point on the line* DE.

74. By reasonings precisely similar it may be proved, that if (Q) (Q') (Q'') (Q''') be quinary symbols for any *four points* PP′P″P‴ in any *common plane,* so that the *four vectors* $\rho\rho'\rho''\rho'''$ are *termino-complanar* (64), then an equation, of the form

$$t(Q) + t'(Q') + t''(Q'') + t'''(Q''') = -u(U),$$

must hold good; and conversely, that if the *fourth symbol* can be expressed as follows,

$$(Q''') = t(Q) + t'(Q') + t''(Q'') + u(U),$$

with any scalar values of t, t', t'', u, then the *fourth point* P‴ is situated *in the plane* PP′P″ of the *other three.* For example, the four points,

(10000), (01000), (00100), (11100),

or A, B, C, D_1 (66), are *complanar;* and the symbol $(t, t', t'', 0, 0)$ may represent *any point in the plane* ABC.

75. When a point P is thus *complanar with three given points,* P_0, P_1, P_2, we have therefore expressions of the following forms, for the *five coefficients* $x, \ldots v$ of its quinary symbol, in terms of the *fifteen* given coefficients of *their* symbols, and of *four* new and arbitrary scalars:

$$x = t_0x_0 + t_1x_1 + t_2x_2 + u; \ldots \qquad v = t_0v_0 + t_1v_1 + t_2v_2 + u.$$

And hence, by *elimination* of these four scalars, $t_0 \ldots u$, we are conducted to a *linear equation* of the form

$$l(x - v) + m(y - v) + n(z - v) + r(w - v) = 0,$$

which may be called the *Quinary Equation of the Plane* $\text{P}_0\text{P}_1\text{P}_2$, or of the supposed *locus of the point* P: because it expresses a *common property* of all the points of that locus; and because the *three ratios* of the *four new coefficients l, m, n, r, determine the position of the plane* in space. It is, however, more *symmetrical*, to write the quinary equation of a plane Π as follows,

$$lx + my + nz + rw + sv = 0,$$

where the *fifth coefficient*, s, is connected with the others by the relation,

$$l + m + n + r + s = 0;$$

and then we may say that $[l, m, n, r, s]$ is (comp. 37) the *Quinary Symbol of the Plane* Π, and may write the *equation*,

$$\Pi = [l, m, n, r, s].$$

For example, the coefficients of the symbol for a point P in the plane ABC may be thus expressed (comp. 74):

$$x = t_0 + u, \qquad y = t_1 + u, \qquad z = t_2 + u, \qquad w = u, \qquad v = u;$$

between which the only relation, *independent of the four arbitrary scalars* $t_0 \ldots u$, is $w - v = 0$; this therefore is the *equation* of the plane ABC, and the *symbol* of that plane is $[0, 0, 0, 1, -1]$; which may (comp. 72) be sometimes written more briefly, without commas, as $[0001\bar{1}]$. It is evident that, in any such symbol, the *coefficients* may all be multiplied by any *common factor*.

76. The symbol of the *plane* $P_0P_1P_2$ having been thus determined, we may next propose to find a symbol for the *point*, P, in which that plane is *intersected* by a given *line* P_3P_4: or to *determine the coefficients* $x \ldots v$, or at least the *ratios* of their *differences* (70), in the quinary symbol of that point,

$$(x, y, z, w, v) = P = P_0P_1P_2 \cdot P_3P_4.$$

Combining, for this purpose, the expressions,

$$x = t_3x_3 + t_4x_4 + u', \ldots \qquad v = t_3v_3 + t_4v_4 + u',$$

(which are included in the symbolical equation (73),

$$(Q) = t_3(Q_3) + t_4(Q_4) + u'(U),$$

and express the *collinearity* PP_3P_4,) with the equations (75),

$$lx + \ldots + sv = 0, \qquad l + \ldots + s = 0,$$

(which express the *complanarity* $PP_0P_1P_2$,) we are conducted to the formula,

$$t_3(lx_3 + \ldots + sv_3) + t_4(lx_4 + \ldots + sv_4) = 0;$$

which determines the *ratio* $t_3 : t_4$, and contains the solution of the problem. For example, if P be a point *on the line* DE, then (comp. 73),

$$x = y = z = u', \qquad w = t_3 + u', \qquad v = t_4 + u';$$

but if it be *also* a point *in the plane* ABC, then $w - v = 0$ (75), and therefore $t_3 - t_4 = 0$; hence

$$(Q) = t_3(00011) + w'(11111), \quad \text{or} \quad (Q) \equiv (00011);$$

which last symbol had accordingly been found (72) to represent the *intersection* (66), $\text{D}_1 = \text{ABC} \cdot \text{DE}$.

77. When the five coefficients, *xyzwv*, of any given quinary *symbol* (*Q*) for a point P, or those of any *congruent* symbol (71), are any *whole numbers* (positive or negative, or zero), we shall say (comp. 42) that the *point* P *is rationally related to the five given points*, A . . E; or briefly, that it is a *Rational Point of the System*, which those five points determine. And in like manner, when the five coefficients, *lmnrs*, of the quinary symbol (75) of a *plane* Π are either *equal* or *proportional* to *integers*, we shall say that the plane is a *Rational Plane* of the same *System*; or that it is *rationally related* to the same five points. On the contrary, when the quinary symbol of a point, or of a plane, has *not* thus already *whole coefficients*, and cannot be *transformed* (comp. 72) so as to have them, we shall say that the point or plane is *irrationally related* to the given points; or briefly, that it is *irrational*. A *right line* which *connects* two *rational points*, or is the *intersection* of two *rational planes*, may be called, on the same plan, a *Rational Line*; and lines which cannot in either of these two ways be constructed, may be said by contrast to be *Irrational Lines*. It is evident from the nature of the *eliminations* employed (comp. again 42), that a *plane*, which is *determined* as *containing three rational points*, is necessarily a *rational plane*; and in like manner, that a *point*, which is determined as the common *intersection of three rational planes*, is always a *rational point*: as is also every point which is obtained by the intersection of a *rational line* with a *rational plane*; or of *two rational lines* with *each other* (when they happen to be complanar).

78. Finally, when *two points*, or *two planes*, differ only by the *arrangement* (or *order*) of the *coefficients* in their quinary *symbols*, those points or planes may be said to have one *common type*; or briefly to be *syntypical*. For example, the *five given points*, A, . . E, are thus syntypical, as being represented by the quinary symbols (10000), . . (00001); and the *ten planes*, obtained by taking all the *ternary combinations* of those five points, have in like manner one common type. Thus, the quinary symbol of the plane ABC has been seen (75) to be $[0001\bar{1}]$; and the analogous symbol $[1\bar{1}000]$ represents the plane CDE, &c. Other examples will present themselves, in a

shortly subsequent Section, on the subject of *Nets in Space*. But it seems proper to say here a few words, respecting those *Anharmonic Co-ordinates*, *Equations*, *Symbols*, and *Types*, for *Space*, which are obtained from the theory and expressions of the present Section, by *reducing* (as we are allowed to do) the *number* of the *coefficients*, in each symbol or equation, from *five* to *four*.

SECTION 3.—*On Anharmonic Co-ordinates in Space.*

79. When we adopt the *second form* (70) for ρ, or suppose (as we may) that the *fifth coefficient* in the *first* form *vanishes*, we get this *other general expression* (comp. 34, 36), for the *vector of a point in space:*

$$\mathrm{OP} = \rho = \frac{xa\alpha + yb\beta + zc\gamma + wd\delta}{xa + yb + zc + wd};$$

and may then write the symbolic *equation* (comp. 36, 71),

$$\mathrm{P} = (x, y, z, w),$$

and call this last the *Quaternary Symbol of the Point* P: although we shall soon see cause for calling it also the *Anharmonic Symbol* of that point. Meanwhile we may remark, that the *only congruent symbols* (71), of this *last form*, are those which differ merely by the introduction of a *common factor:* the *three ratios* of the *four coefficients*, $x \,.\,.\, w$, being *all* required, in order to *determine the position of the point;* whereof those four coefficients may accordingly be said (comp. 36) to be the *Anharmonic Coordinates in Space.*

80. When we thus suppose that $v = 0$, in the quinary *symbol* of the *point* P, we may *suppress the fifth term sv*, in the quinary *equation* of a *plane* Π, $lx + \,.\,.\, + sv = 0$ (75); and therefore may *suppress* also (as here unnecessary) the *fifth coefficient*, s, in the *quinary symbol* of that plane, which is thus reduced to the *quaternary form*,

$$\Pi = [l, m, n, r].$$

This last may also be said (37, 79), to be the *Anharmonic Symbol of the Plane*, of which the *Anharmonic Equation* is

$$lx + my + nz + rw = 0;$$

the *four coefficients*, *lmnr*, which we shall call also (comp. again 37) the *Anharmonic Co-ordinates of that Plane* Π, being not connected *among themselves* by any *general relation* (such as $l + \,.\,.\, + s = 0$): since their *three ratios* (comp. 79) are *all* in general necessary, in order to determine the *position of the plane in space.*

81. If we suppose that the *fourth coefficient*, w, *also vanishes*, in

the recent symbol of a point, that *point* P is *in the plane* ABC; and may then be sufficiently represented (as in 36) by the *Ternary Symbol* (x, y, z). And if we attend only to the points in which an *arbitrary plane* Π *intersects* the *given plane* ABC, we may *suppress its fourth coefficient*, r, as being for *such* points unnecessary. In this manner, then, we are reconducted to the *equation*, $lx + my + nz = 0$, and to the *symbol*, $\Lambda = [l, m, n]$, for a *right line* (37) *in the plane* ABC, considered here as the *trace*, on that plane, of an *arbitrary plane* Π *in space*. If this plane Π be given by its *quinary symbol* (75), we thus obtain the *ternary symbol* for its *trace* Λ, by simply *suppressing the two last coefficients*, r and s.

82. In the more general case, when the point P is *not* confined to the plane ABC, if we denote (comp. 72) its *quaternary symbol* by (Q), the lately established formulæ of *collineation* and *complanarity* (73, 74) will still hold good: provided that we now *suppress* the symbol (U), or suppose its *coefficient* to be *zero*. Thus, the formula,

$$(Q) = t'(Q') + t''(Q'') + t'''(Q'''),$$

expresses that the point P is *in the plane* P′P″P‴; and if the coefficient t''' *vanish*, the equation which then remains, namely,

$$(Q) = t'(Q') + t''(Q''),$$

signifies that P is thus *complanar* with the *two given points* P′, P″, and with an *arbitrary third point*; or, in other words, that it is *on the right line* P′P″; whence (comp. 76) problems of *intersections* of *lines* with *planes* can easily be resolved. In like manner, if we denote briefly by $[R]$ the quaternary symbol $[l, m, n, r]$ for a *plane* Π, the formula

$$[R] = t'[R'] + t''[R''] + t'''[R''']$$

expresses that the plane Π passes *through the intersection* of the *three planes*, Π′, Π″, Π‴; and if we suppose $t''' = 0$, so that

$$[R] = t'[R'] + t''[R''],$$

the formula thus found denotes that the plane Π passes through the point of intersection of the *two* planes, Π′, Π″, with *any third plane*; or (comp. 41), that this plane Π *contains the line of intersection* of Π′, Π″; in which case the *three planes*, Π, Π′, Π″, may be said to be *collinear*. Hence it appears that either of the *two expressions*,

$$\text{I.} \ldots t'(Q') + t''(Q''), \qquad \text{II.} \ldots t'[R'] + t''[R''],$$

may be used as a *Symbol of a Right Line in Space*: according as we consider that *line* Λ either, Ist, as *connecting two given points*, or

IInd, as being the *intersection of two given planes.* The remarks (77) on *rational* and *irrational points*, *planes*, and *lines* require no modification here; and those on *types* (78) adapt themselves as easily to *quaternary* as to *quinary* symbols.

83. From the foregoing general formulæ of collineation and complanarity, it follows that the point P', in which the line AB intersects the plane CDP through CD and any proposed point $P = (xyzw)$ of space, may be denoted thus:

$$P' = AB \cdot CDP = (xy00);$$

for example, $E = (1111)$, and $C' = AB \cdot CDE = (1100)$. In general, if ABCDEF be *any six points of space*, the *four collinear planes* (82), ABC, ABD, ABE, ABF, are said to form a *pencil* through AB; and if this be *cut* by any rectilinear *transversal*, in *four points*, $C^{\backprime}$, $D^{\backprime}$, $E^{\backprime}$, $F^{\backprime}$, then (comp. 35) the *anharmonic function* of this *group* of *points* (25) is called also the *Anharmonic of the Pencil of Planes:* which may be thus denoted,

$$(AB \,.\, CDEF) = (C^{\backprime}D^{\backprime}E^{\backprime}F^{\backprime}).$$

Hence (comp. again 25, 35), by what has just been shown respecting C' and P', we may establish the important formula:

$$(CD \,.\, AEBP) = (AC'BP') = \frac{x}{y};$$

so that this *ratio of coefficients*, in the symbol $(xyzw)$ for a *variable point* P (79), represents the *anharmonic of a pencil of planes*, of which the *variable plane* CDP is *one;* the *three other* planes of this pencil being *given.* In like manner,

$$(AD \,.\, BECP) = \frac{y}{z}, \quad \text{and} \quad (BD \,.\, CEAP) = \frac{z}{x};$$

so that (comp. 36) the *product* of these three last anharmonics is *unity.* On the same plan we have also,

$$(BC \,.\, AEDP) = \frac{x}{w}, \qquad (CA \,.\, BEDP) = \frac{y}{w}, \qquad (AB \,.\, CEDP) = \frac{z}{w};$$

so that the *three ratios*, of the three first coefficients xyz to the fourth coefficient w, suffice to *determine the three planes*, BCP, CAP, ABP, whereof the *point* P is the common *intersection*, by means of the *anharmonics of three pencils of planes*, to which the three planes respectively belong. And thus we see a *motive* (besides that of *analogy* to expressions already used for *points in a given plane*), for calling the *four coefficients*, $xyzw$, in the *quaternary symbol* (79) for a *point in space*, the *Anharmonic Co-ordinates of that Point.*

84. In general, if there be *any four collinear points*, $P_0, \ldots P_3$, so

that (comp. 82) their *symbols* are connected *by two linear equations*, such as the following,

$$(Q_1) = t(Q_0) + u(Q_2), \qquad (Q_3) = t'(Q_0) + u'(Q_2),$$

then the anharmonic of their *group* may be expressed (comp. 25, 44) as follows:

$$(\mathrm{P_0P_1P_2P_3}) = \frac{ut'}{tu'};$$

as appears by considering the *pencil* (CD . P₀P₁P₂P₃), and the *transversal* AB (83). And in like manner, if we have (comp. again 82) the two other symbolic equations, connecting *four collinear planes* $\Pi_0 \,..\, \Pi_3$,

$$[R_1] = t[R_0] + u[R_2], \qquad [R_3] = t'[R_0] + u'[R_2],$$

the anharmonic of their *pencil* (83) is expressed by the precisely similar formula,

$$(\Pi_0\Pi_1\Pi_2\Pi_3) = \frac{ut'}{tu'};$$

as may be proved by supposing the pencil to be cut by the same transversal line AB.

85. It follows that if $f(xyzw)$ and $f_1(xyzw)$ be any two homogeneous and linear functions of x, y, z, w; and if we determine four collinear planes $\Pi_0 \,..\, \Pi_3$ (82), by the four equations,

$$f = 0, \qquad f_1 = f, \qquad f_1 = 0, \qquad f_1 = kf,$$

where k is any scalar; we shall have the following value of the anharmonic function, of the pencil of planes thus determined:

$$(\Pi_0\Pi_1\Pi_2\Pi_3) = k = \frac{f_1}{f}.$$

Hence we derive this *Theorem*, which is important in the application of the present system of co-ordinates to space:—

"*The Quotient of any two given homogeneous and linear Functions, of the anharmonic Co-ordinates* (79) *of a variable Point* P *in space, may be expressed as the Anharmonic* $(\Pi_0\Pi_1\Pi_2\Pi_3)$ *of a Pencil of Planes;* whereof *three* are *given*, while the *fourth* passes *through the variable point* P, and *through a given right line* Λ which is *common* to the *three former planes*."

86. And in like manner may be proved this other but analogous Theorem:—

"*The Quotient of any two given homogeneous and linear Functions, of the anharmonic Co-ordinates* (80) *of a variable Plane* Π, *may be expressed as the Anharmonic* (P₀P₁P₂P₃) *of a Group of Points;* whereof *three* are *given* and *collinear*, and the *fourth* is the *intersection*, Λ · Π, of their *common* and *given right line* Λ, *with the variable plane* Π."

More fully, if the two given functions of *lmnr* be F and F_1, and if we determine three points $P_0P_1P_2$ by the equations (comp. 57) $F=0$, $F_1=F$, $F_1=0$, and denote by P_3 the intersection of their common line Λ with Π, we shall have the quotient,

$$\frac{F_1}{F}=(P_0P_1P_2P_3).$$

For example, if we suppose that

$$A_2=(1001),\qquad B_2=(0101),\qquad C_2=(0011),$$
$$A'_2=(10\bar{0}1),\qquad B'_2=(010\bar{1}),\qquad C'_2=(001\bar{1}),$$

so that

$$A_2=DA\cdot BCE,\ \&c.,\quad \text{and}\quad (DA_2AA'_2)=-1,\ \&c.,$$

we find that the three ratios of *l*, *m*, *n* to *r*, in the symbol $\Pi=[lmnr]$, may be expressed (comp. 39) under the form of anharmonics of groups, as follows:

$$\frac{l}{r}=(DA'_2AQ);\qquad \frac{m}{r}=(DB'_2BR);\qquad \frac{n}{r}=(DC'_2CS);$$

where Q, R, S denote the intersections of the plane Π with the three given right lines, DA, DB, DC. And thus we have a *motive* (comp. 83) besides that of *analogy* to *lines* in a given plane (37), for calling (as above) the *four coefficients* *l*, *m*, *n*, *r*, in the *quaternary symbol* (80) for a *plane* Π, the *Anharmonic Co-ordinates of that Plane in Space.*

87. It may be added, that if we denote by L, M, N the points in which the same plane Π is cut by the three given lines BC, CA, AB, and retain the notations A″, B″, C″ for those other points on the same three lines which were so marked before (in 31, &c.), so that we may now write (comp. 36)

$$A''=(01\bar{1}0),\qquad B''=(\bar{1}010),\qquad C''=(1\bar{1}00),$$

we shall have (comp. 39, 83) these three other anharmonics of groups, with their product equal to unity:

$$\frac{m}{n}=(CA''BL);\qquad \frac{n}{l}=(AB''CM);\qquad \frac{l}{m}=(BC''AN);$$

and the *six given points*, A″, B″, C″, A'_2, B'_2, C'_2, are all *in one given plane* [E], of which the *equation* and *symbol* are:

$$x+y+z+w=0;\qquad [E]=[11111].$$

The *six groups* of points, of which the anharmonic functions thus represent the *six ratios* of the four anharmonic co-ordinates, *lmnr*, of a *variable plane* Π, are therefore situated *on the six edges* of the *given pyramid*, ABCD; *two points* in each group being *corners* of that pyramid, and the *two others* being the *intersections* of the *edge* with the *two planes*, [E] and Π. Finally, the *plane* [E] is (in a known modern sense) the *plane of homology*,* and the point E is the *centre of homology*, of the *given* pyramid ABCD, and of an *inscribed pyramid* $A_1B_1C_1D_1$, where $A_1 = EA \cdot BCD$, &c.; so that D_1 retains its recent signification (66, 76); and we may write the anharmonic symbols,

$$A_1 = (0111), \qquad B_1 = (1011), \qquad C_1 = (1101), \qquad D_1 = (1110).$$

And if we denote by $A'_1B'_1C'_1D'_1$ the harmonic conjugates to these last points, with respect to the lines EA, EB, EC, ED, so that

$$(EA_1AA'_1) = \ldots = (ED_1DD'_1) = -1,$$

we have the corresponding symbols,

$$A'_1 = (2111), \qquad B'_1 = (1211), \qquad C'_1 = (1121) \qquad D'_1 = (1112).$$

Many other relations of position exist, between these various points, lines, and planes, of which some will come naturally to be noticed, in that theory of *nets in space* to which in the following Section we shall proceed.

SECTION 4.—*On Geometrical Nets in Space.*

88. When we have (as in 65) five given points A . . E, whereof no four are complanar, we can *connect* any *two* of them by a right *line*, and the *three* others by a *plane*, and determine the *point* in which these last *intersect* one another: *deriving* thus a system of *ten lines* Λ_1, *ten planes* Π_1, and *ten points* P_1, from the *given* system of *five points* P_0, by what may be called (comp. 34) a *First Construction.* We may next propose to determine all the *new* and *distinct lines*, Λ_2, and *planes*, Π_2, which connect the ten derived points P_1 with the five given points P_0, and with each other; and may then inquire what *new* and *distinct points* P_2 arise (at this stage) as intersections of *lines with planes*, or of *lines in one plane* with *each other:* all *such* new lines, planes, and points being said (comp. again 34) to belong to a *Second Construction.* And then we might proceed to a *Third Construction* of the same kind, and so on for ever : building up thus what has been called† a *Geometrical Net in Space.* To *express* this geometrical process by *quinary symbols* (71, 75, 82) of *points*, *planes*, and *lines*, and by *quinary types* (78), so far at least as to the end of the *second construction*, will be found to be an useful exercise in the

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* See Poncelet's *Traité des Propriétés Projectives* (Paris, 1822).

† By Möbius, in p. 291 of his already cited *Barycentric Calculus.*

application of principles lately established: and therefore ultimately in that METHOD OF VECTORS, which is the subject of the present Book. And the *quinary form* will *here* be more convenient than the *quaternary*, because it will exhibit more clearly the geometrical dependence of the *derived* points and planes on the *five* given points, and will thereby enable us, through a principle of *symmetry*, to *reduce* the *number* of distinct *types*.

89. Of the *five given points*, P_0, the *quinary type* has been seen (78) to be (10000); while of the *ten derived points* P_1, of *first* construction, the corresponding *type* may be taken as (00011); in fact, considered as *symbols*, these two represent the points A and D_1. The nine other points P_1 are $A'B'C'A_1B_1C_1A_2B_2C_2$; and we have now (comp. 83, 87, 86) the symbols,

$$A' = BC \cdot ADE = (01100), \qquad A_1 = EA \cdot BCD = (10001),$$
$$A_2 = DA \cdot BCE = (10010);$$

also, in any symbol or equation of the present form, it is permitted to change A, B, C to B, C, A, provided that we at the same time write the third, first, and second co-efficients, in the places of the first, second, and third: thus, $B' = CA \cdot BDE = (10100)$, &c. The symbol $(xy000)$ represents an *arbitrary point on the line* AB; and the symbol $[00nrs]$, with $n + r + s = 0$, represents an *arbitrary plane through that line: each* therefore may be regarded (comp. 82) as a *symbol* also of the *line* AB *itself*, and at the same time as a *type* of the *ten lines* Λ_1; while the symbol $[0001\bar{1}]$, of the plane ABC (75), may be taken (78) as a type of the *ten planes* Π_1. Finally, the *five pyramids*,

BCDE, CADE, ABDE, ABCE, ABCD,

and the *ten triangles*, such as ABC, whereof each is a *common face* of *two* such pyramids, may be called *pyramids* R_1, and *triangles* T_1, of the *First Construction*.

90. Proceeding to a *Second Construction* (88), we soon find that the *lines* Λ_2 may be arranged in *two distinct groups; one* group consisting of *fifteen lines* $\Lambda_{2,1}$, such as the line* $AA'D_1$, whereof each *connects two points* P_1, and passes also *through one point* P_0, being the *intersection* of *two planes* Π_1 through that point, as here of ABC, ADE; while the *other group* consists of *thirty lines* $\Lambda_{2,2}$, such as $B'C'$, each connecting two points P_1, but *not* passing through any point P_0, and being one of the *thirty edges* of *five new pyramids* R_2, namely,

$$C'B'A_2A_1, \qquad A'C'B_2B_1, \qquad B'A'C_2C_1, \qquad A_2B_2C_2D_1, \qquad A_1B_1C_1D_1:$$

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* AB_1C_2, AB_2C_1, $DA'A_1$, $EA'A_2$, are other lines of this group.

which pyramids R_2 may be said (comp. 87) to be *inscribed homologues* of the five former pyramids R_1, the *centres of homology* for these *five pairs of pyramids* being the five given points A . . E; and the *planes of homology* being five planes [A] . . [E], whereof the last has been already mentioned (87), but which belong properly to a *third* construction (88). The *planes* Π_2, of *second* construction, form in like manner *two groups; one* consisting of *fifteen planes* $\Pi_{2,1}$, such as the plane of the *five points*, $AB_1B_2C_1C_2$, whereof each passes through *one point* P_0, and through *four points* P_1, and contains *two lines* $\Lambda_{2,1}$, as here the lines AB_1C_2, AC_1B_2, besides containing *four lines* $\Lambda_{2,2}$, as here B_1B_2, &c.; while the *other group* is composed of *twenty planes* $\Pi_{2,2}$, such as $A_1B_1C_1$, namely, the *twenty faces* of the five recent pyramids R_2, whereof each contains *three points* P_1, and *three lines* $\Lambda_{2,2}$, but does not pass through any point P_0. It is now required to *express* these *geometrical conceptions** of the *forty-five lines* Λ_2; the *thirty-five planes* Π_2; and the *five planes of homology* of pyramids, [A] . . . [E], by quinary *symbols* and *types*, before proceeding to determine the *points* P_2 of *second* construction.

91. An arbitrary *point* on the right line $AA'D_1$ (90) may be represented by the symbol (*tuu*00); and an arbitrary *plane* through that line by this other symbol, $[0m\bar{m}r\bar{r}]$, where $\bar{m}$ and $\bar{r}$ are written (to save commas) instead of $-m$ and $-r$; hence these two symbols may also (comp. 82) denote the *line* $AA'D_1$ itself, and may be used as types (78) to represent the *group* of lines $\Lambda_{2,1}$. The particular symbol $[01\bar{1}1\bar{1}]$, of the last form, represents that particular plane through the last-mentioned line, which contains also the line AB_1C_2 of the same group; and may serve as a type for the group of planes $\Pi_{2,1}$. The line $B'C'$, and the group $\Lambda_{2,2}$, may be represented by (*stu*00) and $[\bar{t}ttu\bar{s}]$, if we agree† to write $s = t + u$, and $\bar{s} = -s$; while the plane $B'C'A_2$, and the group $\Pi_{2,2}$, may be denoted by $[\bar{1}111\bar{2}]$. Finally, the plane [E] has for its symbol $[1111\bar{4}]$; and the four other planes [A], &c., of homology of pyramids (90), have this last for their common type.

92. The *points* P_2, of *second* construction (88), are more nume-

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* Möbius (in his *Barycentric Calculus*, p. 284, &c.) has very clearly pointed out the existence and chief properties of the foregoing *lines* and *planes;* but besides that his *analysis* is altogether different from ours, he does not appear to have aimed at *enumerating*, or even at *classifying*, all the *points* of what has been above called (88) the *second construction*, as we propose shortly to do.

† With this convention, the line AB, and the group Λ_1, may be denoted by the *plane-symbol* [00*tus*] their *point-symbol* being (*tu*000).

rous than the *lines* Λ_2 and *planes* Π_2 of that construction: yet with the help of *types*, as above, it is not difficult to classify and to enumerate them. It will be sufficient here to write down these types, which are found to be *eight*, and to offer some remarks respecting them; in doing which we shall avail ourselves of the eight following *typical points*, whereof the two first have already occurred, and which are all situated in the plane of ABC:

$$A'' = (01\bar{1}00); \quad A''' = (21100); \quad A^{\text{IV}} = (\bar{2}1100); \quad A^{\text{V}} = (02100);$$
$$A^{\text{VI}} = (02\bar{1}00); \quad A^{\text{VII}} = (12\bar{1}00); \quad A^{\text{VIII}} = (32100); \quad A^{\text{IX}} = (23\bar{1}00);$$

the second and third of these having $(\bar{1}0011)$ and (30011) for *congruent symbols* (71). It is easy to see that these *eight types* represent, respectively, ten, thirty, thirty, twenty, twenty, sixty, sixty, and sixty distinct points, belonging to *eight groups*, which we shall mark as $P_{2,1}, \ldots P_{2,8}$; so that the total number of the points P_2 is 290. If then we consent (88) to *close* the present inquiry, at the end of what we have above *defined* to be the *Second Construction*, the *total number of* the *net points*, P_1, P_2, which are thus *derived* by *lines* and *planes* from the *five given points* P_0, is found to be exactly *three hundred:* while the *joint* number of the *net-lines*, Λ_1, Λ_2, and of the *net-planes*, Π_1, Π_2, has been seen to be *one hundred*, so far.

(1.) To the type $P_{2,1}$ belong the *ten points*,

$$A''B''C'', \qquad A'_2B'_2C'_2, \qquad A'_1B'_1C'_1D'_1,$$

with the quinary symbols,

$$A'' = (01\bar{1}00), \ldots \quad A'_2 = (100\bar{1}0), \ldots \quad A'_1 = (1000\bar{1}), \ldots \quad D'_1 = (0001\bar{1}),$$

which are the *harmonic conjugates* of the ten points P_1, namely, of

$$A'B'C', \qquad A_2B_2C_2, \qquad A_1B_1C_1D_1,$$

with respect to the ten lines Λ_1, on which those points are situated; so that we have ten harmonic equations, $(BA'CA'') = -1$, &c., as already seen (31, 86, 87). Each point $P_{2,1}$ is the *common intersection* of a line Λ_1 with *three lines* $\Lambda_{2,2}$; thus we may establish the four following *formulæ of concurrence* (equivalent, by 89, to *ten* such formulæ):

$$A'' = BC \cdot B'C' \cdot B_1C_1 \cdot B_2C_2; \qquad A'_2 = DA \cdot D_1A_1 \cdot B'C_2 \cdot C'B_2;$$
$$A'_1 = EA \cdot D_1A_2 \cdot B'C_1 \cdot C'B_1; \qquad D'_1 = DE \cdot A_1A_2 \cdot B_1B_2 \cdot C_1C_2.$$

Each point $P_{2,1}$ is also situated in *three planes* Π_1; in *three* other planes, of the group $\Pi_{2,1}$; and in *six* planes $\Pi_{2,2}$; for example, A'' is a point common to the *twelve* planes,

$$ABC,\ BCD,\ BCE; \qquad AB_1C_2C_1B_2, \qquad DB'B_1C'C_1, \qquad EB'B_2C'C_2;$$
$$B'C'A_1, \qquad B_1C_1A_1, \qquad B_2C_2A_2, \qquad B'C'A_2, \qquad B_1C_1D_1, \qquad B_2C_2D_1.$$

Each line, Λ_1 or $\Lambda_{2,2}$, contains *one* point $P_{2,1}$; but no line $\Lambda_{2,1}$ contains any. Each plane, Π_1 or $\Pi_{2,2}$, contains *three* such points; and each plane $\Pi_{2,1}$ contains *two*, which are the *intersections* of *opposite sides* of a *quadrilateral* Q_2 in that plane, whereof the *diagonals* intersect in a point P_0: for example, the diagonals B_1C_2, B_2C_1 of the quadrilateral $B_1B_2C_2C_1$, which is (by 90) in one of the planes $\Pi_{2,1}$, intersect* each other in the point A; while the opposite sides C_1B_1, B_2C_2 intersect in A''; and the two other opposite sides, B_1B_2, C_2C_1 have the point D'_1 for their intersection. The *ten points* $P_{2,1}$ are also ranged, *three by three*, on *ten lines* of *third construction* Λ_3, namely, on the *axes of homology*,

$$A''B'_1C'_1, \,.\,.\, A''B'_2C'_2, \,.\,.\, A'_1A'_2D'_1, \,.\,.\, A''B''C'',$$

of *ten pairs of triangles* T_1, T_2, which are situated in the ten planes Π_1, and of which the *centres* of homology are the ten points P_1: for example, the dotted line $A''B''C''$, in Fig. 21, is the axis of homology of the two triangles, ABC, $A'B'C'$, whereof the latter is *inscribed* in the former, with the point O in that figure (replaced by D_1 in Fig. 29), to represent their centre of homology. The same ten *points* $P_{2,1}$ are also ranged *six by six*, and the ten last *lines* Λ_3 are ranged *four by four*, in *five planes* Π_3, namely in the *planes of homology* of *five pairs of pyramids*, R_1, R_2, already mentioned (90): for example, the plane [E] contains (87) the six points $A''B''C''A'_2B'_2C'_2$, and the four right lines,

$$A''B'_2C'_2, \qquad B''C'_2A'_2, \qquad C''A'_2B'_2, \qquad A''B''C'';$$

which latter are the intersections of the *four faces*,

$$DCB, \qquad DAC, \qquad DBA, \qquad ABC,$$

of the pyramid $ABCD$, with the corresponding faces,

$$D_1C_1B_1, \qquad D_1A_1C_1, \qquad D_1B_1A_1, \qquad A_1B_1C_1,$$

of its *inscribed homologue* $A_1B_1C_1D_1$; and are contained, besides, in the four other planes,

$$A_2B'C', \qquad B_2C'A', \qquad C_2A'B', \qquad A_2B_2C_2:$$

the three triangles, ABC, $A_1B_1C_1$, $A_2B_2C_2$, for instance, being *all homologous*, although *in different planes*, and having the line $A''B''C''$ for their *common axis* of homology. We may also say, that this line $A''B''C''$ is the *common trace* (81) of *two planes* $\Pi_{2,2}$, namely of $A_1B_1C_1$ and $A_2B_2C_2$, on the plane ABC; and in like manner, that the *point* A'' is the common *trace*, on that plane Π_1, of *two lines* $\Lambda_{2,2}$, namely of B_1C_1 and B_2C_2: being also the common trace of the two lines $B'_1C'_1$ and $B'_2C'_2$, which belong to the *third* construction.

(2.) On the whole, *these ten points*, of *second* construction, A'' . . ., may be considered to be already well known to geometers, in connexion with the theory of *transversal*† *lines and planes* in space: but it is important here to observe, with what simplicity and clearness their geometrical relations are *expressed* (88), by the *quinary symbols* and quinary *types* employed. For example, the *collinearity* (82) of the *four planes*, ABC, $A_1B_1C_1$, $A_2B_2C_2$, and [E], becomes evident from mere *inspection* of their *four symbols*,

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* Compare the Note to page 68.

† The collinear, complanar, and harmonic relations between the ten points, which we have above marked as $P_{2,1}$, and which have been considered by Möbius also, in connexion with his theory of *nets in space*, appear to have been first noticed by Carnot, in a Memoir upon *transversals*.

$$[0001\bar{1}], \qquad [111\bar{2}\bar{1}], \qquad [111\bar{1}\bar{2}], \qquad [1111\bar{4}],$$

which represent (75) the *four quinary equations*,

$$w-v=0, \quad x+y+z-2w-v=0, \quad x+y+z-w-2v=0, \quad x+y+z+w-4v=0;$$

with this additional consequence, that the *ternary symbol* (81) of the *common trace*, of the three latter on the former, is [111]: so that this trace is (by 38) the line A″B″C″ of Fig. 21, as above. And if we briefly denote the quinary symbols of the four planes, taken in the same form and order as above, by $[R_0]$ $[R_1]$ $[R_2]$ $[R_3]$, we see that they are connected by the two relations,

$$[R_1] = -[R_0] + [R_2]; \qquad [R_3] = 2[R_0] + [R_2];$$

whence if we denote the planes themselves by Π_1, Π_2, Π'_2, Π_3, we have (comp. 84) the following value for the *anharmonic of their pencil*,

$$(\Pi_1\Pi_2\Pi'_2\Pi_3) = -2;$$

a result which can be very simply verified, for the case when ABCD is a *regular pyramid*, and E (comp. 29) is its *mean point*: the plane Π_3, or [E], becoming in this case (comp. 38) the *plane at infinity*, while the three other planes, ABC, $A_1B_1C_1$, $A_2B_2C_2$, are *parallel*; the *second* being *intermediate* between the other two, but *twice as near* to the *third* as to the *first*.

(3.) We must be a little more concise in our remarks on the *seven other types* of points P_2, which indeed, if not so well known,* are perhaps also, on the whole, not quite so interesting: although it seems that some circumstances of their arrangement in space may deserve to be noted here, especially as affording an additional *exercise* (88), in the present system of symbols and types. The *type* $P_{2,2}$ represents, then, a *group* of *thirty points*, of which A‴, in Fig. 21, is an example; each being the intersection of a line $A_{2,1}$ with a line $A_{2,2}$, as A‴ is the point in which AA′ intersects B′C′: but each belonging to no other line, among those which have been hitherto considered. But without aiming to describe here *all* the lines, planes, and points, of what we have called the *third* construction, we may already see that they must be expected to be numerous: and that the planes Π_3, and the lines A_3, of *that* construction, as well as the pyramids R_2, and the triangles T_2, of the *second* construction, above noticed, can only be regarded as *specimens*, which in a closer study of the subject, it becomes necessary to mark more fully, on the present plan, as $\Pi_{3,1}, \ldots T_{2,1}$. Accordingly it is found that not only is each point $P_{2,2}$ one of the corners of a *triangle* $T_{3,1}$ of *third* construction (as A‴ is of A‴B‴C‴ in Fig. 21), the *sides* of which new triangle are lines $A_{3,2}$, passing each through one point $P_{2,1}$ and through two points $P_{2,2}$ (like the dotted line A″B‴C‴ of Fig. 21); but also each such point $P_{2,2}$ is the intersection of *two new lines* of third construction, $A_{3,3}$, whereof each connects a point P_0 with a point $P_{2,1}$. For example, the point A''' is the *common trace* (on the plane ABC) of the two new lines, DA'_1, EA'_2: because, if we adopt for this point A''' the second of its two congruent symbols, we have (comp. 73, 82) the expressions,

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* It does not appear that any of these *other types*, or *groups*, of points P_2, have hitherto been noticed, in connexion with the *net in space*, except the one which we have ranked as the *fifth*, $P_{2,5}$, and which represents *two points* on each line A_1, as the type $P_{2,1}$ has been seen to represent *one* point on each of those ten lines of first construction: but that *fifth group*, which may be exemplified by the intersections of the line DE with the two planes $A_1B_1C_1$ and $A_2B_2C_2$, has been indicated by Möbius (in page 290 of his already cited work), although with a *different notation*, and as the result of a *different analysis*.

$$A''' = (\bar{1}0011) = (D) - (A'_1) = (E) - (A'_2).$$

We may therefore establish the *formula of concurrence* (comp. the first sub-article):

$$A''' = AA' \cdot B'C' \cdot DA'_1 \cdot EA'_2\,;$$

which represents a system of *thirty* such formulæ.

(4.) It has been remarked that the point A''' may be represented, not only by the quinary symbol (21100), but also by the congruent symbol, $(\bar{1}0011)$; if then we write,

$$A_0 = (\bar{1}1100), \qquad B_0 = (1\bar{1}100), \qquad C_0 = (11\bar{1}00),$$

these three new points $A_0B_0C_0$, in the plane of ABC, must be considered to be *syntypical*, in the *quinary* sense (78), with the three points $A'''B'''C'''$, or to belong to the *same group* $P_{2,2}$, although they have (comp. 88) a different *ternary type*. It is easy to see that, while the triangle $A'''B'''C'''$ is (comp. again Fig. 21) an *inscribed homologue* $T_{3,1}$ of the triangle $A'B'C'$, which is *itself* (comp. sub-article 1) an inscribed homologue $T_{2,1}$ of a triangle T_1, namely of ABC, with $A''B''C''$ for their *common axis* of homology, the new triangle $A_0B_0C_0$ is on the contrary an *exscribed homologue* $T_{3,2}$, with the *same axis* $\Lambda_{3,1}$, of the same given triangle T_1. But from the *syntypical relation*, existing as above for *space* between the points A''' and A_0, we may expect to find that these two points $P_{2,2}$ admit of being *similarly constructed*, when the *five* points P_0 are treated as entering *symmetrically* (or similarly), as *geometrical elements*, into the constructions. The point A_0 must therefore be situated, not only on a line $\Lambda_{2,1}$, namely, on AA', but also on a line $\Lambda_{2,2}$, which is easily found to be A_1A_2, and on two lines $\Lambda_{3,2}$, each connecting a point P_0 with a point $P_{2,1}$; which latter lines are soon seen to be BB'' and CC''. We may therefore establish the formula of *concurrence* (comp. the last sub-article):

$$A_0 = AA' \cdot A_1A_2 \cdot BB'' \cdot CC''\,;$$

and may consider the three points A_0, B_0, C_0 as the *traces* of the three lines A_1A_2, B_1B_2, C_1C_2: while the three new lines AA'', BB'', CC'', which coincide in position with the sides of the exscribed triangle $A_0B_0C_0$, are the traces $\Lambda_{3,3}$ of three *planes* $\Pi_{2,1}$, such as $AB_1C_2B_2C_1$, which pass through the three given points A, B, C, but do not contain the lines $\Lambda_{2,1}$ whereon the six points $P_{2,2}$ in their plane Π_1 are situated. Every *other* plane Π_1 contains, in like manner, *six* points P_2 of the present group; every plane $\Pi_{2,1}$ contains *eight* of them; and every plane $\Pi_{2,2}$ contains *three;* each *line* $\Lambda_{2,1}$ passing through *two* such points, but each line $\Lambda_{2,2}$ only through *one*. But besides being (as above) the intersection of *two lines* Λ_2, each point of this group $P_{2,2}$ is common to *two planes* Π_1, *four planes* $\Pi_{2,1}$, and *two planes* $\Pi_{2,2}$; while each of these thirty points is also a *common corner* of *two* different *triangles* of *third* construction, of the lately mentioned kinds $T_{3,1}$ and $T_{3,2}$, situated respectively in the two planes of *first* construction which contain the point itself. It may be added that each of the two points $P_{2,2}$, on a line $\Lambda_{2,1}$, is the *harmonic conjugate* of *one* of the two points P_1, with respect to the point P_0, and to the *other* point P_1 on that line; thus we have here the two harmonic equations,

$$(AA'D_1A''') = (AD_1A'A_0) = -1,$$

by which the positions of the two points A''' and A_0 might be determined.

(5.) A *third group*, $P_{2,3}$, of *second construction*, consists (like the preceding group) of *thirty points*, ranged *two by two* on the fifteen lines $\Lambda_{2,1}$, and *six by six* on the ten planes Π_1, but so that each is common to *two* such planes; each is also situated in *two planes* $\Pi_{2,1}$, in *two* planes $\Pi_{2,2}$, and on *one line* $\Lambda_{3,1}$ in which (by sub-art. 1) these two last planes intersect each other, and two of the five planes $\Pi_{3,1}$; each plane $\Pi_{2,1}$ contains *four* such points, and each plane $\Pi_{2,2}$ contains *three* of them; but no point of this group is on any line Λ_1, or $\Lambda_{2,2}$. The *six points* $P_{2,3}$, which are *in the plane* ABC, are represented (like the corresponding points of the last group) by *two ternary types*, namely by $(\bar{2}11)$ and (311); and may be exemplified by the two following points, of which these last are the ternary symbols:

$$A^{IV} = AA' \cdot A''B''C'' = AA' \cdot A_1B_1C_1 \cdot A_2B_2C_2;$$
$$A_1^{IV} = AA' \cdot D'_1A'_2A_1 = AA' \cdot B'C_1C_2 \cdot C'B_1B_2.$$

The three points of the first sub-group A^{IV} . . are collinear; but the three points A_1^{IV} . . of the second sub-group are the corners of a *new triangle*, $T_{3,3}$, which is homologous to the triangle ABC, and to all the other triangles in its plane which have been hitherto considered, as well as to the two triangles $A_1B_1C_1$ and $A_2B_2C_2$; the line of the three former points being their *common axis* of homology; and the *sides* of the new triangle, $A_1^{IV}B_1^{IV}C_1^{IV}$, being the *traces* of the *three planes* (comp. 90) of homology of pyramids, $[A]$, $[B]$, $[C]$; as (comp. sub-art. 2) the line $A^{IV}B^{IV}C^{IV}$ or $A''B''C''$ is the *common trace* of the *two other planes* of the same group $\Pi_{3,1}$, namely of $[D]$ and $[E]$. We may also say that the *point* A_1^{IV} is the *trace* of the *line* $A'_1A'_2$; and because the *lines* $B'C_0$, $C'B_0$ are the traces of the two *planes* $\Pi_{2,2}$ in which that point is contained, we may write the formula of concurrence,

$$A_1^{IV} = AA' \cdot A'_1A'_2 \cdot B'C_0 \cdot C'B_0.$$

(6.) It may be also remarked, that each of the two points $P_{2,3}$, on any line $\Lambda_{2,1}$, is the harmonic conjugate of a point $P_{2,2}$, with respect to the point P_0, and to *one* of the two points P_1 on that line; being *also* the harmonic conjugate of this last point, with respect to the same point P_0, and the *other* point $P_{2,2}$: thus, on the line $AA'D_1$, we have the *four harmonic equations*, which are not however all *independent*, since two of them can be deduced from the two others, with the help of the two analogous equations of the fourth sub-article:

$$(AA'''A'A^{IV}) = (AA'A_0A^{IV}) = (AA_0D_1A_1^{IV}) = (AD_1A'''A_1^{IV}) = -1.$$

And the *three pairs of derived points* P_1, $P_{2,2}$, $P_{2,3}$, on any such line $\Lambda_{2,1}$, will be found (comp. 26) to compose an *involution*, with the *given point* P_0 on the line for *one* of its two *double points* (or *foci*): the *other* double point of this involution being a point P_3 of *third* construction; namely, the point in which the line $\Lambda_{2,1}$ meets *that one* of the *five planes* of homology $\Pi_{3,1}$, which *corresponds* (comp. 90) to the particular point P_0 as *centre*. Thus, in the present example, if we denote by A^{X} the point in which the line AA' meets the plane $[A]$, of which (by 81, 91) the trace on ABC is the line $[\bar{4}11]$, and therefore is (as has been stated) the side $B_1^{IV}C_1^{IV}$ of the lately mentioned triangle $T_{3,3}$, so that

$$A^{X} = (122) = AA' \cdot BC''' \cdot CB''' \cdot B_1^{IV}C_1^{IV},$$

we shall have the *three* harmonic equations,

$$(AA'A^{X}D_1) = (AA'''A^{X}A_0) = (AA^{IV}A^{X}A_1^{IV}) = -1;$$

which express that this new point A^{X} is the *common harmonic conjugate* of the given point A, with respect to the *three pairs of points*, $A'D_1$, $A'''A_0$, $A^{IV}A_1^{IV}$; and therefore that *these three pairs* form (as has been said) an *involution*, with A and A^x for its two *double points*.

(7.) It will be found that we have now exhausted all the types of points of second construction, which are situated upon lines $\Lambda_{2,1}$; there being only *four* such *points* on each such *line*. But there are still to be considered two new groups of points P_2 on lines Λ_1, and three others on lines $\Lambda_{2,3}$. Attending first to the former set of lines, we may observe that *each* of the *two new types*, $P_{2,4}$, $P_{2,5}$, represents *twenty points*, situated *two by two* on the ten lines of *first* construction, but not on any line Λ_2; and therefore *six by six* in the *ten planes* Π_1, each point however being common to *three* such planes: also each point $P_{2,4}$ is common to *three* planes $\Pi_{2,2}$, and each point $P_{2,5}$ is situated in *one* such plane; while each of these last planes contains *three* points $P_{2,4}$, but only *one* point $P_{2,5}$. If we attend only to points in the plane ABC, we can represent these *two new groups* by the *two ternary types*, (021) and $(02\bar{1})$, which as *symbols* denote the two typical points,

$$A^{V} = BC \cdot C'A_1A_2 \cdot D_1A_1B_1 \cdot D_1A_2B_2; \qquad A^{VI} = BC \cdot C'B_1B_2 = BC \cdot C'B_0;$$

we have also the concurrence,

$$A^{V} = BC \cdot C'A_0 \cdot D_1C'' \cdot AB'''.$$

It may be noted that A^{V} is the harmonic conjugate of C', with respect to A_0 and B_1^{IV}, which last point is on the same trace $C'A_0$, of the plane $C'A_1A_2$; and that A^{VI} is harmonically conjugate to B_1^{V}, with respect to C' and B_0, on the trace of the plane $C'B_1B_2$, where B_1^{V} denotes (by an analogy which will soon become more evident) the intersection of that trace with the line CA: so that we have the two equations,

$$(A_0C'B_1^{IV}A^{V}) = (B_0B_1^{V}C'A^{VI}) = -1.$$

(8.) *Each line* Λ_1, contains thus two points P_2, of each of the two last new groups, besides the point $P_{2,1}$, the point P_1, and the two points P_0, which had been previously considered: it contains therefore *eight points* in all, if we still abstain (88) from proceeding beyond the *Second Construction*. And it is easy to prove that these *eight points* can, *in two distinct modes*, be so arranged as to form (comp. sub-art. 6) an *involution*, with *two* of them for the two *double* points thereof. Thus, if we attend only to points on the line BC, and represent them by ternary symbols, we may write,

$$B = (010), \qquad C = (001), \qquad A' = (011), \qquad A'' = (01\bar{1});$$
$$A^{V} = (021), \qquad A^{VI} = (02\bar{1}), \qquad A_1^{V} = (012), \qquad A_1^{VI} = (0\bar{1}2);$$

and the resulting harmonic equations

$$\text{I.} \ldots (BA'CA'') = (BA^{V}CA^{VI}) = (BA_1^{V}CA_1^{VI}) = -1,$$
$$\text{II.} \ldots (A'BA''C) = (A'A^{V}A''A_1^{V}) = (A'A^{VI}A''A_1^{VI}) = -1,$$

will then suffice to show: Ist., that *the two points* P_0, *on any line* Λ_1, are the *double points of an involution, in which the points* P_1, $P_{2,1}$ *form one pair of conjugates*, while the *two other pairs* are of the *common form*, $P_{2,4}$, $P_{2,5}$; and IInd., that *the two points* P_1 and $P_{2,1}$, *on any such line* Λ_1, *are the double points of a second involution, obtained by pairing the two points of each of the three other groups.* Also each of the two points P_0, on a line Λ_1, is the harmonic conjugate of one of the two points $P_{2,5}$ on that line, with respect to the other point of the same group, and to the point P_1 on the same line; thus,

$$(BA'A_1^{VI}A^{VI}) = (CA'A^{VI}A_1^{VI}) = -1.$$

(9.) It remains to consider briefly *three other groups* of points P_2, *each group* containing *sixty points*, which are situated, two by two, on the thirty lines $\Lambda_{2,2}$, and six by six in the ten planes Π_1. Confining our attention to those which are in the plane ABC, and denoting them by their ternary symbols, we have thus, on the line $B'C'$, the three new typical points, of the three remaining groups, $P_{2.6}$, $P_{2,7}$, $P_{2,8}$:

$$A^{VII} = (12\bar{1})\,; \qquad A^{VIII} = (321)\,; \qquad A^{IX} = (23\bar{1})\,;$$

with which may be combined these three others, of the same three types, and on the same line $B'C'$:

$$A_1^{VII} = (1\bar{1}2)\,; \qquad A_1^{VIII} = (312)\,; \qquad A_1^{IX} = (2\bar{1}3).$$

Considered as intersections of a line $\Lambda_{2,2}$ with lines Λ_3 in the same plane Π_1, or with planes Π_3 (in which *latter* character alone they belong to the *second* construction), the three points A^{VII}, &c., may be thus denoted:

$$A^{VII} = B'C' \cdot BB'' \cdot CB''' \cdot AA^{VI} = B'C' \cdot BC_1A_2A_1C_2\,;$$
$$A^{VIII} = B'C' \cdot D_1B'' \cdot AB'''A^{V} = B'C' \cdot D_1C_1A_1 \cdot D_1C_2A_2\,;$$
$$A^{IX} = B'C' \cdot A'C_0B_1^{IV}C_1^{V}B^{VI} \cdot BA^{IV}B_1^{VI}B_1^{VII} = B'C' \cdot A'C_1C_2\,;$$

with the harmonic equation,

$$(C_0A'C_1^{V}A^{IX}) = -1,$$

and with analogous expressions for the three other points, A_1^{VII}, &c. The line $B'C'$ thus intersects *one* plane $\Pi_{2,1}$ (or its trace BB'' on the plane ABC), in the point A^{VII}; it intersects *two* planes $\Pi_{2,2}$ (or their common trace D_1B'') in A^{VIII}; and *one* other plane $\Pi_{2,2}$ (or its trace $A'C_0$) in A^{IX}: and similarly for the other points, A_1^{VII}, &c., of the same three groups. *Each plane* $\Pi_{2,1}$ contains *twelve points* $P_{2,6}$, *eight* points $P_{2,7}$, and *eight* points $P_{2,8}$; while every plane $\Pi_{2,2}$ contains *six* points $P_{2,6}$, *twelve* points $P_{2,7}$, and *nine* points $P_{2,8}$. *Each point* $P_{2,6}$ is contained in *one plane* Π_1; in *three* planes $\Pi_{2,1}$; and in *two* planes $\Pi_{2,2}$. Each point $P_{2,7}$ is in *one* plane Π_1, in *two* planes $\Pi_{2,1}$, and in *four* planes $\Pi_{2,2}$. And each point $P_{2,8}$ is situated in *one* plane Π_1, in *two* planes $\Pi_{2,1}$, and in *three* planes $\Pi_{2,2}$.

(10.) The points of the three last groups are situated *only* on lines $\Lambda_{2,2}$; but, on each such line, *two* points of *each* of those three groups are situated; which, along with *one* point of each of the *two* former groups, $P_{2,1}$ and $P_{2,2}$, and with the *two* points P_1, whereby the line itself is determined, make up a system of *ten points* upon that line. For example, the line $B'C'$ contains, besides the *six* points mentioned in the last sub-article, the *four* others:

$$B' = (101)\,; \qquad C' = (110)\,; \qquad A'' = (01\bar{1})\,; \qquad A''' = (211).$$

Of these *ten* points, the *two* last mentioned, namely the points $P_{2,1}$ and $P_{2,2}$ upon the line $\Lambda_{2,2}$, are the *double points* (comp. sub-art. 8) of a *new involution*, in which the *two points* of *each* of the *four other groups* compose a conjugate pair, as is expressed by the harmonic equations,

$$(A''B'A'''C') = (A''A^{VII}A'''A_1^{VII}) = (A''A^{VIII}A'''A_1^{VIII}) = (A''A^{IX}A'''A_1^{IX}) = -1.$$

And the analogous equations,

$$(B'A''C'A''') = (B'A^{VII}C'A^{VIII}) = (B'A_1^{VII}C'A_1^{VIII}) = -1,$$

show that the two points P_1 on any line $\Lambda_{2,2}$ are the double points of of *another involution* (comp. again sub-art. 8), whereof the two points $P_{2,1}$, $P_{2,2}$ on that line form one conjugate pair, while each of the two points $P_{2,6}$ is paired with one of the points $P_{2,7}$ as its conjugate. In fact, the *eight-rayed pencil* $(A\,.\,C'B'A'''A''A^{VIII}A^{VII}A_1{}^{VIII}A_1{}^{VII})$ coincides in position with the pencil $(A\,.\,BCA'A''A^{V}A^{VI}A_1{}^{V}A_1{}^{VI})$, and may be said to be a *pencil in double involution;* the third and fourth, the fifth and sixth, and the seventh and eighth rays forming *one* involution, whereof the first and second are the two *double* rays;* while the first and second, the fifth and seventh, and the sixth and eighth rays compose *another* involution, whereof the double rays are the third and fourth of the pencil.

(11.) If we proceeded to connect systematically the points P_2 among themselves, and with the points P_1 and P_0, we should find many remarkable lines and planes of *third* construction (88), besides those which have been incidentally noticed above; for example, we should have a group $\Pi_{3,2}$ of *twenty new planes*, exemplified by the two following,

$$[E_D]=[1110\bar{3}], \qquad [D_E]=[111\bar{3}0],$$

which have the same common trace $\Lambda_{3,1}$, namely the line $A''B''C''$, on the plane ABC, as the two planes $A_1B_1C_1$, $A_2B_2C_2$, and the two planes [D], [E], of the groups $\Pi_{2,2}$ and $\Pi_{3,1}$, which have been considered in former sub-articles; and each of these new planes $\Pi_{3,2}$ would be found to contain *one* point P_0, *three* points $P_{2,1}$, *six* points $P_{2,2}$, and *three* points $P_{2,3}$. It might be proved also that these twenty new planes are the *twenty faces* of *five new pyramids* R_3, which are the *exscribed homologues* of the five old pyramids R_1 (89), with the five given points P_0 for the corresponding *centres* of homology. But it would lead us beyond the proposed limits, to pursue this discussion further: although a few additional remarks may be useful, as serving to establish the *completeness* of the *enumeration* above given, of the lines, planes, and points of *second* construction.

93. In general, if there be any n *given points*, whereof no four are situated in any common plane, the *number* N of the *derived* points, which are immediately obtained from them, as *intersections* $\Lambda\cdot\Pi$ of *line* with *plane* (each *line* being drawn through *two* of the given points, and each *plane* through *three others*), or the number of points of the *form* AB·CDE, is easily seen to be,

$$N=f(n)=\frac{n(n-1)\,(n-2)\,(n-3)\,(n-4)}{2\,.\,2\,.\,3};$$

so that $N=10$, as before, when $n=5$. But if we were to apply this formula to the case $n=15$, we should find, for that case, the value,

$$N=f(15)=15\,.\,14\,.\,13\,.\,11=30030;$$

and thus *fifteen given and independent points of space* would conduct, by what might (relatively to *them*) be called a *First Construction* (comp. 88), to a system of *more* than *thirty thousand* points. Yet it has been lately stated (92), that from the fifteen points above called P_0, P_1, there can be derived, in this way, *only* two hundred and ninety points P_2, as intersections of the form* $\Lambda\cdot\Pi$; and therefore *fewer* than *three hundred.* That this *reduction* of the *number* of *derived points*, at the end of what has been called (88) the *Second Construction* for the *net in space*, arising from the *dependence* of the *ten* points P_1 on the *five* points P_0, would be found to be *so considerable*, might not perhaps have been anticipated; and although the foregoing examination proves that *all* the *eight types* (92) do really represent points P_2, it may appear *possible*, at this stage, that some *other type* of such points has been *omitted.* A study of the manner in which the *types of points* result, from those of the *lines* and *planes* of which they are the intersections, would indeed decide this question; and it was, in fact, in that way that the eight types, or groups, $P_{2,1}\,..\,P_{2,8}$, of points of second construction for space, were investigated, and found to be sufficient: yet it may be useful (compare the last subart.) to *verify*, as below, the *completeness* of the foregoing *enumeration.*

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* Compare page 172 of the *Géométrie Supérieure* of M. Chasles.

(1.) The *fifteen points*, P_0, P_1, admit of 105 *binary*, and of 455 *ternary combinations;* but these are far from determining so many distinct *lines* and *planes*. In fact, those 15 points are connected by 25 *collineations*, represented by the 25 lines Λ_1, $\Lambda_{2,1}$; which *lines* therefore *count* as 75, among the 105 *binary* combinations of points: and there *remain* only 30 combinations of this sort, which are constructed by the 30 *other* lines, $\Lambda_{2,2}$. Again, there are 25 *ternary* combinations of points, which are represented (as above) by *lines*, and therefore do not *determine* any *plane*. Also, in *each* of the *ten* planes Π_1, there are 29 (= 35 − 6) *triangles* T_1, T_2, because each of those planes contains 7 *points* P_0, P_1, connected by 6 *relations* of collinearity. In like manner, each of the *fifteen* planes $\Pi_{2,1}$ contains 8 (= 10 − 2) *other triangles* T_2, because it contains 5 *points* P_0, P_1, connected by *two* collineations. There remain therefore only 20 (= 455 − 25 − 290 − 120) ternary combinations of points to be accounted for; and these are represented by the 20 planes $\Pi_{2,2}$. The *completeness* of the enumeration of the *lines* and *planes* of the *second* construction is therefore *verified;* and it only remains to verify that the 305 *points*, P_0, P_1, P_2, above considered, *represent all* the *intersections* $\Lambda\cdot\Pi$, of the 55 lines Λ_1, Λ_2, with the 45 planes Π_1, Π_2.

(2.) Each plane Π_1 *contains* three lines of each of the three groups, Λ_1, $\Lambda_{2,1}$, $\Lambda_{2,2}$; each plane $\Pi_{2,1}$ contains two lines $\Lambda_{2,1}$, and four lines $\Lambda_{2,2}$; and each plane $\Pi_{2,2}$ contains three lines $\Lambda_{2,2}$. Hence (or because each line Λ_1 is *contained* in three planes Π_1; each line $\Lambda_{2,1}$ in two planes Π_1, and in two planes $\Pi_{2,1}$; and each line $\Lambda_{2,2}$ in one plane Π_1, in two planes $\Pi_{2,1}$, and in two planes $\Pi_{2,2}$), it follows that, without going beyond the *second* construction, there are 240 (= 30 + 30 + 30 + 30

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* The *definition* (88) of the points P_2 admits, indeed, *intersections* $\Lambda\cdot\Lambda$ *of complanar lines*, when they are not already points P_0 or P_1; but all *such* intersections are *also* points of the form $\Lambda\cdot\Pi$; so that *no generality is lost*, by confining ourselves to this *last form*, as in the present discussion we propose to do.

$+ 60 - 60$) *cases of coincidence* of line and plane; so that the number of *cases of intersection* is *reduced*, hereby, from $55 . 45 = 2475$, to 2235 ($= 2475 - 240$).

(3.) Each point P_0 represents *twelve* intersections of the form $A_1 \cdot \Pi_1$; because it is common to *four lines* A_1, and to *six planes* Π_1, each plane *containing two* of those four lines, but being *intersected* by the *two others* in that point P_0; as the plane ABC, for example, is intersected in A by the two lines, AD and AE. Again, each point P_0 is common to *three planes* $\Pi_{2,1}$, no one of which contains any of the four lines A_1 through that point; it represents therefore a system of *twelve other intersections*, of the form $A_1 \cdot \Pi_{2,1}$. Again, each point P_0 is common to *three lines* $A_{2,1}$, each of which is *contained in two* of the six planes Π_1, but *intersects the four others* in that point P_0; which therefore counts as *twelve* intersections, of the form $A_{2,1} \cdot \Pi_1$. Finally, each of the points P_0 represents *three* intersections, $A_{2,1} \cdot \Pi_{2,1}$; and it represents *no other* intersection, of the form $A \cdot \Pi$, within the limits of the present inquiry. Thus, *each* of the *five given points* is to be considered as representing, or constructing, *thirty-nine* ($= 12 + 12 + 12 + 3$) intersections of line with plane; and there remain only 2040 ($= 2235 - 195$) *other* cases of such intersection $A \cdot \Pi$, to be *accounted for* (in the present verification) by the 300 *derived points*, P_1, P_2.

(4.) For this purpose, the *nine columns*, headed as I. to IX. in the following *Table*, contain the *numbers* of such *intersections* which belong respectively to the *nine forms*,

$$A_1 \cdot \Pi_1, \quad A_1 \cdot \Pi_{2,1}, \quad A_1 \cdot \Pi_{2,2}; \qquad A_{2,1} \cdot \Pi_1, \quad A_{2,1} \cdot \Pi_{2,1}, \quad A_{2,1} \cdot \Pi_{2,2};$$
$$A_{2,2} \cdot \Pi_1, \quad A_{2,2} \cdot \Pi_{2,1}, \quad A_{2,2} \cdot \Pi_{2,2},$$

for each of the *nine typical derived points*, $A' \dots A^{IX}$, of the *nine groups* P_1, $P_{2,1}$, .. $P_{3,8}$. Column X. contains, for each point, the *sum* of the *nine numbers*, thus tabulated in the preceding columns; and expresses therefore the entire number of intersections, which any *one* such *point* represents. Column XI. states the *number of the points* for each *type*; and column XII. contains the *product* of the two last numbers, or the number of intersections $A . \Pi$ which are represented (or constructed) by the *group*. Finally, the *sum* of the numbers in each of the *two last columns* is written at its foot; and because the 300 *derived points*, of first and second constructions, are thus found to represent the 2040 *intersections* which were to be accounted for, the *verification* is seen to be *complete*: and *no new type*, of points P_2, *remains to be discovered*.

(5.) TABLE *of Intersections* $A \cdot \Pi$.

Type.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
A'	1	6	6	6	12	18	18	24	24	115	10	1150
A''	0	3	6	0	0	0	6	3	12	30	10	300
A'''	0	0	0	0	2	2	1	2	0	7	30	210
A^{IV}	0	0	0	0	0	2	0	0	0	2	30	60
A^{V}	0	0	3	0	0	0	0	0	0	3	20	60
A^{VI}	0	0	1	0	0	0	0	0	0	1	20	20
A^{VII}	0	0	0	0	0	0	0	1	0	1	60	60
A^{VIII}	0	0	0	0	0	0	0	0	2	2	60	120
A^{IX}	0	0	0	0	0	0	0	0	1	1	60	60
											300	2040

(6.) It is to be remembered that we have *not admitted*, by our *definition* (88), any *points* which can *only* be determined by *intersections of three planes* Π_1, Π_2, as belonging to the *second construction*: nor have we counted, as *lines* Λ_2 of *that* construction, any lines which can *only* be found as intersections of *two* such planes. For example, we do not regard the *traces* AA'', &c., of certain *planes* $\Lambda_{2,1}$ considered in recent sub-articles, as among the lines of *second* construction, although they would present themselves early in an enumeration of the lines Λ_3 of the *third*. And any point in the plane ABC, which can *only* be determined (at the present stage) as the intersection of *two* such *traces*, is not regarded as a point P_2. A student might find it however to be not useless, as an exercise, to investigate the expressions for *such* intersections; and for that reason it may be noted here, that the *ternary types* (comp. 81) of the *forty-four traces* of *planes* Π_1, Π_2, on the plane ABC, which are found to compose a system of only *twenty-two distinct lines* in that plane, whereof *nine* are lines Λ_1, Λ_2, are the seven following (comp. 88):

$$[100],\quad [01\bar{1}],\quad [\bar{1}11],\quad [111],\quad [011],\quad [\bar{2}11],\quad [\bar{2}1\bar{1}];$$

which, as ternary *symbols*, represent the *seven lines*,

$$BC,\qquad AA',\qquad B'C',\qquad A''B''C'',\qquad AA'',\qquad D_1A'',\qquad A'C_0.$$

(7.) Again, on the same principle, and with reference to the same definition, that new point, say F, which may be denoted by *either* of the two *congruent quinary symbols* (71),

$$F = (43210) \equiv (01234),$$

and which, as a *quinary type* (78), represents a *new group* of *sixty points of space* (and of *no more*, on account of this last *congruence*, whereas a quinary type, with *all* its *five* coefficients *unequal*, represents *generally* a group of 120 distinct points), is *not* regarded by us as a point P_2; although this new point F is easily seen to be the *intersection of three planes of second construction*, namely, of the three following, which all belong to the group $\Pi_{2,1}$:

$$[01\bar{1}\bar{1}1],\qquad [1\bar{1}0\bar{1}1],\qquad [1\bar{1}\bar{1}10],$$

or $AA'D_1C_1B_2$, $CC'D_1B_1A_2$, $EB'B_2C'C_2$. It may, however, be remarked in passing, that *each plane* $\Pi_{2,1}$ contains *twelve points* P_3 of this new group: every such point being common (as is evident from what has been shown) to *three* such planes.

94. From the foregoing discussion it appears that the *five given points* P_0, and the *three hundred derived points* P_1, P_2, are *arranged in space*, upon the *fifty-five lines* Λ_1, Λ_2, and in the *forty-five planes* Π_1, Π_2, as follows. Each *line* Λ_1 contains *eight* of the 305 points, forming on it what may be called (see the sub-article (8.) to 92) a *double involution*. Each line $\Lambda_{2,1}$ contains *seven* points, whereof *one*, namely the *given* point, P_0, has been seen (in the earlier sub-art. (6.)) to be a *double point* of *another* involution, to which the *three derived pairs* of points, P_1, P_2, on the same line belong. And each line $\Lambda_{2,2}$ contains *ten* points, forming on it a *new involution;* while *eight* of these ten points, with a different *order* of succession, compose still *another*

involution[*] (92, (10.)). Again, each *plane* Π_1 contains *fifty-two points*, namely three *given* points, four points of *first*, and 45 points of *second* construction. Each plane $\Pi_{2,1}$ contains *forty-seven* points, whereof *one* is a given point, four are points P_1, and 42 are points

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[*] These theorems respecting the *relations of involution*, of given and derived points on lines of *first* and *second* constructions, for a *net in space*, are perhaps new; although some of the *harmonic relations*, above mentioned, have been noticed under other forms by Möbius: to whom, indeed, as has been stated, the *conception* of such a *net* is due. Thus, if we consider (compare the Note to page 72) the two intersections,

$$\mathrm{E}_1 = \mathrm{DE}\cdot\mathrm{A}_1\mathrm{B}_1\mathrm{C}_1, \qquad \mathrm{E}_2 = \mathrm{DE}\cdot\mathrm{A}_2\mathrm{B}_2\mathrm{C}_2,$$

we easily find that they may be denoted by the quinary symbols,

$$\mathrm{E}_1 = (000\bar{1}2), \qquad \mathrm{E}_2 = (0002\bar{1});$$

they are, therefore, by Art. 92, the two points $\mathrm{P}_{2,6}$ on the line DE: and consequently, by the theorem stated at the end of sub-art. 8, the harmonic conjugate of *each*, taken with respect to the *other* and to the point D_1, must be one of the two points D, E on that line. Accordingly, we soon derive, by comparison of the symbols of these *five points*, $\mathrm{DED}_1\mathrm{E}_1\mathrm{E}_2$, the two following harmonic equations, which belong to the *same type* as the *two last* of that sub-art. 8:

$$(\mathrm{D}_1\mathrm{DE}_2\mathrm{E}_1) = -1; \qquad (\mathrm{D}_1\mathrm{EE}_1\mathrm{E}_2) = -1;$$

but *these* two equations have been assigned (with notations slightly different) in the formerly cited page 290 of the Barycentric Calculus. (Comp. again the recent Note to page 72.) The *geometrical meaning* of the last equation may be illustrated, by conceiving that ABCD is a *regular pyramid*, and that E is its *mean point;* for then (comp. 92, sub-art. (2.)), D_1 is the mean point of the *base* ABC; $\mathrm{D}_1\mathrm{D}$ is the *altitude* of the pyramid; and the *three segments* $\mathrm{D}_1\mathrm{E}$, $\mathrm{D}_1\mathrm{E}_1$, $\mathrm{D}_1\mathrm{E}_2$ are, respectively, the *quarter*, the *third part*, and the *half* of that altitude: they compose therefore (as the formula expresses) a *harmonic progression;* or D_1 and E_1 are *conjugate points*, with respect to E and E_2. But in order to exemplify the *double involution* of the same sub-art. (8.), it would be necessary to consider *three other points* P_2, on the *same line* DE; whereof *one*, above called D'_1, belongs to a *known group* $\mathrm{P}_{2,1}$ (92, (2.)); but the *two others* are of the group $\mathrm{P}_{2,4}$, and do not seem to have been previously noticed. As an example of an involution on a line of *third* construction, it may be remarked that on each line of the group $\Lambda_{3,3}$, or on each of the sides of any one of the ten triangles $T_{3,2}$, in addition to one given point P_0, and one derived point $\mathrm{P}_{2,1}$, there are two points $\mathrm{P}_{2,2}$, and two points $\mathrm{P}_{2,6}$; and that the two first points are the double points of an involution, to which the two last pairs belong: thus, on the side $\mathrm{A}_0\mathrm{BC}_0$ of the exscribed triangle $\mathrm{A}_0\mathrm{B}_0\mathrm{C}_0$, or on the trace of the plane $\mathrm{BC}_1\mathrm{A}_2\mathrm{A}_1\mathrm{C}_2$, we have the two harmonic equations,

$$(\mathrm{BA}_0\mathrm{B}''\mathrm{C}_0) = (\mathrm{BA}^{\mathrm{VII}}\mathrm{B}''\mathrm{C}_1{}^{\mathrm{VII}}) = -1.$$

Again, on the trace $\mathrm{A}'\mathrm{C}_0$ of the plane $\mathrm{A}'\mathrm{C}_1\mathrm{C}_2$, (which latter trace is a line not passing through any one of the given points,) C_0 and $\mathrm{B}_1{}^{\mathrm{IV}}$ are the double points of an involution, wherein A' is conjugate to $\mathrm{C}_1{}^{\mathrm{V}}$ and A^{IX} to B^{VI}. But it would be tedious to multiply such instances.

P_2: of which last, 38 are situated *on the six lines* Λ_2 in the plane, but four are *intersections* of that plane $\Pi_{2,1}$ with *four other* lines of second construction. Finally, each plane $\Pi_{2,2}$ passes through *no given point*, but contains *forty-three derived points*, whereof 40 are points of second construction. And because the planes of *first* construction *alone* contain *specimens* of *all the ten groups of points*, P_0, P_1, $P_{2,1}, \ldots P_{2,6}$, given or derived, and of *all* the *three groups of lines*, Λ_1, $\Lambda_{2,1}$, $\Lambda_{2,2}$, at the close of that *second* construction (since the types $P_{2,4}$, $P_{2,5}$, Λ_1 are not represented by any points or lines in any plane $\Pi_{2,1}$, nor are the types P_0, Λ_1, $\Lambda_{2,1}$ represented in a plane $\Pi_{2,2}$), it has been thought convenient to prepare the annexed *diagram* (Fig. 30), which may serve to illustrate, by some selected instances, the *arrangement* of the *fifty-two points* P_0, P_1, P_2 in a plane Π_1, namely, in the plane ABC; as well as the arrangement of the *nine lines* Λ_1, Λ_2 in that plane, and the *traces* Λ_3 of other planes upon it.

View of the Arrangement of the Principal Points and Lines in a Plane of First Construction.

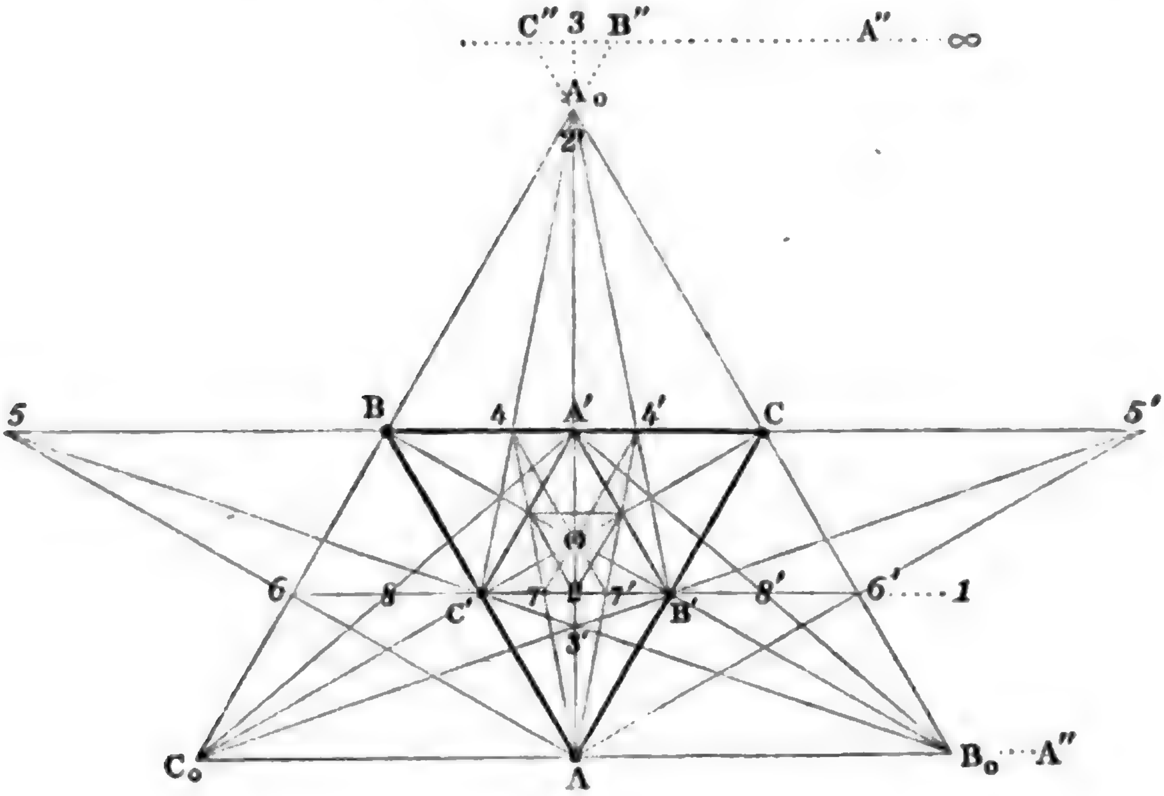

Fig. 30.

In this Figure, the triangle ABC is suppposed, for simplicity, to be the *equilateral base* of a *regular pyramid* ABCD (comp. sub-art. (2,) to 92); and D_1, again replaced by O, is supposed to be its *mean point* (29). The *first inscribed triangle*, A'B'C', therefore, *bisects* the three sides; and the *axis of homology* A''B''C'' is the *line at infinity* (38): the number 1, on the line C'B' prolonged, being designed to suggest that the *point* A'', to which that *line tends*, is of the type $P_{2,1}$, or belongs to the *first group* of points of *second construction*. A *second* inscribed *triangle*, $A'''B'''C'''$, for which Fig. 21 may be consulted, is only *indicated* by the number 2 placed at the middle of the side $B'C'$, to suggest that *this* bisecting point A''' belongs to the *second group* of points P_2. The *same number* 2, but with an *accent*, 2′, is placed near the corner A_0 of the *exscribed* triangle $A_0B_0C_0$, to remind us that this corner *also* belongs (by a syntypical relation in space) to the group $P_{2,2}$. The point A^{IV}, which is now infinitely distant, is indicated by the number 3, on the dotted line at the top; while the same number with an accent, lower down, marks the position of the point A_1^{IV}. Finally, the ten other numbers, unaccented or accented, 4, 4′, 5, 5′, 6, 6′, 7, 7′, 8, 8′, denote the places of the ten points, A^{V}, A_1^{V}, A^{VI}, A_1^{VI}, A^{VII}, A_1^{VII}, A^{VIII}, A_1^{VIII} A^{IX}, A_1^{IX}. And the principal *harmonic relations*, and relations of *involution*, above mentioned, may be verified by inspection of this Diagram.

95. However far the series of construction of the net in space may be continued, we may now regard it as evident, at least on comparison with the analogous property (42) of the *plane* net, that *every point*, *line*, or *plane*, to which such constructions can conduct, must necessarily be *rational* (77); or that it must be *rationally related* to the system of the *five given points:* because the *anharmonic co-ordinates* (79, 80) of every *net-point*, and of every *net-plane*, are equal or proportional to *whole numbers*. Conversely (comp. 43) *every point*, *line*, or *plane*, in space, which is thus *rationally related* to the system of points ABCDE, is a point, line, or plane of the *net*, which those five points determine. Hence (comp. again 43), every *irrational* point, line, or plane (77), is indeed incapable of being *rigorously constructed*, by any processes of the kind above described; but it admits of being *indefinitely approximated to*, by points, lines, or planes of the net. *Every anharmonic ratio*, whether of a *group of net-points*, or of a *pencil of net-lines*, or of *net-planes*, has a *rational value* (comp. 44), which depends *only* on the *processes* of *linear construction* employed, in the generation of that group or pencil, and is entirely *independent* of the *arrangement*, or *configuration*, of the five given points in space. Also, all *relations* of *collineation*, and of *complanarity*, are *preserved*, in the passage from *one* net to *another*, by a change of the given system of points: so that it may be briefly said (comp. again 44) that *all geometrical nets in space are homographic figures*. Finally, *any five points*[*] of such a net, of which no four are in one plane, are *sufficient* (comp.

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[*] These *general properties* (95) of the *space-net* are in substance taken from Möbius, although (as has been remarked before) the *analysis* here employed appears to be new: as do also most of the *theorems* above given, respecting the *points of second construction* (92), at least after we pass beyond the *first group* $P_{2,1}$ of *ten* such points, which (as already stated) have been known comparatively long.

45) for the determination of the *whole net:* or for the *linear construction* of *all* its points, including the five *given* ones.

(1.) As an Example, let the five points $A_1B_1C_1D_1$ and E be now supposed to be *given;* and let it be required to *derive* the four points $ABCD$, by linear constructions, from these new data. In other words, we are now required to ***exscribe*** a ***pyramid*** $ABCD$ to a given pyramid $A_1B_1C_1D_1$, so that it may be ***homologous*** thereto, with the point E for their ***given centre*** of homology. An obvious process is (comp. 45) to ***inscribe*** another homologous pyramid, $A_3B_3C_3D_3$,, so as to have $A_3 = EA_1 \cdot B_1C_1D_1$, &c.; and then to determine the ***intersections*** of ***corresponding faces***, such as $A_1B_1C_1$ and $A_3B_3C_3$; for these ***four lines*** of intersection will be in the ***common plane*** $[E]$, of homology of the ***three pyramids***, and will be the ***traces*** on that plane of the ***four sought planes***, ABC, &c., drawn through the four given points D_1, &c. If it were only required to construct *one corner* A of the exscribed pyramid, we might find the point above called A^{IV} as the common intersection of ***three planes***, as follows,

$$A^{IV} = A_1B_1C_1 \cdot A_1D_1E \cdot A_3B_3C_3\,;$$

and then should have this other formula of intersection,

$$A = EA_1 \cdot D_1A^{IV}.$$

Or the point A might be determined by the anharmonic equation,

$$(EAA_1A_3) = 3,$$

which for a regular pyramid is easily verified.

(2.) As regards the general *passage* from *one* net in space to *another*, let the symbols $P_1 = (x_1 \,..\, v_1), \,..\, P_5 = (x_5 \,..\, v_5)$ denote *any five given points*, whereof no four are complanar; and let $a'b'c'd'e'$ and u' be six coefficients, of which the five ratios are such as to satisfy the symbolical equation (comp. 71, 72),

$$a'(P_1) + b'(P_2) + c'(P_3) + d'(P_4) + e'(P_5) = -u'(U):$$

or the five ordinary equations which it includes, namely,

$$a'x_1 + \,..\, + e'x_5 = \,..\, = a'v_1 + \,..\, + e'v_5 = -u'.$$

Let P' be any sixth point of space, of which the quinary symbol satisfies the equation,

$$(P') = xa'(P_1) + yb'(P_2) + zc'(P_3) + wd'(P_4) + ve'(P_5) + u(U)\,;$$

then it will be found that this last point P' can be ***derived*** from the five points $P_1 \,..\, P_5$ by precisely the ***same constructions***, as those by which the point $P = (xyzwv)$ is derived from the five points $ABCDE$. As an example, if $v' = x + y + z + w - 3v$, then the point $(xyzwv')$ is derived from $A_1B_1C_1D_1E$, by the same constructions as $(xyzwv)$ from $ABCDE$; thus A itself may be constructed from $A_1 \,..\, E$, as the point $P = (30001)$ is from $A \,..\, E$; which would conduct anew to the anharmonic equation of the last sub-article.

(3.) It may be briefly added here, that instead of ***anharmonic ratios***, as connected with a net in space, or indeed generally in relation to ***spatial problems***, we are permitted (comp. 68) to substitute products (or quotients) of ***quotients of volumes*** of pyramids; as a ***specimen*** of which substitution, it may be remarked, that the anharmonic relation, just referred to, admits of being replaced by the following equation, involving ***one*** such quotient of pyramids, but introducing ***no auxiliary point:***

$$\mathrm{EA} : \mathrm{A_1A} = 3\mathrm{EB_1C_1D_1} : \mathrm{A_1B_1C_1D_1}.$$

In general, if $xyzw$ be (as in 79, 83) the *anharmonic co-ordinates* of a point P in space, we may write,

$$\frac{x}{y} = \frac{\mathrm{PBCD}}{\mathrm{PCDA}} : \frac{\mathrm{EBCD}}{\mathrm{ECDA}};$$

with other equations of the same type, on which we cannot here delay.

SECTION 5.—*On Barycentres of Systems of Points; and on Simple and Complex Means of Vectors.*

96. In general, when the *sum* $\Sigma\alpha$ of any number of co-initial vectors,

$$\alpha_1 = \mathrm{OA}_1, \ldots \qquad \alpha_m = \mathrm{OA}_m,$$

is *divided* (16) by their *number*, m, the resulting *vector*,

$$\mu = \mathrm{OM} = \frac{1}{m}\,\Sigma\alpha = \frac{1}{m}\,\Sigma\mathrm{OA},$$

is said to be the *Simple Mean* of those m vectors; and the *point* M, in which this *mean vector* terminates, and of which the *position* (comp. 18) is easily seen to be *independent* of the position of the common *origin* O, is said to be the *Mean Point* (comp. 29), of the *system* of the m *points*, $\mathrm{A}_1, \ldots \mathrm{A}_m$. It is evident that we have the equation,

$$0 = (\alpha_1 - \mu) + \ldots + (\alpha_m - \mu) = \Sigma(\alpha - \mu) = \Sigma\mathrm{MA};$$

or that the *sum* of the m vectors, drawn *from the mean point* M, *to* the points A of the system, is equal to *zero*. And hence (comp. 10, 11, 30), it follows, Ist., that these m vectors are equal to the m *successive sides* of a *closed polygon*; IInd., that if the system and its mean point be *projected*, by any *parallel ordinates*, on any assumed *plane* (or *line*), the *projection* M′, of the *mean point* M, is the *mean point of the projected system*: and IIIrd., that the *ordinate* MM′, of the *mean point*, is the *mean* of all the *other ordinates*, $\mathrm{A_1A'_1}, \ldots \mathrm{A_mA'_m}$. It follows, also, that if N be the mean point of *another system*, $\mathrm{B}_1, \ldots \mathrm{B}_n$; and if S be the mean point of the *total system*, $\mathrm{A}_1 \ldots \mathrm{B}_n$, of the $m + n = s$ points obtained by *combining* the *two* former, considered as *partial systems*; while ν and σ may denote the *vectors*, ON and OS, of these two last mean points: then we shall have the equations,

$$m\mu = \Sigma\alpha, \qquad n\nu = \Sigma\beta, \qquad s\sigma = \Sigma\alpha + \Sigma\beta = m\mu + n\nu,$$
$$m(\sigma - \mu) = n(\nu - \sigma), \qquad m\,.\,\mathrm{MS} = n\,.\,\mathrm{SN};$$

so that the *general mean point*, S, is situated *on the right line* MN, which connects the two *partial mean points*, M and N; and *divides* *that line* (internally), into *two segments* MS and SN, which are *inversely proportional* to the two *whole numbers*, m and n.

(1.) As an Example, let ABCD be a *gauche quadrilateral*, and let E be its *mean point;* or more fully, let

$$\mathrm{OE} = \tfrac{1}{4}(\mathrm{OA} + \mathrm{OB} + \mathrm{OC} + \mathrm{OD}),$$

or

$$\varepsilon = \tfrac{1}{4}(\alpha + \beta + \gamma + \delta);$$

that is to say, let $a = b = c = d$, in the equations of Art. 65. Then, with notations lately used, for certain *derived points* D_1, &c., if we write the *vector formulæ*,

$$\begin{aligned} &\mathrm{OA}_1 = \alpha_1 = \tfrac{1}{3}(\beta + \gamma + \delta), \;.\,. &&\delta_1 = \tfrac{1}{3}(\alpha + \beta + \gamma), \\ &\mathrm{OA}_2 = \alpha_2 = \tfrac{1}{2}(\alpha + \delta), \;.\,. &&\gamma_2 = \tfrac{1}{2}(\gamma + \delta), \\ &\mathrm{OA}' = \alpha' = \tfrac{1}{2}(\beta + \gamma), \;.\,. &&\gamma' = \tfrac{1}{2}(\alpha + \beta), \end{aligned}$$

we shall have *seven* different expressions for the *mean vector*, ε; namely, the following:

$$\begin{aligned} \varepsilon &= \tfrac{1}{4}(\alpha + 3\alpha_1) = .\,. = \tfrac{1}{4}(\delta + 3\delta_1) \\ &= \tfrac{1}{2}(\alpha' + \alpha_2) = .\,. = \tfrac{1}{2}(\gamma' + \gamma_2). \end{aligned}$$

And these conduct to the seven equations between *segments*,

$$\begin{aligned} &\mathrm{AE} = 3\mathrm{EA}_1, \;.\,. &&\mathrm{DE} = 3\mathrm{ED}_1; \\ &\mathrm{A'E} = \mathrm{EA}_2, \;.\,. &&\mathrm{C'E} = \mathrm{EC}_2; \end{aligned}$$

which prove (what is otherwise known) that the *four right lines*, here denoted by $\mathrm{AA}_1, .\,.\, \mathrm{DD}_1$, whereof each connects a *corner* of the *pyramid* ABCD with the mean point of the opposite *face*, *intersect* and *quadrisect* each other, in one *common point*, E; and that the *three common bisectors* $\mathrm{A'A}_2$, $\mathrm{B'B}_2$, $\mathrm{C'C}_2$, of *pairs of opposite edges*, such as BC and DA, intersect and *bisect* each other, in the *same* mean point: so that the *four middle points*, C′, A′, C_2, A_2, of the four successive *sides* AB, &c., of the gauche quadrilateral ABCD, are situated in one *common plane*, which *bisects also* the *common bisector*, $\mathrm{B'B}_2$, of the *two diagonals*, AC and BD.

(2.) In this example, the number s of the *points* A .. D being *four*, the number of the derived *lines*, which thus cross each other in their general mean point E is seen to be *seven;* and the number of the derived *planes* through that point is *nine:* namely, in the notation lately used for the *net* in space, four lines Λ_1, three lines $\Lambda_{2,1}$, six planes Π_1, and three planes $\Pi_{2,1}$. Of these *nine planes*, the *six former* may (in the present connexion) be called *triple planes*, because each contains *three lines* (as the plane ABE, for instance, contains the lines AA_1, BB_1, $\mathrm{C'C}_2$), all passing through the mean point E; and the *three latter* may be said, by contrast, to be *non-triple planes*, because each contains only *two lines* through that point, determined on the foregoing principles.

(3.) In general, let $\phi(s)$ denote the *number* of the *lines*, through the *general mean point* S of a *total system* of s given points, which is thus, in all possible ways, decomposed into *partial systems;* let $f(s)$ denote the number of the *triple planes*, obtained by grouping the given points into *three* such partial systems; let $\psi(s)$ denote the number of *non-triple* planes, each determined by grouping those s points in two different ways into *two* partial systems; and let $\mathrm{F}(s) = f(s) + \psi(s)$ represent the entire number of distinct planes through the point S: so that

$$\phi(4) = 7, \qquad f(4) = 6, \qquad \psi(4) = 3, \qquad \mathrm{F}(4) = 9.$$

Then it is easy to perceive that if we introduce a *new point* C, each *old line* MN furnishes *two new lines*, according as we group the new point with one or other of the two old partial systems, (M) and (N); and that there is, besides, *one other new line*, namely CS: we have, therefore, the *equation in finite differences*,

$$\phi(s+1) = 2\phi(s) + 1;$$

which, with the *particular value* above assigned for $\phi(4)$, or even with the simpler and more obvious value, $\phi(2) = 1$, conducts to the *general expression*,

$$\phi(s) = 2^{s-1} - 1.$$

(4.) Again, if (M) (N) (P) be any *three partial systems*, which jointly make up the old or given *total* system (S); and if, by grouping a *new point* C with each of these in turn, we form *three new partial systems*, (M') (N') (P'); then *each old triple plane* such as MNP, will furnish *three new triple planes*,

$$\text{M'NP}, \qquad \text{MN'P}, \qquad \text{MNP'};$$

while *each old line*, KL, will give *one new* triple plane, CKL: nor can any new triple plane be obtained in any other way. We have, therefore, this new *equation in differences*:

$$f(s+1) = 3f(s) + \phi(s).$$

But we have seen that

$$\phi(s+1) = 2\phi(s) + 1;$$

if then we write, for a moment,

$$f(s) + \phi(s) = \chi(s),$$

we have this other equation in finite differences,

$$\chi(s+1) = 3\chi(s) + 1.$$

Also,

$$f(3) = 1, \qquad \phi(3) = 3, \qquad \chi(3) = 4:$$

therefore,

$$2\chi(s) = 3^{s-1} - 1,$$

and

$$2f(s) = 3^{s-1} - 2^s + 1.$$

(5.) Finally, it is clear that we have the relation,

$$3f(s) + \psi(s) = \tfrac{1}{2}\phi(s).(\phi(s) - 1) = (2^{s-1} - 1)(2^{s-2} - 1);$$

because the *triple* planes, each treated as *three*, and the *non*-triple planes, each treated as *one*, must jointly represent all the *binary combinations* of the *lines*, drawn through the mean point S of the whole system. Hence,

$$2\psi(s) = 2^{2s-2} + 3.2^{s-1} - 3^s - 1;$$

and

$$\text{F}(s) = 2^{2s-3} + 2^{s-2} - 3^{s-1};$$

so that

$$\text{F}(s+1) - 4\text{F}(s) = 3^{s-1} - 2^{s-1},$$

and

$$\psi(s+1) - 4\psi(s) = 3f(s);$$

which last equation in finite differences admits of an independent geometrical interpretation.

(6.) For instance, these general expressions give,

$$\phi(5) = 15; \qquad f(5) = 25; \qquad \psi(5) = 30; \qquad \text{F}(5) = 55;$$

so that if we assume a *gauche pentagon*, or a system of *five points in space*, A . . E, and determine the **mean point** F of this system, there will in general be a set of ***fifteen lines***, of the kind above considered, all passing through this sixth point F: and these will be arranged generally in ***fifty-five distinct planes***, whereof ***twenty-five*** will be what we have called ***triple***, the ***thirty others*** being of the ***non-triple*** kind.

97. More generally, if $\alpha_1 \ldots \alpha_m$ be, as before, a system of *m given* and *co-initial vectors*, and if $a_1, \ldots a_m$ be any system of *m given scalars* (17), then that *new co-initial vector* β, or OB, which is deduced from these by the formula,

$$\beta = \frac{a_1\alpha_1 + \ldots + a_m\alpha_m}{a_1 + \ldots + a_m} = \frac{\Sigma a\alpha}{\Sigma a}, \text{ or } \mathrm{OB} = \frac{\Sigma a\mathrm{OA}}{\Sigma a},$$

or by the equation

$$\Sigma a(\alpha - \beta) = 0, \quad \text{or } \Sigma a\mathrm{BA} = 0,$$

may be said to be the *Complex Mean* of those *m* given *vectors* α, or OA, considered as *affected* (or combined) with that system of given *scalars*, a, as *coefficients*, or as *multipliers* (12, 14). It may also be said that the *derived point* B, of which (comp. 96) the *position* is independent of that of the *origin* O, is the *Barycentre* (or *centre of gravity*) of the given system of points $\mathrm{A}_1 \ldots$, considered as *loaded* with the given *weights* $a_1 \ldots$; and theorems of *intersections* of *lines* and *planes* arise, from the comparison of these *complex means*, or *barycentres*, of *partial* and *total systems*, which are entirely analogous to those lately considered (96), for *simple means* of *vectors* and of *points*.

(1.) As an Example, in the case of Art. 24, the point C is the barycentre of the system of the *two* points, A and B, with the weights a and b; while, under the conditions of 27, the origin O is the barycentre of the *three* points A, B, C, with the three weights a, b, c; and if we use the formula for ρ, assigned in 34 or 36, the same three given points A, B, C, when loaded with xa, yb, zc as weights, have the point P in their plane for their barycentre. Again, with the equations of 65, E is the barycentre of the system of the *four* given points, A, B, C, D, with the weights a, b, c, d; and if the expression of 79 for the vector OP be adopted, then xa, yb, zc, wd are equal (or proportional) to the weights with which the same four points A . . D must be loaded, in order that the point P of space may be their barycentre. In all these cases, the ***weights*** are thus ***proportional*** (by 69) to certain ***segments***, or ***areas***, or ***volumes***, of kinds which have been already considered; and what we have called the ***anharmonic co-ordinates*** of a ***variable point*** P, in a ***plane*** (36), or in ***space*** (79), may be said, on the same plan, to be ***quotients of quotients of weights***.

(2.) The circumstance that the ***position*** of a ***barycentre*** (97), like that of a simple ***mean point*** (96), is ***independent*** of the position of the assumed ***origin*** of vectors, might induce us (comp. 69) to ***suppress*** the ***symbol*** O of that ***arbitrary*** and ***foreign point;*** and therefore to write* simply, under the lately supposed conditions,

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* We should thus have some of the principal ***notations*** of the ***Barycentric Calculus:*** but used mainly with a reference to ***vectors***. Compare the Note to page 56.

$$\text{B} = \frac{\Sigma aA}{\Sigma a} \quad \text{or} \quad bB = \Sigma aA, \text{ if } b = a.$$

It is easy to prove (comp. 96), by principles already established, that the *ordinate of the barycentre* of any given system of points is the *complex mean* (in the sense above defined, and with the same system of *weights*), *of the ordinates of the points* of that system, with reference to *any given plane:* and that the *projection of the barycentre*, on any such plane, is the *barycentre of the projected system.*

(3.) Without any reference to *ordinates*, or to any foreign *origin*, the *barycentric notation* $\text{B} = \frac{\Sigma a\text{A}}{\Sigma a}$ may be *interpreted*, by means of our *fundamental convention* (Art. 1) respecting the geometrical signification of the *symbol* B − A, considered as denoting the *vector* from A to B: together with the rules for *multiplying* such vectors by *scalars* (14, 17), and for taking the *sums* (6, 7, 8, 9) of those (generally new) vectors, which are (15) the *products* of such multiplications. For we have only to write the formula as follows,

$$\Sigma a(\text{A} - \text{B}) = 0,$$

in order to perceive that it may be considered as signifying, that the system of the *vectors from the barycentre* B, to the system of the *given points* A_1, A_2, .. when multiplied respectively by the *scalars* (or coefficients) of the given system a_1, a_2, .. becomes (generally) a new system of vectors with a *null sum:* in such a manner that these last vectors, $a_1 . \text{BA}_1$, $a_2 . \text{BA}_2$. . . can be made (10) the *successive sides of a closed polygon*, by transports without rotation.

(4.) Thus if we meet the formula,

$$\text{B} = \tfrac{1}{2}(\text{A}_1 + \text{A}_2),$$

we may indeed interpret it as an *abridged form* of the equation,

$$\text{OB} = \tfrac{1}{2}(\text{OA}_1 + \text{OA}_2);$$

which implies that if O be any *arbitrary point*, and if O′ be the point which *completes* (comp. 6) the *parallelogram* $\text{A}_1\text{OA}_2\text{O}'$, then B is the point which *bisects the diagonal* OO′, and therefore also the *given line* A_1A_2, which is here the *other diagonal.* But we may *also* regard the formula as a mere *symbolical transformation* of the equation,

$$(\text{A}_2 - \text{B}) + (\text{A}_1 - \text{B}) = 0;$$

which (by the earliest principles of the present Book) expresses that the *two vectors*, from B to the two given points A_1 and A_2, have a *null sum;* or that they are *equal in length*, but *opposite in direction:* which can only be, by B bisecting A_1A_2, as before.

(5.) Again, the formula, $\text{B}_1 = \tfrac{1}{3}(\text{A}_1 + \text{A}_2 + \text{A}_3)$, *may* be interpreted as an *abridgment* of the equation,

$$\text{OB}_1 = \tfrac{1}{3}(\text{OA}_1 + \text{OA}_2 + \text{OA}_3),$$

which expresses that the point B *trisects the diagonal* OO′ of the *parallelepiped* (comp. 62), which has OA_1, OA_2, OA_3 for *three co-initial edges.* But the same formula may also be considered to express, in full consistency with the foregoing interpretation, that the *sum of the three vectors*, from B to the three points A_1, A_2, A_3, *vanishes:* which is the characteristic property (30) of the *mean point* of the *triangle* $\text{A}_1\text{A}_2\text{A}_3$. And similarly in more complex cases: the *legitimacy* of such *transformations* being here regarded as a consequence of the original *interpretation* (1) of the *symbol* B − A, and of the rules for *operations on vectors*, so far as as they have been hitherto established.

SECTION 6.—*On Anharmonic Equations, and Vector-Expressions, of Surfaces and Curves in Space.*

98. When, in the expression 79 for the vector ρ of a *variable point* P of space, the four variable scalars, or anharmonic co-ordinates, $xyzw$, are *connected* (comp. 46) by a given algebraic equation,

$$f_p(x, y, z, w) = 0, \text{ or briefly } f = 0,$$

supposed to be rational and integral, and homogeneous of the pth dimension, then the point P has for its *locus* a *surface of the pth order*, whereof $f = 0$ may be said (comp. 56) to be the *local equation*. For if we substitute instead of the co-ordinates $x \ldots w$, expressions of the forms,

$$x = tx_0 + ux_1, \ldots \qquad w = tw_0 + uw_1,$$

to indicate (82) that P is *collinear* with two given points, P_0, P_1, the resulting algebraic *equation* in $t : u$ is of the pth *degree*; so that (according to a received modern mode of speaking), the *surface* may be said to be *cut in p points* (distinct or coincident, and real or imaginary*), *by any arbitrary right line*, P_0P_1. And in like manner, when the four anharmonic co-ordinates $lmnr$ of a *variable plane* Π (80) are connected by an algebraical equation, of the form,

$$F_q(l, m, n, r) = 0, \text{ or briefly } F = 0,$$

where F denotes a rational and integral function, supposed to be homogeneous of the qth dimension, then this plane Π has for its *envelope* (comp. 56) a *surface of the qth class*, with $F = 0$ for its *tangential equation*: because if we make

$$l = tl_0 + ul_1, \ldots \qquad r = tr_0 + ur_1,$$

to express (comp. 82) that the variable plane Π passes *through a given right line* $\Pi_0 \cdot \Pi_1$, we are conducted to an algebraical *equation of the qth degree*, which gives q (real or imaginary) values for the ratio $t : u$, and thereby assigns q (real or imaginary†) *tangent planes to the sur-*

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* It is to be observed, that *no interpretation is here proposed*, for *imaginary intersections* of this kind, such as those of a *sphere* with a *right line*, which is *wholly external* thereto. The *language* of modern *geometry* requires that *such* imaginary *intersections* should be *spoken of*, and even that they should be *enumerated*: exactly as the *language* of *algebra* requires that we should *count* what are called the *imaginary roots* of an equation. But it would be an error to confound *geometrical imaginaries*, of *this* sort, with those *square roots of negatives*, for which it will soon be seen that the *Calculus of Quaternions* supplies, from the outset, a *definite* and *real interpretation*.

† As regards the *uninterpreted character* of such *imaginary contacts* in geometry, the preceding Note to the present Article, respecting *imaginary intersections*, may be consulted.

face, drawn through any such given but arbitrary right line. We may add (comp. 51, 56), that if the functions f and F be only *homogeneous* (without *necessarily* being *rational* and *integral*), then

$$[D_x f,\ D_y f,\ D_z f,\ D_w f]$$

is the *anharmonic symbol* (80) of the *tangent plane* to the surface $f=0$, at the point $(xyzw)$; and that

$$(D_l F,\ D_m F,\ D_n F,\ D_r F)$$

is in like manner, a symbol for the *point of contact* of the plane $[lmnr]$, with its *enveloped surface*, $F=0$; $D_x, \ldots D_l, \ldots$ being characteristics of *partial derivation.*

(1.) As an Example, the *surface of the second order*, which passes through the *nine points* called lately

$$A,\quad C',\quad B,\quad A',\quad C,\quad C_2,\quad D,\quad A_2,\quad E,$$

has for its *local equation*,

$$0=f=xz-yw;$$

which gives, by differentiation,

$$l=D_x f=z;\qquad m=D_y f=-w;$$
$$n=D_z f=x;\qquad r=D_w f=-y:$$

so that

$$[z,-w,\quad x,-y]$$

is a symbol for the *tangent plane*, at the point (x, y, z, w).

(2.) In fact, the *surface* here considered is the *ruled* (or *hyperbolic*) *hyperboloid*, on which the *gauche quadrilateral* ABCD is *superscribed*, and which passes also through the point E. And if we write

$$P=(xyzw),\qquad Q=(xy00),\qquad R=(0yz0),\qquad S=(00zw),\qquad T=(x00w),$$

then QS and RT (see the annexed Figure 31), namely, the lines drawn through P to intersect the two pairs, AB, CD, and BC, DA, of opposite sides of that quadrilateral ABCD, are the two generating lines, or *generatrices*, through that point; so that their plane, QRST, is the *tangent plane* to the surface, at the point P. If, then, we denote that tangent plane by the symbol $[lmnr]$, we have the equations of condition,

$$0=lx+my=my+nz=nz+rw=rw+lx;$$

whence follows the proportion,

$$l:m:n:r=x^{-1}:-y^{-1}:z^{-1}:-w^{-1};$$

or, because $xz=yw$,

$$l:m:n:r=z:-w:x:-y,$$

as before.

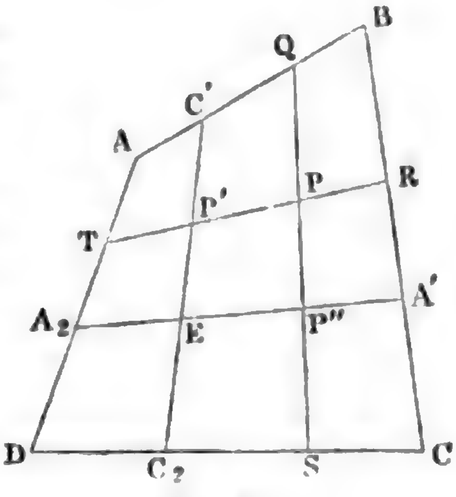

Fig. 31.

(3.) At the same time we see that

$$(AC'BQ)=\frac{x}{y}=\frac{w}{z}=(DC_2CS);$$

so that the *variable generatrix* QS *divides* (as is known) *the two fixed generatrices* AB and DC *homographically** ; AD, BC, and C′C₂ being three of its positions. Conversely, if it were proposed *to find the locus of the right line* QS, which thus *divides* homographically (comp. 26) *two given right lines in space*, we might take AB and DC for those two given lines, and AD, BC, C′C₂ (with the recent meanings of the letters) for three given positions of the variable line; and then should have, for the two variable but *corresponding* (or *homologous*) *points* Q, S themselves, and for any arbitrary point P *collinear* with them, anharmonic symbols of the forms,

$$\text{Q} = (s, u, 0, 0), \qquad \text{S} = (0, 0, u, s), \qquad \text{P} = (st, tu, uv, vs);$$

because, by 82, we should have, between these three symbols, a relation of the form,

$$(\text{P}) = t(\text{Q}) + v(\text{S}):$$

if then we write $\text{P} = (x, y, z, w)$, we have the anharmonic equation $xz = yw$, as before; so that the *locus*, whether of the *line* QS, or of the *point* P, is (as is known) a ruled *surface of the second order*.

(4.) As regards the known *double generation* of that surface, it may suffice to observe that if we write, in like manner,

$$\text{R} = (0tv0), \qquad \text{T} = (t00v), \qquad (\text{P}) = u(\text{R}) + s(\text{T}),$$

we shall have *again* the expression,

$$\text{P} = (st, tu, uv, vs), \quad \text{giving} \quad xz = yw,$$

as before: so that the *same* hyperboloid is *also* the *locus* of that *other line* RT, which divides the *other pair* of *opposite sides* BC, AD of the same gauche quadrilateral ABCD homographically; BA, CD, and A′A₂ being three of its positions; and the lines A′A₂, C′C₂ being still supposed to intersect each other in the given point E.

(5.) The symbol of an arbitrary point on the variable line RT is (by sub-art. 2) of the form, $t(0, y, z, 0) + u(x, 0, 0, w)$, or (ux, ty, tz, uw); while the symbol of an arbitrary point on the given line C′C₂ is (t', t', u', u'). And these two symbols represent one *common point* (comp. Fig. 31),

$$\text{P}' = \text{RT} \cdot \text{C}'\text{C}_2 = (y, y, z, z),$$

when we suppose

$$t' = y, \quad u' = z, \quad t = 1, \quad u = \frac{y}{x} = \frac{z}{w}.$$

Hence the known theorem results, that a *variable generatrix*, RT, *of one system, intersects three fixed lines*, BC, AD, C′C₂, which are *generatrices of the other system*. Conversely, by the same comparison of symbols, for points on the two lines RT and C′C₂, we should be conducted to the equation $xz = yw$, as the *condition* for their *intersection;* and thus should obtain this other known theorem, that *the locus of a right line, which intersects three given right lines in space*, is generally an *hyperboloid* with those three lines for *generatrices*. A similar analysis shows that QS intersects A′A₂, in a point (comp. again Fig. 31) which may be thus denoted:

$$\text{P}'' = \text{QS} \cdot \text{A}'\text{A}_2 = (xyyx).$$

(6.) As another example of the treatment of surfaces by their anharmonic and local equations, we may remark that the recent symbols for P′ and P″, combined with those of sub-art. 2 for P, Q, R, S, T; with the symbols of 83, 86 for C′, A′, C_2, A_2, E; and with the equation $xz = yw$, give the expressions:

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* Compare p. 298 of the *Géométrie Supérieure*.

$$(\mathrm{P}) = (\mathrm{Q}) + (\mathrm{S}) = (\mathrm{R}) + (\mathrm{T}); \qquad (\mathrm{P}') = y(\mathrm{C}') + z(\mathrm{C}_2) = (\mathrm{R}) + \frac{y}{x}(\mathrm{T});$$

$$(\mathrm{E}) = (\mathrm{C}') + (\mathrm{C}_2) = (\mathrm{A}') + (\mathrm{A}_2); \qquad (\mathrm{P}'') = y(\mathrm{A}') + x(\mathrm{A}_2) = (\mathrm{Q}) + \frac{y}{z}(\mathrm{S});$$

whence it follows (84) that the two points P′, P″, and the sides of the quadrilateral ABCD, divide the four generating lines through P and E in the following anharmonic ratios:

$$(\mathrm{C}'\mathrm{EC}_2\mathrm{P}') = (\mathrm{QP}''\mathrm{SP}) = \frac{y}{z} = (\mathrm{BA}'\mathrm{CR}) = (\mathrm{AA}_2\mathrm{DT});$$

$$(\mathrm{A}'\mathrm{EA}_2\mathrm{P}'') = (\mathrm{RP}'\mathrm{TP}) = \frac{y}{x} = (\mathrm{BC}'\mathrm{AQ}) = (\mathrm{CC}_2\mathrm{DS});$$

so that (as again is known) the *variable* generatrices, as well as the *fixed* ones, of the hyperboloid, are *all* divided *homographically*.

(7.) The *tangential equation* of the present surface is easily found, by the expressions in sub-art. 1 for the co-ordinates *lmnr* of the tangent plane, to be the following:

$$0 = \mathrm{F} = ln - mr;$$

which may be interpreted as expressing, that this hyperboloid is the *surface of the second class*, which *touches the nine planes*,

[1000], [0100], [0010], [0001], [1100], [0110], [0011], [1001], [1111];

or with the literal symbols lately employed (comp. 86, 87),

BCD, CDA, DAB, ABC, CDC″, DAA″, ABC'_2, BCA'_2, and [E].*

Or we may interpret the same tangential equation $\mathrm{F} = 0$ as expressing (comp. again 86, 87, where Q, L, N are now replaced by T, R, Q), that the surface is the *envelope of a plane* QRST, which satisfies *either* of the *two* connected *conditions* of *homography*:

$$(\mathrm{BC}'\mathrm{AQ}) = -\frac{l}{m} = -\frac{r}{n} = (\mathrm{CC}_2\mathrm{DS});$$

$$(\mathrm{CA}'\mathrm{BR}) = -\frac{m}{n} = -\frac{l}{r} = (\mathrm{DA}_2\mathrm{AT});$$

a *double generation* of the hyperboloid thus showing itself in a new way. And as regards the *passage* (or *return*), from the *tangential* to the *local equation* (comp. 56), we have in the present example the formulæ:

$$x = \mathrm{D}_l\mathrm{F} = n; \qquad y = \mathrm{D}_m\mathrm{F} = -r; \qquad z = \mathrm{D}_n\mathrm{F} = l; \qquad w = \mathrm{D}_r\mathrm{F} = -m;$$

whence

$$xz - yw = 0,$$

as before.

(8.) More generally, when the surface is of the *second order*, and therefore also of the *second class*, so that the two functions f and F, when presented under rational and integral forms, are both homogeneous of the *second dimension*, then whether we derive $l \,..\, r$ from $x \,..\, w$ by the formulæ,

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* In the anharmonic symbol of Art. 87, for the plane of homology [E], the *coefficient* 1 occurred, through inadvertence, *five times*.

$$l = \mathrm{D}_x f, \qquad m = \mathrm{D}_y f, \qquad n = \mathrm{D}_z f, \qquad r = \mathrm{D}_w f,$$

or $x \,.\,.\, w$ from $l \,.\,.\, r$ by the *converse* formulæ,

$$x = \mathrm{D}_l \mathrm{F}, \qquad y = \mathrm{D}_m \mathrm{F}, \qquad z = \mathrm{D}_n \mathrm{F}, \qquad w = \mathrm{D}_r \mathrm{F},$$

the *point* $\mathrm{P} = (xyzw)$ is, relatively to that surface, what is usually called (comp. 52) the *pole* of the *plane* $\Pi = [lmnr]$; and conversely, the plane Π is the *polar* of the point P; *wherever in space* the point P and plane Π, thus *related to each other*, may be situated. And because the *centre* of a surface of the second order is known to be (comp. again 52) the *pole* of (what is called) the *plane at infinity;* while (comp. 38) the *equation* and the *symbol* of this last *plane* are, respectively,

$$ax + by + cz + dw = 0, \quad \text{and} \quad [a, b, c, d],$$

if the four constants *abcd* have still the same significations as in 65, 70, 79, &c., with reference to the system of the five given points ABCDE: it follows that we may denote this *centre* by the symbol,

$$\mathrm{K} = (\mathrm{D}_a \mathrm{F}_0, \ \mathrm{D}_b \mathrm{F}_0, \ \mathrm{D}_c \mathrm{F}_0, \ \mathrm{D}_d \mathrm{F}_0);$$

where F_0 denotes, for abridgment, the function $\mathrm{F}(abcd)$, and d is still a scalar constant.

(9.) In the recent example, we have $\mathrm{F}_0 = ac - bd$; and the anharmonic symbol for the centre of the hyperboloid becomes thus,

$$\mathrm{K} = (c, -d, a, -b).$$

Accordingly if we assume (comp. sub-arts. 8, 4),

$$\mathrm{P} = (st, tu, uv, vs), \qquad \mathrm{P}' = (s't', -t'u', u'v', = v's'),$$

where s, t, u, v are any four scalars, and P' is a new point, while

$$s' = bt + cv, \qquad t' = cu + ds, \qquad u' = dv + at, \qquad v' = as + bu;$$

if also we write, for abridgment,

$$e' = ac - bd, \qquad w' = ast + btu + cuv + dvs;$$

we shall then have the symbolic relations,

$$e'(\mathrm{P}) + (\mathrm{P}') = w'(\mathrm{K}), \qquad e'(\mathrm{P}) - (\mathrm{P}') = (\mathrm{P}''),$$

if $\mathrm{P}'' = (x''y''z''w'')$ be that new point, of which the co-ordinates are,

$$x'' = 2e'st - cw', \qquad y'' = 2e'tu + dw', \qquad z'' = 2e'uv - aw', \qquad w'' = 2e'vs + bw',$$

and therefore,

$$ax'' + by'' + cz'' + dw'' = 0.$$

That is to say, if PP' be *any chord of the hyperboloid*, which passes *through the fixed point* K, and if P'' be the *harmonic conjugate* of that fixed point, with respect to that variable chord, so that $(\mathrm{PKP'P''}) = -1$, then this conjugate point P'' is *on the infinitely distant plane* $[abcd]$: or in other words, *the fixed point* K *bisects all the chords* PP' *which pass through it*, and is therefore (as above asserted) the *centre* of the surface.

(10.) With the same meanings (65, 79) of the constants a, b, c, d, the *mean point* (96) of the quadrilateral ABCD, or of the system of its corners, may be denoted by the symbol,

$$\mathrm{M} = (a^{-1}, b^{-1}, c^{-1}, d^{-1});$$

if then this mean point be *on the surface*, so that

$$ac - bd = 0,$$

the *centre* K is *on the plane* $[a, b, c, d]$; or in other words, it is *infinitely distant:* so that the surface becomes, in this case, a *ruled* (or *hyperbolic*) *paraboloid.* In general (comp. sub-art. 8), if $\mathrm{F}_0 = 0$, the surface of the second order is a *paraboloid* of *some* kind, because its *centre* is then *at infinity*, in virtue of the equation

$$(a\mathrm{D}_a + b\mathrm{D}_b + c\mathrm{D}_c + d\mathrm{D}_d)\,\mathrm{F}_0 = 0\,;$$

or because (comp. 50, 58) the *plane* [*abcd*] *at infinity* is then one of its *tangent planes*, as satisfying its *tangential equation*, $\mathrm{F} = 0$.

(11.) It is evident that a *curve in space* may be represented by a system of *two* anharmonic and *local equations;* because it may be regarded as the *intersection* of *two surfaces.* And then its *order*, or the *number of points* (real or imaginary*), in which it is *cut by an arbitrary plane*, is obviously the *product* of the *orders* of those two surfaces; or the *product* of the *degrees* of their two local equations, supposed to be rational and integral.

(12.) A *curve of double curvature* may also be considered as the *edge of regression* (or *arête de rebroussement*) of a *developable surface*, namely of the *locus of the tangents to the curve;* and this surface may be supposed to be *circumscribed* at once *to two given surfaces*, which are *envelopes of variable planes* (98), and are represented, as such, by their *tangential equations.* In *this* view, a *curve* of double curvature may *itself* be represented by a system of *two* anharmonic and *tangential* equations; and if the *class* of such a curve be defined to be the *number of its osculating planes*, which pass *through an arbitrary point of space*, then this class is the *product of the classes* of the *two curved surfaces* just now mentioned: or (what comes to the same thing) it is the *product of the dimensions* of the two tangential *equations*, by which the curve is (on this plan) symbolized. But we cannot enter further into these details; the *mechanism* of calculation respecting which would indeed be found to be the same, as that employed in the known method (comp. 41) of *quadriplanar co-ordinates.*

99. Instead of anharmonic co-ordinates, we may consider *any other system* of *n variable scalars*, $x_1, \ldots x_n$, which enter into the expression of a *variable vector*, ρ; for example, into an expression of the form (comp. 96, 97),

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots = \Sigma x\alpha.$$

And then, if those *n scalars* x be all *functions of one independent and variable scalar*, t, we may regard this *vector* ρ as being *itself* a *function* of that *single scalar;* and may write,

$$\text{I.} \ldots \rho = \phi(t).$$

But if the n scalars $x \ldots$ be functions of *two* independent and scalar variables, t and u, then ρ becomes a function of those *two scalars*, and we may write accordingly,

$$\text{II.} \ldots \rho = \phi(t, u).$$

In the Ist case, the *term* P (comp. 1) of the variable vector ρ has

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* Compare the Notes to page 90.

generally for its *locus* a *curve in space*, which may be plane or of double curvature, or may even become a *right line*, according to the *form* of the *vector-function* ϕ; and ρ may be said to be *the vector of this line, or curve*. In the IInd case, ρ is *the vector of a surface*, plane or curved, according to the form of $\phi(t, u)$; or to the manner in which this *vector* ρ depends on the *two independent scalars* that enter into its expression.

(1.) As Examples (comp. 25, 63), the expressions,

$$\text{I.} \ldots \rho = \frac{\alpha + t\beta}{1 + t}; \qquad \text{II.} \ldots \rho = \frac{\alpha + t\beta + u\gamma}{1 + t + u},$$

signify, Ist, that ρ is the vector of a variable point P *on the right line* AB; or that it is *the vector of that line itself*, considered as the *locus of a point;* and IInd, that ρ is the *vector of the plane* ABC, considered in like manner as the locus of an arbitrary point P thereon.

(2.) The equations,

$$\text{I.} \ldots \rho = x\alpha + y\beta, \qquad \text{II.} \ldots \rho = x\alpha + y\beta + z\gamma,$$

with

$$x^2 + y^2 = 1 \text{ for the Ist, and } x^2 + y^2 + z^2 = 1 \text{ for the IInd,}$$

signify Ist, that ρ is the *vector of an ellipse*, and IInd, that it is the *vector of an ellipsoid*, with the *origin* O for their common *centre*, and with OA, OB, or OA, OB, OC, for *conjugate semi-diameters*.

(3.) The equation (comp. 46),

$$\rho = t^2\alpha + u^2\beta + (t + u)^2\gamma,$$

expresses that ρ is the *vector of a cone of the second order*, with O for its *vertex* (or *centre*), which is *touched by the three planes* OBC, OCA, OAB; the *section* of this *cone*, made by the plane ABC, being an *ellipse* (comp. Fig. 25), which is *inscribed in the triangle* ABC; and the *middle points* A′, B′, C′, of the *sides* of that triangle, being the *points of contact* of those sides with that conic.

(4.) The equation (comp. 53),

$$\rho = t^{-1}\alpha + u^{-1}\beta + v^{-1}\gamma, \text{ with } t + u + v = 0,$$

expresses that ρ is the *vector of another cone of the second order*, with O still for *vertex*, but with OA, OB, OC for three of its *sides* (or *rays*). The *section* by the plane ABC is a new *ellipse*, *circumscribed* to the triangle ABC, and having its *tangents* at the *corners* of that triangle respectively *parallel* to the *opposite sides* thereof.

(5.) The equation (comp. 54),

$$\rho = t^3\alpha + u^3\beta + v^3\gamma, \text{ with } t + u + v = 0,$$

signifies that ρ is the vector of a *cone of the third order*, of which the vertex is still the origin; its section (comp. Fig. 27) by the plane ABC being a *cubic curve*, whereof the sides of the triangle ABC are at once the *asymptotes*, and the three (real) *tangents of inflexion;* while the *mean point* (say O′) of that *triangle* is a *conjugate point* of the *curve;* and therefore the *right line* OO′, from the vertex O to that mean point, may be said to be a *conjugate ray* of the *cone*.

(6.) The equation (comp. 98, sub-art. (3.)),

$$\rho = \frac{sta\alpha + tub\beta + uvc\gamma + vsd\delta}{sta + tub + uvc + vsd},$$

in which $\frac{s}{u}$ and $\frac{t}{v}$ are two variable scalars, while a, b, c, d are still four constant scalars, and α, β, γ, δ are four constant vectors, but ρ is still a variable vector, expresses that ρ is the *vector of a ruled* (or *single-sheeted*) *hyperboloid*, on which the gauche quadrilateral ABCD is superscribed, and which passes through the given point E, whereof the vector ε is assigned in 65.

(7.) If we make (comp. 98, sub-art (9.)),

$$\rho' = \frac{s't'a\alpha - t'u'b\beta + u'v'c\gamma - v's'd\delta}{s't'a - t'u'b + u'v'c - v's'd},$$

where

$$s' = bt + cv, \qquad t' = cu + ds, \qquad u' = dv + at, \qquad v' = as + bu,$$

then $\rho' = \text{OP}'$ is the vector of *another point* P′ on the *same hyperboloid;* and because it is found that the *sum* of these two last vectors is *constant*,

$$\rho + \rho' = 2\kappa, \text{ if } \kappa = \frac{ac(\alpha + \gamma) - bd(\beta + \delta)}{2(ac - bd)},$$

it follows that κ is the vector of a *fixed point* K, which *bisects every chord* PP′ that passes through it: or in other words (comp. 52), that this point K is the *centre of the surface.*

(8.) The three vectors,

$$\kappa, \qquad \frac{\alpha + \gamma}{2}, \qquad \frac{\beta + \delta}{2},$$

are *termino-collinear* (24); if then a gauche quadrilateral ABCD be superscribed on a ruled hyperboloid, *the common bisector of the two diagonals*, AC, BD, *passes through the centre* K.

(9.) When $ac = bd$, or when we have the equation,

$$\rho = \frac{st\alpha + tu\beta + uv\gamma + vs\delta}{st + tu + uv + vs},$$

or simply,

$$\rho = st\alpha + tu\beta + uv\gamma + vs\delta, \text{ with } s + u = t + v = 1,$$

ρ is then the *vector of a ruled paraboloid*, of which the *centre* (comp. 52, and 98, sub-art. (10.)), is *infinitely distant*, but upon which the quadrilateral ABCD is still *superscribed.* And this surface passes *through the mean point* M of that quadrilateral, or of the system of the four given points A . . D; because, when $s = t = u = v = \frac{1}{2}$, the variable vector ρ takes the value (comp. 96, sub-art. (1.)),

$$\mu = \tfrac{1}{4}(\alpha + \beta + \gamma + \delta).$$

(10.) In general, it is easy to prove, *from the last vector-expression* for ρ, that this paraboloid is the *locus of a right line*, which *divides similarly* the two *opposite sides* AB and DC of the same gauche quadrilateral ABCD; *or* the *other pair* of opposite sides, BC and AD.

SECTION 7.—*On Differentials of Vectors.*

100. The equation (99, I.),

$$\rho = \phi(t),$$

in which $\rho = \text{OP}$ is *generally* the vector of a point P of a *curve in space*, PQ . . ., gives evidently, for the vector OQ of *another point* Q of the same curve, an expression of the form

$$\rho + \Delta\rho = \phi(t + \Delta t);$$

so that the *chord* PQ, regarded as being itself a *vector*, comes thus to be represented (4) by the *finite difference*,

$$\text{PQ} = \Delta\rho = \Delta\phi(t) = \phi(t + \Delta t) - \phi(t).$$

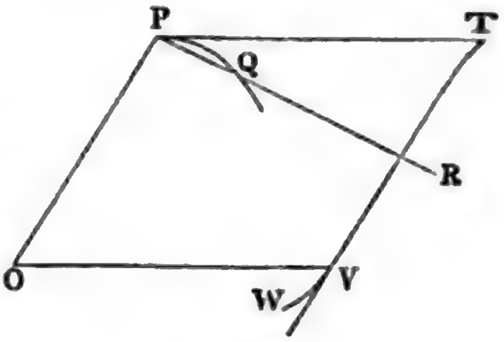

Fig. 32.

Suppose now that the *other* finite difference, Δt, is the nth *part* of a *new scalar*, u; and that the chord $\Delta\rho$, or PQ, is in like manner (comp. Fig. 32), the nth part of a *new vector*, σ_n, or PR; so that we may write,

$$n\Delta t = u, \text{ and } n\Delta\rho = n\,.\,\text{PQ} = \sigma_n = \text{PR}.$$

Then, if we treat the *two scalars*, t and u, as *constant*, but the *number* n as *variable* (the *form* of the *vector-function* ϕ, and the *origin* O, being *given*), the *vector* ρ and the *point* P will be *fixed*: but the *two points* Q and R, the *two differences* Δt and $\Delta\rho$, and the *multiple vector* $n\Delta\rho$, or σ_n, will (in general) *vary together*. And if this number n be *indefinitely increased*, or made to *tend to infinity*, then *each* of the two differences Δt, $\Delta\rho$ will in general *tend to zero*; such being the *common limit*, of $n^{-1}u$, and of $\phi(t + n^{-1}u) - \phi(t)$: so that the *variable point* Q of the curve will *tend to coincide* with the *fixed point* P. But although the *chord* PQ will thus be *indefinitely shortened*, its nth *multiple*, PR or σ_n, will *tend* (generally) to a *finite limit*,* depending on the supposed *continuity* of the *function* $\phi(t)$; namely, to a certain *definite vector*, PT, or σ_∞, or (say) τ, which vector PT will evidently be (in general) *tangential to the curve*: or, in other words, the *variable point* R will *tend to a fixed position* T, *on the tangent* to that curve at P. We shall thus have a *limiting equation*, of the form

$$\tau = \text{PT} = \lim.\ \text{PR} = \sigma_\infty = \lim_{n=\infty}.\ n\Delta\phi(t), \text{ if } n\Delta t = u;$$

t and u being, as above, *two given* and (generally) *finite scalars*. And if we then agree to *call* the *second* of these two given *scalars* the *differential* of the first, and to *denote* it by the *symbol* dt, we shall *define* that the *vector-limit*, τ or σ_∞, is the (corresponding) *differential of the vector* ρ, and shall *denote* it by the *corresponding symbol*, dρ; so as to have, under the supposed conditions,

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* Compare Newton's *Principia*.

$$u = \mathrm{d}t, \text{ and } \tau = \mathrm{d}\rho.$$

Or, eliminating the two symbols u and τ, and *not necessarily supposing* that P is a *point of a curve*, we may *express* our *Definition*[*] *of the Differential of a Vector* ρ, considered as a *Function* ϕ *of a Scalar* t, by the following *General Formula:*

$$\mathrm{d}\rho = \mathrm{d}\phi(t) = \lim_{n=\infty} n \left\{ \phi\left(t + \frac{\mathrm{d}t}{n}\right) - \phi(t) \right\},$$

in which t and dt are *two arbitrary and independent scalars, both generally finite;* and dρ is, *in general*, a *new and finite vector, depending on those* TWO *scalars*, according to a *law* expressed by the *formula*, and *derived* from that *given law*, whereby the *old* or *former vector*, ρ or $\phi(t)$, *depends* upon the *single scalar*, t.

(1.) As an example, let the given *vector-function* have the form,

$$\rho = \phi(t) = \tfrac{1}{2}t^2 a, \text{ where } a \text{ is a given vector.}$$

Then, making $\Delta t = \frac{u}{n}$, where u is *any given scalar*, and n is a *variable whole number*, we have

$$\Delta\rho = \Delta\phi(t) = \frac{a}{2}\left\{\left(t + \frac{u}{n}\right)^2 - t^2\right\} = \frac{au}{n}\left(t + \frac{u}{2n}\right);$$

$$\sigma_n = n\Delta\rho = au\left(t + \frac{u}{2n}\right); \quad \sigma_\infty = atu;$$

and finally, writing dt and dρ for u and σ_∞,

$$\mathrm{d}\rho = \mathrm{d}\phi(t) = \mathrm{d}\left(\frac{t^2 a}{2}\right) = at\mathrm{d}t.$$

(2.) In general, let $\phi(t) = af(t)$, where a is still a given or *constant vector*, and $f(t)$ denotes a *scalar function* of the *scalar variable*, t. Then because a is a *common factor* within the brackets { } of the recent general formula (100) for dρ, we may write,

$$\mathrm{d}\rho = \mathrm{d}\phi(t) = \mathrm{d}\,.\,af(t) = a\mathrm{d}f(t);$$

provided that we now *define* that the *differential of a scalar function*, $f(t)$, is a *new scalar function of two independent scalars*, t and dt, determined by the precisely similar formula:

$$\mathrm{d}f(t) = \lim_{n=\infty} n \left\{ f\left(t + \frac{\mathrm{d}t}{n}\right) - f(t) \right\};$$

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[*] Compare the Note to page 39.

which can easily be proved to *agree*, in all its *consequences*, with the *usual rules* for *differentiating functions of one variable*.

(3.) For example, if we write $\mathrm{d}t = nh$, where h is a *new variable scalar*, namely, the nth *part* of the *given* and (generally) *finite differential*, $\mathrm{d}t$, we shall thus have the equation,

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \lim_{h=0} \frac{f(t+h)-f(t)}{h};$$

in which the first member is *here* considered as the actual *quotient of two finite scalars*, $\mathrm{d}f(t):\mathrm{d}t$, and *not merely* as a *differential coefficient*. We *may*, however, as usual, consider this *quotient*, from the expression of which the *differential* $\mathrm{d}t$ *disappears*, as a *derived function* of the *former variable*, t; and may *denote* it, as such, by either of the *two usual symbols*,

$$f'(t) \text{ and } \mathrm{D}_t f(t).$$

(4.) In like manner we may write, for the *derivative of a vector-function*,* $\phi(t)$, the *formula*:

$$\rho' = \phi'(t) = \mathrm{D}_t\rho = \mathrm{D}_t\phi(t) = \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}\phi(t)}{\mathrm{d}t};$$

these two last forms denoting that *actual and finite vector*, ρ' or $\phi'(t)$, which is obtained, or *derived*, by *dividing* (comp. 16) the *not less actual* (or finite) *vector*, $\mathrm{d}\rho$ or $\mathrm{d}\phi(t)$, *by the finite scalar*, $\mathrm{d}t$. And if again we denote the nth *part* of this last scalar by h, we shall thus have the *equally general formula*:

$$\mathrm{D}_t\rho = \mathrm{D}_t\phi(t) = \lim_{h=0} \frac{\phi(t+h)-\phi(t)}{h};$$

with the equations,

$$\mathrm{d}\rho = \mathrm{D}_t\rho \,.\, \mathrm{d}t = \rho'\mathrm{d}t; \qquad \mathrm{d}\phi(t) = \mathrm{D}_t\phi(t) \,.\, \mathrm{d}t = \phi'(t) \,.\, \mathrm{d}t,$$

exactly as if the *vector-function*, ρ or ϕ, were a *scalar function*, f.

(5.) The particular value, $\mathrm{d}t = 1$, gives thus $\mathrm{d}\rho = \rho'$; so that the *derived vector* ρ' is (with our definitions) a particular but important *case* of the *differential of a vector*. In applications to *mechanics*, if t denote the *time*, and if the *term* P of the *variable vector* ρ be considered as a *moving point*, this *derived* vector ρ' may be called the *Vector of Velocity*: because its *length* represents the *amount*, and its *direction* is the *direction* of the velocity. And if, by setting off *vectors* OV $= \rho'$ (comp. again Fig. 32) *from one origin*, to represent thus the *velocities* of a point moving in space according to any supposed law, expressed by the equation $\rho = \phi(t)$, we construct a *new curve* VW.. of which the corresponding equation may be written as $\rho' = \phi'(t)$, then this *new* curve has been defined to be the HODOGRAPH,† as the *old* curve PQ.. may be called the *orbit* of the motion, or of the moving point.

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* In the theory of *Differentials of Functions of Quaternions*, a *definition* of the *differential* $\mathrm{d}\phi(q)$ will be proposed, which is expressed by an equation of precisely the *same form* as those above assigned, for $\mathrm{d}f(t)$, and for $\mathrm{d}\phi(t)$; but it will be found that, *for quaternions*, the *quotient* $\mathrm{d}\phi(q):\mathrm{d}q$ is *not generally independent of* $\mathrm{d}q$; and consequently that it cannot properly be called a *derived function*, such as $\phi'(q)$, *of the quaternion q alone*. (Compare again the Note to page 39.)

† The subject of the *Hodograph* will be resumed, at a subsequent stage of this work. In fact, it almost requires the assistance of *Quaternions*, to connect it, in what appears to be the best mode, with Newton's Law of Gravitation.

(6.) We may *differentiate a vector-function twice* (or oftener), and so obtain its *successive differentials*. For example, if we *differentiate the derived vector* ρ', we obtain a result of the form,

$$d\rho' = \rho'' dt, \text{ where } \rho'' = \mathrm{D}_t\rho' = \mathrm{D}_t^2\rho,$$

by an obvious extension of notation; and *if we suppose that the second differential*, ddt or d^2t, *of the scalar t is zero*, then the *second differential of the vector* ρ is,

$$d^2\rho = dd\rho = d\,.\,\rho' dt = d\rho'\,.\,dt = \rho''.\,dt^2\,;$$

where dt^2, as usual, denotes $(dt)^2$; and where it is important to observe that, with the definitions adopted, $d^2\rho$ is *as finite a vector* as $d\rho$, or as ρ itself. In applications to *motion*, if t denote the *time*, ρ'' may be said to be the *Vector of Acceleration*.

(7.) We may also say that, in mechanics, the *finite differential* $d\rho$, *of the Vector of Position* ρ, represents, in length and in direction, the *right line* (suppose PT in Fig. 32) *which would have been described, by a freely moving point* P, *in the finite interval of time* dt, immediately *following the time* t, *if* at the end of this time t *all foreign forces had ceased to act.*[*]

(8.) In *geometry*, if $\rho = \phi(t)$ be the equation of a *curve of double curvature*, regarded as the *edge of regression* (comp. 98, (12.)) of a *developable surface*, then the equation of that *surface* itself, considered as the *locus of the tangents* to the curve, may be thus written (comp. 99, II.):

$$\rho = \phi(t) + u\phi'(t)\,; \text{ or simply, } \rho = \phi(t) + d\phi(t),$$

if it be remembered that u, or dt, may be *any arbitrary scalar*.

(9.) If *any other curved surface* (comp. again 99, II.) be represented by an equation of the form, $\rho = \phi(x, y)$, where ϕ now denotes a *vector-function of two independent and scalar variables*, x and y, we may then *differentiate* this *equation*, or this expression for ρ, with respect to *either variable separately*, and so obtain what may be called *two partial* (but *finite*) *differentials*, $d_x\rho$, $d_y\rho$, and *two partial derivatives*, $\mathrm{D}_x\rho$, $\mathrm{D}_y\rho$, whereof the former are connected with the latter, and with the *two arbitrary* (but *finite*) *scalars*, dx, dy, by the relations,

$$d_x\rho = \mathrm{D}_x\rho\,.\,dx\,; \qquad d_y\rho = \mathrm{D}_y\rho\,.\,dy.$$

And these two differentials (or derivatives) of the vector ρ of the surface denote *two tangential vectors*, or at least two vectors *parallel* to *two tangents* to that surface at the point P: so that *their plane* is (or is parallel to) the *tangent plane* at that point.

(10.) The *mechanism* of all such *differentiations of vector-functions* is, at the present stage, precisely the *same* as in the *usual* processes of the *Differential Calculus;* because the *most general form* of such a *vector-function*, which has been considered in the present Book, is that of a *sum of products* (comp. 99) of the form xa, where a is a *constant vector*, and x is a *variable scalar:* so that we have only to *operate on these scalar coefficients* x.., by the *usual rules* of the calculus, the *vectors* a.. being treated as *constant factors* (comp. sub-art. 2). But when we shall come to consider *quotients* or *products of vectors*, or generally those *new functions of vectors* which can only be expressed (in our system) by *Quaternions*, then some few *new rules of differentiation* become necessary, although deduced from the *same* (or nearly the same) *definitions*, as those which have been established in the present Section.

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[*] As is well illustrated by Atwood's machine.

(11.) As an example of *partial differentiation* (comp. sub-art. 9), of a *vector function* (the word "vector" being *here* used as an *adjective*) of *two scalar variables*, let us take the equation,

$$\rho = \phi(x, y) = \tfrac{1}{2}\{x^2\alpha + y^2\beta + (x+y)^2\gamma\};$$

in which ρ (comp. 99, (8.)) is the vector of a certain *cone of the second order;* or more precisely, the vector of *one sheet* of such a cone, if x and y be supposed to be *real* scalars. Here, the two partial derivatives of ρ are the following:

$$\mathrm{D}_x\rho = x\alpha + (x+y)\gamma; \qquad \mathrm{D}_y\rho = y\beta + (x+y)\gamma;$$

and therefore,

$$2\rho = x\mathrm{D}_x\rho + y\mathrm{D}_y\rho;$$

so that the *three vectors*, ρ, $\mathrm{D}_x\rho$, $\mathrm{D}_y\rho$, if drawn (18) from one *common origin*, are contained (22) in one *common plane;* which implies that the *tangent plane* to the surface, at any point P, passes *through the origin* O: and thereby verifies the *conical* character of the *locus* of that *point* P, in which the *variable vector* ρ, or OP, *terminates*.

(12.) If, in the same example, we make $x = 1$, $y = -1$, we have the values,

$$\rho = \tfrac{1}{2}(\alpha + \beta), \qquad \mathrm{D}_x\rho = \alpha, \qquad \mathrm{D}_y\rho = -\beta;$$

whence it follows that the middle point, say C′, of the right line AB, is one of the points of the conical locus; and that (comp. again the sub-art. 3 to Art. 99, and the recent sub-art. 9) the right lines OA and OB are parallel to two of the tangents to the surface at that point; so that the *cone* in question is *touched by the plane* AOB, *along the side* (or *ray*) OC′. And in like manner it may be proved, that the *same cone* is touched by the *two other planes*, BOC and COA, at the middle points A′ and B′ of the *two other lines* BC and CA; and therefore along *the two other sides* (or rays), OA′ and OB′: which again agrees with former results.

(13.) It will be found that a *vector function* of the *sum of two scalar variables*, t and $\mathrm{d}t$, may generally be *developed*, by an extension of *Taylor's Series*, under the form,

$$\phi(t + \mathrm{d}t) = \phi(t) + \mathrm{d}\phi(t) + \tfrac{1}{2}\mathrm{d}^2\phi(t) + \frac{1}{2.3}\mathrm{d}^3\phi(t) + \ldots$$

$$= \left(1 + \mathrm{d} + \frac{\mathrm{d}^2}{2} + \frac{\mathrm{d}^3}{2.3} + \ldots\right)\phi(t) = \epsilon^{\mathrm{d}}\phi(t);$$

it being supposed that $\mathrm{d}^2t = 0$, $\mathrm{d}^3t = 0$, &c. (comp. sub-art. 6). Thus, if $\phi t = \tfrac{1}{2}\alpha t^2$, (as in sub-art. 1), where α is a *constant vector*, we have $\mathrm{d}\phi t = \alpha t\mathrm{d}t$, $\mathrm{d}^2\phi t = \alpha\mathrm{d}t^2$, $\mathrm{d}^3\phi t = 0$, &c.; and

$$\phi(t + \mathrm{d}t) = \tfrac{1}{2}\alpha(t + \mathrm{d}t)^2 = \tfrac{1}{2}\alpha t^2 + \alpha t\mathrm{d}t + \tfrac{1}{2}\alpha\mathrm{d}t^2,$$

rigorously, without any supposition that $\mathrm{d}t$ is *small*.

(14.) When we thus suppose $\Delta t = \mathrm{d}t$, and *develope the finite difference*, $\Delta\phi(t) = \phi(t + \mathrm{d}t) - \phi(t)$, the *first term* of the development so obtained, or the term of *first dimension* relatively to $\mathrm{d}t$, is hence (by a *theorem*. which holds good for *vector-functions*, as well as for *scalar* functions) the *first differential* $\mathrm{d}\phi t$ *of the function;* but we do not choose to *define* that this *Differential is* (or *means*) that *first term:* because the Formula (100), which we prefer, does not postulate the *possibility*, nor even suppose the *conception*, of any such *development*. Many recent remarks will perhaps appear more clear, when we shall come to *connect* them, at a later stage, with that theory of *Quaternions*, to which we next proceed.

BOOK II.

ON QUATERNIONS, CONSIDERED AS QUOTIENTS OF VECTORS, AND AS INVOLVING ANGULAR RELATIONS.

CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING QUOTIENTS OF VECTORS.

SECTION 1.—*Introductory Remarks; First Principles adopted from Algebra.*

ART. 101. The only *angular relations*, considered in the foregoing Book, have been those of *parallelism* between *vectors* (Art. 2, &c.); and the only *quotients*, hitherto employed, have been of the three following kinds:

I. *Scalar quotients of scalars*, such as the *arithmetical fraction* $\frac{n}{m}$ in Art. 14;

II. *Vector quotients*, of *vectors divided by scalars*, as $\frac{\beta}{x} = a$ in Art. 16;

III. *Scalar quotients of vectors*, with *directions* either *similar* or *opposite*, as $\frac{\beta}{a} = x$ in the last cited Article. But we now propose to treat of *other geometric* QUOTIENTS (or *geometric Fractions*, as we shall also call them), such as

$$\frac{\mathrm{OB}}{\mathrm{OA}} = \frac{\beta}{a} = q, \text{ with } \beta \textit{ not} \parallel a \text{ (comp. 15)};$$

for each of which the *Divisor* (or *denominator*), a or OA, and the *Dividend* (or *numerator*), β or OB, shall not only *both* be

Vectors, but shall also be *inclined* to each other at an ANGLE, *distinct* (in general) from *zero*, and from *two** *right angles*.

102. In introducing this *new conception*, of a *General Quotient of Vectors*, with *Angular Relations* in a given plane, or in space, it will obviously be necessary to employ some properties of *circles* and *spheres*, which were not wanted for the purpose of the former Book. But, on the other hand, it will be possible and useful to suppose a much less degree of acquaintance with many important theories† of *modern geometry*, than that of which the possession was assumed, in several of the foregoing Sections. Indeed it is hoped that a very moderate amount of geometrical, algebraical, and trigonometrical preparation will be found sufficient to render the present Book, as well as the early parts of the preceding one, fully and easily intelligible to any attentive reader.

103. It may be proper to premise a few general principles respecting quotients of vectors, which are indeed *suggested* by *algebra*, but are here *adopted* by *definition*. And Ist, it is evident that the *supposed operation* of *division* (whatever its *full geometrical import* may afterwards be found to be), by which we here conceive ourselves to pass from a given *divisor-line* α, and from a given *dividend-line* β, to what we have called (provisionally) their *geometric quotient*, q, may (or rather must) be *conceived* to correspond to *some converse act* (as yet not *fully* known) of *geometrical multiplication*: in which new act the former *quotient*, q, becomes a FACTOR, and *operates on the line* α, so as to *produce* (or *generate*) the *line* β. We shall therefore *write*, as in algebra,

$$\beta = q \cdot \alpha, \text{ or simply, } \beta = q\alpha, \text{ when } \beta : \alpha = q;$$

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* More generally speaking, from *every even multiple* of a *right angle*.

† Such as *homology*, *homography*, *involution*, and generally whatever depends on *anharmonic ratio:* although all that is needful to be known respecting such ratio, for the applications subsequently made, may be learned, without reference to any other treatise, from the *definitions* incidentally given, in Art. 25, &c. It was, perhaps, not strictly *necessary* to introduce any of these modern geometrical theories, in any part of the present work; but it was thought that it might interest one class, at least, of students, to see how they could be *combined* with that fundamental *conception* of the VECTOR, which the First Book was designed to develope.

even if the two lines α and β, or OA and OB, be supposed to be *inclined* to each other, as in Fig. 33. And this very simple and natural *notation* (comp. 16) will then allow us to treat as *identities* the two following formulæ:

$$\left(\frac{\beta}{\alpha}\cdot\alpha=\right)\frac{\beta}{\alpha}\alpha=\beta;\qquad \frac{q\alpha}{\alpha}=q;$$

although we shall, *for the present*, *abstain* from writing *also* such formulæ* as the following:

$$\frac{\beta\alpha}{\alpha}=\beta,\qquad \frac{q}{\alpha}\alpha=q,$$

where α, β still denote *two vectors*, and q denotes their geometrical *quotient:* because we have not *yet* even *begun* to consider the *multiplication of one vector by another*, or the *division of a quotient by a line.*

104. As a IInd general principle, suggested by algebra, we shall next lay it down, that if

$$\frac{\beta'}{\alpha'}=\frac{\beta}{\alpha},\quad\text{and}\quad \alpha'=\alpha,\quad\text{then}\quad \beta'=\beta;$$

or in words, and under a slightly varied form, that *unequal vectors, divided by equal vectors, give unequal quotients.* The importance of this very natural and obvious assumption will soon be seen in its applications.

105. As a IIIrd principle, which indeed may be considered to pervade the whole of *mathematical language*, and without adopting which we could not usefully *speak*, in any case, of EQUALITY as existing between any two geometrical quotients, we shall next assume that two *such quotients can never be equal to the same third*† *quotient, without being* at the same time *equal to each other:* or in symbols, that

$$\text{if }q'=q,\quad\text{and}\quad q''=q,\quad\text{then}\quad q''=q'.$$

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* It will be seen, however, at a later stage, that these two formulæ are permitted, and even required, in the development of the Quaternion System.

† It is scarcely necessary to add, what is indeed *included* in this IIIrd principle, in virtue of the *identity* $q=q$, that if $q'=q$, then $q=q'$; or in words, that we shall never admit that any *two* geometrical quotients, q and q', are *equal* to each other *in one order*, without at the same time admitting that they are *equal*, in the *opposite order also.*

106. In the IVth place, as a preparation for *operations on geometrical quotients*, we shall say that any two such quotients, or *fractions* (101), which have a *common divisor-line*, or (in more familiar words) a *common denominator*, are *added, subtracted*, or *divided*, among themselves, by adding, subtracting, or dividing their *numerators:* the common denominator being *retained*, in each of the two former of these three cases. In symbols, we thus define (comp. 14) that, *for any three* (actual) *vectors*, α, β, γ,

$$\frac{\gamma}{\alpha}+\frac{\beta}{\alpha}=\frac{\gamma+\beta}{\alpha}; \qquad \frac{\gamma}{\alpha}-\frac{\beta}{\alpha}=\frac{\gamma-\beta}{\alpha};$$

and

$$\frac{\gamma}{\alpha}:\frac{\beta}{\alpha}=\frac{\gamma}{\beta};$$

aiming still at agreement with algebra.

107. Finally, as a Vth principle, designed (like the foregoing) to assimilate, so far as can be done, the present Calculus to Algebra, in its *operations* on geometrical quotients, we shall define that the following formula holds good:

$$\left(\frac{\gamma}{\beta}\cdot\frac{\beta}{\alpha}=\right)\frac{\gamma}{\beta}\frac{\beta}{\alpha}=\frac{\gamma}{\alpha};$$

or that *if two geometrical fractions*, q and q', *be so related*, that the *denominator*, β, of the *multiplier* q' (here written towards the left-hand) is equal to the *numerator* of the *multiplicand* q, then the *product*, $q'\cdot q$ or $q'q$, is that *third fraction*, whereof the numerator is the *numerator* γ *of the multiplier*, and the denominator is the *denominator* α *of the multiplicand:* all such denominators, or *divisor-lines*, being still supposed (16) to be *actual* (and not *null*) *vectors*.

SECTION 2.—*First Motive for naming the Quotient of two Vectors a Quaternion.*

108. Already we may see grounds for the application of the *name*, QUATERNION, to such a *Quotient of two Vectors* as has been spoken of in recent articles. In the first place, such a quotient cannot *generally* be what we have called (17) a SCA-

LAR: or in other words, it cannot generally be equal to any of the (so-called) *reals of algebra*, whether of the *positive* or of the *negative* kind. For let x denote any such (actual*) *scalar*, and let a denote any (actual) *vector;* then we have seen (15) that the *product* xa denotes *another* (actual) *vector*, say β', which is either *similar* or *opposite* in direction to a, according as the scalar coefficient, or *factor*, x, is positive or negative; in *neither* case, then, can it represent any vector, such as β, which is *inclined* to a, at any actual *angle*, whether acute, or right, or obtuse: or in other words (comp. 2), the *equation* $\beta' = \beta$, or $xa = \beta$, is impossible, under the conditions here supposed. But we have agreed (16, 103) to write, as in algebra, $\frac{xa}{a} = x$; we must, therefore (by the IInd principle of the foregoing Section, stated in Art. 104), *abstain* from writing *also* $\frac{\beta}{a} = x$, under the same conditions: x still denoting a *scalar*. Whatever *else* a *quotient of two inclined vectors* may be found to be, it is thus, at least, a NON-SCALAR.

109. Now, in forming the conception of the *scalar itself*, as the *quotient of two parallel*† *vectors* (17), we took into account not only *relative length*, or *ratio* of the usual kind, but also *relative direction*, under the form of *similarity* or *opposition*. In passing from a to xa, we *altered* generally (15) the *length* of the line a, in the ratio of $\pm x$ to 1; and we *preserved* or *reversed* the *direction* of that line, according as the *scalar coefficient* x was *positive* or *negative*. And in like manner, in proceeding to form, more definitely than we have yet done, the conception of the *non-scalar quotient* (108), $q = \beta : a =$ OB : OA, of *two inclined vectors*, which for simplicity may be supposed (18) to be *co-*

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* By an *actual scalar*, as by an *actual vector* (comp. 1), we mean here one that is *different from zero*. An *actual vector*, multiplied by a *null scalar*, has for product (15) a *null vector;* it is therefore unnecessary to prove that the *quotient* of *two actual vectors* cannot be a *null scalar*, or zero.

† It is to be remembered that we have proposed (15) to extend the use of this term *parallel*, to the case of two vectors which are (in the *usual* sense of the word) parallel to one *common line*, even when they happen to be *parts* of one and the *same* right line.

initial, we have *still* to take account both of the *relative length*, and of the *relative direction*, of the two lines compared. But while the *former element* of the *complex relation* here considered, between these two lines or vectors, is *still* represented by a simple RATIO (of the kind commonly considered in geometry), or by a *number** expressing that ratio; the *latter element* of the same complex relation is *now* represented by an ANGLE, AOB: and not simply (as it was before) by an *algebraical sign*, + or −.

110. Again in estimating this *angle*, for the purpose of *distinguishing* one quotient of vectors from another, we must consider not only its *magnitude* (or *quantity*), but also its PLANE: since otherwise, in violation of the principle stated in Art. 104, we should have $OB' : OA = OB : OA$, if OB and OB′ were *two distinct rays* or sides of a *cone* of revolution, with OA for its *axis;* in which case (by 2) they would necessarily be *unequal vectors.* For a similar reason, we must attend also to the *contrast* between two *opposite angles*, of equal magnitudes, and in one *common plane*. In short, for the purpose of knowing *fully* the *relative direction* of two co-initial lines OA, OB in *space*, we ought to know not only *how many degrees*, or other *parts* of some *angular unit*, the *angle* AOB contains; but also (comp. Fig. 33) the *direction of the rotation* from OA to OB: including a knowledge of the *plane*, *in which* the rotation is performed; and of the *hand* (as *right* or *left*, when *viewed* from a known *side* of the plane), *towards which* the rotation is *directed*.

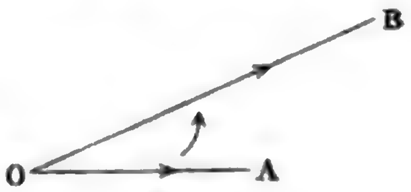

Fig. 33.

111. Or, if we agree to *select* some *one fixed hand* (suppose the *right*† hand), and to call all *rotations positive* when they

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* This *number*, which we shall presently call the *tensor* of the quotient, may be *whole* or *fractional*, or even *incommensurable* with unity; but it may always be *equated*, in calculation, to a *positive scalar:* although it might perhaps more *properly* be said to be a *signless number*, as being derived solely from comparison of *lengths*, without any reference to *directions*.

† If *right-handed rotation* be thus considered as *positive*, then the *positive axis* of the rotation AOB, in Fig. 33, must be conceived to be directed *downward*, or *below* the plane of the paper.

are directed towards *this* selected hand, but all rotations *negative* when they are directed towards the *other hand*, then, for *any given angle* AOB, supposed for simplicity to be less than two right angles, and considered as representing a *rotation in a given plane* from OA to OB, we may speak of *one perpendicular* OC *to that plane* AOB as being the *positive axis* of that rotation; and of the *opposite perpendicular* OC′ to the same plane as being the *negative axis* thereof: the rotation round the positive axis being *itself* positive, and *vice versâ*. And then the *rotation* AOB may be considered to be entirely *known*, if we know, Ist, its *quantity*, or the *ratio* which it bears to a *right rotation;* and IInd, the *direction* of its *positive axis*, OC: but not without a knowledge of these *two* things, or of some data equivalent to them. But whether we consider the *direction of an* AXIS, or the *aspect of a* PLANE, we find (as indeed is well known) that the *determination* of such a *direction*, or of such an *aspect*, depends on TWO *polar co-ordinates**, or other *angular elements*.

112. It appears, then, from the foregoing discussion, that *for the complete determination*, of what we have called the *geometrical* QUOTIENT *of two co-initial Vectors*, a *System of Four Elements*, admitting each separately of numerical expression, *is generally required*. Of these four elements, *one* serves (109) to determine the *relative length* of the two lines compared; and the other *three* are in general necessary, in order to determine *fully* their *relative direction*. Again, of these three latter elements, *one* represents the mutual *inclination*, or *elongation*, of the two lines; or the *magnitude* (or quantity) of the *angle* between them; while the *two others* serve to determine the *direction* of the *axis*, perpendicular to their common *plane*, round which a *rotation* through that angle is to be performed, in a *sense* previously selected as the *positive* one (or towards a fixed and previously selected *hand*), for the purpose of *passing* (in the simplest way, and therefore in the plane of the two lines) *from* the *direction* of the *divisor-line*, *to* the direction of

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* The actual (or at least the frequent) *use* of *such* co-ordinates is foreign to the spirit of the present System: but the *mention* of them here seems likely to assist a student, by suggesting an appeal to results, with which his previous reading can scarcely fail to have rendered him familiar.

the *dividend-line*. And *no more than four* numerical *elements* are necessary, for our present purpose: because the *relative length* of two lines is not changed, when their two lengths are altered *proportionally*, nor is their *relative direction* changed, when the *angle* which they form is merely *turned* about, *in its own plane*. On account, then, of this *essential connexion* of that *complex relation* (109) between two lines, which is *compounded* of a *relation of lengths*, and of a *relation of directions*, and to which we have given (by an *extension* from the theory of *scalars*) the name of a *geometrical quotient*, with a *System of* FOUR *numerical Elements*, we have already a *motive*[*] for saying, that "*the Quotient of two Vectors is generally a Quaternion.*"

SECTION 3.—*Additional Illustrations.*

113. Some additional light may be thrown, on this first *conception* of a *Quaternion*, by the annexed Figure 34. In that Figure, the letters CDEFG are designed to indicate corners of a prismatic *desk*, resting upon a horizontal *table*. The *angle* HCD (supposed to be one of *thirty* degrees) represents a (left-handed) *rotation*, whereby the horizontal *ledge* CD of the desk is conceived to be *elongated* (or removed) from a *given horizontal line* CH, which may be imagined to be an *edge* of the table. The angle GCF (supposed here to contain *forty* degrees) represents the *slope*[†] of the desk, or the amount of its *inclination* to the table. On the *face* CDEF of the desk are drawn two *similar* and *similarly turned triangles*, AOB and A′O′B′, which are supposed to be halves of two equilateral triangles; in such a manner that *each*

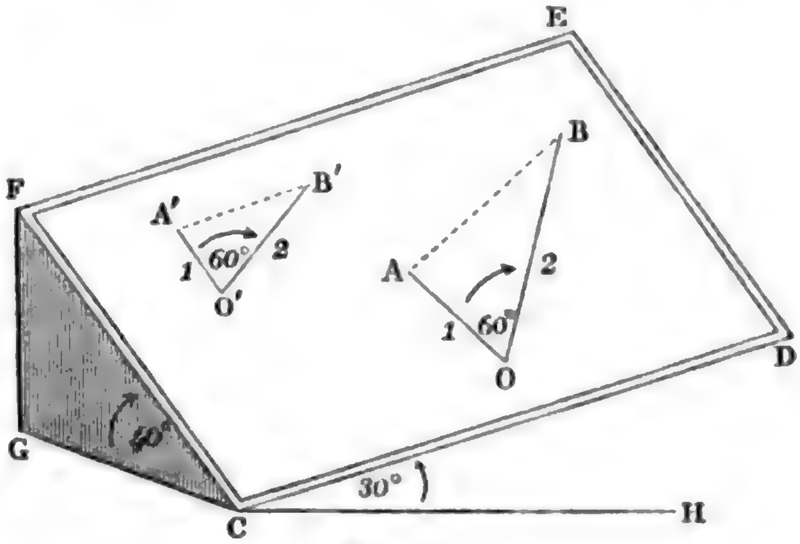

Fig. 34.

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[*] Several other reasons for thus speaking will offer themselves, in the course of the present work.

[†] *These two angles*, HCD and GCF, may thus be considered to correspond to *longitude of node*, and *inclination of orbit*, of a planet or comet in astronomy.

rotation, AOB or A′O′B′ is one of *sixty* degrees, and is directed towards *one common hand* (namely the *right* hand in the Figure): while if *lengths* alone be attended to, the *side* OB is to the side OA, in *one triangle*, as the side O′B′ is to the side O′A′, in the *other;* or as the *number two* to *one*.

114. Under these conditions of construction, we consider the *two quotients*, or the *two geometric fractions*,

$$\mathrm{OB:OA} \text{ and } \mathrm{OB'}:\mathrm{OA'}, \text{ or } \frac{\mathrm{OB}}{\mathrm{OA}} \text{ and } \frac{\mathrm{O'B'}}{\mathrm{O'A'}},$$

as being *equal to each other;* because we regard the *two lines*, OA and OB, as having the *same relative length*, and the *same relative direction*, as the *two other lines*, O′A′ and O′B′. And we consider and speak of *each Quotient*, or *Fraction*, as a *Quaternion:* because its *complete construction* (or *determination*) depends, for all that is *essential* to its *conception*, and requisite to *distinguish* it from others, on a system of *four numerical elements* (comp. 112); which are, in this Example, the *four numbers*,

2, 60, 30, and 40.

115. Of these four *elements* (to recapitulate what has been above supposed), the Ist, namely the number 2, expresses that the *length* of the *dividend-line*, OB or O′B′, is *double* of the *length* of the *divisor-line*, OA or O′A′. The IInd numerical element, namely 60, expresses here that the *angle* AOB or A′O′B′, is one of *sixty degrees;* while the corresponding *rotation*, from OA to OB, or from O′A′ to O′B′, is *towards a known hand* (in this case the *right* hand, as seen by a person looking at the *face* CDEF of the desk), which *hand* is the *same for both* of these two *equal angles*. The IIIrd element, namely 30, expresses that the horizontal *ledge* CD of the desk makes an angle of *thirty degrees* with a known horizontal line CH, being removed from it, by that angular quantity, in a known direction (which in this case happens to be towards the *left* hand, as seen from above). Finally, the IVth element, namely 40, expresses here that the desk has an *elevation* of *forty degrees* as before.

116. Now an *alteration in any one of these Four Elements*, such as an alteration of the *slope* or *aspect* of the *desk*, would make (in the view here taken) *an essential change in the Quaternion*, which is (in the same view) *the Quotient of the two lines* compared: although (as the Figure is in part designed to suggest) *no such change is conceived to take place*, when the *triangle* AOB is merely *turned about, in its own plane*, without being *turned over* (comp. Fig. 36); or when the *sides* of that triangle are *lengthened* or *shortened proportionally*, so as to *preserve the ratio* (in the old sense of that word), of any one to any other of those sides. We may then briefly say, in this mode of *illustrating* the *notion* of a QUATERNION* in *geometry*, by refe-

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* As to the mere *word*, *Quaternion*, it signifies primarily (as is well known), like its Latin original, "Quaternio," or the Greek noun τετρακτύς, a *Set of Four:* but it is obviously used *here*, and elsewhere in the present work, *in a technical sense*.

rence to an *angle on a desk*, that the *Four Elements* which it involves are the following:

Ratio, *Angle*, *Ledge*, and *Slope*;

although the *two latter elements* are in fact *themselves angles also*, but are not *immediately* obtained as such, from the simple comparison of the *two lines*, of which the *Quaternion* is the *Quotient*.

SECTION 4.—*On Equality of Quaternions; and on the Plane of a Quaternion.*

117. It is an immediate consequence of the foregoing *conception* of a Quaternion, that *two quaternions*, or *two quotients of vectors*, supposed for simplicity to be all *co-initial* (18), are regarded as being EQUAL to each other, or that the EQUATION,

$$\frac{\delta}{\gamma}=\frac{\beta}{\alpha}, \quad \text{or} \quad \frac{\mathrm{OD}}{\mathrm{OC}}=\frac{\mathrm{OB}}{\mathrm{OA}},$$

is by us considered and *defined* to hold good, *when the two triangles*, AOB and COD, *are similar and similarly turned*, and *in one common plane*, as represented in the annexed Fig. 35: the RELATIVE LENGTH (109), and the RELATIVE DIRECTION (110), of the two lines, OA, OB, being then in all respects the *same* as the relative length and the relative direction of the *two other lines*, OC, OD.

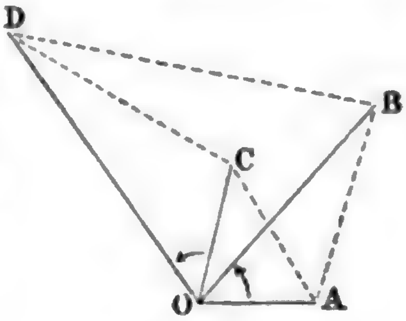

Fig. 35.

118. Under the same conditions, we shall write the following *formula of direct similitude*,

$$\Delta\,\mathrm{AOB} \propto \mathrm{COD};$$

reserving this *other* formula,

$$\Delta\,\mathrm{AOB} \propto' \mathrm{AOB'}, \text{ or } \Delta\,\mathrm{A'OB} \propto' \mathrm{A'OB'},$$

which we shall call a *formula of inverse similitude*, to denote that the two triangles, AOB and AOB′, or A′OB and A′OB′, although otherwise *similar* (and even, in this case, *equal*,* on account of their having a *common side*, OA or OA′), are

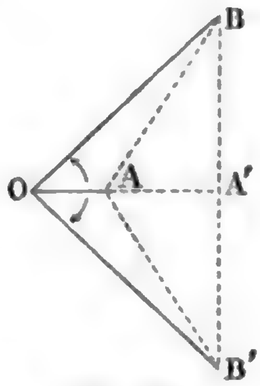

Fig. 36.

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* That is to say, *equal* in *absolute amount* of *area*, but with *opposite algebraic signs* (28). The *two quotients* OB : OA, and OB′ : OA, although *not equal* (110), will soon be defined to be *conjugate quaternions*. Under the same conditions, we shall write also the formula,

$$\Delta\,\mathrm{AOB'} \propto' \mathrm{COD}.$$

oppositely turned (comp. Fig. 36), as if one were the *reflexion* of the other in a mirror; or as if the one triangle were *derived* (or generated) from the other, by a *rotation of its plane* through *two right angles.* We may therefore write,

$$\frac{\text{OB}}{\text{OA}} = \frac{\text{OD}}{\text{OC}}, \text{ if } \Delta\, \text{AOB} \propto \text{COD}.$$

119. When the vectors are thus all drawn from one common origin O, the *plane* AOB of *any two* of them may be called the *Plane of the Quaternion* (or of the Quotient), OB : OA; and of course also the plane of the *inverse* (or *reciprocal*) *quaternion* (or of the inverse quotient), OA : OB. And *any two quaternions*, which have a *common plane* (through O), may be said to be *Complanar** *Quaternions*, or complanar quotients, or fractions; but any two quaternions (or quotients), which have *different planes* (*intersecting* therefore in a right line through the origin), may be said, by contrast, to be *Diplanar*.

120. *Any two quaternions*, considered as *geometric fractions* (101), can be *reduced to a common denominator* without change of the *value*† of either of them, as follows. Let $\frac{\text{OB}}{\text{OA}}$ and $\frac{\text{OD}}{\text{OC}}$ be the two given fractions, or quaternions; and if they be *complanar* (119), let OE be *any line* in their *common plane;* but if they be *diplanar* (see again 119), then let OE be *any assumed part* of the *line of intersection* of the two planes: so that, in *each* case, the line OE is situated at once in the plane AOB, and *also* in the plane COD. We can then always conceive *two other lines*, OF, OG, to be determined so as to satisfy the *two conditions* of direct similitude (118),

$$\Delta\, \text{EOF} \propto \text{AOB}, \qquad \Delta\, \text{EOG} \propto \text{COD};$$

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* It is, however, convenient to extend the use of this word, *complanar*, so as to include the case of quaternions represented by *angles in parallel planes*. Indeed, as all *vectors* which have *equal lengths*, and *similar directions*, are equal (2), so the *quaternion*, which is a *quotient* of *two* such vectors, ought not to be considered as undergoing *any change*, when either vector is merely changed in *position*, by a *transport without rotation*.

† That is to say, the *new* or *transformed* quaternions will be respectively *equal* to the old or *given* ones.

and therefore also the *two equations* between quotients (117, 118),

$$\frac{\text{OF}}{\text{OE}} = \frac{\text{OB}}{\text{OA}}, \qquad \frac{\text{OG}}{\text{OE}} = \frac{\text{OD}}{\text{OC}};$$

and thus the required *reduction* is effected, OE being the *common denominator* sought, while OF, OG are the new or *reduced numerators.* It may be added that if H be a new point in the plane AOB, such that Δ HOE ∝ AOB, we shall have also,

$$\frac{\text{OE}}{\text{OH}} = \frac{\text{OB}}{\text{OA}} = \frac{\text{OF}}{\text{OE}};$$

and therefore, by 106, 107,

$$\frac{\text{OD}}{\text{OC}} \pm \frac{\text{OB}}{\text{OA}} = \frac{\text{OG} \pm \text{OF}}{\text{OE}}; \qquad \frac{\text{OD}}{\text{OC}} : \frac{\text{OB}}{\text{OA}} = \frac{\text{OG}}{\text{OF}}; \qquad \frac{\text{OD}}{\text{OC}} \cdot \frac{\text{OB}}{\text{OA}} = \frac{\text{OG}}{\text{OH}};$$

whatever two geometric quotients (complanar or diplanar) may be represented by OB : OA and OD : OC.

121. If now the two triangles AOB, COD are not only *complanar* but *directly similar* (118), so that Δ AOB ∝ COD, we shall evidently have Δ EOF ∝ EOG; so that we may write OF = OG (or F = G, by 20), the *two new lines* OF, OG (or the two new *points* F, G) in this case *coinciding*. The general construction (120), for the reduction to a common denominator, gives therefore here only *one new triangle*, EOF, and *one new quotient*, OF : OE, to which in this case *each* (comp. 105) of the *two* given *equal* and *complanar* quotients, OB : OA and OD : OC, is equal.

122. But if these two latter symbols (or the *fractional forms* corresponding) denote *two diplanar*[*] *quotients*, then the *two new numerator lines*, OF and OG, have *different directions*, as being situated *in two different planes*, drawn *through the new denominator-line* OE, without having either the direction of that line *itself*, or the direction *opposite* thereto; they are therefore (by 2) *unequal vectors*, even if they should happen to be *equally long;* whence it follows (by 104) that the *two new quotients*, and therefore also (by 105) that the two *old* or *given* quotients, are *unequal*, as a consequence of their *diplanarity*.

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[*] And therefore *non scalar* (108); for a *scalar*, considered as a *quotient* (17), has *no determined plane*, but must be considered as *complanar with every geometric quotient;* since it may be represented (or constructed) by the quotient of two similarly or oppositely directed lines, *in any proposed plane* whatever.

It results, then, from this analysis, that *diplanar quotients of vectors*, and therefore that *Diplanar Quaternions* (119), *are always unequal;* a new and comparatively *technical* process thus *confirming* the conclusion, to which we had arrived by general considerations, and in (what might be called) a *popular* way before, and which we had sought to *illustrate* (comp. Fig. 34) by the consideration of *angles on a desk:* namely, that a *Quaternion*, considered as the quotient of *two mutually inclined lines in space*, involves generally a *Plane*, as an *essential part* (comp. 110) of its constitution, and as necessary to the *completeness* of its *conception.*

123. We propose to use the mark

$$|||$$

as a *Sign of Complanarity*, whether of *lines* or of *quotients;* thus we shall write the formula,

$$\gamma \,|||\, \alpha, \beta,$$

to express that the *three vectors*, α, β, γ, supposed to be (or to be made) *co-initial* (18), are situated *in one plane;* and the analogous formula,

$$q' \,|||\, q, \quad \text{or} \quad \frac{\delta}{\gamma} \,|||\, \frac{\beta}{\alpha},$$

to express that the *two quaternions*, denoted here by q and q', and therefore that the *four vectors*, α, β, γ, δ, are *complanar* (119). And because we have just found (122) that *diplanar* quotients are *unequal*, we see that *one equation of quaternions includes two complanarities of vectors;* in such a manner that we may write,

$$\gamma \,|||\, \alpha, \beta, \quad \text{and} \quad \delta \,|||\, \alpha, \beta, \quad \text{if} \quad \frac{\delta}{\gamma} = \frac{\beta}{\alpha};$$

the *equation of quotients*, $\frac{\text{OD}}{\text{OC}} = \frac{\text{OB}}{\text{OA}}$, being impossible, *unless all the four lines* from O be *in one common plane.* We shall also employ the notation

$$\gamma \,|||\, q,$$

to express that the *vector* γ is *in* (or *parallel to*) *the plane of the quaternion* q.

124. With the same notation for complanarity, we may write generally,

$$x\alpha \mathrel{|||} \alpha, \beta;$$

α and β being *any two vectors*, and x being *any scalar*; because, if $\alpha = \mathrm{OA}$ and $\beta = \mathrm{OB}$ as before, then (by 15, 17) $x\alpha = \mathrm{OA}'$, where A' is *some point on the indefinite right line* through the points O and A: so that the *plane* AOB contains the *line* OA'. For a similar reason, we have generally the following formula of *complanarity of quotients*,

$$\frac{y\beta}{x\alpha} \mathrel{|||} \frac{\beta}{\alpha},$$

whatever *two scalars* x and y may be; α and β still denoting any two *vectors*.

125. It is evident (comp. Fig. 35) that

$$\text{if } \Delta\,\mathrm{AOB} \propto \mathrm{COD}, \quad \text{then } \Delta\,\mathrm{BOA} \propto \mathrm{DOC}, \quad \text{and } \Delta\,\mathrm{AOC} \propto \mathrm{BOD};$$

whence it is easy to infer that for quaternions, as well as for ordinary or algebraic quotients,

$$\text{if } \frac{\beta}{\alpha} = \frac{\delta}{\gamma}, \text{ then, } \textit{inversely}, \ \frac{\alpha}{\beta} = \frac{\gamma}{\delta}, \text{ and } \textit{alternately}, \ \frac{\gamma}{\alpha} = \frac{\delta}{\beta};$$

it being permitted *now* to establish the *converse* of the last formula of 118, or to say that

$$\text{if } \frac{\mathrm{OB}}{\mathrm{OA}} = \frac{\mathrm{OD}}{\mathrm{OC}}, \text{ then } \Delta\,\mathrm{AOB} \propto \mathrm{COD}.$$

Under the same condition, by *combining* inversion with alternation, we have also this other equation, $\frac{\alpha}{\gamma} = \frac{\beta}{\delta}$.

126. If the *sides*, OA, OB, of a triangle AOB, or those sides either way prolonged, be *cut* (as in Fig. 37) by any *parallel*, $\mathrm{A}'\mathrm{B}'$ or $\mathrm{A}''\mathrm{B}''$, to the *base* AB, we have evidently the relations of *direct similarity* (118),

$$\Delta\,\mathrm{A}'\mathrm{OB}' \propto \mathrm{AOB}, \quad \Delta\,\mathrm{A}''\mathrm{OB}'' \propto \mathrm{AOB};$$

whence (comp. Art. 13 and Fig. 12) it follows that we may write, for quaternions as in algebra, the general equation, or identity,

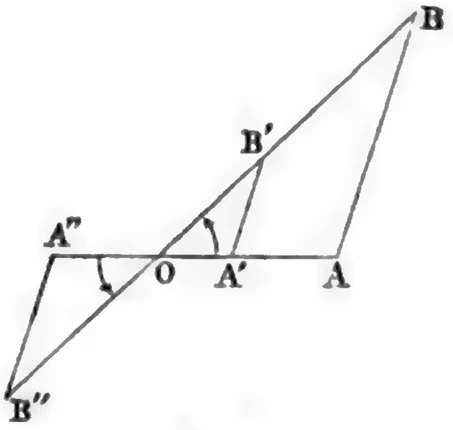

Fig. 37.

$$\frac{x\beta}{x\alpha} = \frac{\beta}{\alpha};$$

where x is again *any scalar*, and α, β are *any two vectors*. It is easy also to see, that for any quaternion q, and any scalar x, we have the *product* (comp. 107),

$$xq = \frac{x\beta}{\beta} \cdot \frac{\beta}{\alpha} = \frac{x\beta}{\alpha} = \frac{\beta}{x^{-1}\alpha} = \frac{\beta}{\alpha} \cdot \frac{\alpha}{x^{-1}\alpha} = qx;$$

so that, in the *multiplication of a quaternion by a scalar* (as in the multiplication of a *vector* by a *scalar*, 15), the *order of the factors* is *indifferent*.

SECTION 5.—*On the Axis and Angle of a Quaternion; and on the Index of a Right Quotient, or Quaternion.*

127. From what has been already said (111, 112), we are naturally led to define that the AXIS, or more fully that the *positive axis, of any quaternion* (or geometric *quotient*) OB : OA, is *a right line perpendicular to the plane* AOB of that quaternion; and is such that the *rotation round this axis, from the divisor-line* OA, *to the dividend-line* OB, is *positive:* or (as we shall henceforth assume) directed *towards the right-hand*,* like the motion of the hands of a watch.

128. To render still more *definite* this conception of the *axis of a quaternion*, we may add, Ist, that the *rotation*, here spoken of, is supposed (112) to be the *simplest possible*, and therefore to be *in the plane of the two lines* (or of the quaternion), being also *generally less than a semi-revolution* in that plane; IInd, that the axis shall be usually supposed to be a line OX drawn *from the assumed origin* O; and IIIrd, that the *length* of this line shall be supposed to be *given*, or *fixed*, and to be equal to some assumed *unit* of length: so that the *term* X, of this *axis* OX, is situated (by its construction) *on a given spheric surface* described about the *origin* O as *centre*, which surface we may call the surface of the UNIT-SPHERE.

129. In this manner, for every given *non-scalar quotient*

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* This is, of course, merely conventional, and the reader may (if he pleases) substitute the *left*-hand throughout.

(108), or for every given *quaternion* q which does not reduce itself (or degenerate) to a mere *positive or negative number*, the *axis* will be an entirely *definite vector*, which may be called an UNIT-VECTOR, on account of its assumed *length*, and which we shall *denote**, for the present, by the *symbol* $\mathrm{Ax}\,.\,q$. Employing then the usual *sign of perpendicularity*, $\perp$, we may now write, for any two vectors α, β, the formula:

$$\mathrm{Ax}.\frac{\beta}{\alpha} \perp \alpha; \quad \mathrm{Ax}.\frac{\beta}{\alpha} \perp \beta; \quad \text{or briefly,} \quad \mathrm{Ax}.\frac{\beta}{\alpha} \perp \begin{cases}\beta\\ \alpha\end{cases}.$$

130. The ANGLE *of a quaternion*, such as OB : OA, shall simply be, with us, the *angle* AOB *between the two lines*, of which the quaternion is the quotient; *this* angle being supposed here to be one of the *usual kind* (such as are considered by Euclid): and therefore being *acute*, or *right*, or *obtuse* (but not of any class *distinct* from these), when the quaternion is a *non-scalar* (108). We shall *denote* this *angle of a quaternion* q, by the *symbol*, $\angle q$; and thus shall have, *generally*, the two *inequalities*† following:

$$\angle q > 0; \qquad \angle q < \pi;$$

where π is used as a symbol for *two right angles*.

131. When the *general quaternion*, q, *degenerates* into a *scalar*, x, then the *axis* (like the *plane*‡) becomes entirely *indeterminate* in its *direction*; and the *angle* takes, at the same time, either *zero* or *two right angles* for its value, according as the scalar is *positive* or *negative*. Denoting then, as above, any such *scalar* by x, we have:

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* At a later stage, reasons will be assigned for denoting this *axis*, $\mathrm{Ax}\,.\,q$, of a quaternion q, by the *less arbitrary* (or more systematic) *symbol*, $\mathrm{UV}q$; but for the present, the notation in the text may suffice.

† In some investigations respecting *complanar quaternions*, and *powers* or *roots* of quaternions, it is convenient to consider *negative angles*, and angles *greater than two* right angles: but these may then be called AMPLITUDES; and the word "Angle," like the word "Ratio," may thus be restricted, at least for the present, to its *ordinary geometrical sense*.

‡ Compare the Note to page 114. The *angle*, as well as the *axis*, becomes *indeterminate*, when the quaternion reduces itself to *zero*; unless we happen to know a *law*, according to which the dividend-line *tends* to become *null*, in the transition from $\frac{\beta}{\alpha}$ to $\frac{0}{\alpha}$.

$$\mathrm{Ax} \,.\, x = \text{an indeterminate unit-vector};$$
$$\angle x = 0, \text{ if } x > 0; \quad \angle x = \pi, \text{ if } x < 0.$$

132. Of *non-scalar quaternions*, the most important are those of which the *angle* is *right*, as in the annexed Figure 38; and when we have thus,

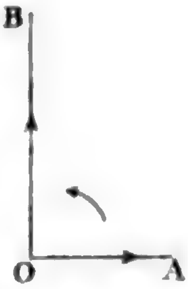

Fig. 38.

$$q = \frac{\mathrm{OB}}{\mathrm{OA}}, \quad \text{and} \quad \mathrm{OB} \perp \mathrm{OA}, \quad \text{or} \quad \angle q = \frac{\pi}{2},$$

the quaternion q may then be said to be a RIGHT QUOTIENT;* or sometimes, a *Right Quaternion.*

(1.) If then $\alpha = \mathrm{OA}$ and $\rho = \mathrm{OP}$, where O and A are *two given* (or *fixed*) *points*, but P is a *variable point*, the *equation*

$$\angle \frac{\rho}{\alpha} = \frac{\pi}{2}$$

expresses that the *locus* of this point P is the *plane through* O, *perpendicular to the line* OA; for it is equivalent to the *formula of perpendicularity* $\rho \perp \alpha$ (129).

(2.) More generally, if $\beta = \mathrm{OB}$, B being any *third given point*, the equation,

$$\angle \frac{\rho}{\alpha} = \angle \frac{\beta}{\alpha}$$

expresses that the *locus* of P is *one sheet of a cone of revolution*, with O for *vertex*, and OA for *axis*, and passing *through the point* B; because it implies that the *angles* AOB and AOP are *equal* in *amount*, but *not necessarily* in one *common plane.*

(3.) The equation (comp. 128, 129),

$$\mathrm{Ax} \,.\, \frac{\rho}{\alpha} = \mathrm{Ax} \,.\, \frac{\beta}{\alpha},$$

expresses that the *locus* of the variable point P is the *given plane* AOB; or rather the indefinite *half-plane*, which contains all the points P that are at once *complanar* with the *three given points* O, A, B, and are also *at the same side of the indefinite right line* OA, *as the point* B.

(4.) The system of the *two equations*,

$$\angle \frac{\rho}{\alpha} = \angle \frac{\beta}{\alpha}, \qquad \mathrm{Ax} \,.\, \frac{\rho}{\alpha} = \mathrm{Ax} \,.\, \frac{\beta}{\alpha},$$

expresses that the point P is situated, *either* on the *finite right line* ~~OA~~ B, *or* on that line *prolonged through* ~~A~~ B, but *not through* O; so that the *locus* of P may in this case be said to be the *indefinite half-line*, or *ray*, which sets out from O in the *direction* of the *vector* OB or β; and we may write $\rho = x\beta$, $x > 0$ (x being understood to be a *scalar*), instead of the equations assigned above.

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* Reasons will afterwards be assigned, for *equating such a quotient*, or quaternion, *to a Vector;* namely to the *line* which will presently (133) be called the *Index of the Right Quotient.*

(5.) This *other* system of *two* equations,

$$\angle \frac{\rho}{\alpha} = \pi - \angle \frac{\beta}{\alpha}, \qquad \mathrm{Ax} . \frac{\rho}{\alpha} = - \mathrm{Ax} . \frac{\beta}{\alpha},$$

expresses that the *locus* of P is the *opposite ray* from O; or that P is situated *on the prolongation of the revector* BO (1); or that $\rho = x\beta$, $x < 0$; or that

$$\rho = x\beta', \ x > 0, \quad \text{if } \beta' = \mathrm{OB}' = -\beta.$$

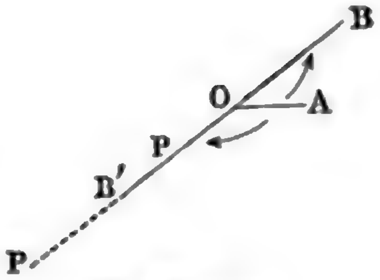

Fig. 33, *bis.*

(Comp. Fig. 33, *bis.*)

(6.) *Other notations*, for representing these and other geometric *loci*, will be found to be supplied, in great abundance, by the Calculus of Quaternions; but it seemed proper to point out these, at the present stage, as serving already to show that even the *two symbols* of the present Section, Ax. and ∠, when considered as *Characteristics of Operation* on *quotients of vectors*, enable us to *express*, very simply and concisely, several useful geometrical *conceptions*.

133. If a *third line*, OI, be drawn in the *direction* of the *axis* OX of such a right quotient (and therefore *perpendicular*, by 127, 129, to *each* of the *two given rectangular lines*, OA, OB); and if the *length* of this new line OI bear to the *length* of that axis OX (and therefore also, by 128, to the assumed *unit* of length) the *same ratio*, which the *length of the dividend-line*, OB, bears *to the length of the divisor-line*, OA; then the line OI, thus determined, is said to be the INDEX *of the Right Quotient*. And it is evident, from this definition of such an *Index*, combined with our general definition (117, 118) of *Equality between Quaternions*, that *two right quotients are equal or unequal to each other, according as their two index-lines* (or *indices*) *are equal or unequal vectors*.

SECTION 6.—*On the Reciprocal, Conjugate, Opposite, and Norm of a Quaternion; and on Null Quaternions.*

134. The RECIPROCAL (or the *Inverse*, comp. 119) of a quaternion, such as $q = \frac{\beta}{\alpha}$, is that *other* quaternion,

$$q' = \frac{\alpha}{\beta},$$

which is formed by *interchanging the divisor-line and the dividend-line;* and in thus passing from any *non-scalar* quaternion to its *reciprocal*, it is evident that the *angle* (as lately defined in 130) remains *unchanged*, but that the *axis* (127, 128) is *reversed* in direction: so that we may write generally,

$$\angle\frac{\alpha}{\beta}=\angle\frac{\beta}{\alpha};\qquad \mathrm{Ax}.\frac{\alpha}{\beta}=-\mathrm{Ax}.\frac{\beta}{\alpha}.$$

135. The *product* of *two reciprocal quaternions* is always equal to *positive unity;* and *each* is equal to the *quotient* of *unity divided by the other;* because we have, by 106, 107,

$$1:\frac{\beta}{\alpha}=\frac{\alpha}{\alpha}:\frac{\beta}{\alpha}=\frac{\alpha}{\beta},\quad\text{and}\quad \frac{\alpha}{\beta}.\frac{\beta}{\alpha}=\frac{\alpha}{\alpha}=1.$$

It is therefore unnecessary to introduce any *new* or peculiar *notation*, to express the mutual relation existing between a quaternion and its *reciprocal;* since, if *one* be denoted by the symbol q, the *other* may (in the present System, as in Algebra) be denoted by the connected symbol,[*] $1:q$, or $\frac{1}{q}$. We have thus the two general formulæ (comp. 134):

$$\angle\frac{1}{q}=\angle q;\qquad \mathrm{Ax}.\frac{1}{q}=-\mathrm{Ax}.q.$$

136. Without *yet* entering on the *general* theory of multiplication and division of quaternions, beyond what has been done in Art. 120, it may be here remarked that if any two quaternions q and q' be (as in 134) *reciprocal* to each other, so that $q'.q=1$ (by 135), and if q'' be *any third* quaternion, then (as in algebra), we have the general formula,

$$q'':q=q''.q'=q''.\frac{1}{q};$$

because if (by 120) we *reduce* q and q'' to a *common denominator* α, and denote the *new numerators* by β and γ, we shall have (by the *definitions* in 106, 107),

$$q'':q=\frac{\gamma}{\alpha}:\frac{\beta}{\alpha}=\frac{\gamma}{\beta}=\frac{\gamma}{\alpha}.\frac{\alpha}{\beta}=q''.q'.$$

137. When *two complanar triangles* AOB, AOB′, with a *com-*

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[*] The symbol q^{-1}, for the *reciprocal* of a quaternion q, is also permitted in the present Calculus; but we defer the use of it, until its legitimacy shall have been established, in connexion with a general theory of powers of Quaternions.

mon side OA, are (as in Fig. 36) *inversely similar* (118), so that the formula $\Delta\,\mathrm{AOB}' \propto' \mathrm{AOB}$ holds good, then the *two unequal quotients,** $\frac{\mathrm{OB}}{\mathrm{OA}}$ and $\frac{\mathrm{OB}'}{\mathrm{OA}}$, are said to be CONJUGATE QUATERNIONS; and if the *first* of them be still denoted by q, then the *second*, which is thus *the conjugate* of that *first*, or of any *other* quaternion which is *equal* thereto, is denoted by the *new symbol*, Kq: in which the letter K may be said to be the *Characteristic of Conjugation*. Thus, with the construction above supposed (comp. again Fig. 36), we may write,

$$\frac{\mathrm{OB}}{\mathrm{OA}} = q; \qquad \frac{\mathrm{OB}'}{\mathrm{OA}} = \mathrm{K}q = \mathrm{K}\frac{\mathrm{OB}}{\mathrm{OA}}.$$

138. From this definition of conjugate quaternions, it follows, Ist, that if the equation $\frac{\mathrm{OB}'}{\mathrm{OA}} = \mathrm{K}\,\frac{\mathrm{OB}}{\mathrm{OA}}$ hold good, then the *line* OB′ may be called (118) the *reflexion of the line* OB (and conversely, the *latter* line the reflexion of the *former*), *with respect to the line* OA; IInd, that, under the same condition, the line OA (prolonged if necessary) *bisects perpendicularly the line* BB′, in some point A′ (as represented in Fig. 36); and IIIrd, that *any two conjugate quaternions* (like any two *reciprocal* quaternions, comp. 134, 135) have *equal angles*, but *opposite axes:* so that we may write, generally,

$$\angle\,\mathrm{K}q = \angle\,q; \qquad \mathrm{Ax.K}q = -\,\mathrm{Ax}.q;$$

and therefore† (by 135),

$$\angle\,\mathrm{K}q = \angle\frac{1}{q}; \qquad \mathrm{Ax.K}q = \mathrm{Ax}.\frac{1}{q}.$$

139. The *reciprocal* of a *scalar*, x, is simply *another scalar*, $\frac{1}{x}$, or x^{-1}, having the *same* algebraic *sign*, and in all other respects related to x as in algebra. But the *conjugate* Kx, of a *scalar* x, considered as a *limit* of a *quaternion*, is equal to that scalar x *itself;* as may be seen by supposing the *two equal* but opposite angles, AOB and AOB′, in Fig. 36, to tend together to

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* Compare the Note to page 112.

† It will soon be seen that these two last equations (138) express, that the *conjugate* and the *reciprocal*, of any proposed quaternion q, have always *equal versors*, although they have in general *unequal tensors*.

zero, or to two right angles. We may therefore write, generally,

$$\mathrm{K}x = x, \text{ if } x \text{ be any scalar;}$$

and conversely*,

$$q = \text{a scalar, if } \mathrm{K}q = q;$$

because then (by 104) we must have $\text{OB} = \text{OB}'$, $\text{BB}' = 0$; and therefore each of the two (now coincident) points, B, B′, must be situated somewhere on the indefinite right line OA.

140. In general, by the construction represented in the same Figure, the *sum* (comp. 6) of the *two numerators* (or *dividend-lines*, OB and OB′), of the *two conjugate fractions* (or quotients, or quaternions), q and $\mathrm{K}q$ (137), is equal to the *double* of the line OA′; whence (by 106), the *sum of those two conjugate quaternions* themselves is,

$$\mathrm{K}q + q = q + \mathrm{K}q = \frac{2\text{OA}'}{\text{OA}};$$

this sum is therefore *always scalar*, being *positive* if the *angle* $\angle q$ be *acute*, but *negative* if that angle be *obtuse*.

141. In the *intermediate case*, when the angle AOB is *right*, the *interval* OA′ between the origin O and the line BB′ *vanishes;* and the two lately mentioned *numerators*, OB, OB′, become two *opposite vectors*, of which the *sum* is *null* (5). Now, in general, it is natural, and will be found useful, or rather *necessary* (for consistency with *former* definitions), to admit that a *null vector, divided by an actual vector,* gives always a NULL QUATERNION as the *quotient;* and to *denote* this *null quotient* by the usual symbol for *Zero.* In fact, we have (by 106) the equation,

$$\frac{0}{a} = \frac{a - a}{a} = \frac{a}{a} - \frac{a}{a} = 1 - 1 = 0;$$

the zero in the numerator of the *left*-hand fraction representing here a *null line* (or a *null vector*, 1, 2); but the zero on the *right*-hand side of the equation denoting a *null quotient* (or quaternion). And thus we are entitled to infer that the *sum*,

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* Somewhat later it will be seen that the equation $\mathrm{K}q = q$ may also be written as $\mathrm{V}q = 0$; and that this last is another mode of expressing that *the quaternion*, q, *degenerates* (131) *into a scalar*.

$\mathrm{K}q + q$, or $q + \mathrm{K}q$, of a *right-angled quaternion*, or *right quotient* (132), and of its *conjugate*, is always equal to *zero.*

142. We have, therefore, the three following formulæ, whereof the *second* exhibits a *continuity* in the transition from the *first* to the *third:*

$$\text{I.} \ldots q + \mathrm{K}q > 0, \quad \text{if} \quad \angle q < \frac{\pi}{2};$$

$$\text{II.} \ldots q + \mathrm{K}q = 0, \quad \text{if} \quad \angle q = \frac{\pi}{2};$$

$$\text{III.} \ldots q + \mathrm{K}q < 0, \quad \text{if} \quad \angle q > \frac{\pi}{2}.$$

And because a quaternion, or *geometric quotient*, with an *actual* and *finite divisor-line* (as here OA), *cannot become equal to zero unless its dividend-line vanishes*, because (by 104) the equation

$$\frac{\beta}{\alpha} = 0 = \frac{0}{\alpha} \text{ requires the equation } \beta = 0,$$

if α be any actual and finite vector, we may infer, conversely, that *the sum* $q + \mathrm{K}q$ *cannot vanish*, without the line OA′ *also* vanishing; that is, without the lines OB, OB′ becoming *opposite vectors*, and therefore the quaternion q becoming a *right quotient* (132). We are therefore entitled to establish the three following *converse* formulæ (which indeed result from the three former):

$$\text{I}'. \ldots \text{if } q + \mathrm{K}q > 0, \quad \text{then} \quad \angle q < \frac{\pi}{2};$$

$$\text{II}'. \ldots \text{if } q + \mathrm{K}q = 0, \quad \text{then} \quad \angle q = \frac{\pi}{2};$$

$$\text{III}'. \ldots \text{if } q + \mathrm{K}q < 0, \quad \text{then} \quad \angle q > \frac{\pi}{2}.$$

143. When *two opposite vectors* (1), as β and $-\beta$, are both divided by *one* common (and actual) vector, α, we shall say that the *two quotients*, thus obtained are OPPOSITE QUATERNIONS; so that the *opposite* of any quaternion q, or of any quotient $\beta : \alpha$, may be denoted as follows (comp. 4):

$$\frac{-\beta}{\alpha} = \frac{0-\beta}{\alpha} = \frac{0}{\alpha} - \frac{\beta}{\alpha} = 0 - q = -q;$$

while the quaternion q *itself* may, on the same plan, be denoted (comp. 7) by the symbol $0+q$, or $+q$. The *sum* of any two opposite quaternions is *zero*, and their *quotient* is *negative unity;* so that we may write, as in algebra (comp. again 7),

$$(-q)+q=(+q)+(-q)=0\,; \quad (-q):q=-1\,; \quad -q=(-1)q\,;$$

because, by 106 and 141,

$$\frac{-\beta}{\alpha}+\frac{\beta}{\alpha}=\frac{\beta-\beta}{\alpha}=\frac{0}{\alpha}=0, \qquad \frac{-\beta}{\alpha}:\frac{\beta}{\alpha}=\frac{-\beta}{\beta}=-1, \text{ \&c.}$$

The *reciprocals* of *opposite* quaternions are themselves *opposite ;* or in symbols (comp. 126),

$$\frac{1}{-q}=-\frac{1}{q}, \text{ because } \frac{\alpha}{-\beta}=\frac{-\alpha}{\beta}=-\frac{\alpha}{\beta}.$$

Opposite quaternions have *opposite axes*, and *supplementary angles* (comp. Fig. 33, *bis*); so that we may establish (comp. 132, (5.)) the two following general formulæ,

$$\angle(-q)=\pi-\angle q\,; \qquad \text{Ax.}(-q)=-\text{Ax}.q.$$

144. We may also now write, in full consistency with the recent formulæ II. and II'. of 142, the equation,

$$\text{II}''\ldots \text{K}q=-q, \quad \text{if} \quad \angle q=\frac{\pi}{2};$$

and conversely* (comp. 138),

$$\text{II}'''\ldots \text{if K}q=-q, \quad \text{then} \quad \angle \text{K}q=\angle q=\frac{\pi}{2}.$$

In words, the *conjugate of a right quotient*, or of a *right-angled* (or *right*) *quaternion* (132), is the right quotient *opposite* thereto; and conversely, *if an actual quaternion* (that is, one which is *not null*) *be opposite to its own conjugate*, it must be a *right quotient.*

(1.) If then we meet the equation,

$$\text{K}\frac{\rho}{\alpha}=-\frac{\rho}{\alpha}, \quad \text{or} \quad \frac{\rho}{\alpha}+\text{K}\frac{\rho}{\alpha}=0,$$

we shall know that $\rho \perp \alpha$; and therefore (if $\alpha=\text{OA}$, and $\rho=\text{OP}$, as before), that the

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[*] It will be seen at a later stage, that the equation $\text{K}q=-q$, or $q+\text{K}q=0$, may be transformed to this other equation, $\text{S}q=0$; and that, under this last form, it expresses that the *scalar part* of the quaternion q *vanishes :* or that this quaternion is a *right quotient* (132).

locus of the *point* P is the *plane through* O, *perpendicular to the line* OA (as in 132, (1.)).

(2.) On the other hand, the equation,

$$\mathrm{K}\frac{\rho}{\alpha} = +\frac{\rho}{\alpha}, \quad \text{or} \quad \frac{\rho}{\alpha} - \mathrm{K}\frac{\rho}{\alpha} = 0,$$

expresses (by 139) that the quotient $\rho : \alpha$ is a *scalar;* and therefore (by 131) that its *angle* $\angle (\rho : \alpha)$ is either 0 or π; so that in *this* case, the *locus* of P is the *indefinite right line* through the two points O and A.

145. As the *opposite of the opposite*, or the *reciprocal of the reciprocal*, so also the *conjugate of the conjugate*, of any quaternion, is that quaternion *itself;* or in symbols,

$$-(-q) = +q; \qquad 1:(1:q) = q; \qquad \mathrm{KK}q = q = 1q;$$

so that, by *abstracting from the subject of the operation*, we may write briefly,

$$\mathrm{K}^2 = \mathrm{KK} = 1.$$

It is easy also to prove, that the *conjugates of opposite quaternions* are themselves *opposite* quaternions; and that the *conjugates of reciprocals* are *reciprocal:* or in symbols, that

$$\text{I.} \ldots \mathrm{K}(-q) = -\mathrm{K}q, \quad \text{or} \quad \mathrm{K}q + \mathrm{K}(-q) = 0;$$

and

$$\text{II.} \ldots \mathrm{K}\frac{1}{q} = 1 : \mathrm{K}q, \quad \text{or} \quad \mathrm{K}q \,.\, \mathrm{K}\frac{1}{q} = 1.$$

(1.) The equation $\mathrm{K}(-q) = -\mathrm{K}q$ is included (comp. 143) in this more general formula, $\mathrm{K}(xq) = x\mathrm{K}q$, where x is *any scalar;* and this last equation (comp. 126) may be proved, by simply conceiving that the two lines OB, OB′, in Fig. 36, are multiplied by any common scalar; or that they are both cut by any parallel to the line BB′.

(2.) To prove that conjugates of reciprocals are reciprocal, or that $\mathrm{K}q \,.\, \mathrm{K}\frac{1}{q} = 1$, we may conceive that, as in the annexed Figure 36, *bis*, while we have *still* the relation of *inverse* similitude,

$$\Delta\, \mathrm{AOB}' \propto' \mathrm{AOB} \ (118, 137),$$

as in the former Figure 36, a new point C is determined, either on the line OA itself, or on that line prolonged through A, so as to satisfy either of the two following connected conditions of *direct* similitude:

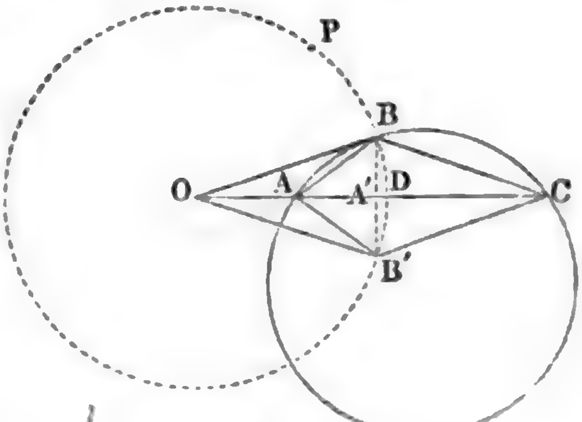

Fig. 36, *bis*.

$$\Delta\, \mathrm{BOC} \propto \mathrm{AOB}'; \qquad \Delta\, \mathrm{B'OC} \propto \mathrm{AOB};$$

or simply, as a relation between the *four points* O, A, B, C, the formula,

$$\Delta\, \mathrm{BOC} \propto' \mathrm{AOB}.$$

For then we shall have the transformations,

$$\mathrm{K}\frac{1}{q} = \mathrm{K}\frac{\mathrm{OA}}{\mathrm{OB}} = \mathrm{K}\frac{\mathrm{OB}'}{\mathrm{OC}} = \frac{\mathrm{OB}}{\mathrm{OC}} = \frac{\mathrm{OA}}{\mathrm{OB}'} = \frac{1}{\mathrm{K}q}.$$

(3.) The two quotients, OB : OA, and OB : OC, that is to say, the *quaternion* q *itself*, and the *conjugate of its reciprocal*, or* the *reciprocal of its conjugate*, have the *same angle*, and the *same axis;* we may therefore write, generally,

$$\angle\,\mathrm{K}\frac{1}{q} = \angle\,q; \qquad \mathrm{Ax}\,.\,\mathrm{K}\frac{1}{q} = \mathrm{Ax}\,.\,q.$$

(4.) Since OA : OB and OA : OB′ have thus been *proved* (by sub-art. 2) to be a pair of *conjugate quotients*, we can now infer this *theorem*, that *any two geometric fractions*, $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta'}$, which have a *common numerator* α, are *conjugate quaternions*, if the *denominator* β' of the *second* be the *reflexion* of the *denominator* β of the *first*, *with respect* to that common *numerator* (comp. 138, I.); whereas it had only been previously *assumed*, as a *definition* (137), that such *conjugation* exists, under the same geometrical condition, between the two *other* (or *inverse*) fractions, $\frac{\beta}{\alpha}$ and $\frac{\beta'}{\alpha}$; the three vectors α, β, β' being supposed to be all *co-initial* (18).

(5.) Conversely, if we meet, in any investigation, the formula

$$\mathrm{OA} : \mathrm{OB}' = \mathrm{K}\,(\mathrm{OA} : \mathrm{OB}),$$

we shall know that the *point* B′ is the *reflexion* of the *point* B, with respect to the *line* OA; or that this line, OA, prolonged if necessary in either of two opposite directions, *bisects at right angles the line* BB′, in some point A′, as in either of the two Figures 36 (comp. 138, II.).

(6.) Under the recent conditions of construction, it follows from the most elementary principles of geometry, that the *circle*, which passes *through the three points* A, B, C, is *touched at* B, *by the right line* OB; and that this line is, *in length*, a *mean proportional* between the lines OA, OC. Let then OD be such a *geometric mean*, and let it be set off from O in the common *direction* of the two last mentioned lines, so that the *point* D falls *between* A and C; also let the vectors OC, OD be denoted by the symbols, γ, δ; we shall then have expressions of the forms,

$$\delta = a\alpha, \qquad \gamma = a^2\alpha,$$

where a is some positive scalar, $a > 0$; and the vector β of B will be connected (comp. sub-art. 2) with this scalar a, and with the vector α, by the formula,

$$\frac{\mathrm{OB}}{\mathrm{OC}} = \mathrm{K}\frac{\mathrm{OA}}{\mathrm{OB}}, \quad \text{or} \quad \frac{\mathrm{OC}}{\mathrm{OB}} = \mathrm{K}\frac{\mathrm{OB}}{\mathrm{OA}}, \quad \text{or} \quad \frac{a^2\alpha}{\beta} = \mathrm{K}\frac{\beta}{\alpha}.$$

(7.) Conversely, if we still suppose that $\gamma = a^2\alpha$, this last formula expresses the *inverse similitude of triangles*, $\Delta\,\mathrm{BOC} \propto' \mathrm{AOB}$; and it expresses *nothing more:* or in other words, it is satisfied by the vector β of *every point* B, which gives that inverse similitude. But for this purpose it is only requisite that the *length* of OB should be (as above) a *geometric mean* between the lengths of OA, OC; or that the *two lines*, OB, OD (sub-art. 6), should be *equally long:* or finally, that B should be situated *somewhere on the surface of a sphere*, which is described so as to pass *through the point* D (in Fig. 36, *bis*), and to have the *origin* O for its *centre*.

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* It will be seen afterwards, that the *common value* of these two equal quaternions, $\mathrm{K}\frac{1}{q}$ and $\frac{1}{\mathrm{K}q}$, may be represented by either of the two new symbols, $\mathrm{U}q : \mathrm{T}q$, or $q : \mathrm{N}q$; or in words, that it is equal to the *versor divided by the tensor;* and also to the *quaternion* itself *divided by the norm.*

(8). If then we meet an equation of the form,

$$\frac{a^2\alpha}{\rho} = \mathrm{K}\frac{\rho}{\alpha}, \quad \text{or} \quad \frac{\rho}{\alpha}\,\mathrm{K}\,\frac{\rho}{\alpha} = a^2,$$

in which $\alpha = \mathrm{OA}$, $\rho = \mathrm{OP}$, and a is a scalar, as before, we shall know that the *locus of the point* P is a *spheric surface*, with its *centre* at the point O, and with the vector $a\alpha$ for a *radius;* and also that if we determine a point C by the equation $\mathrm{OC} = a^2\alpha$, this *spheric locus* of P is a *common orthogonal* to *all the circles* APC, which can be described, so as to pass *through the two fixed points*, A and C: because *every radius* OP of the *sphere* is a *tangent*, at the variable point P, *to the circle* APC, exactly as OB is to ABC in the recent Figure.

(9.) In the same Fig. 36, *bis*, the similar triangles show (by elementary principles) that the *length* of BC is to that of AB in the *sub-duplicate ratio* of OC to OA; or in the *simple ratio* of OD to OA; or as the scalar a to 1. If then we meet, in any research, the recent equation in ρ (sub-art. 8), we shall know that

$$\textit{length of } (\rho - a^2\alpha) = a \times \textit{length of } (\rho - \alpha);$$

while the recent interpretation of the same equation gives this *other* relation of the same kind:

$$\textit{length of } \rho = a \times \textit{length of } \alpha.$$

(10.) At a subsequent stage, it will be shown that the *Calculus of Quaternions* supplies *Rules of Transformation*, by which we can pass from any one to any other of these last equations respecting ρ, *without* (at the time) *constructing any Figure*, or (*immediately*) appealing to *Geometry:* but it was thought useful to point out, already, *how much geometrical meaning** is contained in *so simple a formula*, as that of the last sub-art. 8.

(11.) The *product of two conjugate quaternions* is said to be their *common* NORM,† and is *denoted* thus:

$$q\mathrm{K}q = \mathrm{N}q.$$

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* A student of ancient geometry may recognise, in the two equations of sub-art. 9, a sort of *translation*, into the *language of vectors*, of a celebrated *local theorem* of APOLLONIUS of Perga, which has been preserved through a citation made by his early commentator, Eutocius, and may be thus enunciated: Given any two points (as here A and C) in a plane, and any ratio of inequality (as here that of 1 to a), it is possible to construct a circle in the plane (as here the circle BDB′), such that the (lengths of the) two right lines (as here AB and CB, or AP and CP), which are inflected from the two given points to any common point (as B or P) of the circumference, shall be to each other in the given ratio. (*Δύο δοθέντων σημείων, κ. τ. λ.* Page 11 of Halley's Edition of Apollonius, Oxford, MDCCX.)

† This *name*, NORM, and the corresponding *characteristic*, N, are here adopted, as suggestions from the *Theory of Numbers;* but, in the present work, they will not

It follows that $\mathrm{NK}q = \mathrm{N}q$; and that the *norm of a quaternion* is generally a *positive scalar:* namely, the *square of the quotient of the lengths* of the two lines, of which (*as vectors*) the quaternion *itself* is the *quotient* (112). In fact we have, by sub-art. 6, and by the definition of a *norm*, the transformations:

$$\mathrm{N}\frac{\mathrm{OB}}{\mathrm{OA}} = \mathrm{N}\frac{\mathrm{OB}'}{\mathrm{OA}} = \frac{\mathrm{OC}}{\mathrm{OB}'}\cdot\frac{\mathrm{OB}'}{\mathrm{OA}} = \frac{\mathrm{OC}}{\mathrm{OB}}\cdot\frac{\mathrm{OB}}{\mathrm{OA}} = \frac{\mathrm{OC}}{\mathrm{OA}} = \left(\frac{\mathrm{OD}}{\mathrm{OA}}\right)^2;$$

$$\mathrm{N}q = \mathrm{N}\frac{\beta}{\alpha} = \frac{\beta}{\alpha}\,\mathrm{K}\frac{\beta}{\alpha} = \left(\frac{\textit{length of } \beta}{\textit{length of } \alpha}\right)^2.$$

As a *limit*, we may say that *the norm of a null quaternion is zero;* or in symbols, $\mathrm{N}0 = 0$.

(12.) With this notation, the *equation of the spheric locus* (sub-art. 8), which has the point O for its centre, and the vector $a\alpha$ for one of its radii, assumes the shorter form:

$$\mathrm{N}\frac{\rho}{\alpha} = a^2; \quad \text{or} \quad \mathrm{N}\frac{\rho}{a\alpha} = 1.$$

SECTION 7.—*On Radial Quotients; and on the Square of a Quaternion.*

146. It was early seen (comp. Art. 2, and Fig. 4) that *any two radii*, AB, AC, of any *one circle*, or *sphere*, are necessarily *unequal vectors;* because their *directions differ*. On the other hand, when we are attending *only* to *relative direction* (110), we may suppose that *all* the vectors compared are not merely *co-initial* (18), but are also *equally long;* so that if their *common length* be taken for the *unit*, they are *all radii*, OA, OB, . . of what we have called the *Unit-Sphere* (128), described round the *origin* as *centre;* and may *all* be said to be *Unit-Vectors* (129). And then the quaternion, which is the quotient of any one such vector divided by any other, or generally the *quotient of any two equally long vectors*, may be called a *Radial Quotient;* or sometimes simply a RADIAL. (Compare the annexed Figure 39.)

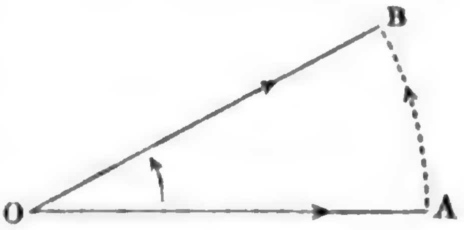

Fig. 39.

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be often *wanted*, although it may occasionally be convenient to employ them. For we shall soon introduce the conception, and the characteristic, of the *Tensor*, Tq, of a quaternion, which is of greater *geometrical* utility than the *Norm*, but of which it will be proved that this norm is simply the *square*,

$$q\mathrm{K}q = \mathrm{N}q = (\mathrm{T}q)^2.$$

Compare the Note to sub-art. 3.

147. The two *Unit-Scalars*, namely, *Positive and Negative Unity*, may be considered as *limiting cases* of *radial quotients*, corresponding to the *two extreme values*, 0 and π, of the *angle* AOB, or $\angle q$ (131). In the *intermediate case*, when AOB is a *right angle*, or $\angle q = \frac{\pi}{2}$, as in Fig. 40, the resulting quotient, or quaternion, may be called (comp. 132) a *Right Radial Quotient;* or simply, a RIGHT RADIAL. The consideration of such *right radials* will be found to be of great importance, in the whole theory and practice of Quaternions.

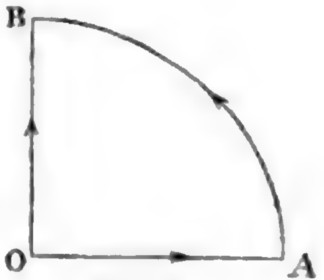

Fig. 40.

148. The most important *general property* of the quotients last mentioned is the following: that *the Square of every Right Radial is equal to Negative Unity;* it being understood that we write generally, as in algebra,

$$q \cdot q = qq = q^2,$$

and call this *product of two equal quaternions* the SQUARE of each of them. For if, as in Fig. 41, we describe a *semicircle* ABA′, with O for *centre*, and with OB for the *bisecting radius*, then the two right quotients, OB : OA, and OA′ : OB, are *equal* (comp. 117); and therefore their *common square* is (comp. 107) the *product*,

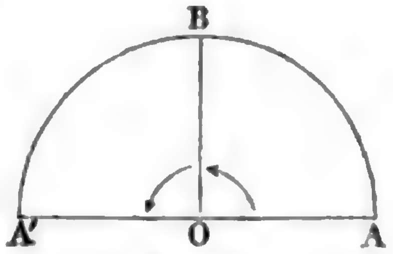

Fig. 41.

$$\left(\frac{\text{OB}}{\text{OA}}\right)^2 = \frac{\text{OA}'}{\text{OB}} \cdot \frac{\text{OB}}{\text{OA}} = \frac{\text{OA}'}{\text{OA}} = -1\,;$$

where OA and OB may represent *any two equally long, but mutually rectangular lines.* More generally, *the Square of every Right Quotient* (132) *is equal to a Negative Scalar;* namely, *to the negative of the square of the number*, which represents the *ratio of the lengths** of the two rectangular lines compared; or to *zero*

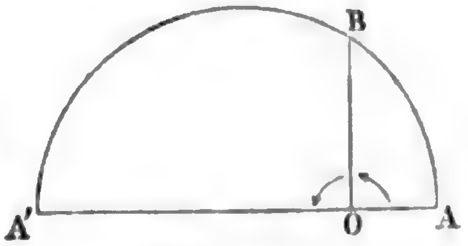

Fig. 41, *bis*.

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* Hence, by 145, (11.), $q^2 = -\text{N}q$, if $\angle q = \frac{\pi}{2}$.

minus the square of the number which denotes (comp. 133) the *length of the Index* of that Right Quotient: as appears from Fig. 41, *bis*, in which OB is only an *ordinate*, and *not* (as before) a *radius*, of the semicircle ABA′; for we have thus,

$$\left(\frac{\mathrm{OB}}{\mathrm{OA}}\right)^2 = \frac{\mathrm{OA}'}{\mathrm{OA}} = -\left(\frac{\textit{length of } \mathrm{OB}}{\textit{length of } \mathrm{OA}}\right)^2, \text{ if } \mathrm{OB} \perp \mathrm{OA}.$$

149. Thus *every Right Radial* is, in the present System, *one of the Square Roots of Negative Unity;* and may therefore be said to be *one of the Values of the Symbol* $\sqrt{-1}$; which celebrated *symbol* has thus a certain degree of *vagueness*, or at least of *indetermination*, of *meaning* in this theory, on account of which we shall not often employ it. For although it thus admits of a *perfectly clear* and *geometrically real Interpretation*, as denoting what has been above called a *Right Radial Quotient*, yet the *Plane of that Quotient is arbitrary;* and therefore the *symbol itself* must be considered to have (in the present system) *indefinitely many values;* or in other words the *Equation*,

$$q^2 = -1,$$

has (in the Calculus of Quaternions) *indefinitely many Roots*,* which are *all Geometrical Reals: besides* any *other roots*, of a *purely symbolical character*, which the same equation may be conceived to possess, and which may be called *Geometrical Imaginaries*.† Conversely, if q be any *real quaternion*, which

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* It will be subsequently shown, that if x, y, z be *any three scalars, of which the sum of the squares is unity*, so that

$$x^2 + y^2 + z^2 = 1;$$

and if i, j, k be *any three right radials, in three mutually rectangular planes;* then the *expression*,

$$q = ix + jy + kz,$$

denotes *another right radial*, which satisfies (*as such*, and by symbolical *laws* to be assigned) the *equation* $q^2 = -1$; and is therefore *one of the geometrically real values of the symbol* $\sqrt{-1}$.

† *Such imaginaries* will be found to offer themselves, in the treatment by *Quaternions* (or rather by what will be called *Biquaternions*), of *ideal intersections*, and of *ideal contacts*, in *geometry;* but we confine our attention, for the present, to *geometrical reals* alone. Compare the Notes to page 90.

satisfies the equation $q^2 = -1$, *it must be a right radial;* for if, as in Fig. 42, we suppose that $\Delta\,\mathrm{AOB} \propto \mathrm{BOC}$, we shall have

$$q^2 = \left(\frac{\mathrm{OB}}{\mathrm{OA}}\right)^2 = \frac{\mathrm{OC}}{\mathrm{OB}} \cdot \frac{\mathrm{OB}}{\mathrm{OA}} = \frac{\mathrm{OC}}{\mathrm{OA}};$$

and this *square* of q cannot become equal to *negative unity*, except by OC being $= -$ OA, or $=$ OA′ in Fig. 41; that is, by the line OB being *at right angles* to the line OA, and being at the same time *equally long*, as in Fig. 40.

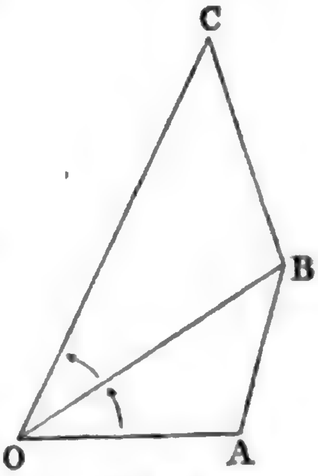

Fig. 42.

(1.) If then we meet the equation,

$$\left(\frac{\rho}{a}\right)^2 = -1,$$

where $a = \mathrm{OA}$, and $\rho = \mathrm{OP}$, as before, we shall know that the *locus* of the point P is the *circumference of a circle*, with O for its *centre*, and with a *radius* which has the *same length* as the line OA; while the *plane* of the circle is *perpendicular* to that given line. In other words, the locus of P is a *great circle*, on a *sphere* of which the centre is the origin; and the given point A, on the same spheric surface, is *one of the poles* of that circle.

(2.) In general, the equation $q^2 = -a^2$, where a is *any* (real) *scalar*, requires that the quaternion q (if real) should be *some right quotient* (132); the number a denoting the *length of the index* (133), of that right quotient or quaternion (comp. Art. 148, and Fig. 41, *bis*). But the *plane* of q is *still* entirely *arbitrary;* and therefore the equation

$$q^2 = -a^2,$$

like the equation $q^2 = -1$, which it includes, must be considered to have (in the present system) *indefinitely many geometrically real roots.*

(3.) Hence the equation,

$$\left(\frac{\rho}{a}\right)^2 = -a^2,$$

in which we may suppose that $a > 0$, expresses that the *locus* of the point P is a (new) *circular circumference*, with the line OA for its *axis*,* and with a *radius* of which the *length* $= a \times$ the length of OA.

150. It may be added that the *index* (133), and the *axis* (128), of a *right radial* (147), are the *same;* and that its *reciprocal* (134), its *conjugate* (137), and its *opposite* (143), are all *equal to each other.* Conversely, *if the reciprocal of a given quaternion q be equal to the opposite*

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* It being understood, that the *axis of a circle* is a right line perpendicular to the plane of that circle, and passing through its centre.

of that quaternion, then q *is a right radial;* because its *square,* q^2, is then equal (comp. 136) to the quaternion itself, *divided by its opposite;* and therefore (by 143) to *negative unity.* But the *conjugate of every radial quotient* is equal to the *reciprocal* of *that* quotient; because if, in Fig. 36, we conceive that the *three lines* OA, OB, OB′ are *equally long*, or if, in Fig. 39, we *prolong* the *arc* BA, by an *equal arc* AB′, we have the equation,

$$\mathrm{K}q = \frac{\mathrm{OB}'}{\mathrm{OA}} = \frac{\mathrm{OA}}{\mathrm{OB}} = \frac{1}{q}.$$

And conversely,[*]

$$\text{if } \mathrm{K}q = \frac{1}{q}, \quad \text{or if } q\mathrm{K}q = 1,$$

then the quaternion q is a radial quotient.

SECTION 8.—*On the Versor of a Quaternion, or of a Vector; and on some General Formulæ of Transformation.*

151. When a quaternion $q = \beta : \alpha$ is thus a *radial quotient* (146), or when the *lengths* of the two lines α and β are *equal*, the *effect* of this quaternion q, considered as a FACTOR (103), in the equation $q\alpha = \beta$, is simply the *turning* of the *multiplicand-line* α, *in the plane* of q (119), and *towards the hand* determined by the direction of the *positive axis* $\mathrm{Ax}\,.\,q$ (129), *through the angle* denoted by $\angle\, q$ (130); so as to bring that line α (or a revolving line which *had* coincided therewith) *into a new direction:* namely, into that of the *product-line* β. And with reference to this conceived *operation of turning*, we shall now say that *every Radial Quotient is a* VERSOR.

152. A *Versor* has thus, in general, a *plane*, an *axis*, and an *angle;* namely, those of the *Radial* (146) to which it *corresponds*, or is *equal:* the *only difference* between them being a difference in the *points of view*[†] from which they are respectively regarded; namely, the *radial* as the *quotient*, q, in the

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[*] Hence, in the *notation of norms* (145, (11.)), if $\mathrm{N}q = 1$, then q is a *radial;* and conversely, the *norm of a radial quotient* is always equal to *positive unity.*

[†] In a slightly *metaphysical* mode of expression it may be said, that the *radial quotient* is the *result* of an *analysis*, wherein *two radii* of one sphere (or circle) are *compared*, as regards their *relative direction;* and that the *equal versor* is the *instrument* of a corresponding *synthesis*, wherein *one radius* is conceived to be *generated*, by a certain *rotation*, from the *other*.

formula, $q = \beta : \alpha$; and the *versor* as the (equal) *factor*, q, in the *converse formula*, $\beta = q \, . \, \alpha$; where it is still supposed that the two vectors, α and β, are *equally long*.

153. A *versor*, like a *radial* (147), cannot *degenerate* into a *scalar*, except by its *angle* acquiring one or other of the two *limit-values*, 0 and π. In the first case, it becomes *positive unity*; and in the second case, it becomes *negative unity*: each of these two *unit-scalars* (147) being here regarded as a *factor* (or *coefficient*, comp. 12), which *operates on a line*, to *preserve* or to *reverse* its *direction*. In this view, we may say that -1 is an *Inversor*; and that *every Right Versor* (or versor with an angle $= \frac{\pi}{2}$) is a *Semi-inversor*:* because it *half-inverts* the *line* on which it *operates*, or *turns it through half of two right angles* (comp. Fig. 41). For the same reason, we are led to consider *every right versor* (like *every right radial*, 149, from which indeed we have just seen, in 152, that it differs only as *factor* differs from *quotient*), as being *one of the square-roots of negative unity*: or as *one of the values of the symbol* $\sqrt{-1}$.

154. In fact we may observe that the *effect* of a *right versor*, considered as *operating on a line* (in its own plane), is *to turn that line, towards a given hand, through a right angle*. If then q be *such* a *versor*, and if $q\alpha = \beta$, we shall have also (comp. Fig. 41), $q\beta = -\alpha$; so that, if α be any line in the plane of a right versor q, we have the equation,

$$q \, . \, q\alpha = -\alpha;$$

whence it is natural to write, under the same condition,

$$q^2 = -1,$$

as in 149. On the other hand, *no versor, which is not right-angled, can be a value of* $\sqrt{-1}$; or can satisfy the equation $q^2\alpha = -\alpha$, as Fig. 42 may serve to illustrate. For it is included in the meaning of this last equation, as applied to the theory of *versors*, that a *rotation through* $2 \angle q$, or through the *double of the angle* of q itself, is equivalent to an *inversion* of *direction;* and therefore to a rotation through *two right angles.*

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* This *word*, "semi-inversor," will not be often used; but the introduction of it here, in passing, seems adapted to throw light on the *view* taken, in the present work, of the *symbol* $\sqrt{} -1$, when regarded as denoting a certain important class (149) of *Reals in Geometry*. There *are* uses of that symbol, to denote *Geometrical Imaginaries* (comp. again Art. 149, and the Notes to page 90), considered as connected with *ideal intersections*, and with *ideal contacts*; but *with such uses* of $\sqrt{} -1$ we have, *at present*, *nothing to do*.

155. In general, if α be *any vector*, and if a be used as a temporary* symbol for the *number* expressing its *length;* so that a is here a *positive scalar*, which bears to *positive unity*, or to the scalar $+1$, the *same ratio* as that which the length of the line α bears to the assumed *unit of length* (comp. 128); then the *quotient* $\alpha : a$ denotes generally (comp. 16) a *new vector*, which has the *same direction* as the *proposed* vector α, but has its *length* equal to that assumed *unit:* so that it is (comp. 146) *the Unit-Vector in the direction of* α. We shall *denote this unit-vector* by the *symbol*, $\mathrm{U}\alpha$; and so shall write, generally,

$$\mathrm{U}\alpha = \frac{\alpha}{a}, \quad \text{if } a = \textit{length of } \alpha;$$

that is, more fully, if a be, as above supposed, the *number* (commensurable or incommensurable, but *positive*) which *represents* that length, with reference to some selected standard.

156. Suppose now that $q = \beta : \alpha$ is (as at first) a *general quaternion*, or the *quotient of any two vectors*, α and β, whether *equal* or *unequal* in *length*. *Such* a *Quaternion* will *not* (generally) be a *Versor* (or at least *not simply such*), according to the definition lately given; because its *effect*, when operating as a *factor* (103) on α, will *not* in general be *simply* to *turn* that line (151): but will (generally) *alter* the *length*,† as well as the *direction*. But if we *reduce* the two proposed vectors, α and β, *to the two unit-vectors* $\mathrm{U}\alpha$ and $\mathrm{U}\beta$ (155), and form the *quotient of these*, we shall *then* have taken account of *relative direction alone:* and the *result* will therefore be a *versor*, in the sense lately defined (151). We propose to call the quotient, or the versor, *thus* obtained, the *versor-element*, or briefly, *the* VERSOR, *of the Quaternion* q; and shall find it convenient to em-

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* We shall soon propose a general *notation* for representing the *lengths of vectors*, according to which the symbol $\mathrm{T}\alpha$ will denote what has been above called a; but are unwilling to introduce more than *one new characteristic of operation*, such as K, or T, or U, &c., *at one time.*

† By what we shall soon call call an *act of tension*, which will lead us to the consideration of the *tensor* of a quaternion.

ploy the *same** *Characteristic*, U, to denote the *operation* of *taking the versor of a quaternion*, as that employed above to denote the operation (155) of *reducing a vector to the unit of length*, without any change of its *direction*. On this plan, the *symbol* Uq will denote the *versor of* q; and the foregoing *definitions* will enable us to establish the *General Formula*:

$$\mathrm{U}q = \mathrm{U}\frac{\beta}{a} = \frac{\mathrm{U}\beta}{\mathrm{U}a};$$

in which the *two unit-vectors*, Ua and Uβ, may be called, by analogy, and for other reasons which will afterwards appear, *the versors*† *of the vectors*, a and β.

157. In thus passing from a given *quaternion*, q, to its *versor*, Uq, we have only changed (in general) the *lengths* of the two lines compared, namely, by *reducing* each to the assumed *unit* of length (155, 156), without making any change in their *directions*. Hence the *plane* (119), the *axis* (127, 128), and the *angle* (130), of the quaternion, remain *unaltered* in this passage; so that we may establish the two following general formulæ:

$$\angle\,\mathrm{U}q = \angle\,q; \qquad \mathrm{Ax}\,.\,\mathrm{U}q = \mathrm{Ax}\,.\,q.$$

More generally we may write,

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* For the moment, this double use of the characteristic U, to assist in denoting *both* the *unit-vector* Ua derived from a given *line* a, and *also* the *versor* Uq derived from a *quaternion* q, may be regarded as established *here* by arbitrary *definition*; but as *permitted*, because the *difference* of the *symbols*, as here a and q, which serve for the present to denote *vectors* and *quaternions*, considered as the *subjects* of these *two operations* U, will prevent such *double use* of that *characteristic* from giving rise to any *confusion*. But we shall further find that several important *analogies* are by anticipation *expressed*, or at least *suggested*, when the proposed *notation* is employed. Thus it will be found (comp. the Note to page 119), that *every vector* a may usefully be *equated to that right quotient*, of which it is (133) the *index*; and that *then* the *unit-vector* Ua may be, on the same plan, *equated to that right radial* (147), which is (in the sense lately defined) *the versor of that right quotient*. We shall also find ourselves led to regard *every unit-vector* as the *axis of a quadrantal* (or *right*) *rotation*, in a plane perpendicular to that axis; which will supply another inducement, to *speak* of every *such* vector as a *versor*. On the whole, it appears that there will be no inconvenience, but rather a prospective advantage, in our already *reading* the symbol Ua as "*versor of* a;" just as we may *read* the analogous symbol Uq, as "*versor of* q."

† Compare the Note immediately preceding.

$$\angle q' = \angle q, \text{ and } \mathrm{Ax}\,.\,q' = \mathrm{Ax}\,.\,q, \text{ if } \mathrm{U}q' = \mathrm{U}q;$$

the *versor of a quaternion depending solely on*, but conversely being sufficient to *determine*, the *relative direction* (156) of the *two lines*, of which (as *vectors*) the quaternion itself is the *quotient* (112); or the axis and angle of the *rotation*, in the plane of those two lines, from the divisor to the dividend (128): so that any two quaternions, which have *equal versors*, must also have *equal angles*, and *equal* (or coincident) *axes*, as is expressed by the last written formula. Conversely, from this dependence of the *versor* Uq on *relative direction** *alone*, it follows that any two quaternions, of which the angles and the axes are equal, have also *equal versors;* or in symbols, that

$$\mathrm{U}q' = \mathrm{U}q, \quad \text{if} \quad \angle q' = \angle q, \quad \text{and} \quad \mathrm{Ax}.q' = \mathrm{Ax}.q.$$

For example, we saw (in 138) that the *conjugate* and the *reciprocal* of any quaternion have thus their *angles* and their *axes* the *same;* it follows, therefore, that the *versor of the conjugate* is always *equal* to the *versor of the reciprocal;* so that we are permitted to establish the following general formula,†

$$\mathrm{UK}q = \mathrm{U}\frac{1}{q}.$$

158. Again, because

$$\mathrm{U}\left(1 : \frac{\beta}{\alpha}\right) = \mathrm{U}\frac{\alpha}{\beta} = \frac{\mathrm{U}\alpha}{\mathrm{U}\beta} = 1 : \frac{\mathrm{U}\beta}{\mathrm{U}\alpha} = 1 : \mathrm{U}\frac{\beta}{\alpha},$$

it follows that the *versor of the reciprocal* of any quaternion is, at the same time, the *reciprocal of the versor;* so that we may write,

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* The *unit-vector* Uα, which we have recently proposed (156) to call *the versor of the vector* α, depends in like manner *on the direction of that vector alone;* which *exclusive reference*, in *each* of these two cases, to DIRECTION, may serve as an additional *motive* for employing, as we have lately done, one *common name*, VERSOR, and one *common characteristic*, U, to assist in describing or denoting *both* the *Unit-Vector* Uα *itself, and* the *Quotient of two such Unit-Vectors*, $\mathrm{U}q = \mathrm{U}\beta : \mathrm{U}\alpha$; all danger of *confusion* being sufficiently guarded against (comp. the Note to Art. 156), by the *difference of the two symbols*, α and q, employed to denote the *vector* and the *quaternion*, which are respectively the *subjects of the two operations* U; while those two *operations agree* in this *essential* point, that *each* serves to *eliminate the quantitative element*, of absolute or relative *length*.

† Compare the Note to Art. 138.

$$\mathrm{U}\frac{1}{q} = \frac{1}{\mathrm{U}q}; \quad \text{or} \quad \mathrm{U}q.\mathrm{U}\frac{1}{q} = 1.$$

Hence, by the recent result (157), we have also, generally,

$$\mathrm{UK}q = \frac{1}{\mathrm{U}q}; \quad \text{or,} \quad \mathrm{U}q.\mathrm{UK}q = 1.$$

Also, because the *versor* Uq is always a *radial* quotient (151, 152), it is (by 150) the *conjugate of its own reciprocal;* and therefore at the same time (comp. 145), the *reciprocal of its own conjugate;* so that the *product of two conjugate versors*, or what we have called (145, (11.)) their *common* NORM, is always equal to *positive unity;* or in symbols (comp. 150),

$$\mathrm{NU}q = \mathrm{U}q.\mathrm{KU}q = 1.$$

For the same reason, the *conjugate of the versor* of any quaternion is equal to the *reciprocal of that versor*, or (by what has just been seen) to the *versor of the reciprocal* of that quaternion; and therefore also (by 157), to the *versor of the conjugate;* so that we may write generally, as a summary of recent results, the formula:

$$\mathrm{KU}q = \frac{1}{\mathrm{U}q} = \mathrm{U}\frac{1}{q} = \mathrm{UK}q;$$

each of these four symbols denoting a *new versor*, which has the *same plane*, and the *same angle*, as the *old* or *given versor* Uq, but has an *opposite axis*, or an *opposite direction of rotation*: so that, with respect to that given *Versor*, it may naturally be called a REVERSOR.

159. As regards the *versor itself*, whether of a vector or of a quaternion, the definition (155) of Uα gives,

$$\mathrm{U}x\alpha = +\mathrm{U}\alpha, \quad \text{or} \quad = -\mathrm{U}\alpha, \quad \text{according as} \quad x > \text{or} < 0;$$

because (by 15) the *scalar coefficient* x *preserves*, in the first case, but *reverses*, in the second case, the *direction* of the vector α; whence also, by the definition (156) of Uq, we have generally (comp. 126, 143),

$$\mathrm{U}xq = +\mathrm{U}q, \quad \text{or} \quad = -\mathrm{U}q, \quad \text{according as} \quad x > \text{or} < 0.$$

The *versor of a scalar*, regarded as the *limit of a quaternion* (131, 139), is equal to *positive* or *negative unity* (comp. 147, 153), according as the scalar itself is positive or negative; or in symbols,

$$\mathrm{U}x = +1, \text{ or } = -1, \quad \text{according as} \quad x > \text{ or } < 0;$$

the *plane* and *axis* of each of these two *unit scalars* (147), considered as *versors* (153), being (as we have already seen) *indeterminate*. The *versor of a null quaternion* (141) must be regarded as *wholly arbitrary*, unless we happen to know a *law*,* according to which the quaternion *tends to zero*, before actually *reaching* that limit; in which latter case, the *plane*, the *axis*, and the *angle* of the *versor*† U0 may *all* become *determined*, as *limits* deduced from that law. The *versor of a right quotient* (132), or of a *right-angled quaternion* (141), is always a *right radial* (147), or a *right versor* (153); and therefore is, as such, *one of the square roots of negative unity* (149), or *one of the values of the symbol* $\sqrt{-1}$; while (by 150) the *axis* and the *index* of *such* a versor *coincide*; and in like manner its *reciprocal*, its *conjugate*, and its *opposite* are all *equal* to each other.

160. It is evident that if a proposed *quaternion* q be *already* a *versor* (151), in the sense of being a *radial* (146), the operation of *taking its versor* (156) produces *no change*; and in like manner that, if a given *vector* α be *already* an *unit-vector*, it remains the *same* vector, when it is *divided* (155) *by its own length*; that is, in this case, by the *number one*. For example, we have assumed (128, 129), that the *axis* of *every* quaternion is an *unit-vector*; we may therefore write, generally, in the notation of 155, the equation,

$$\mathrm{U}(\mathrm{Ax} \,.\, q) = \mathrm{Ax} \,.\, q.$$

A *second operation* U leaves thus the *result* of the *first* operation U *unchanged*, whether the *subject* of such successive operations be a *line*, or a *quaternion*; we have therefore the two following general formulæ, differing only in the symbols of that subject:

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* Compare the Note to Art. 131.

† When the *zero* in this *symbol*, U0, is considered as denoting a *null vector* (2), the symbol itself denotes *generally*, by the foregoing principles, an *indeterminate unit-vector*; although the *direction* of this unit-vector *may*, in certain questions, *become* determined, as a *limit* resulting from a *law*.

$$\mathrm{UU}\alpha = \mathrm{U}\alpha; \qquad \mathrm{UU}q = \mathrm{U}q;$$

whence, by *abstracting* (comp. 145) *from the subject* of the operation, we may write, briefly and symbolically,

$$\mathrm{U}^2 = \mathrm{UU} = \mathrm{U}.$$

161. Hence, with the help of 145, 158, 159, we easily deduce the following (among other) *transformations of the versor of a quaternion:*

$$\mathrm{U}q = \frac{1}{\mathrm{U}\frac{1}{q}} = \frac{1}{\mathrm{KU}q} = \frac{1}{\mathrm{UK}q} = \mathrm{KU}\frac{1}{q} = \mathrm{K}\frac{1}{\mathrm{U}q} = \mathrm{KUK}q$$

$$= \mathrm{U}\frac{1}{\mathrm{K}q} = \mathrm{UK}\frac{1}{q} = \mathrm{U}^2q = \mathrm{UKU}\frac{1}{q} = \mathrm{UK}\frac{1}{\mathrm{U}q} = (\mathrm{UK})^2q;$$

$$\mathrm{U}q = \mathrm{U}xq, \text{ if } x > 0; \qquad = -\mathrm{U}xq, \text{ if } x < 0.$$

We may also write, generally,

$$\frac{q}{\mathrm{K}q} = \frac{\mathrm{U}q}{\mathrm{KU}q} = (\mathrm{U}q)^2 = \mathrm{U}(q^2) = \mathrm{U}q^2;$$

the *parentheses* being here unnecessary, because (as will soon be more fully seen) the *symbol* $\mathrm{U}q^2$ denotes *one common versor*, whether we interpret it as denoting the *square of the versor*, or as the *versor of the square*, of q. The present Calculus will be found to abound in *General Transformations* of this sort; which all (or nearly all), like the foregoing, depend ultimately on very *simple geometrical conceptions;* but which, notwithstanding (or rather, perhaps, on account of) this extreme *simplicity* of their *origin*, are often *useful*, as *elements* of a new kind of *Symbolical Language* in *Geometry:* and generally, as *instruments of expression*, in all those mathematical or physical researches to which the *Calculus of Quaternions* can be applied. It is, however, by no means necessary that a student of the subject, at the present stage, should make himself *familiar* with *all* the recent transformations of $\mathrm{U}q$; although it may be well that he should satisfy himself of their correctness, in doing which the following remarks will perhaps be found to assist.

(1.) To give a *geometrical illustration*, which may also serve as a *proof*, of the recent equation,

$$q : \mathrm{K}q = (\mathrm{U}q)^2,$$

we may employ Fig. 36, *bis*; in which, by 145, (2.), we have

$$q \cdot \frac{1}{\mathrm{K}q} = \frac{\mathrm{OB}}{\mathrm{OA}} \cdot \frac{\mathrm{OA}}{\mathrm{OB}'} = \frac{\mathrm{OB}}{\mathrm{OB}'} = \left(\frac{\mathrm{OB}}{\mathrm{OD}}\right)^2 = \left(\mathrm{U}\frac{\mathrm{OB}}{\mathrm{OA}}\right)^2 = (\mathrm{U}q)^2.$$

(2.) As regards the equation, $\mathrm{U}(q^2) = (\mathrm{U}q)^2$, we have only to conceive that the three lines OA, OB, OC, of Fig. 42, are cut (as in Fig. 42, *bis*) in three new points, A′, B′, C′, by an *unit-circle* (or by a circle with a radius equal to the unit of length), which is described about their common origin O as centre, and in their common plane; for then if these three lines be called α, β, γ, the three new lines OA′, OB′, OC′ are (by 155) the three unit-vectors denoted by the symbols, $\mathrm{U}\alpha$, $\mathrm{U}\beta$, $\mathrm{U}\gamma$; and we have the transformations (comp. 148, 149),

$$\mathrm{U}(q^2) = \mathrm{U}.\left(\frac{\beta}{\alpha}\right)^2 = \mathrm{U}\frac{\gamma}{\alpha} = \frac{\mathrm{U}\gamma}{\mathrm{U}\alpha} = \frac{\mathrm{OC}'}{\mathrm{OA}'} = \left(\frac{\mathrm{OB}'}{\mathrm{OA}'}\right)^2 = (\mathrm{U}q)^2.$$

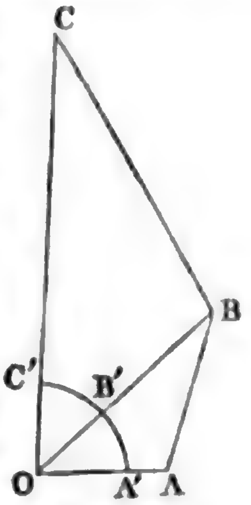

Fig. 42, *bis*.

(3.) As regards *other* recent transformations (161), although we have seen (135) that it is *not necessary* to invent any *new* or peculiar *symbol*, to represent the *reciprocal* of a quaternion, yet if, for the sake of present convenience, and as a merely *temporary notation*, we write

$$\mathrm{R}q = \frac{1}{q},$$

employing thus, for a moment, the letter R as a *characteristic of reciprocation*, or of the operation of *taking the reciprocal*, we shall then have the *symbolical equations* (comp. 145, 158):

$$\mathrm{R}^2 = \mathrm{K}^2 = 1; \qquad \mathrm{RK} = \mathrm{KR}; \qquad \mathrm{RU} = \mathrm{UR} = \mathrm{KU} = \mathrm{UK};$$

but we have also (by 160), $\mathrm{U}^2 = \mathrm{U}$; whence it easily follows that

$$\mathrm{U} = \mathrm{RUR} = \mathrm{RKU} = \mathrm{RUK} = \mathrm{KUR} = \mathrm{KRU} = \mathrm{KUK}$$
$$= \mathrm{URK} = \mathrm{UKR} = \mathrm{UKUR} = \mathrm{UKRU} = (\mathrm{UK})^2 = \&\mathrm{c}.$$

(4.) The equation

$$\mathrm{U}\frac{\rho}{\alpha} = \mathrm{U}\frac{\beta}{\alpha}, \quad \text{or simply,} \quad \mathrm{U}\rho = \mathrm{U}\beta,$$

expresses that the *locus* of the point P is the *indefinite right line*, or *ray* (comp. 132, (4.)), which is drawn *from* O *in the direction of* OB,* but *not* in the *opposite* direction; because it is equivalent to

$$\mathrm{U}\frac{\rho}{\beta} = 1; \quad \text{or} \quad \angle\frac{\rho}{\beta} = 0; \quad \text{or} \quad \rho = x\beta, \ x > 0.$$

(5.) On the other hand the equation,

$$\mathrm{U}\frac{\rho}{\alpha} = -\mathrm{U}\frac{\beta}{\alpha}, \quad \text{or} \quad \mathrm{U}\rho = -\mathrm{U}\beta,$$

expresses (comp. 132, (5.)) that the locus of P is the *opposite ray* from O; or that it is the *indefinite prolongation of the revector* BO; because it may be transformed to

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* In 132, (4.), p. 119, OA and A ought to have been OB and B.

$$\mathrm{U}\frac{\rho}{\beta} = -1; \quad \text{or} \quad \angle\frac{\rho}{\beta} = \pi; \quad \text{or} \quad \rho = x\beta, \ x < 0.$$

(6.) If α, β, γ denote (as in sub-art. 2) the three lines OA, OB, OC of Fig. 42 (or of Fig. 42, *bis*), so that (by 149) we have the equation $\frac{\gamma}{\alpha} = \left(\frac{\beta}{\alpha}\right)^2$, then this other equation,

$$\left(\mathrm{U}\frac{\rho}{\alpha}\right)^2 = \mathrm{U}\frac{\gamma}{\alpha},$$

expresses *generally* that the *locus* of P is the *system* of the *two* last loci; or that it is the *whole indefinite right line*, both ways prolonged, through the two points O and B (comp. 144, (2.)).

(7.) But if it happen that the line γ, or OC, like OA′ in Fig. 41 (or in Fig. 41, *bis*), has the direction *opposite* to that of α, or of OA, so that the last equation takes the *particular form*,

$$\left(\mathrm{U}\frac{\rho}{\alpha}\right)^2 = -1,$$

then $\mathrm{U}\frac{\rho}{\alpha}$ must be (by 154) a *right versor;* and reciprocally, *every right versor*, with a plane containing α, will be (by 153) a value satisfying the equation. In *this* case, therefore, the *locus of the point* P is (as in 132, (1.), or in 144, (1.)) the *plane through* O, *perpendicular to the line* OA; and the recent *equation* itself, if supposed to be satified by a *real** *vector* ρ, may be put under either of these two earlier but equivalent *forms:*

$$\angle\frac{\rho}{\alpha} = \frac{\pi}{2}; \qquad \rho \perp \alpha.$$

SECTION 9.—*On Vector-Arcs, and Vector-Angles, considered as Representatives of Versors of Quaternions; and on the Multiplication and Division of any one such Versor by another.*

162. Since every *unit-vector* OA (129), drawn from the origin O, terminates in some point A on the surface of what we have called the *unit-sphere* (128), that *term* A (1) may be considered as a *Representative Point*, of which the *position* on that *surface* determines, and may be said to *represent*, the *direction of the line* OA *in space;* or of that line *multiplied* (12, 17) by *any positive scalar*. And then the *Quaternion* which is the *quotient* (112) of any *two* such unit-vectors, and which is in one view a *Radial* (146), and in another view a *Versor* (151), may be said to have the *arc of a great circle*, AB, upon the unit sphere, which *connects the terms* of the two

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* Compare 149, (2.); also the second Note to the same Article; and the Notes to page 90.

vectors, for its *Representative Arc.* We may also call this arc a VECTOR ARC, on account of its having a *definite direction* (comp. Art. 1), such as is indicated (for example) by a *curved arrow* in Fig. 39; and as being thus *contrasted* with its own *opposite*, or with what may be called by analogy the *Revector Arc* BA (comp. again 1): this *latter* arc representing, on the present plan, at once the *reciprocal* (134), and the *conjugate* (137), of the former *versor;* because it represents the corresponding *Reversor* (158).

163. This mode of *representation*, of versors of quaternions by vector arcs, would obviously be very imperfect, unless *equals* were to be represented by *equals.* We shall therefore define, as it is otherwise natural to do, that a vector arc, AB, upon the unit sphere, is *equal* to every *other* vector arc CD which can be derived from it, by simply causing (or conceiving) it to *slide** *in its own great circle, without any change of length*, or *reversal of direction.* In fact, the two isosceles and plane triangles AOB, COD, which have the origin O for their common vector, and rest upon the chords of these two arcs as bases, are thus complanar, similar, and similarly turned; so that (by 117, 118) we may here write,

$$\Delta\ \mathrm{AOB} \propto \mathrm{COD}, \qquad \frac{\mathrm{OB}}{\mathrm{OA}} = \frac{\mathrm{OD}}{\mathrm{OC}};$$

the condition of the *equality of the quotients* (that is, here, of the *versors*), *represented* by the two *arcs*, being thus satisfied. We shall sometimes *denote* this sort of *equality of two vector arcs*, AB and CD, by the formula,

$$\frown \mathrm{AB} = \frown \mathrm{CD};$$

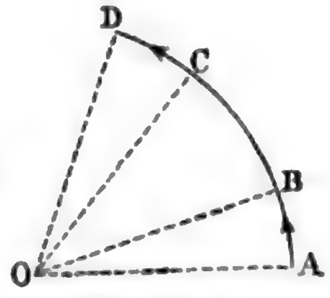

Fig. 35, *bis.*

and then it is clear (comp. 125, and the earlier Art. 3) that we shall also have, by what may be called *inversion* and *alternation*, these two *other* formulæ of *arcual equality*,

$$\frown \mathrm{BA} = \frown \mathrm{DC}; \qquad \frown \mathrm{AC} = \frown \mathrm{BD}.$$

(Compare the annexed Figure 35, *bis.*)

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* Some aid to the conception may here be derived from the inspection of Fig 34; in which *two equal angles* are supposed to be traced on the surface of *one com-*

164. Conversely, *unequal versors* ought to be *represented* (on the present plan) by *unequal vector arcs;* and accordingly, we purpose to regard any *two such arcs*, as being, for the present purpose, *unequal* (comp. 2), even when they *agree* in *quantity*, or contain the *same number of degrees*, provided that they *differ in direction:* which may happen in either of two principal ways, as follows. For, Ist, they may be *opposite arcs* of *one great circle;* as, for example, a *vector* arc AB, and the corresponding *revector* arc BA; and so may *represent* (162) a *versor*, OB : OA, and the corresponding *reversor*, OA : OB, respectively. Or, IInd, the *two arcs* may belong to *different great circles*, like AB and BC in Fig. 43; in which latter case, they represent two *radial quotients* (146) *in different planes;* or (comp. 119) *two diplanar versors*, OB : OA, and OC : OB; but it has been shown generally (122), that *diplanar quaternions* are *always unequal:* we consider therefore, here again the *arcs*, AB and BC, *themselves*, to be (as has been said) *unequal vectors*.

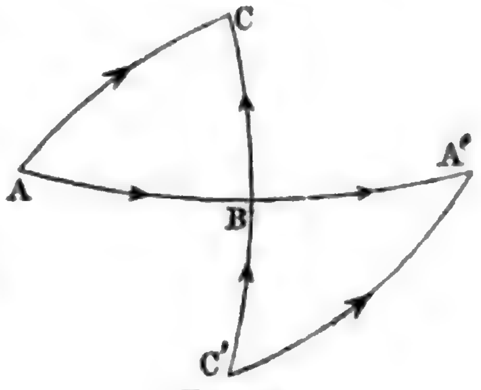

Fig. 43.

165. In this manner, then, we may be led (comp. 122) to regard the *conception of a plane*, or of the *position of a great circle* on the unit sphere, as entering, *essentially*, in general,* into the *conception of a vector-arc*, considered as the *representative of a versor* (162). But even without expressly referring to *versors*, we may see that if, in Fig. 43, we suppose that B is the middle point of an arc AA' of a great circle, so that in a recent notation (163) we may establish the *arcual equation*,

$$\frown \mathrm{AB} = \frown \mathrm{BA}',$$

we ought then (comp. 105) not to write *also*,

$$\frown \mathrm{AB} = \frown \mathrm{BC};$$

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mon desk. Or the four lines OA, OB, OC, OD, of Fig. 35, may now be conceived to be *equally long;* or to be cut by a circle with O for centre, as in the modification of that Figure, which is given in Article 163, a little lower down.

* We say, *in general;* for it will soon be seen that there is a sense in which *all great semicircles*, considered as vector arcs, may be said to be *equal* to each other.

because the *two co-initial arcs*, BA′ and BC, which *terminate differently*, must be considered (comp. 2) to be, *as vector-arcs, unequal*. On the other hand, if we should refuse to admit (as in 163) that any *two complanar arcs*, if *equally long*, and *similarly* (not *oppositely*) *directed*, like AB and CD in the recent Fig. 35, *bis*, are *equal vectors*, we could not usefully speak of *equality* between *vector-arcs* as existing under any circumstances. We are then thus led again to *include*, generally, the conception of a *plane*, or of one *great circle* as *distinguished* from another, as an *element* in the *conception* of a *Vector-Arc*. And hence an *equation* between *two* such arcs must in general be conceived to include *two relations of co-arcuality*. For example, the equation $\frown \mathrm{AB} = \frown \mathrm{CD}$, of Art. 163, includes generally, as a *part* of its signification, the assertion (comp. 123) that the *four points* A, B, C, D belong to one *common great circle* of the unit-sphere; or that *each* of the *two* points, C and D, is *co-arcual* with the *two other* points, A and B.

166. There is, however, a remarkable case of *exception*, in which two vector arcs may be said to be *equal*, although situated in *different planes:* namely, when they are both *great semicircles*. In fact, upon the present plan, *every* great semicircle, AA′, considered as a vector arc, *represents* an *inversor* (153); or it represents *negative unity* $(\mathrm{OA}' : \mathrm{OA} = -a : a = -1)$, considered as one *limit* of a *versor;* but we have seen (159) that *such* a versor has in general an *indeterminate plane*. Accordingly, whereas the *initial and final points*, or (comp. 1) the *origin* A and the *term* B, of a vector arc AB, are *in general* sufficient to determine the plane of that arc, considered as the *shortest* or the *most direct path* (comp. 112, 128) from the one point to the other on the sphere; in the particular *case* when one of the two given points is diametrically *opposite* to the other, as A′ to A, the *direction* of this *path* becomes, on the contrary, *indeterminate*. If then we only attend to the *effect produced*, in the way of *change of position of a point*, by a conceived *vection* (or *motion*) *upon the sphere*, we are permitted to say that *all great semicircles are equal vector arcs;* each serving simply, in the present view, to *transport a point* from *one* position to the *opposite;* and thereby to *reverse* (like the factor −1, of which it is here the *representative*) the *direction* of the *radius* which is drawn to that point of the unit sphere.

(1.) The equation,

$$\frown \text{AA}' = \frown \text{BB}',$$

in which it is here supposed that A′ is opposite to A, and B′ to B, satisfies evidently the general *conditions of co-arcuality* (165); because the *four* points ABA′B′ are all on *one* great circle. It is evident that the same arcual equation admits (as in 163) of *inversion* and *alternation*; so that

$$\frown \text{A}'\text{A} = \frown \text{B}'\text{B}, \quad \text{and} \quad \frown \text{AB} = \frown \text{A}'\text{B}'.$$

(2.) We may also say (comp. 2) that *all null arcs are equal*, as producing *no effect* on the position of a point upon the sphere; and thus may write generally,

$$\frown \text{AA} = \frown \text{BB} = 0,$$

with the *alternate* equation, or identity, $\frown \text{AB} = \frown \text{AB}$.

(3.) Every such *null vector arc* AA is a *representative*, on the present plan, of the *other unit scalar*, namely *positive unity*, considered as another *limit* of a *versor* (153); and its *plane* is again *indeterminate* (159), unless some *law* be given, according to which the arcual *vection* may be conceived to *begin*, from a given point A, to an indefinitely *near* point B upon the sphere.

167. The principal *use* of *Vector Arcs*, in the present theory, is to assist in *representing*, and (so to speak) in *constructing*, by means of a *Spherical Triangle*, the *Multiplication* and *Division* of any two *Diplanar Versors* (comp. 119, 164). In fact, any two such versors of quaternions (156), considered as radial quotients (152), can easily be reduced (by the general process of Art. 120) to the forms,

$$q = \beta : \alpha = \text{OB} : \text{OA}, \qquad q' = \gamma : \beta = \text{OC} : \text{OB},$$

where A, B, C are corners of such a triangle on the unit sphere; and then (by 107), the former quotient multiplied by the latter will give for product:

$$q' . q = \gamma : \alpha = \text{OC} : \text{OA}.$$

If then (on the plan of Art. 1) any *two successive arcs*, as AB and BC in Fig. 43, be called (in relation to each other) *vector* and *provector*; while that *third arc* AC, which is drawn from the initial point of the first to the final point of the second, shall be called (on the same plan) the *transvector*: we may now say that in the multiplication of any one versor (of a quaternion) by any other, if the *multiplicand** q be *represented* (162) by a *vector-arc* AB, and if the *multiplier* q' be in like manner represented by a *provector-arc* BC, which mode of representation is always possible, by what has been already shown, then the *product* $q'.q$, or $q'q$, is *represented*, at the same time, by the *transvector-arc* AC corresponding.

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* Here, as in 107, and elsewhere, we write the symbol of the *multiplier* towards the *left-hand*, and that of the *multiplicand* towards the *right*.

168. One of the most remarkable consequences of this *construction of the multiplication of versors* is the following: that *the value of the product of two diplanar versors* (164) *depends upon the order of the factors*; or that $q'q$ and qq' are *unequal*, unless q' be *complanar* (119) with q. For let AA′ and CC′ be any two arcs of great circles, *in different planes*, bisecting each other in the point B, as Fig. 43 is designed to suggest; so that we have the *two arcual equations* (163),

$$\frown AB = \frown BA', \quad \text{and} \quad \frown BC = \frown C'B;$$

then one or other of the two following alternatives will hold good. Either, Ist, the two mutually *bisecting arcs* will *both* be *semicircles*, in which case the *two new arcs*, AC and C′A′, will indeed both belong to *one great circle*, namely to that of which B is a *pole*, but will have *opposite directions* therein; because, in this case, A′ and C′ will be diametrically *opposite* to A and C, and therefore (by 166, (1.)) the equation

$$\frown AC = \frown A'C',$$

but *not* the equation

$$\frown AC = \frown C'A',$$

will be satisfied. Or, IInd, the arcs AA′ and CC′, which are supposed to bisect each other in B, will *not both* be semicircles, even if *one* of them happen to be such; and in this case, the arcs AC, C′A′ will belong to *two distinct great circles*, so that they will be *diplanar*, and therefore *unequal*, when considered as *vectors*. (Compare the Ist and IInd cases of Art. 164.) In *each* case, therefore, AC and C′A′ are *unequal vector arcs;* but the *former* has been seen (167) to *represent* the product $q'q$; and the *latter* represents, in like manner, the *other* product, qq', of the same two versors taken in the *opposite order*, because it is the *new transvector* arc, when C′B (= BC) is treated as the *new vector* arc, and BA′ (= AB) as the *new provector* arc, as is indicated by the curved arrows in Fig. 43. The *two products*,

$q'q$ and qq', are therefore *themselves unequal*, as above asserted, under the supposed condition of *diplanarity*.

169. On the other hand, when the two factors, q and q', are *complanar versors*, it is easy to prove, in several different ways, that their products, $q'q$ and qq', are *equal*, as in algebra. Thus we may conceive that the arc CC′, in Fig. 43, is made to *turn* round its middle point B, until the spherical angle CBA′ *vanishes;* and then the two *new transvector-arcs*, AC and C′A′, will evidently become not only *complanar* but *equal*, in the sense of Art. 163, as being *still equally long*, and being *now similarly directed*. Or, in Fig. 35, *bis*, of the last cited Article, we may conceive a point E, bisecting the arc BC, and therefore also the arc AD, which is *commedial* therewith (comp. Art. 2, and the second Figure 3 of that Article); and then, if we represent the one versor q by either of the two equal arcs, AE, ED, we may at the same time represent the other versor q' by either of the two other equal arcs, EC, BE; so that the one product, $q'q$, will be represented by the arc AC, and the other product, qq', by the equal arc BD. Or, without reference to *vector arcs*, we may suppose that the two *factors* are,

$$q = \beta : \alpha = \text{OB} : \text{OA}, \qquad q' = \gamma : \alpha = \text{OC} : \text{OA},$$

OA, OB, OC being any three complanar and equally long right lines (see again Fig. 35, *bis*); for thus we have only to determine a fourth line, δ or OD, of the same length, and in the same plane, which shall satisfy the equation $\delta : \gamma = \beta : \alpha$ (117), and therefore also (by 125) the alternate equation, $\delta : \beta = \gamma : \alpha$; and it will then immediately follow* (by 107), that

$$q' \cdot q = \frac{\delta}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\delta}{\alpha} = \frac{\delta}{\gamma} \cdot \frac{\gamma}{\alpha} = q \cdot q'.$$

We may therefore infer, for *any two versors* of quaternions, q and q', the two following reciprocal relations:

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* It is evident that, in this last process of reasoning, we make *no use* of the supposed *equality of lengths* of the four lines compared; so that we might prove, in exactly the same way, that $q'q = qq'$ if $q' \mid\mid\mid q$ (123), *without* assuming that these two *complanar factors*, or *quaternions*, q and q', are *versors*.

$$\text{I.}\ \ldots\ q'q = qq', \quad \text{if} \quad q' \parallel\!\!\!\mid q\ (123);$$
$$\text{II.}\ \ldots\ \text{if}\ q'q = qq', \quad \text{then} \quad q' \parallel\!\!\!\mid q\ (168);$$

convertibility of factors (as regards their *places* in the *product*) being thus at once a *consequence* and a *proof* of *complanarity*.

170. In the Ist case of Art. 168, the *factors* q and q' are both *right versors* (153); and because we have seen that then their *two products*, $q'q$ and qq', are versors *represented* by equally long but oppositely directed arcs of one great circle, as in the Ist case of 164, it follows (comp. 162), that these two products are at once *reciprocal* (134), and *conjugate* (137), to each other; or that they are related as *versor* and *reversor* (158). We may therefore write, generally,

$$\text{I.}\ \ldots\ qq' = \mathrm{K}q'q, \quad \text{and} \quad \text{II.}\ \ldots\ qq' = \frac{1}{q'q},$$

if q and q' be *any two right versors;* because the multiplication of any two *such* versors, in two opposite orders, may always be represented or constructed by a Figure such as that lately numbered 43, in which the bisecting arcs AA′ and CC′ are *semicircles.* The IInd formula may also be thus written (comp. 135, 154):

$$\text{III.}\ \ldots\ \text{if}\ q^2 = -1, \ \text{and}\ q'^2 = -1, \ \text{then}\ q'q \cdot qq' = +1;$$

and under this form it evidently agrees with ordinary algebra, because it expresses that, *under the supposed conditions,*

$$q'q \cdot qq' = q'^2 \cdot q^2;$$

but it will be found that this last equation is *not an identity*, in the general theory of *quaternions.*

171. If the two bisecting semicircles cross each other *at right angles*, the conjugate products are represented by *two quadrants, oppositely turned*, of one great circle. It follows that *if two right versors, in two mutually rectangular planes, be multiplied together in two opposite orders, the two resulting products will be two opposite right versors, in a third plane, rectangular to the two former;* or in symbols, that

$$\text{if}\ q^2 = -1, \ q'^2 = -1, \ \text{and}\ \mathrm{Ax.}\, q' \perp \mathrm{Ax.}\, q,$$

then

$$(q'q)^2 = (qq')^2 = -1, \quad q'q = -qq';$$

and

$$\mathrm{Ax.}\, q'q \perp \mathrm{Ax.}\, q, \quad \mathrm{Ax.}\, q'q \perp \mathrm{Ax.}\, q'.$$

In *this* case, therefore, we have what would be in algebra a *paradox*, namely the equation,

$$(q'q)^2 = -q'^2 \cdot q^2,$$

if q and q' be *any two right versors, in two rectangular planes;* but we see that this result is *not more* paradoxical, in appearance, than the equation

$$q'q = -qq',$$

which exists, *under the same conditions.* And when we come to examine what, in the last analysis, may be said to be the *meaning* of this last equation, we find it to be simply this: that *any two quadrantal* or *right rotations, in planes perpendicular to each other, compound themselves into a third right rotation, as their resultant, in a plane perpendicular to each of them:* and that this *third* or *resultant* rotation has one or other of *two opposite directions,* according to the *order* in which the *two component rotations* are taken, so that one shall be *successive* to the other.

172. We propose to return, in the next Section, to the consideration of such a *System of Right Versors,* as that which we have here briefly touched upon: but desire at present to remark (comp. 167) that a *spherical triangle* ABC may serve to *construct,* by means of *representative arcs* (162), not only the *multiplication,* but also the *division,* of any one of *two diplanar versors* (or radial quotients) by the other. In fact, we have only to conceive (comp. Fig. 43) that the *vector arc* AB represents a given *divisor,* say q, or $\beta : \alpha$, and that the *transvector arc* AC (167) represents a given *dividend,* suppose q'', or $\gamma : \alpha$; for then the *provector arc* BC (comp. again 167) will represent, on the same plan, the *quotient of these two versors,* namely $q'' : q$, or $\gamma : \beta$ (106), or the versor lately called q'; since we have generally, by 106, 107, 120, for quaternions, as in algebra, the two identities:

$$(q'' : q) \cdot q = q''; \qquad q'q : q = q'.$$

173. It is however to be observed that, for reasons already assigned, we must *not* employ, for *diplanar versors,* such an equation as $q \cdot (q'' : q) = q''$; because we have found (168) that, for *such* versors, the ordinary *algebraic identity,* $qq' = q'q$, *ceases to be true.* In fact by 169, we may now establish the two converse formulæ:

$$\text{I.} \;\ldots\; q\,(q'' : q) = q'', \quad \text{if} \quad q'' \,|||\, q \;(123);$$

$$\text{II.} \;\ldots\; \text{if } q\,(q'' : q) = q'', \;\text{then}\; q'' \,|||\, q.$$

Accordingly, in Fig. 43, if q, q', q'' be still represented by the arcs AB, BC, AC, the product $q\,(q'' : q)$, or qq', is *not* represented by

AC, but by the *different arc* C′A′ (168), which *as a vector arc* has been seen to be *unequal* thereto: although it is true that these two last arcs, AC and C′A′, are always *equally long*, and therefore subtend *equal angles* at the *centre* O of the unit sphere; so that we may write, generally, for *any two versors* (or indeed for *any two quaternions*),* q and q'', the formula,

$$\angle\, q\,(q'':q) = \angle\, q''.$$

174. *Another mode* of *Representation of Versors*, or rather *two* such new modes, although intimately connected with each other, may be briefly noticed here.

Ist. We may consider the *angle* AOB, *at the centre* O *of the unit-sphere*, when conceived to have not only a definite *quantity*, but also a determined *plane* (110), and a given *direction* therein (as indicated by one of the curved arrows in Fig. 39, or by the arrow in Fig. 33), as being what may be called by analogy a *Vector-Angle;* and may say that it *represents*, or that it is the *Representative Angle* of, the *Versor* OB : OA, where OA, OB are *radii* of the unit-sphere.

IInd. Or we may replace this *rectilinear* angle AOB at the *centre*, by the equal *Spherical Angle* AC′B, at what may be called the *Positive Pole* of the *representative arc* AB; so that C′A and C′B are *quadrants;* and the *rotation*, at this *pole* C′, from the first of these two quadrants to the second (as seen from a point outside the sphere), has the *direction* which has been selected (111, 127) for the *positive* one, as indicated in the annexed Figure 44: and then we may consider this *spherical angle* as a *new Angular Representative* of the *same versor* q, or OB : OA, as before.

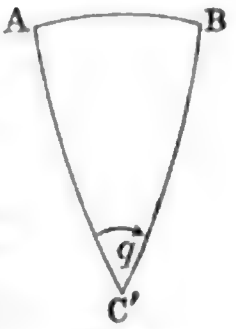

Fig. 44.

175. Conceive now that after employing a *first* spherical triangle ABC, to *construct* (as in 167) the *multiplication* of any one given versor q, by any other given versor q', we form a *second* or *polar* triangle, of which the corners A′, B′, C′ shall be respectively (in the sense just stated) the *positive poles* of the three *successive sides*, BC, CA, AB, of the former triangle; and that then we pass to a *third* triangle A′B″C′, as part of the *same lune* B′B″ with the second, by taking for B″ the point diametrically *opposite* to B′; so that B″ shall be

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* It will soon be seen that several of the formulæ of the present Section, respecting the *multiplication* and *division* of *versors*, considered as *radial quotients* (151), require little or no modification, in the passage to the corresponding *operations on quaternions*, considered as *general quotients of vectors* (112).

the *negative* pole of the arc CA, or the *positive* pole of what was lately called (167) the *transvector-arc* AC: also let C″ be, in like manner, the point opposite to C′ on the unit sphere. Then we may not only write (comp. 129),

$$\mathrm{Ax}.\,q = \mathrm{OC}', \quad \mathrm{Ax}.\,q' = \mathrm{OA}', \quad \mathrm{Ax}.\,q'q = \mathrm{OB}'',$$

but shall also have the equations,

$$\angle\, q = \mathrm{B''C'A'}, \quad \angle\, q' = \mathrm{C'A'B''}, \quad \angle\, q'q = \mathrm{C''B''A'};$$

these three spherical angles, namely the *two base-angles* at C′ and A′, and the *external vertical angle* at B″, of the new or third triangle A′B″C′, will therefore *represent*, respectively, on the plan of 174, II., the *multiplicand*, q, the *multiplier*, q', and the *product*, $q'q$. (Compare the annexed Figure 45.)

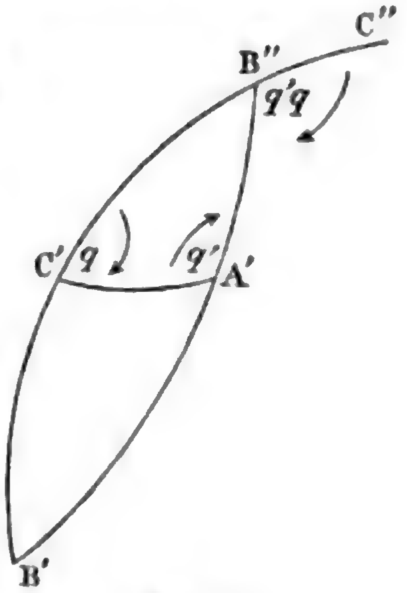

Fig. 45.

176. Without expressly referring to the former triangle ABC, we can *connect* this last *construction of multiplication of versors* (175) with the general formula (107), as follows.

Let α and β be now conceived to be two *unit-tangents*[*] to the sphere at C′, perpendicular respectively to the two arcs C′B″ and C′A′, and drawn towards the same sides of those arcs as the points A′ and B′ respectively; and let two other unit-tangents, equal to these, and denoted by the same letters, be drawn (as in the annexed Figure 45, *bis*) at the points B″ and A′, so as to be normal there to the same arcs C′B″ and C′A′, and to fall towards the same sides of them as before. Let also two other unit-tangents, equal to each other, and each denoted by γ, be drawn at the two last points B″ and A′, so as to be both perpendicular to the arc A′B″, and to fall towards the same side of it as the point C′. Then (comp. 174, II.) the *two quotients*, $\beta : \alpha$ and $\gamma : \beta$, will be equal to the *two versors*, q and q', which were lately *represented* (in Fig. 45) by the

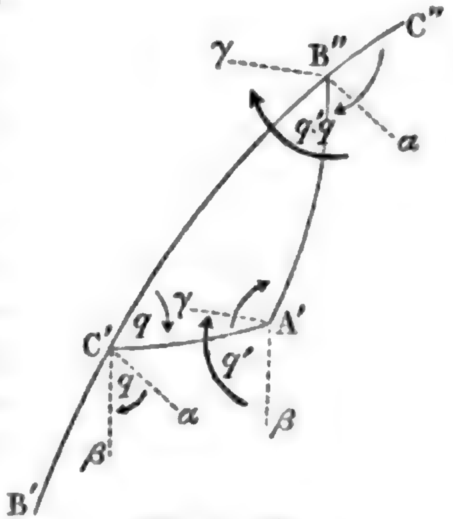

Fig. 45, *bis*.

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[*] By an *unit tangent* is here meant simply an *unit line* (*or unit vector*, 129) so drawn as to be *tangential to the unit-sphere*, and to have its *origin*, or its *initial point* (1), *on the surface* of that sphere, and *not* (as we have *usually* supposed) *at the centre* thereof.

two base angles, at C′ and A′, of the spherical triangle A′B″C′; the *product*, $q'q$, of these two versors, is therefore (by 107) equal to the *third quotient*, $\gamma : \alpha$; and consequently it is *represented*, as before, by the *external vertical angle* C″B″A′ of the same triangle, which is evidently *equal* in *quantity* to the *angle* of this third quotient, and has the *same axis* OB″, and the *same direction of rotation*, as the arrows in Fig. 45, *bis*, may assist to show.

177. In each of the two last Figures, the *internal vertical angle* at B″ is thus equal to the *Supplement*, $\pi - \angle\, q'q$, of the *angle of the product;* and it is important to observe that the corresponding *rotation at the vertex* B″, *from the side* B″A′ *to the side* B″C′, or (as we may briefly express it) *from the point* A′ *to the point* C′, is *positive;* a result which is easily seen to be a *general* one, by the reasoning of the foregoing Article.* We may then infer, generally, that *when the multiplication of any two versors is constructed by a spherical triangle*, of which the two *base angles represent* (as in the two last Articles) the *factors*, while the *external vertical angle* represents the *product*, then the *rotation round the axis* (OB″) *of that product* $q'q$, *from the axis* (OA′) *of the multiplier* q', *to the axis* (OC′) *of the multiplicand* q, is *positive:* whence it follows that the rotation round the axis Ax. q' of the multiplier, from the axis Ax. q of the multiplicand, to the axis Ax. $q'q$ of the product, is *also* positive. Or, to express the same thing more fully, since the only *rotations* hitherto considered have been *plane* ones (as in 128, &c.), we may say that if the two latter *axes* be *projected on a plane perpendicular to the former*, so as still to have a *common origin* O, then the rotation round Ax. q', *from the projection* of Ax. q to the projection of Ax. $q'q$, will be directed (with our conventions) *towards the right hand.*

178. We have therefore thus a *new mode of geometrically exhibiting* the *inequality of the two products*, $q'q$ and qq', of *two diplanar versors* (168), when taken as factors in *two different orders.* For this purpose, let

$$\mathrm{Ax}.q = \mathrm{OP},\quad \mathrm{Ax}.q' = \mathrm{OQ},\quad \mathrm{Ax}.q'q = \mathrm{OR};$$

and prolong to some point S the arc PR of a great circle on the unit sphere. Then, for the spherical triangle PQR, by principles lately established, we shall have (comp. 175) the following values of the two internal base angles at P and Q, and of the external vertical angle at R:

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* If a person be supposed to stand *on the sphere* at B″, and to *look* towards the arc A′C′, it would appear to him to have a *right-handed direction*, which is the one here adopted as positive (127).

$$\text{RPQ} = \angle q\,; \qquad \text{PQR} = \angle q'\,; \qquad \text{SRQ} = \angle q'q\,;$$

and the rotation at Q, from the side QP to the side QR will be right-handed. Let fall an arcual perpendicular, RT, from the vertex R on the base PQ, and prolong this perpendicular to R′, in such a manner as to have

$$\frown \text{RT} = \frown \text{TR}'\,;$$

also prolong PR′ to some point S′. We shall then have a new triangle PQR′, which will be a sort of *reflexion* (comp. 138) of the old one with respect to their common base PQ; and this *new triangle* will serve to *construct* the *new product*, qq'. For the rotation at P from PQ to PR′ will be right-handed, as it ought to be; and we shall have the equations,

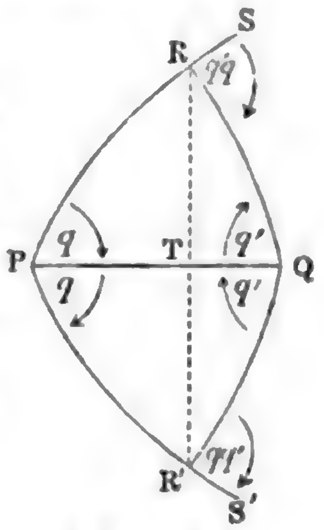

Fig. 46.

$$\text{QPR}' = \angle q\,; \qquad \text{R}'\text{QP} = \angle q'\,; \qquad \text{QR}'\text{S}' = \angle qq'\,; \qquad \text{OR}' = \text{Ax}\,.\,qq'\,;$$

so that the *new external and spherical angle*, QR′S′, will *represent the new versor*, qq', as the *old angle* SRQ represented the *old versor*, $q'q$, obtained from a different *order of the factors*. And although, no doubt, these *two angles*, at R and R′, are always *equal in quantity*, so that we may establish (comp. 173) the *general formula*,

$$\angle q'q = \angle qq',$$

yet *as vector angles* (174), and therefore as *representatives of versors*, they must be considered to be *unequal:* because they have *different planes*, namely, the *tangent planes* to the sphere at the *two vertices* R and R′; or the two planes respectively *parallel* to these, which are drawn *through the centre* O.

179. *Division of Versors* (comp. 172) can be *constructed* by means of *Representative Angles* (174), as well as by *representative arcs* (162). Thus to *divide* q'' by q, or rather to *represent* such division *geometrically*, on a plan entirely similar to that last employed for

multiplication, we have only to determine the two points P and R, in Fig. 46, by the two conditions,

$$\mathrm{OP} = \mathrm{Ax}.\,q, \qquad \mathrm{OR} = \mathrm{Ax}.\,q'',$$

and then to find a third point Q by the two angular equations,

$$\mathrm{RPQ} = \angle\, q, \qquad \mathrm{QRP} = \pi - \angle\, q'',$$

the rotation round P from PR towards PQ being positive; after which we shall have,

$$\mathrm{Ax}.\,(q'' : q) = \mathrm{OQ}; \qquad \angle\,(q'' : q) = \mathrm{PQR}.$$

(1.) Instead of conceiving, in Fig. 46, that the dotted line RTR′, which connects the vertices of the two triangles, with PQ for their common base (178), is an *arc of a great circle*, perpendicularly bisected by that base, we may imagine it to be an *arc of a small circle*, described with the point P for its *positive pole* (comp. 174, II.). And then we may say that the *passage* (comp. 173) *from the versor* q'', or $q'q$, *to the unequal versor* $q\,(q'' : q)$, or qq', *is geometrically performed* by a *Conical Rotation of the Axis* $\mathrm{Ax}.\,q''$, *round the axis* $\mathrm{Ax}.\,q$, *through an angle* $= 2 \angle\, q$, *without any* (*quantitative*) *change of the angle* $\angle\, q''$; so that we have, as before, the general formula (comp. again 173),

$$\angle\, q\,(q'' : q) = \angle\, q''.$$

(2.) Or if we prefer to employ the construction of multiplication and division by representative *arcs*, which Fig. 43 was designed to illustrate, and conceive that a new point C″ is determined in that Figure by the condition $\frown \mathrm{A'C''} = \frown \mathrm{C'A'}$, we may then say that in the passage from the versor q'', which is represented by AC, to the versor $q\,(q'' : q)$, represented by C′A′ or by A′C″, the *representative arc of* q'' is made to *move, without change of length*, so as to preserve a *constant inclination** *to the representative arc* AB *of* q, while *its initial point describes the double of that arc* AB, in passing from A to A′.

(3.) It may be seen, by these few Examples, that if, even independently of some *new characteristics of operation*, such as K and U, *new combinations of old symbols*, such as $q\,(q'' : q)$, occur in the present Calculus, which are not wanted in Algebra, they admit for the most part of *geometrical interpretations*, of an easy and interesting kind; and in fact *represent conceptions*, which cannot well be dispensed with, and which it is useful to be able to *express*, with so much simplicity and conciseness. (Compare the remarks in Art. 161; and the sub-articles to 132, 145.)

180. In connexion with the construction indicated by the two Figures 45, it may be here remarked, that if ABC be *any spherical triangle*, and if A′, B′, C′ be (as in 175) the *positive poles* of its three *successive sides*, BC, CA, AB, then the *rotation* (comp. 177, 179) *round* A′ *from* B′ *to* C′, or that round B′ from

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* In a manner analogous to the motion of the *equator* on the *ecliptic*, by lunisolar *precession*, in astronomy.

C′ to A′, &c., is *positive*. The easiest way, perhaps, of *seeing* the truth of this assertion, is to conceive that if the rotation round A from B to C be not *already positive*, we *make* it such, by passing to the diametrically *opposite triangle* on the sphere, which will not change the *poles* A′, B′, C′. Assuming then that these poles are thus the *near* ones to the corresponding corners of the given triangle, we arrive without any difficulty at the conclusion stated above: which has been virtually employed in our *construction* of *multiplication* (and *division*) of *versors*, by means of *Representative Angles* (175, 176); and which may be otherwise justified (as before), by the consideration of the *unit-tangents* of Fig. 45, *bis*.

(1.) Let then α, β, γ be *any three given unit vectors*, such that the rotation round the first, from the second to the third, is *positive* (in the sense of Art. 177); and let α', β', γ' be three other unit vectors, derived from these by the equations,

$$\alpha' = \mathrm{Ax}.(\gamma:\beta), \qquad \beta' = \mathrm{Ax}.(\alpha:\gamma), \qquad \gamma' = \mathrm{Ax}.(\beta:\alpha);$$

then the rotation round α', from β' to γ', will be positive also; and we shall have the converse formulæ,

$$\alpha = \mathrm{Ax}.(\gamma':\beta'), \qquad \beta = \mathrm{Ax}.(\alpha':\gamma'), \qquad \gamma = \mathrm{Ax}.(\beta':\alpha').$$

(2.) If the rotation round α from β to γ were given to be *negative*, α', β', γ' being still deduced from those three vectors by the same three equations as before, then the *signs* of α, β, γ would all require to be *changed*, in the three *last* (or *reciprocal*) formulæ; but the rotation round α', from β' to γ', would *still* be *positive*.

(3.) Before closing this Section, it may be briefly noticed, that it is sometimes convenient, from motives of analogy (comp. Art. 5), to speak of the *Transvector-Arc* (167), which has been seen to *represent* a *product* of two versors, as being the ARCUAL SUM of the two *successive vector-arcs*, which represent (on the same plan) the *factors; Provector* being still said to be *added* to *Vector:* but the *Order* of such *Addition of Diplanar Arcs* being *not now indifferent* (168), as the corresponding *order* had been early found (in 7) to be, when the *vectors* to be added were *right lines*.

(4.) We may also speak occasionally, by an extension of the same analogy, of the *External Vertical Angle* of a spherical triangle, as being the SPHERICAL SUM *of the two Base Angles* of that triangle, taken in a suitable *order of summation* (comp. Fig. 46); the *Angle* which represents (174) the *Multiplier* being then said to be *added* (as a sort of *Angular Provector*) to that *other Vector-Angle* which represents the *Multiplicand;* whilst what is here called the *sum* of these two angles (and is, with respect to them, a species of *Transvector-Angle*) represents, as has been proved, the *Product*.

(5.) This conception of *angular transvection* becomes perhaps a little more clear, when (on the plan of 174, I.) we assume the *centre* O as the *common vertex* of three angles AOB, BOC, AOC, situated generally in *three different planes*. For then we may conceive a *revolving radius* to be either carried by *two successive angular motions*, from OA to OB, and thence to OC; or to be transported immediately, by *one* such motion, from the *first* to the *third position*.

(6.) Finally, as regards the construction indicated by Fig. 45, *bis*, in which *tangents* instead of *radii* were employed, it may be well to remark distinctly here, that A′B″C′, in that Figure, may be *any given spherical triangle*, for which the rotation round B″ from A′ to C′ is *positive* (177); and that then, if the *two factors*, q and q', be *defined* to be the *two versors*, of which the *internal angles* at C′ and A′ are (in the sense of 174, II.) the *representatives*, the reasonings of Art. 176 will prove, without necessarily referring, *even in thought*, to any *other* triangle (such as ABC), that the *external angle* at B″ is (in the same sense) the *representative of the product*, $q'q$, as before.

SECTION 10.—*On a System of Three Right Versors, in Three Rectangular Planes; and on the Laws of the Symbols, i, j, k.*

181. Suppose that OI, OJ, OK are any three given and co-initial but rectangular unit-lines, the rotation round the first from the second to the third being positive; and let OI′, OJ′, OK′ be the three unit-vectors respectively opposite to these, so that

$$\mathrm{OI}' = -\,\mathrm{OI}, \qquad \mathrm{OJ}' = -\,\mathrm{OJ}, \qquad \mathrm{OK}' = -\,\mathrm{OK}.$$

Let the three new symbols i, j, k denote a *system* (comp. 172) *of three right versors*, in *three mutually rectangular planes*, with the three given lines for their respective *axes;* so that

$$\mathrm{Ax}.i = \mathrm{OI}, \quad \mathrm{Ax}.j = \mathrm{OJ}, \quad \mathrm{Ax}.k = \mathrm{OK},$$

and

$$i = \mathrm{OK} : \mathrm{OJ}, \quad j = \mathrm{OI} : \mathrm{OK}, \quad k = \mathrm{OJ} : \mathrm{OI},$$

as Figure 47 may serve to illustrate. We shall then have these other expressions for the same three versors:

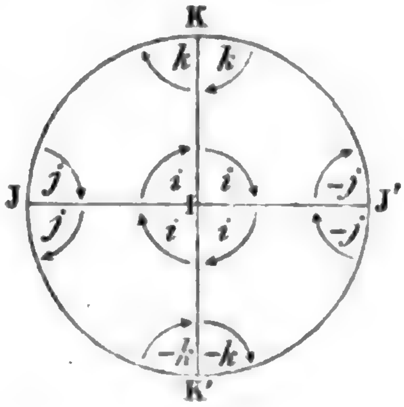

Fig. 47.

$$i = \mathrm{OJ}' : \mathrm{OK} = \mathrm{OK}' : \mathrm{OJ}' = \mathrm{OJ} : \mathrm{OK}';$$
$$j = \mathrm{OK}' : \mathrm{OI} = \mathrm{OI}' : \mathrm{OK}' = \mathrm{OK} : \mathrm{OI}';$$
$$k = \mathrm{OI}' : \mathrm{OJ} = \mathrm{OJ}' : \mathrm{OI}' = \mathrm{OI} : \mathrm{OJ}';$$

while the three respectively *opposite* versors may be thus expressed:

$$-i = \mathrm{OJ} : \mathrm{OK} = \mathrm{OK}' : \mathrm{OJ} = \mathrm{OJ}' : \mathrm{OK}' = \mathrm{OK} : \mathrm{OJ}';$$
$$-j = \mathrm{OK} : \mathrm{OI} = \mathrm{OI}' : \mathrm{OK} = \mathrm{OK}' : \mathrm{OI}' = \mathrm{OI} : \mathrm{OK}';$$
$$-k = \mathrm{OI} : \mathrm{OJ} = \mathrm{OJ}' : \mathrm{OI} = \mathrm{OI}' : \mathrm{OJ}' = \mathrm{OJ} : \mathrm{OI}'.$$

And from the comparison of these different expressions several important symbolical consequences follow, which it will be worth while to enunciate separately here, although some of them are virtually included in the results of former Sections.

182. In the *first* place, since

$$i^2 = (\text{OJ}' : \text{OK}) . (\text{OK} : \text{OJ}) = \text{OJ}' : \text{OJ}, \text{\&c.},$$

we deduce (comp. 148) the following equal values for the *squares* of the new symbols:

$$\text{I.} \ldots i^2 = -1; \qquad j^2 = -1; \qquad k^2 = -1;$$

as might indeed have been at once inferred (154), from the circumstance that the three radial quotients (146), denoted here by i, j, k, are all *right versors* (181).

In the *second* place, since

$$ij = (\text{OJ} : \text{OK}') . (\text{OK}' : \text{OI}) = \text{OJ} : \text{OI}, \text{\&c.},$$

we have the following values for the *products* of the same three symbols, or versors, when taken *two by two*, and in a certain *order of succession* (comp. 168, 171):

$$\text{II.} \ldots ij = k; \qquad jk = i; \qquad ki = j.$$

But in the *third* place (comp. again 171), since

$$j . i = (\text{OI} : \text{OK}) . (\text{OK} : \text{OJ}) = \text{OI} : \text{OJ}, \text{\&c.},$$

we have these other and *contrasted* formulæ, for the *binary products* of the *same* three right versors, when taken as factors with an *opposite order*:

$$\text{III.} \ldots ji = -k; \qquad kj = -i; \qquad ik = -j.$$

Hence, while the *square of each* of the *three right versors*, denoted by these *three new symbols*, ijk, is equal (154) to *negative unity*, the *product of any two* of them is equal either to the *third itself*, or to the *opposite* (171) of that third versor, according as the *multiplier precedes* or *follows* the *multiplicand*, in the *cyclical succession*,

$$i, \ j, \ k, \quad i, \ j, \ \ldots$$

which the annexed Figure 47, *bis*, may give some help towards remembering.

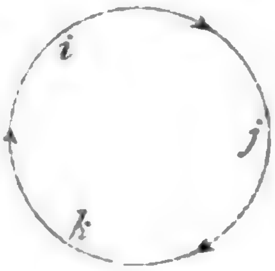

Fig. 47, *bis*.

(1.) To connect such multiplications of i, j, k with the theory of representative arcs (162), and of representative angles (174), we may regard any one of the four *quadrantal arcs*, JK, KJ′, J′K′, K′J, in Fig. 47, or any one of the four *spherical right angles*, JIK, KIJ′, J′IK′, K′IJ, which those arcs subtend at their common pole I, as *representing* the versor i; and similarly for j and k, with the introduction of the point I′ opposite to I, which is to be conceived as being at the back of the Figure.

(2.) The *squaring* of i, or the equation $i^2 = -1$, comes thus to be geometrically *constructed* by the *doubling* (comp. Arts. 148, 154, and Figs. 41, 42) of an *arc*, or of an *angle*. Thus, we may conceive the *quadrant* KJ′ to be *added* to the *equal arc* JK, their sum being the *great semicircle* JJ′, which (by 166) *represents* an *inversor* (153), or *negative unity* considered as a *factor*. Or we may *add* the *right angle* KIJ′ to the *equal angle* JIK, and so obtain a *rotation* through *two* right angles at the *pole* I, or at the *centre* O; which rotation is equivalent (comp. 154, 174) to an *inversion of direction*, or to a passage from the radius OJ, to the opposite radius OJ′.

(3.) The *multiplication* of *j by i*, or the equation $ij = k$, may in like manner be *arcually constructed*, by the *addition* of K′J, as a *provector-arc* (167), to IK′ as a *vector-arc* (162), giving IJ, which is a *representative* of k, as the *transvector-arc*, or *arcual-sum* (180, (3.)). Or the same multiplication may be *angularly* constructed, with the help of the *spherical triangle* IJK; in which the *base-angles* at I and J represent respectively the *multiplier*, i, and the *multiplicand*, j, the rotation round I from J to K being *positive*: while their *spherical sum* (180, (4.)), or the *external vertical angle* at K (comp. 175, 176), represents the same *product*, k, as before.

(4.) The *contrasted* multiplication of i by j, or of j *into** i, may in like manner be *constructed*, or geometrically represented, either by the addition of the arc KI, as a *new provector*, to the arc JK as a new vector, which new process gives JI (instead of IJ) as the *new transvector*; or with the aid of the *new triangle* IJK′ (comp. Figs. 46, 47), in which the rotation round I from J to the *new vertex* K′ is *negative*, so that the angle at I represents now the multiplicand, and the resulting angle at the *new pole* K′ represents the *new* and *opposite product*, $ji = -k$.

183. Since we have thus $ji = -ij$ (as we had $q'q = -qq'$ in 171), we see that the *laws of combination of the new symbols*, i, j, k, are *not in all respects the same* as the corresponding laws in *algebra*; since the *Commutative Property of Multiplication*, or the *convertibility* (169) of the places of the *factors* without change of value of the *product*, does *not here* hold good: which arises (168) from the circumstance, that the factors to be combined are here *diplanar versors* (181). It is therefore important to observe, that there *is* a respect in which

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* A multiplicand is said to be multiplied *by* the multiplier; while, on the other hand, a multiplier is said to be multiplied *into* the multiplicand: a *distinction* of this sort between the *two factors* being necessary, as we have seen, for *quaternions*, although it is not needed for algebra.

the *laws of i, j, k agree* with usual and *algebraic laws:* namely, in the *Associative Property of Multiplication;* or in the property that the new symbols always obey the *associative formula* (comp. 9),

$$\iota . \kappa\lambda = \iota\kappa . \lambda,$$

whichever of them may be substituted for ι, for κ, and for λ; in virtue of which equality of values we may *omit the point*, in any such symbol of a *ternary product* (whether of equal or of unequal factors), and write it simply as $\iota\kappa\lambda$. In particular we have thus,

$$i.jk = i.i = i^2 = -1; \qquad ij.k = k.k = k^2 = -1;$$

or briefly,

$$ijk = -1.$$

We may, therefore, by 182, establish the following important *Formula:*

$$i^2 = j^2 = k^2 = ijk = -1; \tag{A}$$

to which we shall occasionally refer, as to "Formula A," and which we shall find to contain (virtually) *all the laws of the symbols ijk*, and therefore to be a *sufficient symbolical basis* for the whole *Calculus of Quaternions:*[*] because it will be shown that *every quaternion can be reduced to the Quadrinomial Form*,

$$q = w + ix + jy + kz,$$

where w, x, y, z compose a *system of four scalars*, while i, j, k are the same *three right versors* as above.

(1.) A direct proof of the equation, $ijk = -1$, may be derived from the definitions of the symbols in Art. 181. In fact, we have only to remember that those definitions were seen to give,

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[*] This formula (A) was accordingly made the *basis* of that Calculus in the first communication on the subject, by the present writer, to the Royal Irish Academy in 1843; and the letters, i, j, k, continued to be, for some time, the *only peculiar symbols* of the calculus in question. But it was gradually found to be useful to incorporate with these a few other *notations* (such as K and U, &c.), for representing *Operations on Quaternions.* It was also thought to be instructive to establish the *principles* of that Calculus, on a more *geometrical* (or less exclusively *symbolical*) *foundation* than at first; which was accordingly afterwards done, in the volume entitled: *Lectures on Quaternions* (Dublin, 1853); and is again attempted in the present work, although with many differences in the adopted *plan* of exposition, and in the *applications* brought forward, or suppressed.

$$i = \mathrm{OJ'} : \mathrm{OK}, \qquad j = \mathrm{OK} : \mathrm{OI'}, \qquad k = \mathrm{OI'} : \mathrm{OJ};$$

and to observe that, by the general formula of multiplication (107), *whatever four lines* may be denoted by $\alpha, \beta, \gamma, \delta$, we have always,

$$\frac{\delta}{\gamma} . \frac{\gamma}{\beta} \frac{\beta}{\alpha} = \frac{\delta}{\gamma} . \frac{\gamma}{\alpha} = \frac{\delta}{\alpha} = \frac{\delta}{\beta} . \frac{\beta}{\alpha} = \frac{\delta}{\gamma} \frac{\gamma}{\beta} . \frac{\beta}{\alpha};$$

or briefly, as in algebra,

$$\frac{\delta}{\gamma} \frac{\gamma}{\beta} \frac{\beta}{\alpha} = \frac{\delta}{\alpha},$$

the *point* being thus *omitted* without danger of confusion: so that

$$ijk = \mathrm{OJ'} : \mathrm{OJ} = -1, \text{ as before.}$$

Similarly, we have these two other ternary products:

$$jki = (\mathrm{OK'} : \mathrm{OI})(\mathrm{OI} : \mathrm{OJ'})(\mathrm{OJ'} : \mathrm{OK}) = \mathrm{OK'} : \mathrm{OK} = -1;$$
$$kij = (\mathrm{OI'} : \mathrm{OJ})(\mathrm{OJ} : \mathrm{OK'})(\mathrm{OK'} : \mathrm{OI}) = \mathrm{OI'} : \mathrm{OI} = -1.$$

(2.) On the other hand,

$$kji = (\mathrm{OJ} : \mathrm{OI})(\mathrm{OI} : \mathrm{OK})(\mathrm{OK} : \mathrm{OJ}) = \mathrm{OJ} : \mathrm{OJ} = +1;$$

and in like manner,

$$ikj = +1, \quad \text{and} \quad jik = +1.$$

(3.) The equations in 182 give also these other ternary products, in which the law of *association of factors* is still obeyed:

$$i . ij = ik = -j = i^2 j = ii . j, \qquad iij = -j;$$
$$i . ji = i . -k = -ik = j = ki = ij . i, \qquad iji = +j;$$
$$i . jj = i . -1 = -i = kj = ij . j, \qquad ijj = -i;$$

with others deducible from these, by mere *cyclical permutation* of the letters, on the plan illustrated by Fig. 47, *bis*.

(4.) In general, if the *Associative Law of Combination* exist for *any three symbols* whatever of a *given class*, and for a *given mode* of combination, as for *addition of lines* in Art. 9, or for *multiplication of ijk* in the present Article, the *same law* exists for any *four* (or more) symbols of the same class, and combinations of the same kind. For example, if *each* of the four letters $\iota, \kappa, \lambda, \mu$ denote *some one* of the three symbols i, j, k (but not necessarily the *same* one), we have the formula,

$$\iota . \kappa\lambda\mu = \iota . \kappa . \lambda\mu = \iota\kappa . \lambda\mu = \iota\kappa . \lambda . \mu = \iota\kappa\lambda . \mu = \iota\kappa\lambda\mu.$$

(5.) Hence, *any multiple* (or *complex*) *product* of the symbols ijk, in any manner *repeated*, but taken in one given *order*, may be interpreted, with one definite *result*, by *any mode of association*, or of *reduction to partial factors*, which can be performed *without commutation*, or *change of place* of the given factors. For example, the symbol $ijkkji$ may be interpreted in either of the *two* following (among other) ways:

$$ij . kk . ji = ij . -ji = i . -j^2 . i = ii = -1; \qquad ijk . kji = -1 . 1 = -1.$$

184. The formula (A) of 183 *includes* obviously the three equations (I.) of 182. To show that it includes *also* the six other equations, (II.), (III.), of the last cited Article, we may observe that it gives, with the help of the *associative* principle of multiplication (which may be suggested to the memory by the absence of the *point* in the symbol ijk),

$$ij = -ij\,.\,kk = -ijk\,.\,k = +k; \qquad jk = -i\,.\,ijk = +i;$$
$$ji = j\,.\,jk = j^2k = -k; \qquad ik = i\,.\,ij = i^2j = -j;$$
$$kj = ij\,.\,j = ij^2 = -i; \qquad ki = -k^2j = -ji^2 = +j.$$

And then it is easy to prove, *without any reference to geometry*, if the foregoing *laws of the symbols* be admitted, that we have also,

$$jki = kij = -1, \qquad kji = jik = ikj = +1,$$

as otherwise and *geometrically* shown in recent sub-articles. It may be added that the mere *inspection* of the formula (A) is sufficient to show that the *three** *square roots of negative unity*, denoted in it by i, j, k, *cannot be subject to all the ordinary rules of algebra:* because that formula gives, at sight,

$$i^2 j^2 k^2 = (-1)^3 = -1 = -(ijk)^2;$$

the *non-commutative character* (183), of the *multiplication of such roots* among themselves, being thus put in evidence.

SECTION 11.—*On the Tensor of a Vector, or of a Quaternion; and on the Product or Quotient of any two Quaternions.*

185. Having now sufficiently availed ourselves, in the two last Sections, of the conceptions (alluded to, so early as in the First Article of these Elements) of a *vector-arc* (162), and of a *vector-angle* (174), in *illustration*† of the laws of *multiplication* and *division* of *versors* of quaternions; we propose to *return* to that use of the word, VECTOR, with which alone the First Book, and the first eight Sections of this First Chapter of the Second Book, have been concerned: and shall therefore henceforth mean again, *exclusively*, by that word "vector," a *Directed Right Line* (as in 1). And because we have already considered and expressed the *Direction* of any such line, by

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* It is evident that $-i$, $-j$, $-k$ are *also*, on the same principles, values of the symbol $\sqrt{-1}$; because they also are *right versors* (153); or because $(-q)^2 = q^2$. More generally (comp. a Note to page 131), if x, y, z be *any three scalars* which satisfy the condition $x^2 + y^2 + z^2 = 1$, it will be proved, at a later stage, that

$$(ix + jy + kz)^2 = -1.$$

† One of the chief uses of *such* vectors, in connexion with those laws, has been to illustrate the *non-commutative property* (168) of *multiplication of versors*, by exhibiting a corresponding property of what has been called, by analogy to the earlier operation of the same kind on *linear* vectors (5), the *addition* of *arcs* and *angles* on a *sphere*. Compare 180, (3.), (4.).

introducing the conception and notation (155) of the *Unit-Vector*, $\mathrm{U}\alpha$, which has the *same direction* with the line α, and which we have proposed (156) to call *the Versor of that Vector*, α; we now propose to consider and express the *Length* of the same line α, by introducing the *new name* TENSOR, and the *new symbol*,* $\mathrm{T}\alpha$; which latter symbol we shall *read*, as *the Tensor of the Vector* α: and shall *define* it to be, or to denote, *the Number* (comp. again 155) *which represents the Length of that line* α, by expressing the *Ratio* which that length bears to some assumed standard, or *Unit* (128).

186. To connect more closely these two conceptions, of the *versor* and the *tensor* of a *vector*, we may remember that when we employed (in 155) the letter a as a temporary symbol for the number which thus expresses the length of the line α, we had the equation, $\mathrm{U}\alpha = \alpha : a$, as one form of the *definition* of the *unit-vector* denoted by $\mathrm{U}\alpha$. We might therefore have written also these two other forms of equation (comp. 15, 16),

$$\alpha = a.\mathrm{U}\alpha, \qquad a = \alpha : \mathrm{U}\alpha,$$

to express the dependence of the *vector*, α, and of the *scalar*, a, on each other, and on what has been called (156) the *versor*, $\mathrm{U}\alpha$. For example, with the construction of Fig. 42, *bis* (comp. 161, (2.)), we may write the three equations,

$$a = \mathrm{OA} : \mathrm{OA}', \qquad b = \mathrm{OB} : \mathrm{OB}', \qquad c = \mathrm{OC} : \mathrm{OC}',$$

if a, b, c be thus the *three positive scalars*, which denote the *lengths* of the three lines, OA, OB, OC; and these three scalars may then be considered as *factors*, or as *coefficients* (12), by which the *three unit-vectors* $\mathrm{U}\alpha$, $\mathrm{U}\beta$, $\mathrm{U}\gamma$, or OA′, OB′, OC′ (in the cited Figure), are to be respectively *multiplied* (15), in order to change them into the three other vectors α, β, γ, or OA, OB, OC, by *altering their lengths*, without any change in their *directions*. But such an exclusive *Operation*, on the *Length* (or on the *extension*) of a line, may be said to be an *Act of Tension*;† as an operation *on direction alone* may be called (comp. 151) an *act of version*. We have then thus a *motive* for the introduction of the *name*, *Tensor*, as applied to the *positive number* which (as above) represents the *length* of a line. And when the *notation* Ta (instead of a) is employed for such a *tensor*, we see that we may write generally, *for any vector* a, the equations (compare again 15, 16):

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* Compare the Note to Art. 155.

† Compare the Note to Art. 156, in page 135.

$$\mathrm{U}a = a : \mathrm{T}a; \qquad \mathrm{T}a = a : \mathrm{U}a; \qquad a = \mathrm{T}a \,.\, \mathrm{U}a = \mathrm{U}a \,.\, \mathrm{T}a.$$

For example, if a be an unit-vector, so that $\mathrm{U}a = a$ (160), then $\mathrm{T}a = 1$; and therefore, generally, *whatever vector* may be denoted by a, we have always,

$$\mathrm{TU}a = 1.$$

For the same reason, *whatever quaternion* may be denoted by q, we have always (comp. again 160) the equation,

$$\mathrm{T}\,(\mathrm{Ax}\,.\,q) = 1.$$

(1.) Hence the equation

$$\mathrm{T}\rho = 1,$$

where $\rho = $ OP, expresses that the *locus* of the variable point P is the surface of the *unit sphere* (128).

(2.) The equation $\mathrm{T}\rho = \mathrm{T}a$ expresses that the locus of P is the spheric surface with O for centre, which passes through the point A.

(3.) On the other hand, for the sphere through O, which has its *centre* at A, we have the equation,

$$\mathrm{T}\,(\rho - a) = \mathrm{T}a;$$

$AP = OP - OA$

which expresses that the lengths of the two lines, AP, AO, are equal.

$\therefore T(AP) = T($

(4.) More generally, the equation,

$$\mathrm{T}\,(\rho - a) = \mathrm{T}\,(\beta - a),$$

$T(AP) = T(AB)$

expresses that the locus of P is the spheric surface through B, which has its centre at A.

(5.) The equation of the Apollonian* Locus, 145, (8.), (9.), may be written under either of the two following forms:

$OA = \alpha \quad OC = a\alpha$

$$\mathrm{T}\,(\rho - a^2 a) = a\mathrm{T}\,(\rho - a); \qquad \mathrm{T}\rho = a\mathrm{T}a;$$

$T(PC) = a\,T($

from each of which we shall find ourselves able to pass to the other, at a later stage, by general *Rules of Transformation*, without appealing to *geometry* (comp. 145, (10.)).

(6.) The equation,

$$\mathrm{T}\,(\rho + a) = \mathrm{T}\,(\rho - a),$$

expresses that the locus of P is the plane through O, perpendicular to the line OA; because it expresses that if OA′ = − OA, then the point P is equally distant from the two points A and A′. It represents therefore the *same locus* as the equation,

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* Compare the first Note to page 128.

$$\angle \frac{\rho}{\alpha} = \frac{\pi}{2}, \text{ of 132, (1.);}$$

or as the equation,

$$\frac{\rho}{\alpha} + \mathrm{K}\frac{\rho}{\alpha} = 0, \text{ of 144, (1.);}$$

or as

$$\left(\mathrm{U}\frac{\rho}{\alpha}\right)^2 = -1, \text{ of 161, (7.);}$$

or as the simple geometrical formula, $\rho \perp \alpha$ (129). And in fact it will be found possible, by *General Rules* of this Calculus, to *transform any one* of these *five* formulæ into *any other* of them; or into this *sixth form*,

$$\mathrm{S}\frac{\rho}{\alpha} = 0,$$

which expresses that the *scalar part** of the *quaternion* $\frac{\rho}{\alpha}$ is *zero*, and therefore that this quaternion is a *right quotient* (132).

(7.) In like manner, the equation

$$\mathrm{T}(\rho - \beta) = \mathrm{T}(\rho - \alpha)$$

expresses that the locus of P is the plane which perpendicularly bisects the line AB; because it expresses that P is equally distant from the two points A and B.

(8.) The *tensor*, $\mathrm{T}\alpha$, being generally a *positive scalar*, but *vanishing* (as a *limit*) *with* α, we have,

$$\mathrm{T}x\alpha = \pm x\mathrm{T}\alpha, \quad \text{according as} \quad x > \quad \text{or} \quad < 0;$$

thus, in particular,

$$\mathrm{T}(-\alpha) = \mathrm{T}\alpha; \quad \text{and} \quad \mathrm{T}0\alpha = \mathrm{T}0 = 0.$$

(9.) That

$$\mathrm{T}(\beta + \alpha) = \mathrm{T}\beta + \mathrm{T}\alpha, \quad \text{if} \quad \mathrm{U}\beta = \mathrm{U}\alpha,$$

but not otherwise (α and β being any two actual vectors), will be seen, at a later stage, to be a symbolical consequence from the *rules* of the present *Calculus*; but in the mean time it may be *geometrically* proved, by conceiving that while $\alpha = \mathrm{OA}$, as usual, we make $\beta + \alpha = \mathrm{OC}$, and therefore $\beta = \mathrm{OC} - \mathrm{OA} = \mathrm{AC}$ (4); for thus we shall see that while, *in general*, the three points O, A, C are corners of a *triangle*, and therefore the *length* of the *side* OC is *less* than the *sum* of the lengths of the two other sides OA and AC, the former length becomes, on the contrary, *equal* to the latter sum, in the particular *case* when the triangle vanishes, by the point A falling *on the finite line* OC; in which case, OA and AC, or α and β, have one *common direction*, as the equation $\mathrm{U}\alpha = \mathrm{U}\beta$ implies.

(10.) If α and β be any actual vectors, and if their *versors be unequal* ($\mathrm{U}\alpha$ *not* $= \mathrm{U}\beta$), then

$$\mathrm{T}(\beta + \alpha) < \mathrm{T}\beta + \mathrm{T}\alpha;$$

an inequality which results at once from the consideration of the recent *triangle* OAC; but which (as it will be found) may also be *symbolically* proved, by *rules* of the calculus of quaternions.

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* Compare the Note to page 125; and the following Section of the present Chapter.

(11.) If $U\beta = -U\alpha$, then $T(\beta+\alpha) = \pm(T\beta - T\alpha)$, according as $T\beta >$ or $< T\alpha$; but

$$T(\beta+\alpha) > \pm(T\beta - T\alpha), \quad \text{if} \quad U\beta \textit{ not} = -U\alpha.$$

187. The *quotient*, $U\beta : U\alpha$, *of the versors* of the two vectors, α and β, has been called (in 156) the *Versor of the Quotient*, or quaternion, $q = \beta : \alpha$; and has been denoted, as such, by the symbol, Uq. On the same plan, we propose now to call the quotient, $T\beta : T\alpha$, of the *tensors* of the same two vectors, the *Tensor** *of the Quaternion* q, or $\beta : \alpha$, and to denote it by the corresponding *symbol*, Tq. And then, as we have called the letter U (in 156) the characteristic of the operation of *taking the versor*, so we may now speak of T as the *Characteristic of the* (corresponding) *Operation of taking the Tensor*, whether of a *Vector*, α, or of a *Quaternion*, q. We shall thus have, generally,

$T(\beta : \alpha) = T\beta : T\alpha$, as we had $U(\beta : \alpha) = U\beta : U\alpha$ (156);

and may say that as the *versor* Uq depended solely on, but conversely was sufficient to determine, the *relative direction* (157), so the *tensor* Tq depends on and determines the *relative length*† (109), of the two vectors, α and β, of which the *quaternion* q is the *quotient* (112).

(1.) Hence the equation $T\frac{\rho}{\alpha} = 1$, like $T\rho = T\alpha$, to which it is equivalent, expresses that the locus of P is the sphere with O for centre, which passes through the point A.

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* Compare the Note to Art. 109, in page 108; and that to Art. 156, in page 135.

† It has been shown, in Art. 112, and in the *Additional Illustrations* of the third Section of the present Chapter (113–116), that *Relative Length*, as well as *relative direction*, enters as an *essential element* into the very *Conception* of a *Quaternion*. Accordingly, in Art. 117, an *agreement* of relative *lengths* (as well as an agreement of relative directions) was made one of the *conditions of equality*, between any *two* quaternions, considered as quotients of vectors: so that we may now say, that *the tensors* (as well as the *versors*) *of equal quaternions are equal*. Compare the first Note to page 137, as regards what was there called the *quantitative element*, of absolute or relative *length*, which was *eliminated* from α, or from q, by means of the *characteristic* U; whereas the *new* characteristic, T, of the present Section, serves on the contrary to *retain that element alone*, and to eliminate what may be called by contrast the *qualitative* element, of absolute or relative *direction*.

(2.) The equation comp. 186, (6.)),

$$\mathrm{T}\frac{\rho+\alpha}{\rho-\alpha}=1,$$

expresses that the locus of P is the plane through O, perpendicular to the line OA.

(3.) Other examples of the same sort may easily be derived from the sub-articles to 186, by introducing the notation (187) for the *tensor of a quotient*, or quaternion, as additional to that for the *tensor of a vector* (185).

(4.) $\mathrm{T}(\beta:\alpha) >, =, \text{or} < 1$, according as $\mathrm{T}\beta >, =, \text{or} < \mathrm{T}\alpha$.

(5.) The *tensor of a right quotient* (132) is always equal to the tensor of its *index* (133).

(6.) The *tensor of a radial* (146) is always *positive unity;* thus we have, generally, by 156,

$$\mathrm{TU}q=1;$$

and in particular, by 181,

$$\mathrm{T}i=\mathrm{T}j=\mathrm{T}k=1.$$

(7.) $\mathrm{T}xq=\pm x\mathrm{T}q$, according as $x >$ or < 0;

thus, in particular, $\mathrm{T}(-q)=\mathrm{T}q$, or the tensors of *opposite* quaternions are *equal.*

(8.) $\mathrm{T}x=\pm x$, according as $x >$ or < 0;

thus, the tensor of a *scalar* is that scalar *taken positively.*

(9.) Hence,

$$\mathrm{TT}\alpha=\mathrm{T}\alpha, \qquad \mathrm{TT}q=\mathrm{T}q;$$

so that, by abstracting from the *subject* of the operation T (comp. 145, 160), we may establish the symbolical equation,

$$\mathrm{T}^2=\mathrm{TT}=\mathrm{T}.$$

(10.) Because the tensor of a quaternion is generally a positive scalar, such a tensor is *its own conjugate* (139); its *angle* is *zero* (131); and its *versor* (159) is positive *unity:* or in symbols,

$$\mathrm{KT}q=\mathrm{T}q; \qquad \angle\,\mathrm{T}q=0; \qquad \mathrm{UT}q=1.$$

(11.) $\mathrm{T}(1:q)=\mathrm{T}(\alpha:\beta)=\mathrm{T}\alpha:\mathrm{T}\beta=1:\mathrm{T}q$;

or in words, the *tensor of the reciprocal* of a quaternion is equal to the *reciprocal of the tensor*.

(12.) Again, since the two lines, OB and OB′, in Fig. 36, are *equally long,* the definition (137) of a conjugate gives

$$\mathrm{TK}q=\mathrm{T}q;$$

or in words, the tensors of *conjugate* quaternions are *equal.*

(13.) It is scarcely necessary to remark, that any two quaternions which have *equal tensors,* and *equal versors,* are themselves *equal:* or in symbols, that

$$q'=q, \quad \text{if} \quad \mathrm{T}q'=\mathrm{T}q, \quad \text{and} \quad \mathrm{U}q'=\mathrm{U}q.$$

188. Since we have, generally,

$$\frac{\beta}{\alpha}=\frac{\mathrm{T}\beta\,.\,\mathrm{U}\beta}{\mathrm{T}\alpha\,.\,\mathrm{U}\alpha}=\frac{\mathrm{T}\beta}{\mathrm{T}\alpha}\cdot\frac{\mathrm{U}\beta}{\mathrm{U}\alpha}=\frac{\mathrm{U}\beta}{\mathrm{U}\alpha}\cdot\frac{\mathrm{T}\beta}{\mathrm{T}\alpha} \text{ (comp. 126, 186)},$$

we may establish the two following general formulæ of *decom-*

position of a quaternion into two factors, of the *tensor* and *versor* kinds:

$$\text{I.} \ldots q = \mathrm{T}q . \mathrm{U}q; \qquad \text{II.} \ldots q = \mathrm{U}q . \mathrm{T}q;$$

which are exactly analogous to the formulæ (186) for the corresponding decomposition of a *vector*, into *factors* of the same two kinds: namely,

$$\text{I}'. \ldots \alpha = \mathrm{T}\alpha . \mathrm{U}\alpha; \qquad \text{II}'. \ldots \alpha = \mathrm{U}\alpha . \mathrm{T}\alpha.$$

To illustrate this last decomposition of a quaternion, q, or OB : OA, into factors, we may conceive that AA′ and BB′ are two concentric and circular, but oppositely directed arcs, which terminate respectively on the two lines OB and OA, or rather on the longer of those two lines itself, and on the shorter of them prolonged, as in the annexed Figure 48; so that OA′ has the *length* of OA, but the *direction* of OB, while OB′, on the contrary, has the length of OB, but the direction of OA; and that therefore we may write, by what has been defined respecting *versors* and *tensors* of *vectors* (155, 156, 185, 186),

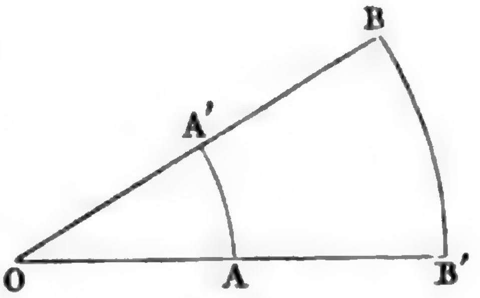

Fig 48.

$$\mathrm{OA}' = \mathrm{T}\alpha . \mathrm{U}\beta; \qquad \mathrm{OB}' = \mathrm{T}\beta . \mathrm{U}\alpha.$$

Then, by the definitions in 156, 187, of the versor and tensor of a *quaternion*,

$$\mathrm{U}q = \mathrm{U}(\mathrm{OB} : \mathrm{OA}) = \mathrm{OA}' : \mathrm{OA} = \mathrm{OB} : \mathrm{OB}';$$
$$\mathrm{T}q = \mathrm{T}(\mathrm{OB} : \mathrm{OA}) = \mathrm{OB}' : \mathrm{OA} = \mathrm{OB} : \mathrm{OA}';$$

whence, by the general formula of multiplication of quotients (107),

$$\text{I.} \ldots q = \mathrm{OB} : \mathrm{OA} = (\mathrm{OB} : \mathrm{OA}') . (\mathrm{OA}' : \mathrm{OA}) = \mathrm{T}q . \mathrm{U}q;$$

and

$$\text{II.} \ldots q = \mathrm{OB} : \mathrm{OA} = (\mathrm{OB} : \mathrm{OB}') . (\mathrm{OB}' : \mathrm{OA}) = \mathrm{U}q . \mathrm{T}q,$$

as above.

189. In words, if we wish to pass from the vector α to the vector β, or from the line OA to the line OB, we are at liberty either, 1st, to *begin by turning*, from OA to OA′, and then to *end by stretching*,

from OA′ to OB, as Fig. 48 may serve to illustrate; or, IInd, to *begin* by stretching, from OA to OB′, and *end* by turning, from OB′ to OB. The *act of multiplication* of a *line* a by a *quaternion* q, considered as a *factor* (103), which affects *both* length and direction (109), may thus be *decomposed* into *two* distinct and *partial acts*, of the kinds which we have called *Version* and *Tension;* and these two acts may be performed, at pleasure, in *either* of two *orders* of succession. And although, if we attended *merely to lengths*, we might be led to say that the *tensor* of a quaternion was a *signless number*,* expressive of a geometrical *ratio* of magnitudes, yet when the recent *construction* (Fig. 48) is adopted, we see, by either of the two resulting expressions (188) for $\mathrm{T}q$, that there is a *propriety* in treating this tensor as a *positive scalar*, as we have lately done, and propose systematically to do.

190. Since $\mathrm{TK}q = \mathrm{T}q$, by 187, (12.), and $\mathrm{UK}q = 1 : \mathrm{U}q$, by 158, we may write, generally, for any quaternion and its conjugate, the two connected expressions:

$$\text{I.} \ldots q = \mathrm{T}q \,.\, \mathrm{U}q; \qquad \text{II.} \ldots \mathrm{K}q = \mathrm{T}q : \mathrm{U}q;$$

whence, by multiplication and division,

$$\text{III.} \ldots q \,.\, \mathrm{K}q = (\mathrm{T}q)^2; \qquad \text{IV.} \ldots q : \mathrm{K}q = (\mathrm{U}q)^2.$$

This last formula had occurred before; and we saw (161) that in it the *parentheses* might be omitted, because $(\mathrm{U}q)^2 = \mathrm{U}(q^2)$. In like manner (comp. 161, (2.)), we have also

$$(\mathrm{T}q)^2 = \mathrm{T}(q^2) = \mathrm{T}q^2,$$

parentheses being again omitted; or in words, the *tensor of the square* of a quaternion is always equal to the *square of the tensor:* as appears (among other ways) from inspection of Fig. 42, *bis*, in which the *lengths* of OA, OB, OC form a *geometrical progression;* whence

$$\mathrm{T}.\left(\frac{\mathrm{OB}}{\mathrm{OA}}\right)^2 = \mathrm{T}\frac{\mathrm{OC}}{\mathrm{OA}} = \frac{\mathrm{T}\,.\,\mathrm{OC}}{\mathrm{T}\,.\,\mathrm{OA}} = \left(\frac{\mathrm{T}\,.\,\mathrm{OB}}{\mathrm{T}\,.\,\mathrm{OA}}\right)^2 = \left(\mathrm{T}\frac{\mathrm{OB}}{\mathrm{OA}}\right)^2.$$

At the same time, we see again that the *product* $q\mathrm{K}q$ *of two conjugate quaternions*, which has been called (145, (11.)) their common *Norm*, and denoted by the symbol $\mathrm{N}q$, represents geometrically the *square of the quotient of the lengths* of the two lines, of which (when considered as *vectors*) the quaternion q is itself the quotient (112). We may therefore write generally,†

$$\text{V.} \ldots q\mathrm{K}q = \mathrm{T}q^2 = \mathrm{N}q; \qquad \text{VI.} \ldots \mathrm{T}q = \sqrt{\mathrm{N}q} = \sqrt{(q\mathrm{K}q)}.$$

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* Compare the Note in page 108, to Art. 109.

† Compare the Note in page 129.

(1.) We have also, by II., the following other general transformations for the tensor of a quaternion:

$$\text{VII.} \ldots \mathrm{T}q = \mathrm{K}q \,.\, \mathrm{U}q; \qquad \text{VIII.} \ldots \mathrm{T}q = \mathrm{U}q \,.\, \mathrm{K}q;$$

of which the geometrical significations might easily be exhibited by a diagram, but of which the validity is sufficiently proved by what precedes.

(2.) Also (comp. 158),

$$\frac{1}{\mathrm{U}q} = \frac{\mathrm{K}q}{\mathrm{T}q} = \mathrm{K}\frac{q}{\mathrm{T}q} = \mathrm{KU}q; \qquad \mathrm{K}\frac{1}{\mathrm{U}q} = \frac{q}{\mathrm{T}q} = \mathrm{U}q.$$

(3.) The reciprocal of a quaternion, and the conjugate* of that reciprocal, may now be thus expressed:

$$\frac{1}{q} = \frac{\mathrm{K}q}{\mathrm{T}q^2} = \frac{\mathrm{K}q}{\mathrm{N}q} = \frac{\mathrm{KU}q}{\mathrm{T}q} = \frac{1}{\mathrm{U}q} \cdot \frac{1}{\mathrm{T}q} = \frac{1}{\mathrm{T}q} \cdot \frac{1}{\mathrm{U}q};$$

$$\mathrm{K}\frac{1}{q} = \frac{q}{\mathrm{N}q} = \frac{q}{\mathrm{T}q^2} = \frac{\mathrm{U}q}{\mathrm{T}q} = \frac{1}{\mathrm{K}q}.$$

(4.) We may also write, generally,

$$\text{IX.} \ldots \mathrm{K}q = \mathrm{T}q \,.\, \mathrm{KU}q = \mathrm{N}q : q.$$

191. In general, let *any two* quaternions, q and q', be considered as multiplicand and multiplier, and let them be reduced (by 120) to the forms $\beta : a$ and $\gamma : \beta$; then the tensor and versor of that *third* quaternion, $\gamma : a$, which is (by 107) their *product* $q'q$, may be thus expressed:

$$\text{I.} \ldots \mathrm{T}q'q = \mathrm{T}(\gamma : a) = \mathrm{T}\gamma : \mathrm{T}a = (\mathrm{T}\gamma : \mathrm{T}\beta).(\mathrm{T}\beta : \mathrm{T}a) = \mathrm{T}q'.\mathrm{T}q;$$

$$\text{II.} \ldots \mathrm{U}q'q = \mathrm{U}(\gamma : a) = \mathrm{U}\gamma : \mathrm{U}a = (\mathrm{U}\gamma : \mathrm{U}\beta).(\mathrm{U}\beta : \mathrm{U}a) = \mathrm{U}q'.\mathrm{U}q;$$

where $\mathrm{T}q'q$ and $\mathrm{U}q'q$ are written, for simplicity, instead of $\mathrm{T}(q'.q)$ and $\mathrm{U}(q'.q)$. Hence, in any such multiplication, the *tensor of the product* is the *product of the tensors*; and the *versor of the product* is the *product of the versors*; the *order* of the factors being generally *retained* for the *latter* (comp. 168, &c.), although it may be *varied* for the *former*, on account of the *scalar* character of a *tensor*. In like manner, for the *division* of any one quaternion q', by any other q, we have the analogous formulæ:

$$\text{III.} \ldots \mathrm{T}(q' : q) = \mathrm{T}q' : \mathrm{T}q; \qquad \text{IV.} \ldots \mathrm{U}(q' : q) = \mathrm{U}q' : \mathrm{U}q;$$

or in words, the *tensor of the quotient* of any two quaternions is equal to the *quotient of the tensors*; and similarly, the *versor of the quotient* is equal to the *quotient of the versors*. And because multiplication and division of *tensors* are performed according to the rules of *algebra*, or rather of *arithme-*

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* Compare Art. 145, and the Note to page 127.

tic (a tensor being always, by what precedes, a *positive number*), we see that the difficulty (whatever it may be) of the general *multiplication and division of quaternions* is thus reduced to that of the corresponding *operations on versors:* for which *latter* operations geometrical *constructions* have been assigned, in the ninth Section of the present Chapter.

(1.) The two products, $q'q$ and qq', of any two quaternions taken as factors in two different orders, are *equal* or *unequal*, according as those two factors are *complanar* or *diplanar;* because such equality (169), or inequality (168), has been already proved to exist, for the case* when each tensor is unity: but we have always (comp. 178),

$$\mathrm{T}q'q = \mathrm{T}qq', \quad \text{and} \quad \angle q'q = \angle qq'.$$

(2.) If $\angle q = \angle q' = \frac{\pi}{2}$, then $qq' = \mathrm{K}q'q$ (170); so that the products of two *right* quotients, or right quaternions (182), taken in *opposite orders*, are always *conjugate* quaternions.

(3.) If $\angle q = \angle q' = \frac{\pi}{2}$, and $\mathrm{Ax}\,.\,q' \perp \mathrm{Ax}\,.\,q$, then $qq' = -q'q$,

$$\angle qq' = \angle q'q = \frac{\pi}{2}, \qquad \mathrm{Ax}\,.\,q'q \perp \mathrm{Ax}\,.\,q, \qquad \mathrm{Ax}\,.\,q'q \perp \mathrm{Ax}\,.\,q' \text{ (171)};$$

so that *the product of two right quaternions, in two rectangular planes, is a third right quaternion, in a plane rectangular to both;* and is *changed to its own opposite*, when the *order* of the factors is *reversed:* as we had $ij = k = -ji$ (182).

(4.) In general, if q and q' be any two *diplanar* quaternions, the *rotation* round $\mathrm{Ax}\,.\,q'$, from $\mathrm{Ax}\,.\,q$ to $\mathrm{Ax}\,.\,q'q$, is *positive* (177).

(5.) Under the same condition, $q(q' : q)$ is a quaternion with the *same tensor*, and *same angle*, as q', but with a *different axis;* and this new axis, $\mathrm{Ax}\,.\,q(q' : q)$, may be derived (179, (1.)) from the old axis, $\mathrm{Ax}\,.\,q'$, by a *conical rotation* (in the positive direction) round $\mathrm{Ax}\,.\,q$, through an angle $= 2\angle q$.

(6.) The product or quotient of two *complanar* quaternions is, in general, a *third* quaternion complanar with both; but if they be both scalar, or both right, then this product or quotient *degenerates* (131) into a scalar.

(7.) Whether q and q' be complanar or diplanar, we have always as in algebra (comp. 106, 107, 136) the two identical equations:

$$\text{V.} \ldots (q' : q) \cdot q = q'; \qquad \text{VI.} \ldots (q' \cdot q) : q = q'.$$

(8.) Also, by 190, V., and 191, I., we have this other general formula:

$$\text{VII.} \ldots \mathrm{N}q'q = \mathrm{N}q' \,.\, \mathrm{N}q;$$

or in words, the *norm of the product* is equal to the *product of the norms*.

192. Let $q = \beta : \alpha$, and $q' = \gamma : \beta$, as before; then

$$1 : q'q = 1 : (\gamma : \alpha) = \alpha : \gamma = (\alpha : \beta) \,.\, (\beta : \gamma) = (1 : q) \,.\, (1 : q');$$

so that the *reciprocal of the product* of any two quaternions is equal to the *product of the reciprocals*, taken in an *inverted order:* or briefly,

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* Compare the Notes to pages 148, 151.

$$\text{I.}\ldots\ \mathrm{R}q'q = \mathrm{R}q\,.\,\mathrm{R}q',$$

if R be again used (as in 161, (3.)) as a (temporary) *characteristic of reciprocation.* And because we have then (by the same sub-article) the symbolical equation, $\mathrm{KU} = \mathrm{UR}$, or in words, the *conjugate of the versor* of any quaternion q is equal (158) to the *versor of the reciprocal* of that quaternion; while the *versor of a product* is equal (191) to the product of the versors: we see that

$$\mathrm{KU}q'q = \mathrm{UR}q'q = \mathrm{UR}q\,.\,\mathrm{UR}q' = \mathrm{KU}q\,.\,\mathrm{KU}q'.$$

But

$$\mathrm{K}q = \mathrm{T}q\,.\,\mathrm{KU}q,\ \text{by 190, IX.; and}\ \mathrm{T}q'q = \mathrm{T}q'.\mathrm{T}q = \mathrm{T}q.\mathrm{T}q',$$

by 191; we arrive then thus at the following other important and general formula:

$$\text{II.}\ldots\ \mathrm{K}q'q = \mathrm{K}q\,.\,\mathrm{K}q';$$

or in words, the *conjugate of the product* of any two quaternions is equal to the *product of the conjugates*, taken (still) in an *inverted order.*

(1.) These two results, I., II., may be illustrated, for *versors* ($\mathrm{T}q = \mathrm{T}q' = 1$), by the consideration of a *spherical triangle* ABC (comp. Fig. 48); in which the sides AB and BC (comp. 167) may *represent* q and q', the arc AC then representing $q'q$. For then the new multiplier $\mathrm{R}q = \mathrm{K}q$ (158) is represented (162) by BA, and the new multiplicand $\mathrm{R}q' = \mathrm{K}q'$ by CB; whence the new product, $\mathrm{R}q\,.\,\mathrm{R}q' = \mathrm{K}q\,.\,\mathrm{K}q'$, is represented by the *inverse arc* CA, and is therefore at once the *reciprocal* $\mathrm{R}q'q$, and the *conjugate* $\mathrm{K}q'q$, of the old product $q'q$.

(2.) If q and q' be *right* quaternions, then $\mathrm{K}q = -q$, $\mathrm{K}q' = -q'$ (by 144); and the recent formula II. becomes, $\mathrm{K}q'q = qq'$, as in 170.

(3.) In general, that formula II. (of 192) may be thus written:

$$\text{III.}\ldots\ \mathrm{K}\frac{\gamma}{\alpha} = \mathrm{K}\frac{\beta}{\alpha}\,.\,\mathrm{K}\frac{\gamma}{\beta};$$

where α, β, γ may denote *any three vectors.*

(4.) Suppose then that, as in the annexed Fig. 49, we have the two following relations of *inverse similitude* of triangles (118),

$$\Delta\,\mathrm{AOB} \propto' \mathrm{BOC}, \qquad \Delta\,\mathrm{BOE} \propto' \mathrm{DOB};$$

and therefore (by 187) the two equations,

$$\frac{\gamma}{\beta} = \mathrm{K}\frac{\beta}{\alpha}, \qquad \frac{\beta}{\varepsilon} = \mathrm{K}\frac{\varepsilon}{\beta};$$

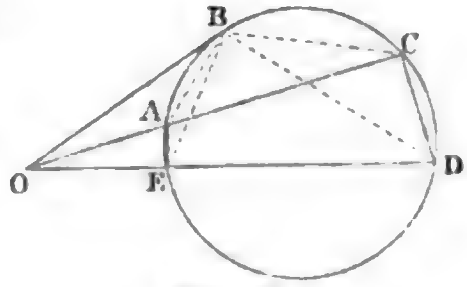

Fig. 49.

we shall have, by III.,

$$\frac{\gamma}{\delta} = \mathrm{K}\frac{\varepsilon}{\alpha}, \quad \text{or} \quad \Delta\,\mathrm{DOC} \propto' \mathrm{AOE};$$

so that this *third* formula of inverse similitude is a *consequence* from the other *two*.

(5.) If then (comp. 145, (6.)) *any two circles*, whether in one plane or in space, *touch* one another at a point B; and if from any point O, on the *common tangent* BO, *two secants* OAC, OED be drawn, to these two circles; the *four points of section*, A, C, D, E, will be *on one common circle*: for such *concircularity* is an easy consequence (through *equal angles*, &c.), from the last *inverse similitude*.

(6.) The same conclusion (respecting concircularity, &c.) may be otherwise and *geometrically* drawn, from the equality of the *two rectangles*, AOC and DOE, each being equal to the *square* of the *tangent* OB; which may serve as an instructive *verification* of the recent formula III., and as an example of the *consistency* of the results, to which calculations with quaternions conduct.

(7.) It may be noticed that the construction would *in general* give *three circles*, although only *one* is drawn in the Figure; but that if the two triangles ABC and DBE be situated *in different planes*, then these three circles, and of course the *five points* ABCDE, are situated *on one common sphere*.

193. An important application of the foregoing general theory of Multiplication and Division, is to the case of *Right Quaternions* (132), taken in connexion with their *Index-Vectors*, or *Indices* (133).

Considering *division* first, and employing the general formula of 106, let β and γ be each $\perp\ \alpha$; and let β' and γ' be the respective indices of the two right quotients, $q = \beta : \alpha$, and $q' = \gamma : \alpha$. We shall thus have the two complanarities, $\beta' \mathbin{|||} \beta, \gamma$, and $\gamma' \mathbin{|||} \beta, \gamma$ (comp. 123), because the four lines β, γ, β', γ' are all perpendicular to α; and within their common plane it is easy to see, from definitions already given, that these four lines form a *proportion of vectors*, in the same sense in which α, β, γ, δ did so, in the fourth Section of the present Chapter: so that we may write the *equation of quotients*,

$$\gamma' : \beta' = \gamma : \beta.$$

In fact, we have (by 133, 185, 187) the following relations of *length*,

$$\mathrm{T}\beta' = \mathrm{T}\beta : \mathrm{T}\alpha, \quad \mathrm{T}\gamma' = \mathrm{T}\gamma : \mathrm{T}\alpha, \text{ and } \therefore \mathrm{T}(\gamma' : \beta') = \mathrm{T}(\gamma : \beta);$$

while the relation of *directions*, expressed by the formula,

$$\mathrm{U}(\gamma' : \beta') = \mathrm{U}(\gamma : \beta), \quad \text{or} \quad \mathrm{U}\gamma' : \mathrm{U}\beta' = \mathrm{U}\gamma : \mathrm{U}\beta,$$

is easily established by means of the equations,

$$\angle(\gamma' : \gamma) = \angle(\beta' : \beta) = \frac{\pi}{2}; \qquad \mathrm{Ax} \, . \, (\gamma' : \gamma) = \mathrm{Ax} \, . \, (\beta' : \beta) = \mathrm{U}\alpha.$$

We arrive, then, at this general Theorem (comp. again 133): that "*the Quotient of any two Right Quaternions is equal to the Quotient of their Indices.*"[*]

(1.) For example (comp. 150, 159, 181), the indices of the right versors i, j, k are the *axes* of those three versors, namely, the lines OI, OJ, OK; and we have the equal quotients,

$$j : i = \mathrm{OI} : \mathrm{OJ}' = k = \mathrm{OJ} : \mathrm{OI}, \text{ \&c.}$$

(2.) In like manner, the indices of $-i, -j, -k$ are OI′, OJ′, OK′; and

$$i : -j = \mathrm{OJ}' : \mathrm{OI}' = k = \mathrm{OI} : \mathrm{OJ}', \text{ \&c.}$$

(3.) In general the *quotient of any two right versors* is equal to the *quotient of their axes*; as the theory of *representative arcs*, and of their *poles*, may easily serve to illustrate.

194. As regards the *multiplication* of two right quaternions, in connexion with their indices, it may here suffice to observe that, by 106 and 107, the *product* $\gamma : \alpha = (\gamma : \beta) \, . \, (\beta : \alpha)$ is equal (comp. 136) to the *quotient*, $(\gamma : \beta) : (\alpha : \beta)$; whence it is easy to infer that "*the Product, $q'q$, of any two Right Quaternions, is equal to the Quotient of the Index of the Multiplier, q', divided by the Index of the Reciprocal of the Multiplicand, q.*"

It follows that the *plane*, whether of the product or of the quotient of two right quaternions, coincides with the *plane of their indices*; and therefore also with the plane of their *axes*; because we have, generally, by principles already established, the transformation,

$$\text{if } \angle q = \frac{\pi}{2}, \text{ then } \textit{Index of } q = \mathrm{T}q \, . \, \mathrm{Ax} \, . \, q.$$

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[*] We have thus a new point of *agreement*, or of *connexion*, between *right quaternions*, and their *index-vectors*, tending to justify the ultimate assumption (not yet made), of *equality* between the former and the latter. In fact, we shall soon *prove* that the *index of the sum* (or difference), of any two right quotients (182), is equal to the *sum* (or difference) *of their indices*; and shall find it convenient subsequently to *interpret* the *product $\beta\alpha$ of any two vectors*, as being the *quaternion-product* (194) *of the two right quaternions*, of which those *two lines* are the *indices* (133): after which, the above-mentioned assumption of *equality* will appear natural, and be found to be useful. (Compare the Notes to pages 119, 136.)

SECTION 12.—*On the Sum or Difference of any two Quaternions; and on the Scalar (or Scalar Part) of a Quaternion.*

195. The *Addition* of any given quaternion q', considered as a geometrical quotient or *fraction* (101), to any other given quaternion q, considered also as a fraction, can always be accomplished by the first general formula of Art. 106, when these two fractions have a *common denominator;* and if they be not already *given* as having such, they can always be *reduced* so as to have one, by the process of Art. 120. And because the *addition of any two lines* was early seen to be a *commutative operation* (7, 9), so that we have always $\gamma + \beta = \beta + \gamma$, it follows (by 106) that the addition of any two quaternions is *likewise* a commutative operation, or in symbols, that

$$\text{I.}\ \ldots\ q + q' = q' + q\,;$$

so that the SUM *of any two* Quaternions* has a *Value*, which is independent of their *Order:* and which (by what precedes) must be considered to be *given*, or at least *known*, or *definite*, when the two *summand* quaternions are given. It is easy also to see that the *conjugate* of any such *sum* is equal to the *sum of the conjugates*, or in symbols, that

$$\text{II.}\ \ldots\ \mathrm{K}(q' + q) = \mathrm{K}q' + \mathrm{K}q.$$

(1.) The important formula last written becomes geometrically evident, when it is presented under the following form. Let OBDC be any parallelogram, and let OA be any right line, drawn from one corner of it, but not generally in its plane. Let the three other corners, B, C, D, be *reflected* (in the sense of 145, (5.)) with respect to that line OA, into three new points, B′, C′, D′; or let the three lines OB, OC, OD be reflected (in the sense of 138) with respect to the same line OA; which thus bisects at right angles the three joining lines, BB′, CC′, DD′, as it does BB′ in Fig. 36. Then *each* of the *lines* OB, OC, OD, and therefore also the whole *plane figure* OBDC, may be considered to have simply *revolved* round the line OA as an *axis*, by a *conical rotation* through *two right angles;* and consequently the *new figure* OB′D′C′, like that *old* one OBDC, must be a *parallelogram*. Thus (comp. 106, 137), we have

$$\mathrm{OD}' = \mathrm{OC}' + \mathrm{OB}', \qquad \delta' = \gamma' + \beta', \qquad \delta' : \alpha = (\gamma' : \alpha) + (\beta' : \alpha)\,;$$

and the recent formula II. is justified.

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* It will be found that this result admits of being extended to the case of *three* (or *more*) quaternions; but, for the moment, we content ourselves with *two*.

(2.) Simple as this last reasoning is, and unnecessary as it appears to be to draw any new Diagram to illustrate it, the reader's attention may be once more invited to the great *simplicity of expression*, with which many important *geometrical conceptions*, respecting *space of three dimensions*, are *stated* in the present Calculus: and are thereby kept *ready* for future application, and for easy combination with *other* results of the same kind. Compare the remarks already made in 132, (6.); 145, (10.); 161; 179, (3.); 192, (6.); and some of the shortly following sub-articles to 196, respecting properties of an *oblique cone* with circular base.

196. One of the most important *cases* of *addition*, is that of *two conjugate summands*, q and $\mathrm{K}q$; of which it has been seen (in 140) that the *sum* is always a *scalar*. We propose now to denote the *half* of this sum by the *symbol*,

$$\mathrm{S}q;$$

thus writing generally,

$$\text{I.} \ldots q + \mathrm{K}q = \mathrm{K}q + q = 2\mathrm{S}q;$$

or *defining* the new symbol $\mathrm{S}q$ by the formula,

$$\text{II.} \ldots \mathrm{S}q = \tfrac{1}{2}(q + \mathrm{K}q); \text{ or briefly, } \text{II}'. \ldots \mathrm{S} = \tfrac{1}{2}(1 + \mathrm{K}).$$

For reasons which will soon more fully appear, we shall also call this new quantity, $\mathrm{S}q$, the *scalar part*, or simply *the* SCALAR, *of the Quaternion*, q; and shall therefore call the letter S, thus used, the *Characteristic of the Operation of taking the Scalar* of a quaternion. (Comp. 132, (6.); 137; 156; 187.) It follows that not only *equal quaternions*, but also *conjugate quaternions*, have *equal scalars*; or in symbols,

$$\text{III.} \ldots \mathrm{S}q' = \mathrm{S}q, \quad \text{if} \quad q' = q; \quad \text{and} \quad \text{IV.} \ldots \mathrm{SK}q = \mathrm{S}q;$$

or briefly,

$$\text{IV}'. \ldots \mathrm{SK} = \mathrm{S}.$$

And because we have seen that $\mathrm{K}q = +q$, if q be a *scalar* (139), but that $\mathrm{K}q = -q$, if q be a *right quotient* (144), we find that the *scalar of a scalar* (considered as a *degenerate quaternion*, 131) is equal to that scalar *itself*, but that the *scalar of a right quaternion* is *zero*. We may therefore now write (comp. 160):

$$\text{V.} \,.\, \mathrm{S}x = x, \text{ if } x \text{ be a scalar}; \qquad \text{VI.} \,.\, .\mathrm{SS}q = \mathrm{S}q, \ \mathrm{S}^2 = \mathrm{SS} = \mathrm{S};$$

and

$$\text{VII.} \ldots \mathrm{S}q = 0, \quad \text{if} \quad \angle q = \frac{\pi}{2}.$$

Again, because OA′ in Fig. 36 is multiplied by x, when OB is multiplied thereby, we may write, generally,

$$\text{VIII.} \ldots \mathrm{S}xq = x\mathrm{S}q, \text{ if } x \text{ be any scalar};$$

and therefore in particular (by 188),

$$\text{IX.} \ldots \mathrm{S}q = \mathrm{S}(\mathrm{T}q.\mathrm{U}q) = \mathrm{T}q.\mathrm{SU}q.$$

Also because $\mathrm{SK}q = \mathrm{S}q$, by IV., while $\mathrm{KU}q = \mathrm{U}\frac{1}{q}$, by 158, we have the general equation,

$$\text{X.} \ldots \mathrm{SU}q = \mathrm{SU}\frac{1}{q}; \quad \text{or} \quad \text{X}'. \ldots \mathrm{SU}\frac{\beta}{\alpha} = \mathrm{SU}\frac{\alpha}{\beta};$$

whence, by IX.,

$$\text{XI.} \ldots \mathrm{S}q = \mathrm{T}q.\mathrm{SU}\frac{1}{q}; \quad \text{or} \quad \text{XI}'. \ldots \mathrm{S}\frac{\beta}{\alpha} = \mathrm{T}\frac{\beta}{\alpha}.\mathrm{SU}\frac{\alpha}{\beta};$$

and therefore also, by 190, (V.), since $\mathrm{T}q.\mathrm{T}\frac{1}{q} = 1$,

$$\text{XII.} \ldots \mathrm{S}q = \mathrm{T}q^2.\mathrm{S}\frac{1}{q} = \mathrm{N}q.\mathrm{S}\frac{1}{q}; \quad \text{XII}'. \ldots \mathrm{S}\frac{\beta}{\alpha} = \mathrm{N}\frac{\beta}{\alpha}.\mathrm{S}\frac{\alpha}{\beta}.$$

The results of 142, combined with the recent definition I. or II., enable us to extend the recent formula VII., by writing,

$$\text{XIII.} \ldots \mathrm{S}q >, =, \text{ or } < 0, \text{ according as } \angle q <, =, \text{ or } > \frac{\pi}{2};$$

and conversely,

$$\text{XIV.} \ldots \angle q <, =, \text{ or } > \frac{\pi}{2}, \text{ according as } \mathrm{S}q >, =, \text{ or } < 0.$$

In fact, if we compare that *definition* I. with the formula of 140, and with Fig. 36, we see at once that because, in that Figure,

$$\mathrm{S}(\mathrm{OB} : \mathrm{OA}) = \mathrm{OA}' : \mathrm{OA},$$

we may write, generally,

$$\text{XV.} \ldots \mathrm{S}q = \mathrm{T}q.\cos\angle q; \quad \text{or} \quad \text{XVI.} \ldots \mathrm{SU}q = \cos\angle q;$$

equations which will be found of great importance, as serving to *connect quaternions with trigonometry;* and which show that

$$\text{XVII.} \ldots \angle q' = \angle q, \quad \text{if} \quad \mathrm{SU}q' = \mathrm{SU}q,$$

the angle $\angle q$ being still taken (as in 130), so as not to fall outside the limits 0 and π; whence also,

$$\text{XVIII.} \ldots \angle q' = \angle q, \quad \text{if} \quad Sq' = Sq, \quad \text{and} \quad Tq' = Tq,$$

the *angle of a quaternion* being thus *given*, when the *scalar* and the *tensor* of that quaternion are given, or known. Finally because, in the same Figure 36 (comp. 15, 103), the *line*,

$$\text{OA}' = (\text{OA}' : \text{OA}) . \text{OA} = \text{OA} . S(\text{OB} : \text{OA}),$$

may be said to be the *projection* of OB on OA, since A′ is the *foot of the perpendicular* let fall from the *point* B upon this latter line OA, we may establish this other general formula:

$$\text{XIX.} \ldots \alpha S\frac{\beta}{\alpha} = S\frac{\beta}{\alpha} . \alpha = \textit{projection of } \beta \textit{ on } \alpha;$$

a result which will be found to be of great utility, in investigations respecting *geometrical loci*, and which may be also written thus:

$$\text{XX.} \ldots \textit{Projection of } \beta \textit{ on } \alpha = U\alpha . T\beta . SU\frac{\beta}{\alpha};$$

with other transformations deducible from principles stated above. It is scarcely necessary to remark that, on account of the *scalar* character of Sq, we have, generally, by 159, and 187, (8.), the expressions,

$$\text{XXI.} \ldots USq = \pm 1; \qquad \text{XXII.} \ldots TSq = \pm Sq;$$

while, for the same reason, we have always, by 139, the equation (comp. IV.),

$$\text{XXIII.} \ldots KSq = Sq; \quad \text{or} \quad \text{XXIII}'. \ldots KS = S;$$

and, by 131,

$$\text{XXIV.} \ldots \angle Sq = 0, \text{ or } = \pi, \text{ unless } \angle q = \frac{\pi}{2};$$

in which last case $Sq = 0$, by VII., and therefore $\angle Sq$ is indeterminate:* USq becoming at the same time indeterminate, by 159, but TSq vanishing, by 186, 187.

(1.) The equation,

$$S\frac{\rho}{\alpha} = 0,$$

is now seen to be equivalent to the formula, $\rho \perp \alpha$; and therefore to denote the

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* Compare the Note in page 118, to Art. 131.

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same plane locus for P, as that which is represented by any one of the four other equations of 186, (6.); or by the equation,

$$T\frac{\rho+\alpha}{\rho-\alpha}=1, \text{ of } 187, (2.).$$

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(2.) The equation,

$$S\frac{\rho-\beta}{\alpha}=0, \quad \text{or} \quad S\frac{\rho}{\alpha}=S\frac{\beta}{\alpha},$$

expresses that BP ⊥ OA; or that the *points* B and P have the *same projection* on OA; or that the *locus* of P is the *plane through* B, *perpendicular to the line* OA.

(3.) The equation,

$$SU\frac{\rho}{\alpha}=SU\frac{\beta}{\alpha},$$

expresses (comp. 132, (2.)) that P is on *one sheet* of a *cone of revolution*, with O for *vertex*, and OA for *axis*, and passing *through the point* B.

(4.) The *other sheet* of the *same cone* is represented by this other equation,

$$SU\frac{\rho}{\alpha}=-SU\frac{\beta}{\alpha};$$

and *both sheets* jointly by the equation,

$$\left(SU\frac{\rho}{\alpha}\right)^2=\left(SU\frac{\beta}{\alpha}\right)^2.$$

(5.) The equation,

$$S\frac{\rho}{\alpha}=1, \quad \text{or} \quad SU\frac{\rho}{\alpha}=T\frac{\alpha}{\rho},$$

||—

expresses that the *locus* of P is the *plane through* A, *perpendicular to the line* OA; because it expresses (comp. XIX.) that the *projection* of OP on OA is the line OA *itself*; or that the *angle* OAP is *right*; or that $S\frac{\rho-\alpha}{\alpha}=0$.

(6.) On the other hand the equation,

$$S\frac{\beta}{\rho}=1, \quad \text{or} \quad SU\frac{\beta}{\rho}=T\frac{\rho}{\beta},$$

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expresses that the projection of OB on OP is OP itself; or that the angle OPB is right; or that the locus of P is that *spheric surface*, which has the line OB for a *diameter*.

(7.) Hence the *system* of the *two equations*,

$$S\frac{\rho}{\alpha}=1, \qquad S\frac{\beta}{\rho}=1,$$

represents the *circle*, in which the *sphere* (6.), with OB for a *diameter*, is *cut* by the *plane* (5.), with OA for the *perpendicular* let fall on it from O.

(8.) And therefore this new equation,

$$S\frac{\rho}{\alpha}\cdot S\frac{\beta}{\rho}=1,$$

obtained by multiplying the two last, represents the *Cyclic*[*] *Cone* (or *cone of the*

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[*] Historically speaking, the *oblique cone with circular base* may deserve to be named the *Apollonian Cone*, from Apollonius of Perga, in whose great work on Co-

second order, but *not generally* of *revolution*), which *rests* on this last *circle* (7.) as its *base*, and has the point O for its *vertex*. In fact, the equation (8.) is evidently satisfied, when the two equations (7.) are so; and therefore *every point* of the *circular circumference*, denoted by those two equations, must be *a point of the locus*, represented by the equation (8.). But the latter equation remains unchanged, at least essentially, when ρ is changed to $x\rho$, x being *any scalar*; the locus (8.) is, therefore, *some conical surface*, with its *vertex* at the *origin*, O; and consequently it can be none other than *that particular cone* (both ways prolonged), which *rests* (as above) on the given *circular base* (7.).

(9.) The system of the *two* equations,

$$\mathrm{S}\frac{\rho}{\alpha}.\mathrm{S}\frac{\beta}{\rho}=1, \qquad \mathrm{S}\frac{\rho}{\gamma}=1,$$

(in writing the first of which the *point* may be omitted,) represents a *conic section*; namely *that* section, in which the *cone* (8.) is *cut* by the new *plane*, which has OC for the *perpendicular* let fall upon it, from the origin of vectors O.

(10.) Conversely, *every plane ellipse* (or other *conic section*) *in space*, of which the plane does not pass through the origin, may be represented by a *system* of two equations, of this last *form* (9.); because the *cone* which rests on any such *conic* as its *base*, and has its *vertex* at *any* given point O, is known to be a *cyclic* cone.

(11.) The *curve* (or rather the *pair of curves*), in which an *oblique* but *cyclic cone* (8.) is cut by a *concentric sphere* (that is to say, a cone resting on a circular base by a sphere which has its centre at the vertex of that cone), has come, in modern times, to be called a *Spherical Conic*. And any *such* conic may, on the foregoing plan, be represented by the system of the two equations,

$$\mathrm{S}\frac{\rho}{\alpha}\,\mathrm{S}\frac{\beta}{\rho}=1, \qquad \mathrm{T}\rho=1;$$

the length of the *radius* of the sphere being here, for simplicity, supposed to be the *unit* of length. But, by writing $\mathrm{T}\rho=a$, where a may denote any constant and positive scalar, we can at once remove this last restriction, if it be thought useful or convenient to do so.

(12.) The equation (8.) may be written, by XII. or XII'., under the form (comp. 191, VII.):

$$\mathrm{S}\frac{\alpha}{\rho}\,\mathrm{S}\frac{\rho}{\beta}=\mathrm{N}\frac{\alpha}{\beta}=\left(\mathrm{T}\frac{\alpha}{\beta}\right)^2;$$

or briefly,

$$\mathrm{S}\frac{\beta'}{\rho}\,\mathrm{S}\frac{\rho}{\alpha'}=1,$$

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nics (*κωνικῶν*), already referred to in a Note to page 128, the properties of *such* a cone appear to have been first treated systematically; although the *cone of revolution* had been studied by Euclid. But the designation "*cyclic cone*" is shorter; and it seems more natural, *in geometry*, to speak of the above-mentioned oblique cone *thus*, for the purpose of marking its connexion with the *circle*, than to call it, as is now usually done, a *cone of the second order*, or of the *second degree:* although these phrases also have their advantages.

$$\text{if } \alpha' = \beta\mathrm{T}\frac{\alpha}{\beta} = \mathrm{T}\alpha\,.\,\mathrm{U}\beta, \quad\text{and}\quad \beta' = \alpha\mathrm{T}\frac{\beta}{\alpha} = \mathrm{T}\beta\,.\,\mathrm{U}\alpha\,;$$

so that α' and β' are here the lines OA′ and OB′, of Art. 188, and Fig. 48.

(13.) Hence the cone (8.) is cut, not only by the plane (5.) in the circle (7.), which is on the sphere (6.), but also by the (generally) *new plane*, $\mathrm{S}\frac{\rho}{\alpha'} = 1$, in the (generally) *new circle*, in which this new plane cuts the (generally) *new sphere*, $\mathrm{S}\frac{\beta'}{\rho} = 1$; or in the circle which is represented by the system of the two equations,

$$\mathrm{S}\frac{\rho}{\alpha'} = 1, \qquad \mathrm{S}\frac{\beta'}{\rho} = 1.$$

(14.) In the *particular case* when $\beta \parallel \alpha$ (15), so that the quotient $\beta : \alpha$ is a *scalar*, which must be positive and greater than unity, in order that the plane (5.) may (*really*) *cut* the sphere (6.), and therefore that the circle (7.) and the cone (8.) may be *real*, we may write

$$\beta = a^2\alpha, \qquad a > 1, \qquad \mathrm{T}(\beta : \alpha) = a^2, \qquad \alpha' = \alpha, \qquad \beta' = \beta\,;$$

and the circle (13.) *coincides* with the circle (7.).

(15.) In the same *case*, the cone is one of *revolution*; every point P of its circular *base* (that is, of the *circumference* thereof) being at one *constant distance from the vertex* O, namely at a distance $= a\mathrm{T}\alpha$. For, in the case supposed, the equations (7.) give, by XII.,

$$\mathrm{N}\frac{\rho}{\alpha} = \mathrm{S}\frac{\rho}{\alpha} : \mathrm{S}\frac{\alpha}{\rho} = 1 : \mathrm{S}\frac{\alpha}{\rho} = a^2 : \mathrm{S}\frac{\beta}{\rho} = a^2\,; \quad\text{or}\quad \mathrm{T}\rho = a\mathrm{T}\alpha.$$

(Compare 145, (12.), and 186, (5.).)

(16.) Conversely, if the cone *be* one of revolution, the equations (7.) must conduct to a result of the form,

$$a^2 = \mathrm{N}\frac{\rho}{\alpha} = \mathrm{S}\frac{\rho}{\alpha} : \mathrm{S}\frac{\alpha}{\rho} = \mathrm{S}\frac{\beta}{\rho} : \mathrm{S}\frac{\alpha}{\rho}, \text{or} \quad (\text{comp. } (2.)),\quad \mathrm{S}\frac{\beta - a^2\alpha}{\rho} = 0\,;$$

which can only be by the line $\beta - a^2\alpha$ vanishing, or by our having $\beta = a^2\alpha$, as in (14.); since otherwise we should have, by XIV., $\rho \perp \beta - a^2\alpha$, and *all the points of the base* would be situated *in one plane* passing *through the vertex* O, which (for any actual cone) would be absurd.

(17.) Supposing, then, that we have *not* $\beta \parallel \alpha$, and therefore *not* $\alpha' = \alpha$, $\beta' = \beta$, as in (14.), nor even $\alpha' \parallel \alpha$, $\beta' \parallel \beta$, we see that the cone (8.) is *not* a cone of *revolution* (or what is often called a *right cone*); but that it is, on the contrary, an *oblique* (or *scalene*) *cone*, although still a *cyclic* one. And we see that *such* a cone is cut in *two distinct series** *of circular sections*, by planes parallel to the two distinct (and mutually non-parallel) planes, (5.) and (13.); or to *two new planes*, drawn *through the vertex* O, which have been called† the *two Cyclic Planes* of the cone, namely, the two following:

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* These *two series* of *sub-contrary* (or *antiparallel*) but *circular sections* of a cyclic cone, appear to have been first discovered by Apollonius: see the Fifth Proposition of his First Book, in which he says, *καλείσθω δὲ ἡ τοιαύτη τομὴ ὑπεναντία* (page 22 of Halley's Edition).

† By M. Chasles.

$$\mathrm{S}\frac{\rho}{\alpha}=0; \qquad \mathrm{S}\frac{\rho}{\beta}=0;$$

while the *two lines* from the vertex, OA and OB, which are *perpendicular to these two planes* respectively, may be said to be the *two Cyclic Normals.*

(18.) Of these two *lines*, α and β, the *second* has been seen to be a *diameter* of the sphere (6.), which may be said to be *circumscribed* to the cone (8.), when that cone is considered as having the circle (7.) for its *base;* the *second cyclic plane* (17.) is therefore the *tangent plane at the vertex* of the cone, to that *first circumscribed sphere* (6.).

(19.) The sphere (13.) may in like manner be said to be circumscribed to the cone, if the latter be considered as resting on the new circle (13.), or as terminated by *that* circle as its *new base;* and the diameter of this *new sphere* is the line OB′, or β', which has by (12.) the *direction* of the line α, or of the *first cyclic normal* (17.); so that (comp. (18.)) the *first cyclic plane* is the *tangent plane* at the vertex, to the *second circumscribed sphere* (13.).

(20.) *Any other sphere* through the vertex, which *touches the first cyclic plane,* and which therefore has its *diameter from the vertex* $=b'\beta'$, where b' is some scalar co-efficient, is represented by the equation,

$$\mathrm{S}\frac{b'\beta'}{\rho}=1, \quad \text{or} \quad \mathrm{S}\frac{\beta'}{\rho}=\frac{1}{b'};$$

it therefore *cuts the cone in a circle*, of which (by (12.)) the *equation of the plane* is

$$\mathrm{S}\frac{\rho}{\alpha'}=b', \quad \text{or} \quad \mathrm{S}\frac{\rho}{b'\alpha'}=1,$$

so that the *perpendicular from the vertex* is $b'\alpha' \parallel \beta$ (comp. (5.)); and consequently this *plane of section* of sphere and cone is *parallel to the second cyclic plane* (17.).

(21.) In like manner *any sphere*, such as

$$\mathrm{S}\frac{b\beta}{\rho}=1, \text{ where } b \text{ is any scalar,}$$

which *touches the second cyclic plane* at the vertex, *intersects the cone* (8.) in a *circle*, of which the plane has for equation,

$$\mathrm{S}\frac{\rho}{b\alpha}=1,$$

and is therefore *parallel to the first cyclic plane.*

(22.) The equation of the cone (by IX., X., XVI.) may also be thus written:

$$\mathrm{SU}\frac{\rho}{\alpha}\,.\,\mathrm{SU}\frac{\beta}{\rho}=\mathrm{T}\frac{\alpha}{\beta}; \quad \text{or,} \quad \cos\angle\frac{\rho}{\alpha}\,.\,\cos\angle\frac{\rho}{\beta}=\mathrm{T}\frac{\alpha}{\beta};$$

it expresses, therefore, that *the product of the cosines of the inclinations, of any variable side* (ρ) *of an oblique cyclic cone, to two fixed lines* (α and β), namely to the *two cyclic normals* (17.), is *constant;* or that the *product of the sines of the inclinations*, of the same variable side (or *ray*, ρ) of the cone, *to two fixed planes*, namely to the *two cyclic planes*, is thus a constant quantity.

(23.) The *two great circles*, in which the *concentric sphere* $\mathrm{T}\rho=1$ is *cut by the two cyclic planes*, have been called the *two Cyclic Arcs** of the *Spherical Conic* (11.), in which that sphere is cut by the cone. It follows (by (22.)) that the *product of the sines of the* (*arcual*) *perpendiculars, let fall from any point* P *of a given spherical conic, on its two cyclic arcs, is constant.*

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* By M. Chasles.

(24.) These properties of *cyclic cones*, and of *spherical conics*, are not put forward as *new;* but they are of importance enough, and have been here deduced with sufficient facility, to show that we are already in possession of a *Calculus*, with its own *Rules** *of Transformation*, whereby *one enunciation* of a geometrical theorem, or problem, or construction, can be *translated* into several others, of which some may be clearer, or simpler, or more elegant, than the one first proposed.

197. Let α, β, γ be any three co-initial vectors, OA, &c., and let $\text{OD} = \delta = \gamma + \beta$, so that OBDC is a parallelogram (6); then, if we write

$$\beta : \alpha = q, \qquad \gamma : \alpha = q', \quad \text{and} \quad \delta : \alpha = q'' = q' + q \ (106),$$

and suppose that B′, C′, D′ are the feet of perpendiculars let fall from the points B, C, D on the line OA, we shall have, by 196, XIX., the expressions,

$$(\text{OB}' =)\ \beta' = \alpha \text{S}q, \qquad \gamma' = \alpha \text{S}q', \qquad \delta' = \alpha \text{S}q'' = \alpha \text{S}\,(q' + q).$$

But also $\text{OB} = \text{CD}$, and therefore $\text{OB}' = \text{C}'\text{D}'$, the *similar projections* of *equal lines* being *equal;* hence (comp. 11) the *sum of the projections* of the lines β, γ must be equal to the *projection of the sum*, or in symbols,

$$\text{OD}' = \text{OC}' + \text{OB}', \qquad \delta' = \gamma' + \beta', \qquad \delta' : \alpha = (\gamma' : \alpha) + (\beta' : \alpha).$$

Hence, generally, for *any two quaternions*, q and q', we have the formula:

$$\text{I.} \ldots \ \text{S}\,(q' + q) = \text{S}q' + \text{S}q;$$

or in words, the *scalar of the sum* is equal to the *sum of the scalars*. It is easy to extend this result to the case of any *three* (or more) quaternions, with their respective scalars; thus, if q'' be a third *arbitrary* quaternion, we may write

$$\text{S}\{q'' + (q' + q)\} = \text{S}q'' + \text{S}\,(q' + q) = \text{S}q'' + (\text{S}q' + \text{S}q);$$

where, on account of the *scalar* character of the summands, the last parentheses may be omitted. We may therefore write, generally,

$$\text{II.} \ldots \ \text{S}\Sigma q = \Sigma \text{S}q, \quad \text{or briefly,} \quad \text{S}\Sigma = \Sigma\text{S};$$

where Σ is used as a *sign of Summation:* and may say that

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* Comp. 145, (10.), &c.

the Operation of taking the Scalar of a Quaternion is a Distributive Operation (comp. 13). As to the general *Subtraction* of any one quaternion from any other, there is no difficulty in reducing it, by the method of Art. 120, to the second general formula of 106; nor in proving that the *Scalar of the Difference** is always equal to the *Difference of the Scalars*. In symbols,

$$\text{III.}\ldots \mathrm{S}(q'-q)=\mathrm{S}q'-\mathrm{S}q;$$

or briefly,

$$\text{IV.}\ldots \mathrm{S}\Delta q=\Delta \mathrm{S}q, \qquad \mathrm{S}\Delta=\Delta\mathrm{S};$$

when Δ is used as the characteristic of the operation of *taking a difference*, by *subtracting* one quaternion, or one scalar, from another.

(1.) It has not *yet* been proved (comp. 195), that the *Addition* of *any number* of Quaternions, $q, q', q'', \ldots$ is an *associative* and a *commutative operation* (comp. 9). But we see, already, that the *scalar of the sum* of any such set of quaternions has a *value*, which is *independent of their order*, and of the mode of *grouping them*.

(2.) If the summands be all *right* quaternions (132), the scalar of *each* separately vanishes, by 196, VII.; wherefore the scalar of their *sum* vanishes also, and that sum is consequently *itself*, by 196, XIV., a *right* quaternion: a result which it is easy to verify. In fact, if $\beta \perp \alpha$ and $\gamma \perp \alpha$, then $\gamma+\beta \perp \alpha$, because α is then perpendicular to the plane of β and γ; hence, by 106, the *sum of any two* right quaternions is a right quaternion, and therefore also the sum of *any number* of such quaternions.

(3.) Whatever two quaternions q and q' may be, we have always, as in algebra, the two *identities* (comp. 191, (7.)):

$$\text{V.}\ldots (q'-q)+q=q'; \qquad \text{VI.}\ldots (q'+q)-q=q'.$$

198. Without yet entering on the general theory of *scalars of products* or *quotients* of quaternions, we may observe here that because, by 196, XV., *the scalar of a quaternion depends only on the tensor and the angle*, and is *independent of the axis*, we are at liberty to write generally (comp. 173, 178, and 191, (1.), (5.)),

$$\text{I.}\ldots \mathrm{S}qq'=\mathrm{S}q'q; \qquad \text{II.}\ldots \mathrm{S}\,.\,q\,(q':q)=\mathrm{S}q';$$

the *two products*, qq' and $q'q$, having thus always *equal scalars*, although they have been seen to have *unequal axes*, for the general case of *diplanarity* (168, 191). It may also be noticed, that in virtue of what was shown in 193, respecting the quotient, and in 194 respecting the product, of any two *right* quaternions (132), in connexion with their *indices* (133), we may now establish, for any *such* quaternions, the formulæ:

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* Examples have already occurred in 196, (2.), (5.), (16.).

$$\text{III.} \ldots \mathrm{S}\,(q' : q) = \mathrm{S}\,(\mathrm{I}q' : \mathrm{I}q) = \mathrm{T}\,(q' : q) \,.\, \cos\angle\,(\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q);$$

$$\text{IV.} \ldots \mathrm{S}q'q = \mathrm{S}\,(q' \,.\, q) = \mathrm{S}\left(\mathrm{I}q' : \mathrm{I}\,\frac{1}{q}\right) = -\,\mathrm{T}q'q \,.\, \cos\angle\,(\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q);$$

where the new symbol $\mathrm{I}q$ is used, as a temporary abridgment, to denote the *Index* of the quaternion q, supposed here (as above) to be a *right* one. With the same supposition, we have therefore also these other and shorter formulæ:

$$\text{V.} \ldots \mathrm{SU}\,(q' : q) = +\cos\angle\,(\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q);$$

$$\text{VI.} \ldots \mathrm{SU}q'q = -\cos\angle\,(\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q);$$

which may, by 196, XVI., be interpreted as expressing that, under the same condition of *rectangularity* of q and q',

$$\text{VII.} \ldots \angle\,(q' : q) = \angle\,(\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q);$$

$$\text{VIII.} \ldots \angle\, q'q = \pi - \angle\,(\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q).$$

In words, *the Angle of the Quotient of two Right Quaternions is equal to the Angle of their Axes;* but the *Angle of the Product*, of two such quaternions, is equal to the *Supplement of the Angle of the Axes.* There is no difficulty in proving these results otherwise, by constructions such as that employed in Art. 193; nor in illustrating them by the consideration of isosceles quadrantal triangles, upon the surface of a sphere.

199. Another important *case* of the scalar of a *product*, is the case of the *scalar of the square* of a quaternion. On referring to Art. 149, and to Fig. 42, we see that while we have always $\mathrm{T}\,(q^2) = (\mathrm{T}q)^2$, as in 190, and $\mathrm{U}\,(q^2) = \mathrm{U}\,(q)^2$, as in 161, we have also,

$$\text{I.} \ldots \angle\,(q)^2 = 2\angle\, q, \quad \text{and} \quad \mathrm{Ax}.\,(q^2) = \mathrm{Ax}.\, q, \quad \text{if} \quad \angle\, q < \frac{\pi}{2};$$

but, by the adopted definitions of $\angle\, q$ (130), and of $\mathrm{Ax}.\, q$ (127, 128),

$$\text{II.} \ldots \angle\,(q^2) = 2\,(\pi - \angle\, q), \quad \mathrm{Ax}.\,(q^2) = -\,\mathrm{Ax}.\, q, \quad \text{if} \quad \angle\, q > \frac{\pi}{2}.$$

In *each* case, however, by 196, XVI., we may write,

$$\text{III.} \ldots \mathrm{SU}\,(q^2) = \cos\angle\,(q^2) = \cos 2\angle\, q;$$

a formula which holds even when $\angle q$ is 0, or $\frac{\pi}{2}$, or π, and which gives,

$$\text{IV.} \ldots \mathrm{SU}(q^2) = 2(\mathrm{SU}q)^2 - 1.$$

Hence, generally, the scalar of q^2 may be put under either of the two following forms:

$$\text{V.} \ldots \mathrm{S}(q^2) = \mathrm{T}q^2 . \cos 2\angle q; \qquad \text{VI.} \ldots \mathrm{S}(q^2) = 2(\mathrm{S}q)^2 - \mathrm{T}q^2;$$

where we see that it would not be safe to *omit the parentheses*, without some *convention* previously made, and to write simply $\mathrm{S}q^2$, without first deciding whether this last symbol shall be understood to signify the *scalar of the square*, or the *square of the scalar* of q: these two things being generally *unequal*. The *latter* of them, however, occurring rather *oftener* than the former, it appears convenient to fix on *it* as that which is to be understood by $\mathrm{S}q^2$, while the *other* may occasionally be written with a *point* thus, $\mathrm{S}\,.\,q^2$; and then, with these conventions respecting *notation*,[*] we may write:

$$\text{VII.} \ldots \mathrm{S}q^2 = (\mathrm{S}q)^2; \qquad \text{VIII.} \ldots \mathrm{S}\,.\,q^2 = \mathrm{S}(q^2).$$

But the *square of the conjugate* of any quaternion is easily seen to be the *conjugate of the square*; so that we have generally (comp. 190, II.) the formula:

$$\text{IX.} \ldots \mathrm{K}q^2 = \mathrm{K}(q^2) = (\mathrm{K}q)^2 = \mathrm{T}q^2 : \mathrm{U}q^2.$$

(1.) A quaternion, like a positive scalar, may be said to have in general *two opposite square roots*; because the *squares of opposite quaternions* are always *equal* (comp. (3.)). But of these two *roots* the *principal* (or *simpler*) one, and that which we shall denote by the symbol $\sqrt{q}$, or $\surd q$, and shall call by eminence *the Square Root* of q, is that which has its *angle acute*, and not *obtuse*. We shall therefore write, generally,

$$\text{X.} \ldots \angle\sqrt{q} = \tfrac{1}{2}\angle q; \qquad \mathrm{Ax}.\sqrt{q} = \mathrm{Ax}.\,q;$$

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[*] As, in the *Differential Calculus*, it is usual to write $\mathrm{d}x^2$ instead of $(\mathrm{d}x)^2$; while $\mathrm{d}(x^2)$ is sometimes written as $\mathrm{d}\,.\,x^2$. But as d^2x denotes a *second differential*, so it seems safest *not* to denote the square of $\mathrm{S}q$ by the symbol S^2q, which *properly* signifies $\mathrm{SS}q$, or $\mathrm{S}q$, as in 196, VI.; the *second scalar* (like the *second tensor*, 187, (9.), or the *second versor*, 160) being equal to the *first*. Still every calculator will of course use his own discretion; and the employment of the notation S^2q for $(\mathrm{S}q)^2$, as $\cos^2 x$ is often written for $(\cos x)^2$, may sometimes cause a *saving of space*.

with the reservation that, when $\angle q = 0$, or $= \pi$, this common *axis* of q and $\sqrt{q}$ becomes (by 131, 149) an *indeterminate* unit-line.

(2.) Hence,

$$\text{XI.} \ldots \mathrm{S}\sqrt{q} > 0, \quad \text{if} \quad \angle q < \pi;$$

while this *scalar of the square root* of a quaternion may, by VI., be thus transformed:

$$\text{XII.} \ldots \mathrm{S}\sqrt{q} = \sqrt{\{\tfrac{1}{2}(\mathrm{T}q + \mathrm{S}q)\}};$$

a formula which holds good, even at the limit $\angle q = \pi$.

(3.) The principle* (1.), that in quaternions, as in algebra, the equation,

$$\text{XIII.} \ldots (-q)^2 = q^2,$$

is an *identity*, may be illustrated by conceiving that, in Fig. 42, a point B′ is determined by the equation $\mathrm{OB}' = \mathrm{BO}$; for then we shall have (comp. Fig. 33, *bis*),

$$(-q)^2 = \left(\frac{\mathrm{OB}'}{\mathrm{OA}}\right)^2 = \frac{\mathrm{OC}}{\mathrm{OA}} = q^2, \text{ because } \Delta\, \mathrm{AOB}' \propto \mathrm{B}'\mathrm{OC}.$$

200. Another useful connexion between *scalars* and *tensors* (or *norms*) of quaternions may be derived as follows. In any plane triangle AOB, we have† the relation,

$$(\mathrm{T}\,.\,\mathrm{AB})^2 = (\mathrm{T}\,.\,\mathrm{OA})^2 - 2(\mathrm{T}\,.\,\mathrm{OA})\,.\,(\mathrm{T}\,.\,\mathrm{OB})\,.\cos \mathrm{AOB} + (\mathrm{T}\,.\,\mathrm{OB})^2;$$

in which the symbols T . OA, &c., denote (by 185, 186) the *lengths* of the sides OA, &c.; but if we still write $q = \mathrm{OB} : \mathrm{OA}$, we have $q - 1 = \mathrm{AB} : \mathrm{OA}$; dividing therefore by $(\mathrm{T}\,.\,\mathrm{OA})^2$, the formula becomes (by 196, &c.),

$$\text{I.} \ldots \mathrm{T}(q-1)^2 = 1 - 2\mathrm{T}q\,.\,\mathrm{SU}q + \mathrm{T}q^2 = \mathrm{T}q^2 - 2\mathrm{S}q + 1;$$

or

$$\text{II.} \ldots \mathrm{N}(q-1) = \mathrm{N}q - 2\mathrm{S}q + 1.$$

But q is here a perfectly *general* quaternion; we may therefore change its *sign*, and write,

$$\text{III.} \ldots \mathrm{T}(1+q)^2 = 1 + 2\mathrm{S}q + \mathrm{T}q^2; \qquad \text{IV.} \ldots \mathrm{N}(1+q) = 1 + 2\mathrm{S}q + \mathrm{N}q.$$

And since it is easy to prove (by 106, 107) that

$$\text{V.} \ldots \left(\frac{q'}{q} + 1\right) q = q' + q,$$

whatever two quaternions q and q' may be, while

$$\text{VI.} \ldots \mathrm{S}\frac{q'}{q}\,.\,\mathrm{N}q = \mathrm{S}\,.\,q'\mathrm{K}q = \mathrm{S}\,.\,q\mathrm{K}q',$$

we easily infer this other general formula,

$$\text{VII.} \ldots \mathrm{N}(q' + q) = \mathrm{N}q' + 2\mathrm{S}\,.\,q\mathrm{K}q' + \mathrm{N}q;$$

which gives, if x be any scalar,

$$\text{VIII.} \ldots \mathrm{N}(q + x) = \mathrm{N}q + 2x\mathrm{S}q + x^2.$$

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* Compare the first Note to page 162.

† By the Second Book of Euclid, or by plane trigonometry.

(1.) We are now prepared to effect, by *rules*[*] *of transformation*, some other *passages* from one mode of *expression* to another, of the kind which has been alluded to, and partly exemplified, in former sub-articles. Take, for example, the formula,

$$\mathrm{T}\,\frac{\rho+\alpha}{\rho-\alpha}=1, \text{ of 187, (2.);}$$

or the equivalent formula,

$$\mathrm{T}\,(\rho+\alpha)=\mathrm{T}\,(\rho-\alpha), \text{ of 186, (6.);}$$

which has been seen, on *geometrical* grounds, to represent a certain *locus*, namely the plane through O, perpendicular to the line OA; and therefor the *same locus* as that which is represented by the equation,

$$\mathrm{S}\,\frac{\rho}{\alpha}=0, \text{ of 196, (1.).}$$

To *pass* now from the former equations to the latter, by *calculation*, we have only to denote the quotient $\rho:\alpha$ by q, and to observe that the first or second form, as just now cited, becomes then,

$$\mathrm{T}\,(q+1)=\mathrm{T}\,(q-1); \quad \text{or} \quad \mathrm{N}\,(q+1)=\mathrm{N}\,(q-1);$$

or finally, by II. and IV.,

$$\mathrm{S}q=0,$$

which gives the third form of equation, as required.

(2.) Conversely, from $\mathrm{S}\,\frac{\rho}{\alpha}=0$, we can *return*, by the same general formulæ II. and IV., to the equation $\mathrm{N}\left(\frac{\rho}{\alpha}-1\right)=\mathrm{N}\left(\frac{\rho}{\alpha}+1\right)$, or by I. and III. to $\mathrm{T}\left(\frac{\rho}{\alpha}-1\right)=\mathrm{T}\left(\frac{\rho}{\alpha}+1\right)$, or to $\mathrm{T}\,(\rho-\alpha)=\mathrm{T}\,(\rho+\alpha)$, or to $\mathrm{T}\,\frac{\rho+\alpha}{\rho-\alpha}=1$, as above; and generally,

$$\mathrm{S}q=0 \quad \text{gives} \quad \mathrm{T}\,(q-1)=\mathrm{T}\,(q+1), \quad \text{or} \quad \mathrm{T}\,\frac{q+1}{q-1}=1;$$

while the latter equations, in turn, involve, as has been seen, the former.

(3.) Again, if we take the Apollonian Locus, 145, (8.), (9.), and employ the *first* of the two forms 186, (5.) of its equation, namely,

$$\mathrm{T}\,(\rho-a^2\alpha)=a\mathrm{T}\,(\rho-\alpha),$$

where a is a given positive scalar different from unity, we may write it as

$$\mathrm{T}\,(q-a^2)=a\mathrm{T}\,(q-1), \quad \text{or as} \quad \mathrm{N}\,(q-a^2)=a^2\mathrm{N}\,(q-1);$$

or by VIII.,

$$\mathrm{N}q-2a^2\mathrm{S}q+a^4=a^2\,(\mathrm{N}q-2\mathrm{S}q+1);$$

or, after suppressing $-2a^2\mathrm{S}q$, transposing, and dividing by a^2-1,

$$\mathrm{N}q=a^2; \quad \text{or,} \quad \mathrm{N}\rho=a^2\mathrm{N}\alpha; \quad \text{or,} \quad \mathrm{T}\rho=a\mathrm{T}\alpha;$$

which last is the *second form* 186, (5.), and is thus *deduced from the first, by calculation alone*, without any immediate appeal to *geometry*, or the construction of any *diagram*.

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[*] Compare 115, (10.); and several subsequent sub-articles.

$N\overline{q-1} = N_q - 2$

(4.) Conversely if we take the equation,

$$\mathrm{N}\frac{\rho}{a} = a^2, \text{ of 145, (12.)},$$

which was there seen to represent the same locus, considered as a spheric surface, with O for centre, and $a\alpha$ for one of its radii, and write it as $\mathrm{N}q = a^2$, we can then *by calculation* return to the form

$$\mathrm{N}(q - a^2) = a^2\mathrm{N}(q-1), \quad \text{or} \quad \mathrm{T}(q - a^2) = a\mathrm{T}(q-1),$$

or finally,

$$\mathrm{T}(\rho - a^2\alpha) = a\mathrm{T}(\rho - \alpha), \text{ as in 186, (5.)};$$

this *first form* of that sub-article being thus deduced *from the second*, namely from $\mathrm{T}\rho = a\mathrm{T}\alpha$, or $\mathrm{T}\frac{\rho}{\alpha} = a$.

(5.) It is far from being the intention of the foregoing remarks, to *discourage attention* to the *geometrical interpretation* of the various *forms of expression*, and general *rules of transformation*, which thus offer themselves in working with quaternions; on the contrary, one main object of the present Chapter has been to establish a firm *geometrical basis*, for all such forms and rules. But *when* such a *foundation* has once been laid, it is, as we see, *not necessary* that we should continually *recur* to the examination of it, in building up the *superstructure*. That *each* of the *two forms*, in 186, (5.), *involves* the *other*, may be *proved*, as above, *by calculation*; but it is interesting to inquire what is the *meaning* of this result: and in seeking to *interpret* it, we should be led anew to the theorem of the *Apollonian Locus*.

(6.) The result (4.) of calculation, that

$$\mathrm{N}(q - a^2) = a^2\mathrm{N}(q-1), \text{ if } \mathrm{N}q = a^2,$$

may be expressed under the form of an *identity*, as follows:

$$\text{IX.} \ldots \mathrm{N}(q - \mathrm{N}q) = \mathrm{N}q \,.\, \mathrm{N}(q-1);$$

in which q may be *any quaternion*.

(7.) Or, by 191, VII., because it will soon be seen that

$$q(q-1) = q^2 - q, \text{ as in algebra,}$$

we may write it as this other identity:

$$\text{X.} \ldots \mathrm{N}(q - \mathrm{N}q) = \mathrm{N}(q^2 - q).$$

(8.) If $\mathrm{T}(q-1) = 1$, then $\mathrm{S}\frac{1}{q} = \frac{1}{2}$; and conversely, the former equation follows from the latter; because each may be put under the form (comp. 196, XII.),

$$\mathrm{N}q = 2\mathrm{S}q.$$

(9.) Hence, if $\mathrm{T}(\rho - \alpha) = \mathrm{T}\alpha$, then $\mathrm{S}\frac{2\alpha}{\rho} = 1$, and reciprocally. In fact (comp. 196, (6.)), each of these two equations expresses that the locus of P is the sphere which passes through O, and has its centre at A; or which has $\mathrm{OB} = 2\alpha$ for a diameter.

(10.) By changing q to $q+1$ in (8), we find that

$$\text{if } \mathrm{T}q = 1, \quad \text{then} \quad \mathrm{S}\frac{q-1}{q+1} = 0, \text{ and reciprocally.}$$

(11.) Hence if $T\rho = T\alpha$, then $S\dfrac{\rho - \alpha}{\rho + \alpha} = 0$, and reciprocally; because (by 106)

$$\frac{\rho - \alpha}{\rho + \alpha} = \frac{\rho - \alpha}{\alpha} : \frac{\rho + \alpha}{\alpha} = \left(\frac{\rho}{\alpha} - 1\right) : \left(\frac{\rho}{\alpha} + 1\right).$$

(12.) Each of these two equations (11.) expresses that the locus of P is the sphere through A, which has its centre at O; and their proved agreement is a recognition, by quaternions, of the elementary geometrical theorem, that the angle in a semicircle is a right angle.

SECTION 13.—*On the Right Part (or Vector Part) of a Quaternion; and on the Distributive Property of the Multiplication of Quaternions.*

201. A given *vector* OB can always be decomposed, in one but in only one way, into two component vectors, of which it is the *sum* (6); and of which *one*, as OB′ in Fig. 50, is *parallel* (15) to another given vector OA, while the *other*, as OB″ in the same Figure, is *perpendicular* to that given line OA; namely, by letting fall the perpendicular BB′ on OA, and drawing OB″ = B′B, so that OB′BB″ shall be a rectangle. In other words, if α and β be any two given, actual, and co-initial vectors, it is always possible to deduce from them, in one definite way, two other co-initial vectors, β' and β'', which need not however *both* be actual (1); and which shall satisfy (comp. 6, 15, 129) the conditions,

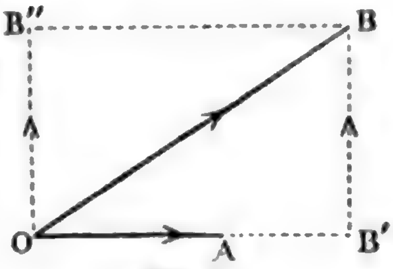

Fig. 50.

$$\beta = \beta' + \beta'' = \beta'' + \beta', \qquad \beta' \parallel \overset{\alpha}{\cancel{\beta}}, \qquad \beta'' \perp \overset{\alpha}{\cancel{\beta}};$$

β' vanishing, when $\beta \perp \alpha$; and β'' being null, when $\beta \parallel \alpha$; but both being (what we may call) *determinate vector-functions* of α and β. And of these two functions, it is evident that β' is the orthographic *projection of* β *on the line* α; and that β'' is the corresponding *projection of* β *on the plane* through O, which is *perpendicular to* α.

202. Hence it is easy to infer, that there is always one, but only one way, of *decomposing a given quaternion*,

$$q = \text{OB} : \text{OA} = \beta : \alpha,$$

into two parts or summands (195), of which *one* shall be, as in

196, a *scalar*, while the *other* shall be a *right quotient* (132). Of these two parts, the *former* has been already called (196) *the scalar part*, or simply *the Scalar* of the Quaternion, and has been denoted by the symbol Sq; so that, with reference to the recent Figure 50, we have

$$\text{I.} \ldots \mathrm{S}q = \mathrm{S}(\mathrm{OB} : \mathrm{OA}) = \mathrm{OB}' : \mathrm{OA}; \quad \text{or,} \quad \mathrm{S}(\beta : \alpha) = \beta' : \alpha.$$

And we now propose to call the *latter* part the RIGHT PART* of the same quaternion, and to denote it by the new symbol

$$\mathrm{V}q;$$

writing thus, in connexion with the same Figure,

$$\text{II.} \ldots \mathrm{V}q = \mathrm{V}(\mathrm{OB} : \mathrm{OA}) = \mathrm{OB}'' : \mathrm{OA}; \quad \text{or,} \quad \mathrm{V}(\beta : \alpha) = \beta'' : \alpha.$$

The *System of Notations*, peculiar to the present Calculus, will thus have been completed; and we shall have the following general *Formula of Decomposition of a Quaternion into two Summands* (comp. 188), of the *Scalar* and *Right* kinds:

$$\text{III.} \ldots q = \mathrm{S}q + \mathrm{V}q = \mathrm{V}q + \mathrm{S}q,$$

or, briefly and symbolically,

$$\text{IV.} \ldots 1 = \mathrm{S} + \mathrm{V} = \mathrm{V} + \mathrm{S}.$$

(1.) In connexion with the same Fig. 50, we may write also,

$$\mathrm{V}(\mathrm{OB} : \mathrm{OA}) = \mathrm{B'B} : \mathrm{OA},$$

because, by construction, $\mathrm{B'B} = \mathrm{OB}''$.

(2.) In like manner, for Fig. 36, we have the equation,

$$\mathrm{V}(\mathrm{OB} : \mathrm{OA}) = \mathrm{A'B} : \mathrm{OA}.$$

(3.) Under the recent conditions,

$$\mathrm{V}(\beta' : \alpha) = 0, \quad \text{and} \quad \mathrm{S}(\beta'' : \alpha) = 0.$$

(4.) In general, it is evident that

$$\text{V.} \ldots q = 0, \quad \text{if} \quad \mathrm{S}q = 0, \quad \text{and} \quad \mathrm{V}q = 0; \text{ and reciprocally.}$$

(5.) More generally,

$$\text{VI.} \ldots q' = q, \quad \text{if} \quad \mathrm{S}q' = \mathrm{S}q, \quad \text{and} \quad \mathrm{V}q' = \mathrm{V}q; \text{ with the converse.}$$

(6.) Also $\quad \text{VII.} \ldots \mathrm{V}q = 0, \quad \text{if} \quad \angle q = 0, \quad \text{or} \quad = \pi;$

or $\quad \text{VIII.} \ldots \mathrm{V}(\beta : \alpha) = 0, \quad \text{if} \quad \beta \parallel \alpha;$

the *right part* of a *scalar* being zero.

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* This *Right Part*, Vq, will come to be also called *the Vector Part*, or simply *the* VECTOR, of the Quaternion; because it will be found possible and useful to identify such part with its own Index-Vector (133). Compare the Notes to pages 119, 136, 174.

(7.) On the other hand,

$$\text{IX.} \ldots \mathrm{V}q = q, \quad \text{if} \quad \angle q = \frac{\pi}{2};$$

a *right quaternion* being *its own right part.*

203. We had (196, XIX.) a formula which may now be written thus,

$$\text{I.} \ldots \mathrm{OB}' = \mathrm{S}(\mathrm{OB} : \mathrm{OA}) \,.\, \mathrm{OA}, \quad \text{or} \quad \beta' = \mathrm{S}\frac{\beta}{\alpha} \cdot \alpha,$$

to express the *projection of* OB *on* OA, or of the vector β on α; and we have evidently, by the definition of the new symbol $\mathrm{V}q$, the analogous formula,

$$\text{II.} \ldots \mathrm{OB}'' = \mathrm{V}(\mathrm{OB} : \mathrm{OA}) \,.\, \mathrm{OA}, \quad \text{or} \quad \beta'' = \mathrm{V}\frac{\beta}{\alpha} \cdot \alpha,$$

to express the *projection of* β *on the plane* (through O), which is drawn so as to be *perpendicular to* α; and which has been considered in several former sub-articles (comp. 186, (6.), and 196, (1.)). It follows (by 186, &c.) that

$$\text{III.} \ldots \mathrm{T}\beta'' = \mathrm{TV}\frac{\beta}{\alpha} \cdot \mathrm{T}\alpha = \textit{perpendicular distance of } \mathrm{B} \textit{ from } \mathrm{OA};$$

this perpendicular being *here* considered with reference to its *length* alone, as the characteristic T of the *tensor* implies. It is to be observed that because the *factor*, $\mathrm{V}\frac{\beta}{\alpha}$, in the recent formula II. for the projection β'', is *not a scalar*, we must write that factor as a *multiplier*, and *not* as a *multiplicand;* although we were at liberty, in consequence of a general convention (15), respecting the *multiplication of vectors and scalars*, to denote the *other* projection β' under the form,

$$\text{I}'. \ldots \beta' = \alpha\mathrm{S}\frac{\beta}{\alpha} \text{ (196, XIX.)}.$$

(1.) The equation,

$$\mathrm{V}\frac{\rho}{\alpha} = 0,$$

expresses that the locus of P is the *indefinite right line* OA.

(2.) The equation,

$$\mathrm{V}\frac{\rho - \beta}{\alpha} = 0, \quad \text{or} \quad \mathrm{V}\frac{\rho}{\alpha} = \mathrm{V}\frac{\beta}{\alpha},$$

expresses that the locus of P is the indefinite right line BB″, in Fig. 50, which is drawn through the point B, parallel to the line OA.

(3.) The equation

$$\mathrm{S}\frac{\rho-\beta}{\alpha}=0, \quad \text{or} \quad \mathrm{S}\frac{\rho}{\alpha}=\mathrm{S}\frac{\beta}{\alpha}, \text{ of 196, (2.),}$$

has been seen to express that the locus of P is the *plane* through B, perpendicular to the line OA; if then we *combine* it with the recent equation (2.), we shall express that the point P is situated at the *intersection* of the two last mentioned loci; or that it *coincides* with the *point* B.

(4.) Accordingly, whether we take the two first or the two last of these recent forms (2.), (3.), namely,

$$\mathrm{V}\frac{\rho-\beta}{\alpha}=0, \quad \mathrm{S}\frac{\rho-\beta}{\alpha}=0, \quad \text{or} \quad \mathrm{V}\frac{\rho}{\alpha}=\mathrm{V}\frac{\beta}{\alpha}, \quad \mathrm{S}\frac{\rho}{\alpha}=\mathrm{S}\frac{\beta}{\alpha},$$

we can infer this position of the point P: in the first case by inferring, through 202, V., that $\frac{\rho-\beta}{\alpha}=0$, whence $\rho-\beta=0$, by 142; and in the second case by inferring, through 202, VI., that $\frac{\rho}{\alpha}=\frac{\beta}{\alpha}$; so that we have in each case (comp. 104), or as a consequence from each system, the equality $\rho=\beta$, or OP = OB; or finally (comp. 20) the *coincidence*, P = B.

(5.) The equation,

$$\mathrm{TV}\frac{\rho}{\alpha}=\mathrm{TV}\frac{\beta}{\alpha},$$

expresses that the locus of the point P is the *cylindric surface of revolution*, which passes through the point B, and has the line OA for its axis; for it expresses, by III., that the *perpendicular distances* of P and B, *from this latter line*, are *equal*.

(6.) The system of the two equations,

$$\mathrm{TV}\frac{\rho}{\alpha}=\mathrm{TV}\frac{\beta}{\alpha}, \quad \mathrm{S}\frac{\rho}{\gamma}=0,$$

expresses that the locus of P is the (generally) *elliptic section* of the cylinder (5.), made by the plane through O, which is perpendicular to the line OC.

(7.) If we employ an analogous decomposition of ρ, by supposing that

$$\rho=\rho'+\rho'', \qquad \rho' \parallel \alpha, \qquad \rho'' \perp \alpha,$$

the three rectilinear or plane loci, (1.), (2.), (3.), may have their equations thus briefly written:

$$\rho''=0; \qquad \rho''=\beta''; \qquad \rho'=\beta':$$

while the combination of the two last of these gives $\rho=\beta$, as in (4.).

(8.) The equation of the cylindric locus, (5.), takes at the same time the form,

$$\mathrm{T}\rho''=\mathrm{T}\beta'';$$

which last equation expresses that the projection P″ of the point P, on the plane through O perpendicular to OA, falls somewhere on the circumference of a circle, with O for centre, and OB″ for radius: and this *circle* may accordingly be considered as the *base of the right cylinder*, in the sub-article last cited.

204. From the mere circumstance that Vq is always a *right quotient* (132), whence UVq is a *right versor* (153), of

which the *plane* (119), and the *axis* (127), coincide with those of q, several general consequences easily follow. Thus we have generally, by principles already established, the relations:

$$\text{I.} \ldots \angle \mathrm{V}q = \frac{\pi}{2}; \qquad \text{II.} \ldots \mathrm{Ax.\,V}q = \mathrm{Ax.\,UV}q = \mathrm{Ax.}\,q;$$

$$\text{III.} \ldots \mathrm{KV}q = -\mathrm{V}q, \quad \text{or} \quad \mathrm{KV} = -\mathrm{V}\ (144);$$

$$\text{IV.} \ldots \mathrm{SV}q = 0, \quad \text{or} \quad \mathrm{SV} = 0\ (196,\ \text{VII.});$$

$$\text{V.} \ldots (\mathrm{UV}q)^2 = -1\ (153,\ 159);$$

and therefore,

$$\text{VI.} \ldots (\mathrm{V}q)^2 = -(\mathrm{TV}q)^2 = -\mathrm{NV}q,\text{*}$$

because, by the general decomposition (188) of a quaternion into *factors*, we have

$$\text{VII.} \ldots \mathrm{V}q = \mathrm{TV}q\,.\,\mathrm{UV}q.$$

We have also (comp. 196, VI.),

$$\text{VIII.} \ldots \mathrm{VS}q = 0, \quad \text{or} \quad \mathrm{VS} = 0\ (202,\ \text{VII.});$$

$$\text{IX.} \ldots \mathrm{VV}q = \mathrm{V}q, \quad \text{or} \quad \mathrm{V}^2 = \mathrm{VV} = \mathrm{V}\ (202,\ \text{IX.});$$

and

$$\text{X.} \ldots \mathrm{VK}q = -\mathrm{V}q, \quad \text{or} \quad \mathrm{VK} = -\mathrm{V},$$

because *conjugate quaternions* have *opposite right parts*, by the definitions in 137, 202, and by the construction of Fig. 36. For the same reason, we have this other general formula,

$$\text{XI.} \ldots \mathrm{K}q = \mathrm{S}q - \mathrm{V}q, \quad \text{or} \quad \mathrm{K} = \mathrm{S} - \mathrm{V};$$

but we had

$$q = \mathrm{S}q + \mathrm{V}q, \quad \text{or} \quad 1 = \mathrm{S} + \mathrm{V}, \text{ by 202, III., IV.};$$

hence not only, by addition,

$$q + \mathrm{K}q = 2\mathrm{S}q, \quad \text{or} \quad 1 + \mathrm{K} = 2\mathrm{S}, \text{ as in 196, I.},$$

but also, by subtraction,

$$\text{XII.} \ldots q - \mathrm{K}q = 2\mathrm{V}q, \quad \text{or} \quad 1 - \mathrm{K} = 2\mathrm{V};$$

whence the *Characteristic*, V, *of the Operation of taking the Right Part of a Quaternion* (comp. 132, (6.); 137; 156; 187; 196), may be *defined* by either of the two following symbolical equations:

$$\text{XIII.} \ldots \mathrm{V} = 1 - \mathrm{S}\ (202,\ \text{IV.}); \qquad \text{XIV.} \ldots \mathrm{V} = \tfrac{1}{2}(1 - \mathrm{K});$$

whereof the former connects it with the characteristic S, and the latter with the characteristic K; while the dependence of K on S and V is expressed by the recent formula XI.; and that of S on K by 196, II′. Again, if the line OB, in Fig. 50, be multiplied (15) by any scalar coefficient, the perpendicular BB′ is evidently multiplied by the same; hence, generally,

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* Compare the Note to page 130.

$$\text{XV.}\ldots\ \mathrm{V}xq = x\mathrm{V}q, \text{ if } x \text{ be any scalar};$$

and therefore, by 188, 191,

$$\text{XVI.}\ldots\ \mathrm{V}q = \mathrm{T}q\,.\mathrm{VU}q, \quad\text{and}\quad \text{XVII.}\ldots\ \mathrm{TV}q = \mathrm{T}q\,.\mathrm{TVU}q.$$

But the consideration of the right-angled triangle, OB′B, in the same Figure, shows that

$$\text{XVIII.}\ldots\ \mathrm{TV}q = \mathrm{T}q\,.\sin\angle q,$$

because, by 202, II., we have

$$\mathrm{TV}q = \mathrm{T}(\mathrm{OB}'' : \mathrm{OA}) = \mathrm{T}.\mathrm{OB}'' : \mathrm{T}.\mathrm{OA},$$

and

$$\mathrm{T}.\mathrm{OB}'' = \mathrm{T}.\mathrm{OB}\,.\sin \mathrm{AOB};$$

we arrive then thus at the following general and useful formula, connecting *quaternions* with *trigonometry* anew:

$$\text{XIX.}\ldots\ \mathrm{TVU}q = \sin\angle q;$$

by combining which with the formula,

$$\mathrm{SU}q = \cos\angle q \ (196, \text{XVI.}),$$

we arrive at the general relation:

$$\text{XX.}\ldots\ (\mathrm{SU}q)^2 + (\mathrm{TVU}q)^2 = 1;$$

which may also (by XVII., and by 196, IX.) be written thus:

$$\text{XXI.}\ldots\ (\mathrm{S}q)^2 + (\mathrm{TV}q)^2 = (\mathrm{T}q)^2;$$

and might have been immediately deduced, *without sines and cosines*, from the right-angled triangle, by the property of the square of the hypotenuse, under the form,

$$(\mathrm{T}.\mathrm{OB}')^2 + (\mathrm{T}.\mathrm{B}'\mathrm{B})^2 = (\mathrm{T}.\mathrm{OB})^2.$$

The same important relation may be expressed in various other ways; for example, we may write,

$$\text{XXII.}\ldots\ \mathrm{N}q = \mathrm{T}q^2 = \mathrm{S}q^2 - \mathrm{V}q^2,$$

where it is assumed, as an abridgment of *notation* (comp. 199, VII., VIII.), that

$$\text{XXIII.}\ldots\ \mathrm{V}q^2 = (\mathrm{V}q)^2, \quad\text{but that}\quad \text{XXIV.}\ldots\ \mathrm{V}.q^2 = \mathrm{V}(q^2),$$

the import of this last symbol remaining to be examined. And because, by the definition of a *norm*, and by the properties of Sq and Vq,

$$\text{XXV.} \ldots \text{NS}q = \text{S}q^2, \quad \text{but} \quad \text{XXVI.} \ldots \text{NV}q = -\text{V}q^2,$$

we may write also,

$$\text{XXVII.} \ldots \text{N}q = \text{N}(\text{S}q + \text{V}q) = \text{NS}q + \text{NV}q;$$

a result which is indeed included in the formula 200, VIII., since that equation gives, generally,

$$\text{XXVIII.} \ldots \text{N}(q + x) = \text{N}q + \text{N}x, \quad \text{if} \quad \angle q = \frac{\pi}{2};$$

x being, as usual, any scalar. It may be added that because (by 106, 143) we have, as in algebra, the identity,

$$\text{XXIX.} \ldots -(q' + q) = -q' - q,$$

the *opposite of the sum* of any two quaternions being thus equal to the *sum of the opposites*, we may (by XI.) establish this other general formula:

$$\text{XXX.} \ldots -\text{K}q = \text{V}q - \text{S}q;$$

the *opposite of the conjugate* of any quaternion q having thus the *same right part* as that quaternion, but an *opposite scalar part.*

(1.) From the last formula it may be inferred, that

$$\text{if} \quad q' = -\text{K}q, \quad \text{then} \quad \text{V}q' = +\text{V}q, \quad \text{but} \quad \text{S}q' = -\text{S}q;$$

and therefore that

$$\text{T}q' = \text{T}q, \quad \text{and} \quad \text{Ax.}\, q' = \text{Ax.}\, q, \quad \text{but} \quad \angle q' = \pi - \angle q;$$

which two last relations might have been deduced from 138 and 143, without the introduction of the characteristics S and V.

(2.) The equation,

$$\left(\text{V}\frac{\rho}{\alpha}\right)^2 = \left(\text{V}\frac{\beta}{\alpha}\right)^2, \quad \text{or (by XXVI.)}, \quad \text{NV}\frac{\rho}{\alpha} = \text{NV}\frac{\beta}{\alpha},$$

like the equation of 203, (5.), expresses that the locus of P is the *right cylinder*, or cylinder of revolution, with OA for its axis, which passes through the point B.

(3.) The system of the two equations,

$$\left(\text{V}\frac{\rho}{\alpha}\right)^2 = \left(\text{V}\frac{\beta}{\alpha}\right)^2, \qquad \text{S}\frac{\rho}{\gamma} = 0,$$

like the corresponding system in 203, (6.), represents generally an *elliptic section* of the same right cylinder; but if it happen that $\gamma \parallel \alpha$, the section then becomes *circular.*

(4.) The system of the two equations,

$$\mathrm{S}\frac{\rho}{\alpha}=x, \qquad \left(\mathrm{V}\frac{\rho}{\alpha}\right)^2=x^2-1, \quad \text{with} \quad x>-1, \quad x<1,$$

represents the *circle*,[*] in which the cylinder of revolution, with OA for axis, and with $(1-x^2)^{\frac{1}{2}}\mathrm{T}\alpha$ for radius, is perpendicularly cut by a plane at a distance $= \pm x\mathrm{T}\alpha$ from O; the vector of the centre of this circular section being $x\alpha$.

(5.) While the scalar x increases (algebraically) from -1 to 0, and thence to $+1$, the connected scalar $\sqrt{(1-x^2)}$ at first increases from 0 to 1, and then decreases from 1 to 0; the *radius* of the circle (4.) at the same time enlarging from zero to a maximum $= \mathrm{T}\alpha$, and then again diminishing to zero; while the position of the *centre* of the circle varies continuously, in one constant direction, from a *first limit-point* A′, if $\mathrm{OA}' = -\alpha$, to the point A, as a *second limit.*

(6.) The *locus* of all such *circles* is the *sphere*, with AA′ for a diameter, and therefore with O for centre; namely, the sphere which has already been represented by the equation $\mathrm{T}\rho = \mathrm{T}\alpha$ of 186, (2.); or by $\mathrm{T}\frac{\rho}{\alpha}=1$, of 187, (1.); or by

$$\mathrm{S}\frac{\rho-\alpha}{\rho+\alpha}=0, \text{ of } 200, (11.);$$

but which now presents itself under the new form,

$$\left(\mathrm{S}\frac{\rho}{\alpha}\right)^2-\left(\mathrm{V}\frac{\rho}{\alpha}\right)^2=1,$$

obtained by *eliminating* x between the two recent equations (4).

(7.) It is easy, however, to *return* from the last form to the second, and thence to the first, or to the third, by rules of calculation already established, or by the general relations between the symbols used. In fact, the last equation (6.) may be written, by XXII., under the form,

$$\mathrm{N}\frac{\rho}{\alpha}=1;$$

whence

$$\mathrm{T}\frac{\rho}{\alpha}=1, \text{ by } 190, \text{ VI.};$$

and therefore also $\mathrm{T}\rho = \mathrm{T}\alpha$, by 187, and $\mathrm{S}\frac{\rho-\alpha}{\rho+\alpha}=0$, by 200, (11.).

(8.) Conversely, the sphere through A, with O for centre, might already have been seen, by the first definition and property of a *norm*, stated in 145, (11.), to admit (comp. 145, (12.)) of being represented by the equation $\mathrm{N}\frac{\rho}{\alpha}=1$; and therefore, by XXII., under the recent form (6.); in which if we write x to denote the variable scalar $\mathrm{S}\frac{\rho}{\alpha}$, as in the first of the two equations (4.), we recover the second of those equations: and thus might be led to consider, as in (6.), the *sphere* in question as the *locus of a variable circle*, which is (as above) the *intersection* of a *variable cylinder*, with a *variable plane* perpendicular to its axis.

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[*] By the word "circle," in these pages, is usually meant a *circumference*, and *not* an *area*; and in like manner, the words "sphere," "cylinder," "cone," &c., are usually here employed to denote *surfaces*, and *not volumes*.

(9.) The same sphere may also, by XXVII., have its equation written thus,

$$\mathrm{N}\left(\mathrm{S}\frac{\rho}{\alpha}+\mathrm{V}\frac{\rho}{\alpha}\right)=1; \quad \text{or} \quad \mathrm{T}\left(\mathrm{S}\frac{\rho}{\alpha}+\mathrm{V}\frac{\rho}{\alpha}\right)=1.$$

(10.) If, in each variable plane represented by the first equation (4.), we conceive the radius of the circle, or that of the variable cylinder, to be multiplied by any constant and positive scalar a, the centre of the circle and the axis of the cylinder remaining unchanged, we shall pass thus to a *new system of circles*, represented by this new system of equations,

$$\mathrm{S}\frac{\rho}{\alpha}=x, \qquad \left(\mathrm{V}\frac{\rho}{a\alpha}\right)^2=x^2-1.$$

(11.) The *locus* of these *new* circles will evidently be a *Spheroid of Revolution*; the *centre* of this new surface being the centre O, and the *axis* of the same surface being the *diameter* AA′, of the *sphere* lately considered: which sphere is therefore either *inscribed* or *circumscribed* to the spheroid, according as the constant $a >$ or < 1; because the *radii* of the new circles are in the first case *greater*, but in the second case *less*, than the radii of the old circles; or because the *radius of the equator* of the spheroid $= a\mathrm{T}\alpha$, while the radius of the sphere $= \mathrm{T}\alpha$.

(12.) The equations of the *two co-axal cylinders* of revolution, which *envelope* respectively the sphere and spheroid (or are *circumscribed* thereto) are:

$$\left(\mathrm{V}\frac{\rho}{\alpha}\right)^2=-1; \qquad \left(\mathrm{V}\frac{\rho}{a\alpha}\right)^2=-1;$$

or

$$\mathrm{NV}\frac{\rho}{\alpha}=1, \qquad \mathrm{NV}\frac{\rho}{\alpha}=a^2;$$

or

$$\mathrm{TV}\frac{\rho}{\alpha}=1, \qquad \mathrm{TV}\frac{\rho}{\alpha}=a.$$

(13.) The system of the two equations,

$$\mathrm{S}\frac{\rho}{\alpha}=x, \qquad \left(\mathrm{V}\frac{\rho}{\beta}\right)^2=x^2-1, \quad \text{with } \beta \textit{ not} \parallel \alpha,$$

represents (comp. (3.)) a *variable ellipse*, if the scalar x be still treated as a variable.

(14.) The result of the elimination of x between the two last equations, namely this new equation,

$$\left(\mathrm{S}\frac{\rho}{\alpha}\right)^2-\left(\mathrm{V}\frac{\rho}{\beta}\right)^2=1;$$

or

$$\mathrm{NS}\frac{\rho}{\alpha}+\mathrm{NV}\frac{\rho}{\beta}=1, \text{ by XXV., XXVI.};$$

or

$$\mathrm{N}\left(\mathrm{S}\frac{\rho}{\alpha}+\mathrm{V}\frac{\rho}{\beta}\right)=1, \text{ by XXVII.};$$

or finally,

$$\mathrm{T}\left(\mathrm{S}\frac{\rho}{\alpha}+\mathrm{V}\frac{\rho}{\beta}\right)=1, \text{ by 190, VI.},$$

represents the *locus of all such ellipses* (13.), and will be found to be an adequate representation, through quaternions, of the *general* ELLIPSOID (with *three unequal axes*): that celebrated surface being here referred to its *centre*, as the *origin* O of vectors to its points; and the *six scalar* (or algebraic) *constants*, which enter into the usual *algebraic equation* (by co-ordinates) of such a *central ellipsoid*, being here virtually included in the *two independent vectors*, α and β, which may be called its *two Vector-Constants.*[*]

(15.) The equation (comp. (12.)),

$$\left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = -1, \quad \text{or} \quad \mathrm{NV}\frac{\rho}{\beta} = 1, \quad \text{or} \quad \mathrm{TV}\frac{\rho}{\beta} = 1,$$

represents a *cylinder of revolution, circumscribed to the ellipsoid*, and touching it *along the ellipse* which answers to the value $x = 0$, in (13.); so that the *plane* of this *ellipse of contact* is represented by the equation,

$$\mathrm{S}\frac{\rho}{\alpha} = 0;$$

the *normal* to this *plane* being thus (comp. 196, (17.)) the vector α, or OA; while the *axis* of the lately mentioned enveloping *cylinder* is β, or OB.

(16.) Postponing any further discussion of the recent *quaternion equation of the ellipsoid* (14.), it may be noted here that we have generally, by XXII., the two following useful transformations for the *squares*, of the *scalar* Sq, and of the *right part* Vq, of any quaternion q:

$$\text{XXXI.}\ldots\ \mathrm{S}q^2 = \mathrm{T}q^2 + \mathrm{V}q^2; \qquad \text{XXXII.}\ldots\ \mathrm{V}q^2 = \mathrm{S}q^2 - \mathrm{T}q^2.$$

(17.) In referring briefly to these, and to the connected formula XXII., upon occasion, it may be somewhat safer to write,

$$(\mathrm{S})^2 = (\mathrm{T})^2 + (\mathrm{V})^2, \qquad (\mathrm{V})^2 = (\mathrm{S})^2 - (\mathrm{T})^2, \qquad (\mathrm{T})^2 = (\mathrm{S})^2 - (\mathrm{V})^2,$$

than $\mathrm{S}^2 = \mathrm{T}^2 + \mathrm{V}^2$, &c.; because these last forms of notation, S^2, &c., have been otherwise interpreted already, in analogy to the known *Functional Notation*, or Notation of the *Calculus of Functions*, or of *Operations* (comp. 187, (9.); 196, VI.; and 204, IX.).

(18.) In pursuance of the same analogy, *any scalar* may be denoted by the *general symbol*,

$$\mathrm{V}^{-1}0;$$

because *scalars* are the *only* quaternions of which the *right parts vanish*.

(19.) In like manner, a *right quaternion*, generally, may be denoted by the symbol,

$$\mathrm{S}^{-1}0;$$

and since this includes (comp. 204, I.) the *right part* of *any* quaternion, we may establish this *general symbolic transformation of a Quaternion*:

$$q = \mathrm{V}^{-1}0 + \mathrm{S}^{-1}0.$$

(20.) With this form of notation, we should have generally, at least for *real*[†] quaternions, the inequalities,

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[*] It will be found, however, that *other pairs* of *vector-constants*, for the *central ellipsoid*, may occasionally be used with advantage.

[†] Compare Art. 149; and the Notes to pages 90, 134.

$$(\mathrm{V}^{-1}0)^2 > 0; \qquad (\mathrm{S}^{-1}0)^2 < 0;$$

so that a (geometrically *real*) *Quaternion* is generally of the form:

Square-root of a Positive, plus Square-root of a Negative.

(21.) The equations 196, XVI. and 204, XIX. give, as a new link between quaternions and *trigonometry*, the formula:

$$\text{XXXIII.} \ldots \tan \angle q = \mathrm{TVU}q : \mathrm{SU}q = \mathrm{TV}q : \mathrm{S}q.$$

(22.) It may not be entirely in accordance with the *theory* of that *Functional* (or *Operational*) *Notation*, to which allusion has lately been made, but it will be found to be convenient in *practice*, to write this last result under one or other of the *abridged forms*:*

$$\text{XXXIV.} \ldots \tan \angle q = \frac{\mathrm{TV}}{\mathrm{S}} \cdot q; \quad \text{or} \quad \text{XXXIV}'. \ldots \tan \angle q = (\mathrm{TV} : \mathrm{S})\, q;$$

which have the advantage of *saving the repetition of the symbol of the quaternion*, when that symbol happens to be a *complex expression*, and not, as here, a single letter, q.

(23.) The transformation 194, for the *index* of a right quotient, gives generally, by II., for any quaternion q, the formulæ:

$$\text{XXXV.} \ldots \mathrm{IV}q = \mathrm{TV}q \,.\, \mathrm{Ax.}\, q; \qquad \text{XXXVI.} \ldots \mathrm{IUV}q = \mathrm{Ax.}\, q;$$

so that we may establish generally the symbolical† equation,

$$\text{XXXVI}'. \ldots \mathrm{IUV} = \mathrm{Ax.}$$

(24.) And because $\mathrm{Ax.}\,(1 : \mathrm{V}q) = -\,\mathrm{Ax.}\,\mathrm{V}q$, by 135, and therefore $= -\,\mathrm{Ax.}\,q$, by II., we may write also, by XXXV.,

$$\text{XXXV}'. \ldots \mathrm{I}(1 : \mathrm{V}q) = -\,\mathrm{Ax.}\,q : \mathrm{TV}q.$$

205. If any parallelogram OBDC (comp. 197) be projected on the plane through O, which is perpendicular to OA, the projected figure OB″D″C″ (comp. 11) is still a parallelogram; so that

$$\mathrm{OD}'' = \mathrm{OC}'' + \mathrm{OB}'' \ (6), \quad \text{or} \quad \delta'' = \gamma'' + \beta'';$$

and therefore, by 106,

$$\delta'' : \alpha = (\gamma'' : \alpha) + (\beta'' : \alpha).$$

Hence, by 120, 202, for *any two quaternions*, q and q', we have the general formula,

$$\text{I.} \ldots \mathrm{V}(q' + q) = \mathrm{V}q' + \mathrm{V}q;$$

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* Compare the Note to Art. 199.

† At a later stage it will be found possible (comp. the Note to page 174, &c.), to write, generally,

$$\mathrm{IV}q = \mathrm{V}q, \qquad \mathrm{IUV}q = \mathrm{UV}q;$$

and then (comp. the Note in page 118 to Art. 129) the recent equations, XXXVI., XXXVI′., will take these shorter forms:

$$\mathrm{Ax.}\, q = \mathrm{UV}q; \qquad \mathrm{Ax.} = \mathrm{UV}.$$

with which it is easy to connect this other,

$$\text{II.} \ldots \mathrm{V}(q' - q) = \mathrm{V}q' - \mathrm{V}q.$$

Hence also, for any *three* quaternions, q, q', q'',

$$\mathrm{V}\{q'' + (q' + q)\} = \mathrm{V}q'' + \mathrm{V}(q' + q) = \mathrm{V}q'' + (\mathrm{V}q' + \mathrm{V}q);$$

and similarly for any greater number of summands: so that we may write generally (comp. 197, II.),

$$\text{III.} \ldots \mathrm{V}\Sigma q = \Sigma \mathrm{V}q, \quad \text{or briefly} \quad \text{III}'. \ldots \mathrm{V}\Sigma = \Sigma\mathrm{V};$$

while the formula II. (comp. 197, IV.) may, in like manner, be thus written,

$$\text{IV.} \ldots \mathrm{V}\Delta q = \Delta\mathrm{V}q, \quad \text{or} \quad \text{IV}'. \ldots \mathrm{V}\Delta = \Delta\mathrm{V};$$

the *order* of the terms added, and the mode of *grouping* them, in III., being *as yet* supposed to remain unaltered, although both those restrictions will soon be removed. We conclude then, that the *characteristic* V, of the operation of *taking the right part* (202, 204) of a quaternion, like the characteristic S of *taking the scalar* (196, 197), and the characteristic K of *taking the conjugate* (137, 195[*]), is a *Distributive Symbol*, or represents a *distributive operation:* whereas the characteristics, Ax., ∠, N, U, T, of the operations of taking respectively the *axis* (128, 129), the *angle* (130), the *norm* (145, (11.)), the *versor* (156), and the *tensor* (187), are *not* thus distributive symbols (comp. 186, (10.), and 200, VII.); or do *not* operate upon a *whole* (or *sum*), by operating on its *parts* (or *summands*).

(1.) We may now recover the symbolical equation $\mathrm{K}^2 = 1$ (145), under the form (comp. 196, VI.; 202, IV.; and 204, IV. VIII. IX. XI.):

$$\text{V.} \ldots \mathrm{K}^2 = (\mathrm{S} - \mathrm{V})^2 = \mathrm{S}^2 - \mathrm{SV} - \mathrm{VS} + \mathrm{V}^2 = \mathrm{S} + \mathrm{V} = 1.$$

(2.) In like manner we can recover each of the expressions for S^2, V^2 from the other, under the forms (comp. again 202, IV.):

$$\text{VI.} \ldots \mathrm{S}^2 = (1 - \mathrm{V})^2 = 1 - 2\mathrm{V} + \mathrm{V}^2 = 1 - \mathrm{V} = \mathrm{S}, \quad \text{as in 196, VI.;}$$

$$\text{VII.} \ldots \mathrm{V}^2 = (1 - \mathrm{S})^2 = 1 - 2\mathrm{S} + \mathrm{S}^2 = 1 - \mathrm{S} = \mathrm{V}, \quad \text{as in 204, IX.;}$$

or thus (comp. 196, II'., and 204, XIV.), from the expressions for S and V in terms of K:

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[*] Indeed, it has only been proved as yet (comp. 195, (1.)), that $\mathrm{K}\Sigma q = \Sigma\mathrm{K}q$, for the case of *two* summands; but this result will soon be extended.

$$\text{VIII.} \ldots \mathrm{S}^2 = \tfrac{1}{4}(1+\mathrm{K})^2 = \tfrac{1}{4}(1+2\mathrm{K}+\mathrm{K}^2) = \tfrac{1}{2}(1+\mathrm{K}) = \mathrm{S};$$

$$\text{IX.} \ldots \mathrm{V}^2 = \tfrac{1}{4}(1-\mathrm{K})^2 = \tfrac{1}{4}(1-2\mathrm{K}+\mathrm{K}^2) = \tfrac{1}{2}(1-\mathrm{K}) = \mathrm{V}.$$

(3.) Similarly,

$$\text{X.} \ldots \mathrm{SV} = \tfrac{1}{4}(1+\mathrm{K})(1-\mathrm{K}) = \tfrac{1}{4}(1-\mathrm{K}^2) = 0, \text{ as in 204, IV.};$$

and
$$\text{XI.} \ldots \mathrm{VS} = \tfrac{1}{4}(1-\mathrm{K})(1+\mathrm{K}) = \tfrac{1}{4}(1-\mathrm{K}^2) = 0, \text{ as in 204, VIII.}$$

206. As regards the *addition* (or subtraction) of such *right parts*, $\mathrm{V}q$, $\mathrm{V}q'$, or generally of any two right quaternions (132), we may *connect* it with the addition (or subtraction) of their *indices* (133), as follows. Let OBDC be again any parallelogram (197, 205), but let OA be now an unit-vector (129) perpendicular to its plane; so that

$$\mathrm{T}\alpha = 1, \quad \angle(\beta : \alpha) = \angle(\gamma : \alpha) = \angle(\delta : \alpha) = \frac{\pi}{2}, \quad \delta = \gamma + \beta.$$

Let OB′D′C′ be another parallelogram in the same plane, obtained by a positive rotation of the former, through a right angle, round OA as an axis; so that

$$\angle(\beta' : \beta) = \angle(\gamma' : \gamma) = \angle(\delta' : \delta) = \frac{\pi}{2};$$

$$\mathrm{Ax.}(\beta' : \beta) = \mathrm{Ax.}(\gamma' : \gamma) = \mathrm{Ax.}(\delta' : \delta) = \alpha.$$

Then the three right quotients, $\beta : \alpha$, $\gamma : \alpha$, and $\delta : \alpha$, may represent *any two right quaternions*, q, q', and their *sum*, $q' + q$, which is always (by 197, (2.)) *itself* a *right* quaternion; and the *indices* of these three right quotients are (comp. 133, 193) the three lines β', γ', δ', so that we may write, under the foregoing conditions of construction,

$$\beta' = \mathrm{I}(\beta : \alpha), \quad \gamma' = \mathrm{I}(\gamma : \alpha), \quad \delta' = \mathrm{I}(\delta : \alpha).$$

But this third index is (by the second parallelogram) the *sum* of the two former indices, or in symbols, $\delta' = \gamma' + \beta'$; we may therefore write,

$$\text{I.} \ldots \mathrm{I}(q' + q) = \mathrm{I}q' + \mathrm{I}q, \quad \text{if} \quad \angle q = \angle q' = \frac{\pi}{2};$$

or in words *the Index of the Sum** *of any two Right Quaternions is equal to the Sum of their Indices.* Hence, generally, for any two quaternions, q and q', we have the formula,

$$\text{II.} \ldots \mathrm{IV}(q' + q) = \mathrm{IV}q' + \mathrm{IV}q,$$

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* Compare the Note to page 174.

because $\mathrm{V}q$, $\mathrm{V}q'$ are *always* right quotients (202, 204), and $\mathrm{V}(q'+q)$ is always their *sum* (205, I.); so that the *index of the right part of the sum of any two quaternions* is the *sum of the indices of the right parts.* In like manner, there is no difficulty in proving that

$$\text{III.}\ldots\ \mathrm{I}(q'-q) = \mathrm{I}q' - \mathrm{I}q, \quad \text{if} \quad \angle q' = \angle q = \frac{\pi}{2};$$

and generally, that

$$\text{IV.}\ldots\ \mathrm{IV}(q'-q) = \mathrm{IV}q' - \mathrm{IV}q;$$

the *Index of the Difference* of any two right quotients, or of the right parts of any two quaternions, being thus equal to the *Difference of the Indices.** We may then *reduce* the *addition* or *subtraction* of any two such quotients, or parts, to the addition or subtraction of their *indices;* a right quaternion being always (by 133) determined, when its index is given, or known.

207. We see, then, that as the MULTIPLICATION *of any two Quaternions* was (in 191) *reduced* to (Ist) the *arithmetical operation* of *multiplying their tensors*, and (IInd) the *geometrical operation* of *multiplying their versors*, which latter was *constructed* by a certain *composition of rotations*, and was *represented* (in either of two distinct but connected ways, 167, 175) by sides or angles of a *spherical triangle:* so the ADDITION *of any two Quaternions* may be reduced (by 197, I., and 206, II.) to, Ist, *the algebraical addition of their scalar parts*, considered as two positive or negative numbers (16); and, IInd, *the geometrical addition of the indices of their right parts*, considered as certain *vectors* (1): this latter *Addition of Lines* being performed according to the *Rule of the Parallelogram* (6.).† In

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* Compare again the Note to page 174.

† It does not fall within the plan of these Notes to allude often to the history of the subject; but it ought to be distinctly stated that this celebrated *Rule*, for what may be called *Geometrical Addition of right lines*, considered as *analogous* to *composition of motions* (or of *forces*), had occurred to several writers, *before* the invention of the quaternions: although the method adopted, in the present and in a former work, of deducing that rule, by algebraical analogies, from the *symbol* B − A (1) for the line AB, may possibly not have been anticipated. The reader may compare the Notes to the Preface to the author's Volume of Lectures on Quaternions (Dublin, 1853).

like manner, as the general *Division of Quaternions* was seen (in 191) to admit of being reduced to an *arithmetical division of tensors*, and a *geometrical division of versors*, so we may now (by 197, III., and 206, IV.) reduce, generally, the *Subtraction of Quaternions* to (Ist) an *algebraical subtraction of scalars*, and (IInd) a *geometrical subtraction of vectors:* this last operation being again constructed by a parallelogram, or even by a *plane triangle* (comp. Art. 4, and Fig. 2). And because the *sum* of any given set of *vectors* was early seen to have a *value* (9), which is independent of their *order*, and of the mode of *grouping* them, we may now infer that the *Sum of any number of given Quaternions* has, in like manner, a *Value* (comp. 197, (1.)), which is *independent of the Order, and of the Grouping of the Summands:* or in other words, that *the general Addition of Quaternions is a Commutative* and an Associative Operation.*

(1.) The formula,

$$\mathrm{V}\Sigma q = \Sigma \mathrm{V}q, \text{ of 205, III.,}$$

is now seen to hold good, for *any number* of quaternions, independently of the *arrangement* of the terms in each of the two sums, and of the manner in which they may be *associated.*

(2.) We can infer anew that

$$\mathrm{K}(q' + q) = \mathrm{K}q' + \mathrm{K}q, \text{ as in 195, II.,}$$

under the form of the equation or identity,

$$\mathrm{S}(q' + q) - \mathrm{V}(q' + q) = (\mathrm{S}q' - \mathrm{V}q') + (\mathrm{S}q - \mathrm{V}q).$$

(3.) More generally, it may be proved, in the same way, that

$$\mathrm{K}\Sigma q = \Sigma \mathrm{K}q, \quad \text{or briefly,} \quad \mathrm{K}\Sigma = \Sigma\mathrm{K},$$

whatever the number of the summands may be.

208. As regards the *quotient or product of the right parts,* Vq and Vq', of any two quaternions, let t and t' denote the *tensors* of those two parts, and let x denote the *angle of their indices*, or of their *axes*, or the mutual *inclination* of the axes, or of the *planes*,† of the two quaternions q and q' *themselves*, so that (by 204, XVIII.),

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* Compare the Note to page 175.

† *Two planes*, of course, make with each other, in general, *two* unequal and supplementary *angles ;* but we here suppose that these are mutually *distinguished*, by taking account of the *aspect* of each plane, as distinguished from the *opposite* aspect : which is most easily done (111.), by considering the *axes* as above.

$$t = \mathrm{TV}q = \mathrm{T}q \,.\, \sin \angle q, \qquad t' = \mathrm{TV}q' = \mathrm{T}q' \,.\, \sin \angle q',$$

and

$$x = \angle (\mathrm{IV}q' : \mathrm{IV}q) = \angle (\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q).$$

Then, by 193, 194, and by 204, XXXV., XXXV'.,

$$\mathrm{I.} \ldots \mathrm{V}q' : \mathrm{V}q = \mathrm{IV}q' : \mathrm{IV}q = +(\mathrm{TV}q' : \mathrm{TV}q) \,.\, (\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q);$$

$$\mathrm{II.} \ldots \mathrm{V}q' . \mathrm{V}q = \mathrm{IV}q' : \mathrm{I}\frac{1}{\mathrm{V}q} = -(\mathrm{TV}q' \,.\, \mathrm{TV}q) \,.\, (\mathrm{Ax}.\, q' : \mathrm{Ax}.\, q);$$

and therefore (comp. 198), with the temporary abridgments proposed above,

$$\mathrm{III.} \ldots \mathrm{S}(\mathrm{V}q' : \mathrm{V}q) = t't^{-1} \cos x; \qquad \mathrm{IV.} \ldots \mathrm{SU}(\mathrm{V}q' : \mathrm{V}q) = + \cos x;$$

$$\mathrm{V.} \ldots \mathrm{S}(\mathrm{V}q' \,.\, \mathrm{V}q) = -t't \cos x; \qquad \mathrm{VI.} \ldots \mathrm{SU}(\mathrm{V}q' \,.\, \mathrm{V}q) = - \cos x;$$

$$\mathrm{VII.} \ldots \angle (\mathrm{V}q' : \mathrm{V}q) = x; \qquad \mathrm{VIII.} \ldots \angle (\mathrm{V}q' \,.\, \mathrm{V}q) = \pi - x.$$

We have also generally (comp. 204, XVIII., XIX.),

$$\mathrm{IX.} \ldots \mathrm{TV}(\mathrm{V}q' : \mathrm{V}q) = t't^{-1} \sin x; \qquad \mathrm{X.} \ldots \mathrm{TVU}(\mathrm{V}q' : \mathrm{V}q) = \sin x;$$

$$\mathrm{XI.} \ldots \mathrm{TV}(\mathrm{V}q' \,.\, \mathrm{V}q) = t't \sin x; \qquad \mathrm{XII.} \ldots \mathrm{TVU}(\mathrm{V}q' \,.\, \mathrm{V}q) = \sin x;$$

and in particular,

$$\mathrm{XIII.} \ldots \mathrm{V}(\mathrm{V}q' : \mathrm{V}q) = 0, \quad \text{and} \quad \mathrm{XIV.} \ldots \mathrm{V}(\mathrm{V}q' \,.\, \mathrm{V}q) = 0,$$
$$\text{if } q' \parallel\!\mid q \ (123);$$

because (comp. 191, (6.), and 204, VI.) the quotient or product of the right parts of two *complanar* quaternions (supposed here to be both *non-scalar* (108), so that t and t' are each > 0) degenerates (131) into a *scalar*, which may be thus expressed:

$$\mathrm{XV.} \ldots \mathrm{V}q' : \mathrm{V}q = +t't^{-1}, \quad \text{and} \quad \mathrm{XVI.} \ldots \mathrm{V}q' . \mathrm{V}q = -t't, \quad \text{if } x = 0;$$

but

$$\mathrm{XVII.} \ldots \mathrm{V}q' : \mathrm{V}q = -t't^{-1}, \quad \text{and} \quad \mathrm{XVIII.} \ldots \mathrm{V}q' . \mathrm{V}q = +t't, \quad \text{if } x = \pi;$$

the first case being that of *coincident*, and the second case that of *opposite axes*. In the more general case of *diplanarity* (119), if we denote by δ the unit-line which is *perpendicular to both their axes*, and therefore *common to their two planes*, or in which those planes *intersect*, and which is so directed that the *rotation* round it from Ax. q to Ax. q' is *positive* (comp. 127, 128), the recent formulæ I., II. give easily,

$$\mathrm{XIX.} \ldots \mathrm{Ax}.\, (\mathrm{V}q' : \mathrm{V}q) = +\delta; \qquad \mathrm{XX.} \ldots \mathrm{Ax}.\, (\mathrm{V}q' . \mathrm{V}q) = -\delta;$$

and therefore (by IX., XI., and by 204, XXXV.), the *indices of the right parts*, of the quotient and product of the right parts of any two diplanar quaternions, may be expressed as follows:

$$\mathrm{XXI.} \ldots \mathrm{IV}(\mathrm{V}q' : \mathrm{V}q) = +\delta . t't^{-1} \sin x;$$
$$\mathrm{XXII.} \ldots \mathrm{IV}(\mathrm{V}q' \,.\, \mathrm{V}q) = -\delta . t't \sin x.$$

(1.) Let ABC be any triangle upon the unit-sphere (128), of which the spherical *angles* and the *corners* may be denoted by the same letters A, B, C, while the *sides* shall as usual be denoted by a, b, c; and let it be supposed that the *rotation* (comp. 177) round A from C to B, and therefore that round B from A to C, &c., is *positive*, as in Fig. 43. Then writing, as we have often done,

$$q = \beta : \alpha, \quad \text{and} \quad q' = \gamma : \beta, \quad \text{where} \quad \alpha = \text{OA}, \text{ \&c.},$$

we easily obtain the the following expressions for the three scalars t, t', x, and for the vector δ:

$$t = \sin c; \qquad t' = \sin a; \qquad x = \pi - \text{B}; \qquad \delta = -\beta.$$

(2.) In fact we have here,

$$\text{T}q = \text{T}q' = 1, \qquad \angle q = c, \qquad \angle q' = a;$$

whence t and t' are as just stated. Also if A′, B′, C′ be (as in 175) the *positive poles* of the three *successive sides* BC, CA, AB, of the given triangle, and therefore the points A, B, C the *negative poles* (comp. 180, (2.)) of the new arcs B′C′, C′A′, A′B′, then

$$\text{Ax}.\, q = \text{OC}', \qquad \text{Ax}.\, q' = \text{OA}';$$

but x and δ are the *angle* and the *axis* of the *quotient* of these two axes, or of the quaternion which is *represented* (162) by the *arc* C′A′; therefore x is, as above stated, the *supplement* of the *angle* B, and δ is directed to the point upon the sphere, which is diametrically *opposite* to the *point* B.

(3.) Hence, by III. V. VII. VIII. IX. XI., for any triangle ABC on the unit-sphere, with $\alpha = \text{OA}$, &c., we have the formulæ:

$$\text{XXIII.} \ldots \text{S}\left(\text{V}\frac{\gamma}{\beta} : \text{V}\frac{\beta}{\alpha}\right) = -\sin a \operatorname{cosec} c \cos \text{B};$$

$$\text{XXIV.} \ldots \text{S}\left(\text{V}\frac{\gamma}{\beta} . \text{V}\frac{\beta}{\alpha}\right) = +\sin a \sin c \cos \text{B};$$

$$\text{XXV.} \ldots \angle\left(\text{V}\frac{\gamma}{\beta} : \text{V}\frac{\beta}{\alpha}\right) = \pi - \text{B}; \qquad \text{XXVI.} \ldots \angle\left(\text{V}\frac{\gamma}{\beta} . \text{V}\frac{\beta}{\alpha}\right) = \text{B};$$

$$\text{XXVII.} \ldots \text{TV}\left(\text{V}\frac{\gamma}{\beta} : \text{V}\frac{\beta}{\alpha}\right) = +\sin a \operatorname{cosec} c \sin \text{B};$$

$$\text{XXVIII.} \ldots \text{TV}\left(\text{V}\frac{\gamma}{\beta} . \text{V}\frac{\beta}{\alpha}\right) = +\sin a \sin c \sin \text{B}.$$

(4.) Also, by XIX. XX. XXI. XXII., if the rotation round B from A to C be still *positive*,

$$\text{XXIX.} \ldots \text{Ax.}\left(\text{V}\frac{\gamma}{\beta} : \text{V}\frac{\beta}{\alpha}\right) = -\beta; \qquad \text{XXX.} \ldots \text{Ax.}\left(\text{V}\frac{\gamma}{\beta} . \text{V}\frac{\beta}{\alpha}\right) = +\beta;$$

$$\text{XXXI.} \ldots \text{IV}\left(\text{V}\frac{\gamma}{\beta} : \text{V}\frac{\beta}{\alpha}\right) = -\beta \sin a \operatorname{cosec} c \sin \text{B};$$

$$\text{XXXII.} \ldots \text{IV}\left(\text{V}\frac{\gamma}{\beta} . \text{V}\frac{\beta}{\alpha}\right) = +\beta \sin a \sin c \sin \text{B}.$$

(5.) If, on the other hand, the rotation round B from A to C were *negative*, then writing for a moment $\alpha_1 = -\alpha$, $\beta_1 = -\beta$, $\gamma_1 = -\gamma$, we should have a new and *opposite triangle*, $\text{A}_1\text{B}_1\text{C}_1$, in which the rotation round B_1 from A_1 to C_1 would be *positive*, but the angle at B_1 equal in magnitude to that at B; so that by treating (as usual) all the angles of a spherical triangle as positive, we should have $\text{B}_1 = \text{B}$, as well as $c_1 = c$, and $a_1 = a$; and therefore, for example, by XXXI.

$$\mathrm{IV}\left(\mathrm{V}\frac{\gamma_1}{\beta_1}:\mathrm{V}\frac{\beta_1}{\alpha_1}\right)=-\beta_1 \sin a_1 \operatorname{cosec} c_1 \sin \mathrm{B}_1,$$

$$\text{or}\quad \mathrm{IV}\left(\mathrm{V}\frac{\gamma}{\beta}:\mathrm{V}\frac{\beta}{\alpha}\right)=+\beta \sin a \operatorname{cosec} c \sin \mathrm{B};$$

the four formulæ of (4.) would therefore still subsist, provided that, for this new direction of rotation in the given triangle, we were to *change the sign of* β, *in the second member of each.*

(6.) Abridging, generally $\mathrm{IV}q:\mathrm{S}q$ to $(\mathrm{IV}:\mathrm{S})q$, as $\mathrm{TV}q:\mathrm{S}q$ was abridged, in 204, XXXIV'., to $(\mathrm{TV}:\mathrm{S})q$, we have by (5.), and by XXIV., XXXII., this other general formula, for any three unit-vectors α, β, γ, considered still as terminating at the corners of a spherical triangle ABC:

$$\text{XXXIII.}\ldots(\mathrm{IV}:\mathrm{S})\left(\mathrm{V}\frac{\gamma}{\beta}.\mathrm{V}\frac{\beta}{\alpha}\right)=\pm\beta\tan \mathrm{B};$$

the upper or the lower sign being taken, according as the rotation round B from A to C, or that round β from α to γ, which might perhaps be denoted by the symbol $\alpha\hat{\beta}\gamma$, and which in quantity is equal to the spherical angle B, is positive or negative.

209. When the *planes* of any *three* quaternions q, q', q'', considered as all passing through the origin O (119), contain any *common line*, those three may then be said to be *Collinear*[*] *Quaternions;* and because the *axis* of each is then perpendicular to that line, it follows that *the Axes of Collinear Quaternions are Complanar:* while conversely, the *complanarity of the axes* insures the *collinearity of the quaternions*, because the *perpendicular to the plane* of the axes is a line common to the planes of the quaternions.

(1.) Complanar quaternions are always collinear; but the converse proposition does not hold good, collinear quaternions being not necessarily complanar.

(2.) Collinear quaternions, considered as *fractions* (101), can always be reduced to a *common denominator* (120); and conversely, if three or more quaternions *can* be so reduced, as to appear under the form of fractions with a common denominator ϵ, those quaternions must be *collinear:* because the line ϵ is then common to all their planes.

(3.) *Any two quaternions* are *collinear with any scalar;* the *plane of* a scalar being *indeterminate*[†] (131).

(4.) Hence the *scalar and right parts*, $\mathrm{S}q$, $\mathrm{S}q'$, $\mathrm{V}q$, $\mathrm{V}q'$, of *any two quaternions*, are always *collinear* with each other.

(5.) The *conjugates* of collinear quaternions are themselves collinear.

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[*] Quaternions of which the planes are *parallel* to any common line may also be said to be *collinear*. Compare the first Note to page 113.

[†] Compare the Note to page 114.

210. Let q, q', q'' be any three collinear quaternions; and let α denote a line common to their planes. Then we may determine (comp. 120) three other lines β, γ, δ, such that

$$q = \frac{\beta}{\alpha}, \qquad q' = \frac{\gamma}{\alpha}, \qquad q'' = \frac{\alpha}{\delta};$$

and thus may conclude that (as in algebra),

$$\text{I.} \ldots (q' + q)\, q'' = q'q'' + qq'',$$

because, by 106, 107,

$$\left(\frac{\gamma}{\alpha} + \frac{\beta}{\alpha}\right)\frac{\alpha}{\delta} = \frac{\gamma+\beta}{\alpha} \cdot \frac{\alpha}{\delta} = \frac{\gamma+\beta}{\delta} = \frac{\gamma}{\delta} + \frac{\beta}{\delta} = \frac{\gamma}{\alpha}\frac{\alpha}{\delta} + \frac{\beta}{\alpha}\frac{\alpha}{\delta}.$$

In like manner, at least under the same condition of collinearity,[*] it may be proved that

$$\text{II.} \ldots (q' - q)\, q'' = q'q'' - qq''.$$

Operating by the characteristic K upon these two equations, and attending to 192, II., and 195, II., we find that

$$\text{III.} \ldots \mathrm{K}q''.(\mathrm{K}q' + \mathrm{K}q) = \mathrm{K}q''.\mathrm{K}q' + \mathrm{K}q''.\mathrm{K}q;$$
$$\text{IV.} \ldots \mathrm{K}q''.(\mathrm{K}q' - \mathrm{K}q) = \mathrm{K}q''.\mathrm{K}q' - \mathrm{K}q''.\mathrm{K}q;$$

where (by 209, (5.)) the three *conjugates* of arbitrary collinears, $\mathrm{K}q$, $\mathrm{K}q'$, $\mathrm{K}q''$, may represent *any three* collinear quaternions. We have, therefore, with the same degree of generality as before,

$$\text{V.} \ldots q''(q' + q) = q''q' + q''q; \qquad \text{VI.} \ldots q''(q' - q) = q''q' - q''q.$$

If, then, q, q', q'', q''' be *any four collinear quaternions*, we may establish the formula (again agreeing with algebra):

$$\text{VII.} \ldots (q''' + q'')\,(q' + q) = q'''q' + q''q' + q'''q + q''q;$$

and similarly for any greater number, so that we may write briefly,

$$\text{VIII.} \ldots \Sigma q' . \Sigma q = \Sigma q'q,$$

where

$$\Sigma q' = q_1 + q_2 + \ldots + q_m, \qquad \Sigma q' = q'_1 + q'_2 + \ldots + q'_n,$$

and

$$\Sigma q'q = q'_1 q_1 + \ldots q_1' q_m + q'_2 q_1 + \ldots + q'_n q_m,$$

m and n being any positive whole numbers. In words (comp. 13), *the Multiplication of Collinear*[†] *Quaternions is a Doubly Distributive Operation.*

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[*] It will soon be seen, however, that this condition is unnecessary.

[†] This *distributive property of multiplication* will soon be found (compare the last Note) to extend to the more general case, in which the quaternions are *not collinear*.

(1.) Hence, by 209, (4.), and 202, III., we have this general transformation, for the *product of any two quaternions*:

$$\text{IX.}\ldots\ q'q = \mathrm{S}q'.\mathrm{S}q + \mathrm{V}q'.\mathrm{S}q + \mathrm{S}q'.\mathrm{V}q + \mathrm{V}q'.\mathrm{V}q.$$

(2.) Hence also, for the *square of any quaternion*, we have the transformation (comp. 126; 199, VII.; and 204, XXIII.):

$$\text{X.}\ldots\ q^2 = \mathrm{S}q^2 + 2\mathrm{S}q.\mathrm{V}q + \mathrm{V}q^2.$$

(3.) *Separating* the scalar and right *parts* of this last expression, we find these other general formulæ:

$$\text{XI.}\ldots\ \mathrm{S}.q^2 = \mathrm{S}q^2 + \mathrm{V}q^2; \qquad \text{XII.}\ldots\ \mathrm{V}.q^2 = 2\mathrm{S}q.\mathrm{V}q;$$

whence also, dividing by $\mathrm{T}q^2$, we have

$$\text{XIII.}\ldots\ \mathrm{SU}(q^2) = (\mathrm{SU}q)^2 + (\mathrm{VU}q)^2; \qquad \text{XIV.}\ldots\ \mathrm{VU}(q^2) = 2\mathrm{SU}q.\mathrm{VU}q.$$

(4.) By supposing $q' = \mathrm{K}q$, in IX., and therefore $\mathrm{S}q' = \mathrm{S}q$, $\mathrm{V}q' = -\mathrm{V}q$, and transposing the two conjugate and therefore complanar factors (comp. 191, (1.)), we obtain this general transformation for a *norm*, or for the *square* of a *tensor* (comp. 190, V.; 202, III.; and 204, XI.):

$$\text{XV.}\ldots\ \mathrm{T}q^2 = \mathrm{N}q = q\mathrm{K}q = (\mathrm{S}q + \mathrm{V}q)(\mathrm{S}q - \mathrm{V}q) = \mathrm{S}q^2 - \mathrm{V}q^2;$$

which had indeed presented itself before (in 204, XXII.) but is now obtained in a new way, and *without any employment of sines, or cosines*, or even of the well-known theorem respecting the *square of the hypotenuse*.

(5.) Eliminating $\mathrm{V}q^2$, by XV., from XI., and dividing by $\mathrm{T}q^2$, we find that

$$\text{XVI.}\ldots\ \mathrm{S}.q^2 = 2\mathrm{S}q^2 - \mathrm{T}q^2; \qquad \text{XVII.}\ldots\ \mathrm{SU}(q^2) = 2(\mathrm{SU}q)^2 - 1;$$

agreeing with 199, VI. and IV., but obtained here without any use of the known formula for the *cosine of the double* of an angle.

(6.) Taking the scalar and right parts of the expression IX., we obtain these other general expressions:

$$\text{XVIII.}\ldots\ \mathrm{S}q'q = \mathrm{S}q'.\mathrm{S}q + \mathrm{S}(\mathrm{V}q'.\mathrm{V}q);$$

$$\text{XIX.}\ldots\ \mathrm{V}q'q = \mathrm{V}q'.\mathrm{S}q + \mathrm{V}q.\mathrm{S}q' + \mathrm{V}(\mathrm{V}q'.\mathrm{V}q);$$

in the latter of which we may (by 126) transpose the two factors, $\mathrm{V}q'$, $\mathrm{S}q$, or $\mathrm{V}q$, $\mathrm{S}q'$. We may also (by 206, 207) write, instead of XIX., this other formula:

$$\text{XIX}'.\ldots\ \mathrm{IV}q'q = \mathrm{IV}q'.\mathrm{S}q + \mathrm{IV}q.\mathrm{S}q' + \mathrm{IV}(\mathrm{V}q'.\mathrm{V}q).$$

(7.) If we suppose, in VII., that $q'' = \mathrm{K}q$, $q''' = \mathrm{K}q'$, and transpose (comp. (4.)) the two complanar (because conjugate) factors, $q' + q$ and $\mathrm{K}(q' + q)$, we obtain the following general expression for the *norm of a sum*:

$$(q' + q)\,\mathrm{K}(q' + q) = q'\mathrm{K}q' + q\mathrm{K}q' + q'\mathrm{K}q + q\mathrm{K}q;$$

or briefly,

$$\text{XX.}\ldots\ \mathrm{N}(q' + q) = \mathrm{N}q' + 2\mathrm{S}.q\mathrm{K}q' + \mathrm{N}q, \text{ as in 200, VII.};$$

because

$$q'\mathrm{K}q = \mathrm{K}.q\mathrm{K}q', \text{ by 192, II., and } (1 + \mathrm{K}).q\mathrm{K}q' = 2\mathrm{S}.q\mathrm{K}q', \text{ by 196, II}'.$$

(8.) By changing q' to x in XX., or by forming the product of $q + x$ and $\mathrm{K}q + x$, where x is any scalar, we find that

$$\text{XXI.}\ldots\ \mathrm{N}(q + x) = \mathrm{N}q + 2x\mathrm{S}q + x^2, \text{ as in 200, VIII.};$$

whence, in particular,

$$\text{XXI}'.\ldots\ \mathrm{N}(q - 1) = \mathrm{N}q - 2\mathrm{S}q + 1, \text{ as in 200, II.}$$

(9.) Changing q to $\beta : \alpha$, and multiplying by the square of $T\alpha$, we get, for any two vectors, α and β, the formula,

$$\text{XXII.}\ldots\ T(\beta-\alpha)^2 = T\beta^2 - 2T\beta\,.\,T\alpha\,.\,SU\frac{\beta}{\alpha} + T\alpha^2,$$

in which $T\alpha^2$ denotes[*] $(T\alpha)^2$; because (by 190, and by 196, IX.),

$$N\left(\frac{\beta}{\alpha} - 1\right) = N\frac{\beta-\alpha}{\alpha} = \left(\frac{T(\beta-\alpha)}{T\alpha}\right)^2, \quad \text{and} \quad S\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}\,SU\frac{\beta}{\alpha}.$$

(10.) In any plane triangle, ABC, with sides of which the *lengths* are as usual denoted by a, b, c, let the vertex C be taken as the origin O of vectors; then

$$\alpha = \text{CA}, \quad \beta = \text{CB}, \quad \beta - \alpha = \text{AB}, \quad T\alpha = b, \quad T\beta = a, \quad T(\beta-\alpha) = c, \quad SU\frac{\beta}{\alpha} = \cos \text{C};$$

we *recover* therefore, from XXII., the *fundamental formula of plane trigonometry*, under the form,

$$\text{XXIII.}\ldots\ c^2 = a^2 - 2ab\cos \text{C} + b^2.$$

(11.) It is important to observe that we have not here been arguing in a circle; because although, in Art. 200, we assumed, for the convenience of the student, a previous knowledge of the last written formula, in order to arrive more rapidly at certain applications, yet in these recent deductions from the *distributive property* VIII. *of multiplication* of (at least) collinear quaternions, we have founded nothing on the results of that former Article; and have made no use of any properties of *oblique-angled triangles*, or even of *right-angled* ones, since the theorem of the square of the hypotenuse has been virtually proved anew in (4.): nor is it necessary to the argument, that any *properties* of trigonometric *functions* should be known, beyond the mere *definition* of a *cosine*, as a certain *projecting factor*, from which the formula 196, XVI. was derived, and which justifies us in writing cos C in the last equation (10.). The geometrical Examples, in the sub-articles to 200, may therefore be read again, and their validity be seen anew, *without any appeal to even plane trigonometry* being now supposed.

(12.) The formula XV. gives $Sq^2 = Tq^2 + Vq^2$, as in 204, XXXI.; and we know that Vq^2, as being generally the *square of a right quaternion*, is equal to a *negative scalar* (comp. 204, VI.), so that

$$\text{XXIV}\ldots\ Vq^2 < 0, \quad \text{unless} \quad \angle q = 0, \quad \text{or} \quad = \pi,$$

in each of which two cases $Vq = 0$, by 202, (6.), and therefore its square vanishes; hence,

$$\text{XXV.}\ldots\ Sq^2 < Tq^2, \qquad (SUq)^2 < 1,$$

in every other case.

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[*] We are not *yet* at liberty to interpret the symbol $T\alpha^2$ as denoting *also* $T(\alpha^2)$; because we have not yet assigned *any* meaning to the *square of a vector*, or generally to the *product of two vectors*. In the Third Book of these Elements it will be shown, that such a square or product can be interpreted as being a *quaternion*: and then it will be found (comp. 190), that

$$T(\alpha^2) = (T\alpha)^2 = T\alpha^2,$$

whatever vector α may be.

(13.) It might therefore have been thus proved, without any use of the transformation $\mathrm{SU}q = \cos \angle q$ (196, XVI.), that (for any *real* quaternion q) we have the inequalities,

$$\text{XXVI.} \ldots \mathrm{SU}q < +1, \quad \mathrm{SU}q > -1, \quad \text{and} \quad \mathrm{S}q < +\mathrm{T}q, \quad \mathrm{S}q > -\mathrm{T}q,$$

unless it happen that $\angle q = 0$, or $= \pi$; $\mathrm{SU}q$ being $= +1$, and $\mathrm{S}q = +\mathrm{T}q$, in the first case; whereas $\mathrm{SU}q = -1$, and $\mathrm{S}q = -\mathrm{T}q$, in the second case.

(14.) Since $\mathrm{T}q^2 = \mathrm{N}q$, and $\mathrm{T}q . \mathrm{T}q' = \mathrm{T} . q\mathrm{K}q' = \mathrm{T} . q'\mathrm{K}q = \mathrm{N}q . \mathrm{T}(q' : q)$, while $\mathrm{S} . q\mathrm{K}q' = \mathrm{S} . q'\mathrm{K}q = \mathrm{N}q . \mathrm{S}(q' : q)$, the formula XX. gives, by XXVI.,

$$\text{XXVII.} \ldots (\mathrm{T}q' + \mathrm{T}q)^2 - \mathrm{T}(q' + q)^2 = 2(\mathrm{T} - \mathrm{S})q\mathrm{K}q' = 2\mathrm{N}q . (\mathrm{T} - \mathrm{S})\ (q' : q) > 0,$$

if we adopt the abridged notation,

$$\text{XXVIII.} \ldots \mathrm{T}q - \mathrm{S}q = (\mathrm{T} - \mathrm{S})q,$$

and suppose that the quotient $q' : q$ is *not* a positive scalar; hence,

$$\text{XXIX.} \ldots \mathrm{T}q' + \mathrm{T}q > \mathrm{T}(q' + q), \quad \text{unless} \quad q' = xq, \quad \text{and} \quad x > 0;$$

in which excepted case, each member of this last inequality becomes $= (1 + x)\mathrm{T}q$.

(15.) Writing $q = \beta : \alpha$, $q' = \gamma : \alpha$, and multiplying by $\mathrm{T}\alpha$, the formula XXIX. becomes,

$$\text{XXX.} \ldots \mathrm{T}\gamma + \mathrm{T}\beta > \mathrm{T}(\gamma + \beta), \quad \text{unless} \quad \gamma = x\beta, \quad x > 0;$$

in which latter case, but not in any other, we have $\mathrm{U}\gamma = \mathrm{U}\beta$ (155). We therefore arrive anew at the results of 186, (9.), (10.), but without its having been necessary to consider any *triangle*, as was done in those former sub-articles.

(16.) On the other hand, with a corresponding abridgment of notation, we have, by XXVI.,

$$\text{XXXI.} \ldots \mathrm{T}q + \mathrm{S}q = (\mathrm{T} + \mathrm{S})q > 0, \quad \text{unless} \quad \angle q = \pi;$$

also, by XX., &c.,

$$\text{XXXII.} \ldots \mathrm{T}(q' + q)^2 - (\mathrm{T}q' - \mathrm{T}q)^2 = 2(\mathrm{T} + \mathrm{S})q\mathrm{K}q' = 2\mathrm{N}q . (\mathrm{T} + \mathrm{S})\ (q' : q);$$

hence,

$$\text{XXXIII.} \ldots \mathrm{T}(q' + q) > \pm(\mathrm{T}q' - \mathrm{T}q), \quad \text{unless} \quad q' = -xq, \quad x > 0;$$

where either sign may be taken.

(17.) And hence, on the plan of (15.), for any two vectors β, γ,

$$\text{XXXIV.} \ldots \mathrm{T}(\gamma + \beta) > \pm(\mathrm{T}\gamma - \mathrm{T}\beta), \quad \textit{unless} \quad \mathrm{U}\gamma = -\mathrm{U}\beta,$$

whichever sign be adopted; but, on the contrary,

$$\text{XXXV.} \ldots \mathrm{T}(\gamma + \beta) = \pm(\mathrm{T}\gamma - \mathrm{T}\beta), \quad \textit{if} \quad \mathrm{U}\gamma = -\mathrm{U}\beta,$$

the upper or the lower sign being taken, according as $\mathrm{T}\gamma >$ or $< \mathrm{T}\beta$: all which agrees with what was inferred, in 186, (11.), from *geometrical* considerations alone, combined with the definition of $\mathrm{T}\alpha$. In fact, if we make $\beta = \mathrm{OB}$, $\gamma = \mathrm{OC}$, and $-\gamma = \mathrm{OC}'$, then OBC' will be in general *a plane triangle*, in which the length of the side BC' *exceeds* the difference of the lengths of the two other sides; but if it happen that the directions of the two lines OB, OC' coincide, or in other words that the lines OB, OC have opposite directions, then the difference of lengths of these two lines becomes *equal* to the length of the line BC'.

(18.) With the representations of q and q', assigned in 208, (1.), by two sides of a *spherical triangle* ABC, we have the values,

$$\mathrm{S}q = \cos c, \qquad \mathrm{S}q' = \cos a, \qquad \mathrm{S}q'q = \mathrm{S}(\gamma : \alpha) = \cos b;$$

the equation XVIII. gives therefore, by 208, XXIV., the *fundamental formula of spherical trigonometry* (comp. (10.)), as follows:

$$\text{XXXVI.} \ldots \cos b = \cos a \cos c + \sin a \sin c \cos \mathrm{B}.$$

(19.) To interpret, with reference to the same spherical triangle, the connected equation XIX., or XIX'., let it be now supposed, as in 208, (5.), that the rotation round B from C to A is positive, so that B and B' are situated at the same side of the arc CA, if B' be still, as in 208, (2.), the positive pole of that arc. Then writing $\alpha' = \mathrm{OA}'$, &c., we have

$$\mathrm{IV}q = \gamma' \sin c; \qquad \mathrm{IV}q' = \alpha' \sin a; \qquad \mathrm{IV}q'q = -\beta' \sin b;$$

and $$\mathrm{IV}(\mathrm{V}q' . \mathrm{V}q) = -\beta \sin a \sin c \sin \mathrm{B} \text{ (comp. 208, (5.))},$$

with the recent values (18.), for $\mathrm{S}q$ and $\mathrm{S}q'$; thus the formula XIX'. becomes, by transposition of the two terms last written:

$$\text{XXXVII.} \ldots \beta \sin a \sin c \sin \mathrm{B} = \alpha' \sin a \cos c + \beta' \sin b + \gamma' \sin c \cos a.$$

(20.) Let $\rho = \mathrm{OP}$ be any unit-vector; then, dividing each term of the last equation by ρ, and taking the scalar of each of the four quotients, we have, by 196, XVI., this new equation:

$$\text{XXXVIII.} \ldots \sin a \sin c \sin \mathrm{B} \cos \mathrm{PB} = \sin a \cos c \cos \mathrm{PA}' + \sin b \cos \mathrm{PB}' + \sin c \cos a \cos \mathrm{PC}';$$

where a, b, c are as usual the sides of the spherical triangle ABC, and A', B', C' are still, as in 208, (2.), the positive poles of those sides; but P is an arbitrary point, upon the surface of the sphere. Also cos PA', cos PB', cos PC', are evidently the sines of the arcual perpendiculars, let fall from that point upon those sides; being positive when P is, relatively to them, in the same hemispheres as the opposite corners of the triangle, but negative in the contrary case; so that cos AA', &c., are positive, and are the sines of the three altitudes of the triangle.

(21.) If we place P at B, two of these perpendiculars vanish, and the last formula becomes, by 208, XXVIII.,

$$\text{XXXIX.} \ldots \sin b \cos \mathrm{BB}' = \sin a \sin c \sin \mathrm{B} = \mathrm{TV}\left(\mathrm{V}\frac{\gamma}{\beta} . \mathrm{V}\frac{\beta}{\alpha}\right);$$

such then is the quaternion expression for the product of the sine of the side CA, multiplied by the sine of the perpendicular let fall upon that side, from the opposite vertex B.

(22.) Placing P at A, dividing by $\sin a \cos c$, and then interchanging B and C, we get this other fundamental formula of spherical trigonometry,

$$\text{XL.} \ldots \cos \mathrm{AA}' = \sin c \sin \mathrm{B} = \sin b \sin \mathrm{C};$$

and we see that this is included in the interpretation of the quaternion equation XIX., or XIX'., as the formula XXXVI. was seen in (18.) to be the interpretation of the connected equation XVIII.

(23.) By assigning other positions to P, other formulæ of spherical trigonometry may be deduced, from the recent equation XXXVIII. Thus if we suppose P to coincide with B', and observe that (by the supplementary* triangle),

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* *No previous knowledge of spherical trigonometry*, properly so called, is here supposed; the supplementary relations of two polar triangles to each other forming rather a part, and a very elementary one, of *spherical geometry*.

$$\mathrm{B'C'} = \pi - \mathrm{A}, \qquad \mathrm{C'A'} = \pi - \mathrm{B}, \qquad \mathrm{A'B'} = \pi - \mathrm{C},$$

while

$$\cos \mathrm{BB'} = \sin a \sin \mathrm{C} = \sin c \sin \mathrm{A}, \text{ by XL.,}$$

we easily deduce the formula,

$$\text{XLI.} \ldots \sin a \sin c \sin \mathrm{A} \sin \mathrm{B} \sin \mathrm{C} = \sin \mathrm{B} - \cos c \cos \mathrm{C} \sin \mathrm{A} - \cos a \cos \mathrm{A} \sin \mathrm{C};$$

which obviously agrees, at the plane limit, with the elementary relation,

$$\mathrm{A} + \mathrm{B} + \mathrm{C} = \pi.$$

(24.) Again, by placing P at A′, the general equation becomes,

$$\text{XLII.} \ldots \sin a \cos c = \sin b \cos \mathrm{C} + \sin c \cos a \cos \mathrm{B};$$

with the verification that, at the plane limit,

$$a = b \cos \mathrm{C} + c \cos \mathrm{B}.$$

But we cannot here delay on such deductions, or verifications: although it appeared to be worth while to point out, that the whole of spherical trigonometry may thus be developed, from the fundamental equation of multiplication of quaternions (107), when that equation is operated on by the two characteristics S and V, and the results interpreted as above.

211. It may next be proved, as follows, that the distributive formula I. of the last Article holds good, when the three quaternions, q, q', q'', which enter into it, without being now necessarily *collinear*, are *right;* in which case their *reciprocals* (135), and their *sums* (197, (2.)), will be right also. Let then

$$\angle q = \angle q' = \angle q'' = \frac{\pi}{2}, \qquad q q_{,} = 1;$$

and therefore,

$$\angle q_{,} = \angle (q'' + q') = \frac{\pi}{2}.$$

We shall then have, by 106, 194, 206,

$$\begin{aligned}(q'' + q')\, q &= \mathrm{I}\,(q'' + q') : \mathrm{I}\, q_{,} \\ &= (\mathrm{I}q'' : \mathrm{I}q_{,}) + (\mathrm{I}q' : \mathrm{I}q_{,}) = q''q + q'q;\end{aligned}$$

and the distributive property in question is proved.

(1.) By taking conjugates, as in 210, it is easy hence to infer, that the *other* distributive formula, 210, V., holds good for any three right quaternions; or that

$$q(q'' + q') = qq'' + qq', \quad \text{if} \quad \angle q = \angle q' = \angle q'' = \frac{\pi}{2}.$$

(2.) For *any three* quaternions, we have therefore the two equations:

$$(\mathrm{V}q'' + \mathrm{V}q') \,.\, \mathrm{V}q = \mathrm{V}q'' \,.\, \mathrm{V}q + \mathrm{V}q' \,.\, \mathrm{V}q;$$
$$\mathrm{V}q \,.\, (\mathrm{V}q'' + \mathrm{V}q') = \mathrm{V}q \,.\, \mathrm{V}q'' + \mathrm{V}q \,.\, \mathrm{V}q'.$$

(3.) The quaternions q, q', q'' being still arbitrary, we have thus, by 210, IX.,

$$\begin{aligned}(q''+q')\,q &= (\mathrm{S}q''+\mathrm{S}q').\mathrm{S}q+(\mathrm{V}q''+\mathrm{V}q').\mathrm{S}q+\mathrm{V}q.(\mathrm{S}q''+\mathrm{S}q')+(\mathrm{V}q''+\mathrm{V}q').\mathrm{V}q\\ &= (\mathrm{S}q''.\mathrm{S}q+\mathrm{V}q''.\mathrm{S}q+\mathrm{V}q.\mathrm{S}q''+\mathrm{V}q''.\mathrm{V}q)+(\mathrm{S}q'.\mathrm{S}q+\mathrm{V}q'.\mathrm{S}q+\mathrm{V}q.\mathrm{S}q'+\mathrm{V}q'.\mathrm{V}q)\\ &= q''q+q'q;\end{aligned}$$

so that the formula 210, I., and therefore also (by conjugates) the formula 210, V., is valid *generally*.

212. The *General* Multiplication of Quaternions* is therefore (comp. 13, 210) a *Doubly Distributive Operation*; so that we may *extend*, to quaternions *generally*, the formula (comp. 210, VIII.),

$$\text{I.}\ldots\ \Sigma q'.\Sigma q=\Sigma q'q:$$

however many the summands of each set may be, and whether they be, or be not, *collinear* (209), or *right* (211).

(1.) Hence, as an extension of 210, XX., we have now,

$$\text{II.}\ldots\ \mathrm{N}\Sigma q=\Sigma\mathrm{N}q+2\Sigma\mathrm{S}q\mathrm{K}q';$$

where the second sign of summation refers to all possible binary combinations of the quaternions q, q', . .

(2.) And, as an extension of 210, XXIX., we have the inequality,

$$\text{III.}\ldots\ \Sigma\mathrm{T}q>\mathrm{T}\Sigma q,$$

unless *all* the quaternions q, q', . . bear *scalar* and *positive* ratios to each other, in which case the two members of this inequality become equal: so that the *sum of the tensors*, of any set of quaternions, is *greater than the tensor of the sum*, in every other case.

(3.) In general, as an extension of 210, XXVII.,

$$\text{IV.}\ldots\ (\Sigma\mathrm{T}q)^2-(\mathrm{T}\Sigma q)^2=2\Sigma(\mathrm{T}-\mathrm{S})\,q\mathrm{K}q'.$$

(4.) The formulæ, 210, XVIII., XIX., admit easily of analogous extensions.

(5.) We have also (comp. 168) the general equation,

$$\text{V.}\ldots\ (\Sigma q)^2-\Sigma(q^2)=\Sigma(qq'+q'q);$$

in which, by 210, IX.,

$$\text{VI.}\ldots\ qq'+q'q=2(\mathrm{S}q.\mathrm{S}q'+\mathrm{V}q.\mathrm{S}q'+\mathrm{V}q'.\mathrm{S}q+\mathrm{S}(\mathrm{V}q'.\mathrm{V}q));$$

because, by 208, we have generally

$$\text{VII.}\ldots\ \mathrm{V}(\mathrm{V}q'.\mathrm{V}q)=-\mathrm{V}(\mathrm{V}q.\mathrm{V}q');$$

or

$$\text{VIII.}\ldots\ \mathrm{V}q'q=-\mathrm{V}qq',\quad\text{if}\quad\angle q=\angle q'=\frac{\pi}{2}.$$

(Comp. 191, (2.), and 204, X.)

213. Besides the advantage which the Calculus of Quaternions gains, from the general establishment (212) of the *Distributive Principle*, or *Distributive Property of Multiplication*, by being, so far,

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* Compare the Notes to page 208.

assimilated to Algebra, in processes which are of continual occurrence, this principle or property will be found to be of great importance, in applications of that calculus to *Geometry;* and especially in questions respecting the (real or ideal*) *intersections of right lines with spheres*, or other surfaces of the second order, including *contacts* (real or ideal), as *limits* of such intersections. The following Examples may serve to give some notion, how the general distributive principle admits of being applied to such questions: in some of which however the less general principle (210), respecting the multiplication of *collinear* quaternions (209), would be sufficient. And first we shall take the case of *chords of a sphere*, drawn from a given point upon its surface.

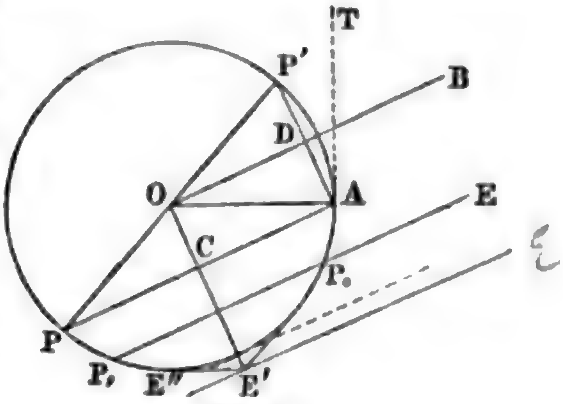

Fig. 51.

(1.) From a point A, of a sphere with O for centre, let it be required to draw a chord AP, which shall be parallel to a given line OB; or more fully, to *assign the vector*, $\rho = $ OP, of the extremity of the chord so drawn, *as a function of the two given vectors*, $\alpha = $ OA, and $\beta = $ OB; or rather of α and $U\beta$, since it is evident that the *length* of the line β cannot affect the result of the construction, which Fig. 51 may serve to illustrate.

(2.) Since AP ∥ OB, or $\rho - \alpha \parallel \beta$, we may begin by writing the expression,

$$\rho = \alpha + x\beta \ (15),$$

which may be considered (comp. 23, 99) as a form of the *equation of the right line* AP; and in which it remains to determine the scalar coefficient x, so as to satisfy the *equation of the sphere*,

$$T\rho = T\alpha \ (186, (2.)).$$

In short, we are to seek to satisfy the equation,

$$T(\alpha + x\beta) = T\alpha,$$

$\therefore N\left(\frac{\alpha}{\beta} + x\right) = N\alpha$

by some scalar x which shall be (in general) different from zero; and then to substitute this scalar in the expression $\rho = \alpha + x\beta$, in order to determine the required vector ρ.

$N\alpha + 2xS\frac{\alpha}{\beta} + x^2 = N\alpha$

(3.) For this purpose, an obvious process is, after dividing both sides by $T\beta$, to square, and to employ the formula 210, XXI., which had indeed occurred before, as 200, VIII., but not then as a consequence of the distributive property of multiplication. In this manner we are conducted to a quadratic equation, which admits of division by x, and gives then,

$$x = -2S\frac{\alpha}{\beta}; \qquad \rho = \alpha - 2\beta S\frac{\alpha}{\beta};$$

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* Compare the Notes to page 90, &c.

the problem (1.) being thus resolved, with the verification that β may be replaced by $\mathrm{U}\beta$, in the resulting expression for ρ.

(4.) As a mere exercise of calculation, we may vary the last process (3.), by dividing the last equation (2.) by $\mathrm{T}\alpha$, instead of $\mathrm{T}\beta$, and then going on as before. This last procedure gives,

$$1 = \mathrm{N}\left(1 + x\frac{\beta}{\alpha}\right) = 1 + 2x\mathrm{S}\frac{\beta}{\alpha} + x^2\mathrm{N}\frac{\beta}{\alpha};$$

and therefore,

$$x = -2\mathrm{S}\frac{\beta}{\alpha} : \mathrm{N}\frac{\beta}{\alpha} = -2\mathrm{S}\frac{\alpha}{\beta} \text{ (by 196, XII'.), as before.}$$

(5.) In general, by 196, II'.,

$$1 - 2\mathrm{S} = -\mathrm{K};$$

hence, by (3.),

$$\frac{\rho}{\beta} = -\mathrm{K}\frac{\alpha}{\beta};$$

and finally,

$$\rho = -\mathrm{K}\frac{\alpha}{\beta}.\beta;$$

a new expression for ρ, in which it is not permitted generally, as it was in (3.), to treat the vector β as the multiplier,* instead of the multiplicand.

(6.) It is now easy to see that the second equation of (2.) is satisfied; for the expression (5.) for ρ gives (by 186, 187, &c.),

$$\mathrm{T}\rho = \mathrm{T}\frac{\alpha}{\beta}.\mathrm{T}\beta = \mathrm{T}\alpha,$$

as was required.

(7.) To interpret the solution (3.), let C in Fig. 51 be the middle point of the chord AP, and let D be the foot of the perpendicular let fall from A on OB; then the expression (3.) for ρ gives, by 196, XIX.,

$$\mathrm{CA} = \tfrac{1}{2}(\alpha - \rho) = \beta\mathrm{S}\frac{\alpha}{\beta} = \mathrm{OD};$$

and accordingly, OCAD is a parallelogram.

(8.) To interpret the expression (5.), which gives

$$\frac{-\rho}{\beta} = \mathrm{K}\frac{\alpha}{\beta}, \quad \text{or} \quad \frac{\mathrm{OP}'}{\mathrm{OB}} = \mathrm{K}\frac{\mathrm{OA}}{\mathrm{OB}}, \quad \text{if} \quad \mathrm{OP}' = \mathrm{PO},$$

we have only to observe (comp. 138) that the angle AOP′ is bisected internally, or the supplementary angle AOP externally, by the indefinite right line OB (see again Fig. 51).

(9.) Conversely, the *geometrical considerations* which have thus served in (7.) and (8.) to *interpret* or to *verify* the two forms of solution (3.), (5.), might have been employed to *deduce* those two forms, if we had not seen how to obtain them, by *rules of calculation*, from the proposed conditions of the question. (Comp. 145, (10.), &c.)

(10.) It is evident, from the nature of that question, that α ought to be deducible from β and ρ, by exactly the same processes as those which have served us to deduce ρ from β and α. Accordingly, the form (3.) of ρ gives,

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* Compare the Note to page 159.

$$\mathrm{S}\frac{\rho}{\beta} = -\mathrm{S}\frac{\alpha}{\beta}, \qquad \alpha = \rho + 2\beta\mathrm{S}\frac{\alpha}{\beta} = \rho - 2\beta\mathrm{S}\frac{\rho}{\beta};$$

and the form (5.) gives,

$$\mathrm{K}\frac{\rho}{\beta} = -\frac{\alpha}{\beta}, \qquad \alpha = -\mathrm{K}\frac{\rho}{\beta}.\beta.$$

And since the first form can be recovered from the second, we see that each leads us back to the parallelism, $\rho - \alpha \parallel \beta$ (2.).

(11.) The solution (3.) for x shows that

$$x = 0, \quad \rho = \alpha, \quad \mathrm{P} = \mathrm{A}, \quad \text{if} \quad \mathrm{S}(\alpha : \beta) = 0, \quad \text{or if} \quad \beta \perp \alpha.$$

And the geometrical meaning of this result is obvious; namely, that a right line drawn at the extremity of a radius OA of a sphere, so as to be perpendicular to that radius, does not (in strictness) *intersect* the sphere, but *touches* it: its *second* point of meeting the surface *coinciding*, in this case, as a *limit*, with the *first*.

(12.) Hence we may infer that the plane represented by the equation,

$$\mathrm{S}\frac{\rho - \alpha}{\alpha} = 0, \quad \text{or} \quad \mathrm{S}\frac{\rho}{\alpha} = 1,$$

is the *tangent plane* (comp. 196, (5.)) to the sphere here considered, at the point A.

(13.) Since β may be replaced by any vector parallel thereto, we may substitute for it $\gamma - \alpha$, if $\gamma = \mathrm{OC}$ be the vector of *any given point* C *upon the chord* AP, whether (as in Fig. 51) the middle point, or not; we may therefore write, by (3.) and (5.),

$$\rho = \alpha - 2(\gamma - \alpha)\,\mathrm{S}\frac{\alpha}{\gamma - \alpha} = -\mathrm{K}\frac{\alpha}{\gamma - \alpha}.(\gamma - \alpha).$$

= − K

214. In the Examples of the foregoing Article, there was no room for the occurrence of *imaginary roots* of an equation, or for *ideal intersections* of line and surface. To give now a case in which such imaginary intersections may occur, we shall proceed to consider the question of drawing a *secant* to a sphere, in a given direction, from a given *external point;* the recent Figure 51 still serving us for illustration.

(1.) Suppose then that ε is the vector of any given point E, through which it is required to draw a chord or secant $\mathrm{EP_0P_1}$, parallel to the same given line β as before. We have now, if $\rho_0 = \mathrm{OP_0}$,

$$\rho_0 = \varepsilon + x_0\beta, \qquad \mathrm{T}\alpha = \mathrm{T}\rho_0 = \mathrm{T}(\varepsilon + x_0\beta),$$

$$x_0^2 + 2x_0\mathrm{S}\frac{\varepsilon}{\beta} + \mathrm{N}\frac{\varepsilon}{\beta} - \mathrm{N}\frac{\alpha}{\beta} = 0,$$

$$x_0 = -\mathrm{S}\frac{\varepsilon}{\beta} \mp \sqrt{\left\{\left(\mathrm{T}\frac{\alpha}{\beta}\right)^2 + \left(\mathrm{V}\frac{\varepsilon}{\beta}\right)^2\right\}},$$

γ = [illegible]

x_0 being a new scalar; and similarly, if $\rho_1 = \mathrm{OP_1}$,

$$\rho_1 = \varepsilon + x_1\beta, \qquad x_1 = -\mathrm{S}\frac{\varepsilon}{\beta} \pm \sqrt{\left\{\left(\mathrm{T}\frac{\alpha}{\beta}\right)^2 + \left(\mathrm{V}\frac{\varepsilon}{\beta}\right)^2\right\}},$$

by transformations* which will easily occur to any one who has read recent articles with attention. And the points P_0, P_1 will be together *real*, or together *imaginary*, according as the quantity under the radical sign is positive or negative; that is, according as we have one or other of the two following inequalities,

$$T\frac{\alpha}{\beta} > \quad \text{or} \quad < TV\frac{\epsilon}{\beta}.$$

(2.) The equation (comp. 203, (5.)),

$$TV\frac{\rho}{\beta} = T\frac{\alpha}{\beta}, \quad \text{or} \quad \left(T\frac{\alpha}{\beta}\right)^2 + \left(V\frac{\rho}{\beta}\right)^2 = 0,$$

represents a cylinder of revolution, with OB for its axis, and with $T\alpha$ for the radius of its base. If E be a point of this cylindric surface, the quantity under the radical sign in (1.) vanishes; and the two roots x_0, x_1 of the quadratic become *equal*. In this case, then, the line through E, which is parallel to OB, *touches* the given sphere; as is otherwise evident geometrically, since the cylinder *envelopes* the sphere (comp. 204, (12.)), and the line is one of its generatrices. If E be *internal* to the cylinder, the intersections P_0, P_1 are *real*; but if E be *external* to the same surface, those intersections are *ideal*, or *imaginary*.

(3.) In this last case, if we make, for abridgment,

$$s = -S\frac{\epsilon}{\beta}, \quad \text{and} \quad t = \sqrt{\left\{\left(TV\frac{\epsilon}{\beta}\right)^2 - \left(T\frac{\alpha}{\beta}\right)^2\right\}},$$

s and t being thus two given and *real scalars*, we may write,

$$x_0 = s - t\sqrt{-1}; \qquad x_1 = s + t\sqrt{-1};$$

where $\sqrt{-1}$ is the *old* and *ordinary imaginary symbol of Algebra*, and is *not invested here* with any sort of *Geometrical Interpretation*.† We merely express thus the *fact of calculation*, that (with these meanings of the symbols α, β, ϵ, s and t) the formula $T\alpha = T(\epsilon + x\beta)$, (1.), *when treated by the rules of quaternions, conducts to the quadratic equation*,

$$(x - s)^2 + t^2 = 0,$$

which has *no real root*; the reason being that *the right line* through E is, in the present case, *wholly external to the sphere*, and therefore *does not really intersect it at all*; although, for the sake of *generalization of language*, we may agree to *say*, as usual, that the line intersects the sphere in *two imaginary points*.

(4.) We must however agree, then, for *consistency of symbolical expression*, to consider these two ideal points as having *determinate* but *imaginary vectors*, namely, the two following:

$$\rho_0 = \epsilon + s\beta - t\beta\sqrt{-1}; \qquad \rho_1 = \epsilon + s\beta + t\beta\sqrt{-1};$$

in which it is easy to prove, Ist, that the *real part* $\epsilon + s\beta$ is the *vector* ϵ' *of the foot* E′ *of the perpendicular* let fall from the centre O on the line through E which is drawn (as above) parallel to OB; and IInd, that the *real tensor* $tT\beta$ *of the coefficient of*

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* It does not seem to be necessary, at the present stage, to supply so many references to former Articles, or Sub-articles, as it has hitherto been thought useful to give; but such may still, from time to time, be given.

† Compare again the Notes to page 90, and Art. 149.

$\sqrt{-1}$ in the ***imaginary part*** of each expression, represents the ***length of a tangent*** E′E″ to the sphere, drawn from that external point, or foot, E′.

(5.) In fact, if we write $\mathrm{OE}' = \epsilon' = \epsilon + s\beta$, we shall have

$$\mathrm{E'E} = \epsilon - \epsilon' = -s\beta = \beta\mathrm{S}\frac{\epsilon}{\beta} = \text{projection of OE on OB};$$

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which proves the Ist assertion (4.), whether the points P_0, P_1 be real or imaginary. And because

$$\left(\mathrm{T}\frac{\epsilon'}{\beta}\right)^2 = \mathrm{N}\frac{\epsilon'}{\beta} = \mathrm{N}\left(\frac{\epsilon}{\beta} + s\right) = \mathrm{N}\frac{\epsilon}{\beta} + 2s\mathrm{S}\frac{\epsilon}{\beta} + s^2$$
$$= \left(\mathrm{T}\frac{\epsilon}{\beta}\right)^2 - \left(\mathrm{S}\frac{\epsilon}{\beta}\right)^2 = \left(\mathrm{TV}\frac{\epsilon}{\beta}\right)^2 = t^2 + \left(\mathrm{T}\frac{\alpha}{\beta}\right)^2,$$

we have, for the case of imaginary intersections,

$$t\mathrm{T}\beta = \sqrt{(\mathrm{T}\epsilon'^2 - \mathrm{T}\alpha^2)} = \mathrm{T}\,.\,\mathrm{E'E''},$$

and the IInd assertion (4.) is justified.

(6.) An expression of the form (4.), or of the following,

$$\rho' = \beta + \sqrt{-1}\gamma,$$

in which β and γ are ***two real vectors***, while $\sqrt{-1}$ is the (scalar) ***imaginary of algebra***, and ***not*** a symbol for a ***geometrically real right versor*** (149, 153), may be said to be a BIVECTOR.

(7.) In like manner, an expression of the form (3.), or $x' = s + t\sqrt{-1}$, where s and t are ***two real scalars***, but $\sqrt{-1}$ is still the ordinary imaginary of ***algebra***, may be said by analogy to be a BISCALAR. ***Imaginary roots of algebraic equations*** are thus, in general, ***biscalars***.

(8.) And if a ***bivector*** (6.) be ***divided by a (real) vector***, the ***quotient***, such as

$$q' = \frac{\rho'}{\alpha} = \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}\sqrt{-1} = q_0 + q_1\sqrt{-1},$$

in which q_0 and q_1 are ***two real quaternions***, but $\sqrt{-1}$ is, as before, ***imaginary***, may be said to be a BIQUATERNION.*

215. The same distributive principle (212) may be employed in investigations respecting *circumscribed cones*, and the *tangents* (real or ideal), which can be drawn to a given sphere from a given point.

(1.) Instead of conceiving that O, A, B are three given points, and that ***limits of position*** of the point E are sought, as in 214, (2.), which shall allow the points of intersection P_0, P_1 to be real, we may suppose that O, A, E (which may be assumed to be collinear, without loss of generality, since α enters only by its tensor) are now the data of the question; and that ***limits of direction*** of the line OB are to be assigned, which shall permit the same reality: $\mathrm{EP_0P_1}$ being still drawn parallel to OB, as in 214, (1.).

(2.) Dividing the equation $\mathrm{T}\alpha = \mathrm{T}(\epsilon + x\beta)$ by $\mathrm{T}\epsilon$, and squaring, we have

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* Compare the second Note to page 131.

$$\mathrm{N}\frac{\alpha}{\epsilon} = \left(\mathrm{N}\left(1 + x\frac{\beta}{\epsilon}\right) = \right) 1 + 2x\mathrm{S}\frac{\beta}{\epsilon} + x^2\mathrm{N}\frac{\beta}{\epsilon};$$

the quadratic in x may therefore be thus written,

$$\left(x\mathrm{T}\frac{\beta}{\epsilon} + \mathrm{SU}\frac{\beta}{\epsilon}\right)^2 = \left(\mathrm{T}\frac{\alpha}{\epsilon}\right)^2 + \left(\mathrm{VU}\frac{\beta}{\epsilon}\right)^2;$$

and its roots are real and unequal, or real and equal, or imaginary, according as

$$\mathrm{TVU}\frac{\beta}{\epsilon} < \text{ or } = \text{ or } > \mathrm{T}\frac{\alpha}{\epsilon};$$

that is, according as

$$\sin \text{EOB} < \text{ or } = \text{ or } > \mathrm{T}.\text{OA} : \mathrm{T}.\text{OE}.$$

(3.) If E be *interior* to the sphere, then $\mathrm{T}\epsilon < \mathrm{T}\alpha$, $\mathrm{T}(\alpha : \epsilon) > 1$; but $\mathrm{TVU}q$ can never exceed unity (by 204, XIX., or by 210, XV., &c.); we have, therefore, in this case, the *first* of the three recent alternatives, and the two roots of the quadratic are *necessarily* real and unequal, whatever the direction of β may be. Accordingly it is evident, geometrically, that *every* indefinite right line, drawn through an internal point, must cut the spheric surface in two distinct and real points.

(4.) If the point E be *superficial*, so that $\mathrm{T}\epsilon = \mathrm{T}\alpha$, $\mathrm{T}(\alpha : \epsilon) = 1$, then the first alternative (2.) still exists, except at the limit for which $\beta \perp \epsilon$, and therefore $\mathrm{TVU}(\beta : \epsilon) = 1$, in which case we have the second alternative. *One* root of the quadratic in x is now $= 0$, for *every* direction of β; and the other root, namely $x = -2\mathrm{S}(\epsilon : \beta)$, is likewise always *real*, but *vanishes* for the case when the angle EOB is right. In short, we have here the same system of chords and of tangents, from a point upon the surface, as in 213; the only difference being, that we now write E for A, or ϵ for α.

(5.) But finally, if E be an *external* point, so that $\mathrm{T}\epsilon > \mathrm{T}\alpha$, and $\mathrm{T}(\alpha : \epsilon) < 1$, then $\mathrm{TVU}(\beta : \epsilon)$ may either fall short of this last tensor, or equal, or exceed it; so that any one of the three alternatives (2.) may come to exist, according to the varying direction of β.

(6.) To illustrate geometrically the law of passage from one such alternative to another, we may observe that the equation,

$$\mathrm{TVU}\frac{\rho}{\epsilon} = \mathrm{T}\frac{\alpha}{\epsilon},$$

or

$$\sin \text{EOP} = \mathrm{T}.\text{OA} : \mathrm{T}.\text{OE},$$

represents (when E is thus external) a *real cone* of revolution, with its vertex at the centre O of the sphere; and according as the line OB lies *inside* this cone, or *on* it, or *outside* it, the first or the second or the third of the three alternatives (2.) is to be adopted; or in other words, the line through E, drawn parallel (as before) to OB, either *cuts* the sphere, or *touches* it, or does not (really) *meet* it at all. (Compare the annexed Fig. 52.)

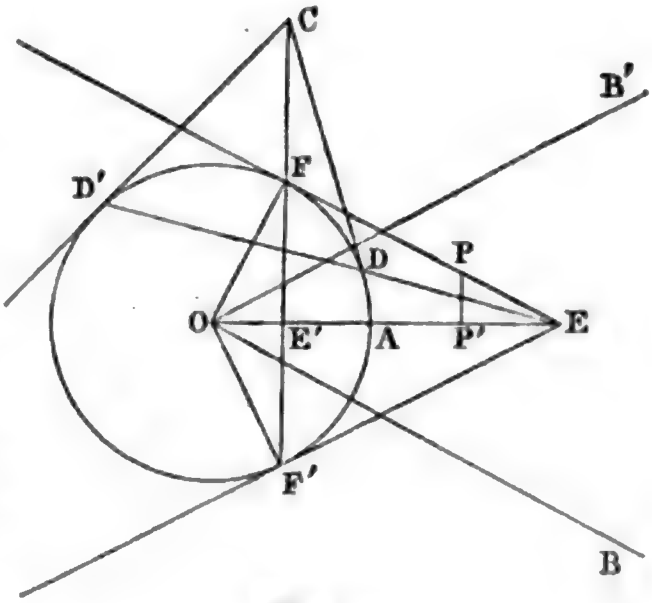

Fig. 52.

(7.) If E be still an external point, the *cone of tangents* which can be drawn from it to the sphere is *real;* and the equation of this *enveloping* or *circumscribed cone*, with its vertex at E, may be obtained from that of the recent cone (6.), by simply changing ρ to $\rho - \epsilon$; it is, therefore, or at least one form of it is,

$$\mathrm{TVU}\frac{\rho-\epsilon}{\epsilon} = \mathrm{T}\frac{\alpha}{\epsilon}; \quad \text{or} \quad \sin \mathrm{OEP} = \mathrm{T.OA} : \mathrm{T.OE}.$$

(8.) In general, if q be any quaternion, and x any scalar,

$$\mathrm{VU}(q+x) = \mathrm{V}q : \mathrm{T}(q+x);$$

the recent equation (7.) may therefore be thus written:

$$\mathrm{T}\frac{\mathrm{V}(\rho:\epsilon).\epsilon}{\rho-\epsilon} = \mathrm{T}\frac{\alpha}{\epsilon};$$

or

$$\mathrm{T.P'P} : \mathrm{T.EP} = \mathrm{T.OA} : \mathrm{T.OE},$$

if P′ be the foot of the perpendicular let fall from P on OE; and in fact the first quotient is evidently $= \sin$ OEP.

(9.) We may also write,

$$\mathrm{TV}\frac{\rho}{\epsilon} = \mathrm{T}\frac{\alpha}{\epsilon}.\mathrm{T}\left(\frac{\rho}{\epsilon}-1\right); \quad \text{or} \quad 0 = \left(\mathrm{S}\frac{\rho}{\epsilon}\right)^2 - \mathrm{N}\frac{\rho}{\epsilon} + \mathrm{N}\frac{\alpha}{\epsilon}\left(\mathrm{N}\frac{\rho}{\epsilon} - 2\mathrm{S}\frac{\rho}{\epsilon} + 1\right);$$

or

$$\left(\mathrm{S}\frac{\rho}{\epsilon} - \mathrm{N}\frac{\alpha}{\epsilon}\right)^2 = \left(1 - \mathrm{N}\frac{\alpha}{\epsilon}\right)\left(\mathrm{N}\frac{\rho}{\epsilon} - \mathrm{N}\frac{\alpha}{\epsilon}\right),$$

as another form of the equation of the circumscribed cone.

(10.) If then we make also

$$\mathrm{N}\frac{\rho}{\alpha} = 1, \quad \text{or} \quad \mathrm{N}\frac{\rho}{\epsilon} = \mathrm{N}\frac{\alpha}{\epsilon},$$

to express that the point P is *on the enveloped sphere*, as well as on the enveloping *cone*, we find the following equation of the *plane of contact*, or of what is called the *polar plane* of the point E, with respect to the given sphere:

$$\left(\mathrm{S}\frac{\rho}{\epsilon} - \mathrm{N}\frac{\alpha}{\epsilon}\right)^2 = 0; \quad \text{or} \quad \mathrm{S}\frac{\rho}{\epsilon} - \mathrm{N}\frac{\alpha}{\epsilon} = 0;$$

[illegible]

while the fact that it *is* a plane of *contact*[*] is exhibited by the occurrence of the exponent 2, or by its equation entering through its *square*.

(11.) The vector,

$$\epsilon' = \epsilon\mathrm{S}\frac{\rho}{\epsilon} = \epsilon\mathrm{N}\frac{\alpha}{\epsilon} = \mathrm{OE}',$$

is that of the point E′ in which the polar plane (10.) of E cuts perpendicularly the right line OE; and we see that

$$\mathrm{T}\epsilon.\mathrm{T}\epsilon' = \mathrm{T}\alpha^2, \quad \text{or} \quad \mathrm{T.OE.T.OE'} = (\mathrm{T.OA})^2,$$

as was to be expected from elementary theorems, of spherical or even of plane geometry.

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[*] In fact a modern geometer would say, that we have here a case of *two coincident planes* of intersection, merged into a single plane of contact.

(12.) The equation (10.), of the polar plane of E, may easily be thus transformed:

$$\mathrm{S}\frac{\epsilon}{\rho} = \left(\mathrm{S}\frac{\rho}{\epsilon}.\mathrm{N}\frac{\epsilon}{\rho} = \right)\mathrm{N}\frac{\alpha}{\rho}, \quad \text{or} \quad \mathrm{S}\frac{\epsilon}{\rho} - \mathrm{N}\frac{\alpha}{\rho} = 0;$$

it continues therefore to hold good, when ϵ and ρ are *interchanged*. If then we take, as the vertex of a *new enveloping cone*, any point C external to the sphere, and situated on the polar plane FF'.. of the former external point E, the *new plane of contact*, or the polar plane DD'.. of the new point C, will pass through the former vertex E: a geometrical relation of *reciprocity*, or of *conjugation*, between the two points C and E, which is indeed well-known, but which it appeared useful for our purpose to prove by quaternions* anew.

(13.) In general, each of the two connected equations,

$$\mathrm{S}\frac{\rho'}{\rho} = \mathrm{N}\frac{\alpha}{\rho}, \qquad \mathrm{S}\frac{\rho}{\rho'} = \mathrm{N}\frac{\alpha}{\rho'},$$

which may also be thus written,

$$1 = \left(\mathrm{S}\frac{\rho'}{\alpha}\frac{\alpha}{\rho}.\mathrm{N}\frac{\rho}{\alpha} = \right)\mathrm{S}.\frac{\rho'}{\alpha}\mathrm{K}\frac{\rho}{\alpha}, \qquad 1 = \mathrm{S}.\frac{\rho}{\alpha}\mathrm{K}\frac{\rho'}{\alpha},$$

may be said to be a form of the *Equation of Conjugation* between any two points P and P' (not those so marked in Fig. 52), of which the vectors satisfy it: because it expresses that those two points are, in a well-known sense, *conjugate* to each other, with respect to the given sphere, $\mathrm{T}\rho = \mathrm{T}\alpha$.

(14.) If *one* of the two points, as P', be *given* by its vector ρ', while the *other* point P and vector ρ are *variable*, the equation then represents a *plane locus;* namely, what is still called the *polar plane* of the given point, whether that point be external or internal, or on the surface of the sphere.

(15.) Let P, P' be thus two conjugate points; and let it be proposed to find the points S, S', in which the right line PP' intersects the sphere. Assuming (comp. 25) that

$$\mathrm{OS} = \sigma = x\rho + y\rho', \qquad x + y = 1, \qquad \mathrm{T}\sigma = \mathrm{T}\alpha,$$

and attending to the equation of conjugation (13.), we have, by 210, XX., or by 200, VII., the following quadratic equation in $y : x$,

$$(x + y)^2 = \mathrm{N}\left(x\frac{\rho}{\alpha} + y\frac{\rho'}{\alpha}\right) = x^2\mathrm{N}\frac{\rho}{\alpha} + 2xy + y^2\mathrm{N}\frac{\rho'}{\alpha};$$

which gives,

$$x^2\left(\mathrm{N}\frac{\rho}{\alpha} - 1\right) = y^2\left(1 - \mathrm{N}\frac{\rho'}{\alpha}\right).$$

(16.) Hence it is evident that, if the points of intersection S, S' are to be *real*, one of the two points P, P' must be interior, and the other must be exterior to the sphere; because, of the *two norms* here occurring, one must be greater and the other less than unity. And because the *two roots* of the quadratic, or the two values of $y : x$, differ

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* In fact, it will easily be seen that the investigations in recent sub-articles are put forward, almost entirely, as exercises in the Language and Calculus of Quaternions, and not as offering any geometrical novelty of result.

only by their *signs*, it follows (by 26) that the right line PP' is *harmonically divided* (as indeed it is well known to be), at the two points S, S' at which it meets the sphere: or that in a notation already several times employed (25, 31, &c.), we have the *harmonic formula*,

$$(\mathrm{PSP'S'}) = -1.$$

(17.) From a real but *internal point* P, we can still *speak* of a *cone of tangents*, as being drawn to the sphere: but if so, we must say that those tangents are *ideal*, or *imaginary*;* and must consider them as terminating on an *imaginary circle of contact*; of which the *real* but wholly *external plane* is, by quaternions, as by modern geometry, recognised as being (comp. (14.)) the polar plane of the supposed internal point.

216. Some readers may find it useful, or at least interesting, to see here a few examples of the application of the General Distributive Principle (212) of multiplication to the *Ellipsoid*, of which some forms of the Quaternion Equation were lately assigned (in 204, (14.)); especially as those forms have been found to conduct† to a Geometrical Construction, previously unknown, for that celebrated and important Surface: or rather to several such constructions. In what follows, it will be supposed that any such reader has made himself already sufficiently familiar with the chief formulæ of the preceding Articles; and therefore comparatively few references‡ will be given, at least upon the present subject.

(1.) To prove, first, that the locus of the variable ellipse,

$$\mathrm{I.} \ldots \mathrm{S}\frac{\rho}{\alpha} = x, \qquad \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = x^2 - 1, \qquad 204, (13.)$$

which locus is represented by the equation,

$$\mathrm{II.} \ldots \left(\mathrm{S}\frac{\rho}{\alpha}\right)^2 - \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = 1, \qquad 204, (14.)$$

the two constant vectors α, β being supposed to be real, and to be inclined to each other at some acute or obtuse (but not right§) angle, is a *surface of the second order*, in the sense that it is cut by an arbitrary rectilinear transversal in *two* (real or imaginary) points, and in *no more* than two, let us assume two points L, M, or their vectors $\lambda = \text{OL}$, $\mu = \text{OM}$, as given; and let us seek to determine the points P (real or imaginary), in which the indefinite right line LM intersects the locus II.; or rather the *number* of such intersections, which will be sufficient for the present purpose.

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* Compare again the second Note to page 90, and others formerly referred to.

† See the Proceedings of the Royal Irish Academy, for the year 1846.

‡ Compare the Note to page 218.

§ If $\beta \perp \alpha$, the system I. represents (not an ellipse but) a *pair of right lines*, real or ideal, in which the *cylinder of revolution*, denoted by the second equation of that system, is cut by a *plane parallel to its axis*, and represented by the first equation.

(2.) Making then $\rho = \frac{y\lambda + z\mu}{y + z}$ (25), we have, for $y : z$, the following quadratic equation,

$$\text{III.} \ldots \left(y\text{S}\frac{\lambda}{\alpha} + z\text{S}\frac{\mu}{\alpha} \right)^2 - \left(y\text{V}\frac{\lambda}{\alpha} + z\text{V}\frac{\mu}{\alpha} \right)^2 = (y + z)^2;$$

without proceeding to *resolve* which, we see already, by its mere *degree*, that the number sought is *two;* and therefore that the locus II. is, as above stated, a surface of the *second* order.

(3.) The equation II. remains unchanged, when $-\rho$ is substituted for ρ; the surface has therefore a *centre*, and this centre is at the *origin* O of vectors.

(4.) It has been seen that the equation of the surface may also be thus written:

$$\text{IV.} \ldots \text{T}\left(\text{S}\frac{\rho}{\alpha} + \text{V}\frac{\rho}{\beta} \right) = 1; \qquad 204, (14.)$$

it gives therefore, for the reciprocal of the radius vector from the centre, the expression,

$$\text{V.} \ldots \frac{1}{\text{T}\rho} = \text{T}\left(\text{S}\frac{\text{U}\rho}{\alpha} + \text{V}\frac{\text{U}\rho}{\beta} \right);$$

and this expression has a real value, which never vanishes,* whatever real value may be assigned to the versor $\text{U}\rho$, that is, whatever direction may be assigned to ρ: the surface is therefore *closed*, and *finite*.

(5.) Introducing two new constant and auxiliary vectors, determined by the two expressions,

$$\text{VI.} \ldots \gamma = \frac{2\beta}{\beta + \alpha} \cdot \alpha, \qquad \delta = \frac{2\beta}{\beta - \alpha} \cdot \alpha,$$

which give (by 125) these other expressions,

$$\text{VI}'. \ldots \gamma = \frac{2\alpha}{\beta + \alpha} \cdot \beta, \qquad \delta = \frac{2\alpha}{\beta - \alpha} \cdot \beta,$$

we have

$$\text{VII.} \ldots \frac{\gamma}{\alpha} + \frac{\gamma}{\beta} = 2, \qquad \frac{\delta}{\alpha} - \frac{\delta}{\beta} = 2;$$

$$\text{VII}'. \ldots \frac{\alpha}{\gamma} + \frac{\alpha}{\delta} = 1, \qquad \frac{\beta}{\gamma} - \frac{\beta}{\delta} = 1;$$

all the vectors
α, β, γ, δ must
be complanar
from assumed
conditions

and under these conditions, γ is said to be the ***harmonic mean*** between the two former vectors, α and β; and in like manner, δ is the harmonic mean between α and $-\beta$; while 2α is the corresponding mean between γ, δ; and 2β is so, between γ and $-\delta$.

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* It is to be remembered that we have excluded in (1.) the case where $\beta \perp \alpha$; in which case it can be shown that the equation II. represents an *elliptic cylinder*.

(6.) Under the same conditions, for any arbitrary vector ρ, we have the transformations,

$$\text{VIII.}\ldots\frac{\rho}{\gamma}=\tfrac{1}{2}\left(\frac{\rho}{\alpha}+\frac{\rho}{\beta}\right);\qquad \frac{\rho}{\delta}=\tfrac{1}{2}\left(\frac{\rho}{\alpha}-\frac{\rho}{\beta}\right);$$

$$\text{IX.}\ldots\frac{\rho}{\gamma}+\mathrm{K}\frac{\rho}{\delta}=\mathrm{S}\frac{\rho}{\alpha}+\mathrm{V}\frac{\rho}{\beta};$$

204 x′ k″

the equation IV. of the surface may therefore be thus written:

$$\text{X.}\ldots\mathrm{T}\left(\frac{\rho}{\gamma}+\mathrm{K}\frac{\rho}{\delta}\right)=1;\quad\text{or thus,}\quad \text{X}'.\ldots\mathrm{T}\left(\frac{\rho}{\delta}+\mathrm{K}\frac{\rho}{\gamma}\right)=1;$$

[illegible]

the geometrical meaning of which new forms will soon be seen.

(7.) The system of the two planes through the origin, which are respectively perpendicular to the new vectors γ and δ, is represented by the equation,

$$\text{XI.}\ldots\mathrm{S}\frac{\rho}{\gamma}\,\mathrm{S}\frac{\rho}{\delta}=0,\quad\text{or}\quad\text{XII.}\ldots\left(\mathrm{S}\frac{\rho}{\alpha}\right)^2=\left(\mathrm{S}\frac{\rho}{\beta}\right)^2;$$

combining which with the equation II. we get

$$\text{XIII.}\ldots 1=\left(\mathrm{S}\frac{\rho}{\beta}\right)^2-\left(\mathrm{V}\frac{\rho}{\beta}\right)^2=\mathrm{N}\frac{\rho}{\beta};\quad\text{or,}\quad\text{XIV.}\ldots\mathrm{T}\rho=\mathrm{T}\beta.$$

These two diametral planes therefore cut the surface in *two circular sections*, with $\mathrm{T}\beta$ for their common radius; and the normals γ and δ, to the same two planes, may be called (comp. 196, (17.)) the *cyclic normals* of the surface; while the planes themselves may be called its *cyclic planes*.

(8.) Conversely, if we seek the intersection of the surface with the concentric sphere XIV., of which the radius is $\mathrm{T}\beta$, we are conducted to the equation XII. of the system of the two cyclic planes, and therefore to the two circular sections (7.); so that every radius vector of the surface, which is *not* drawn in one or other of these two planes, has a length either greater or less than the radius $\mathrm{T}\beta$ of the sphere.

(9.) By all these marks, it is clear that the locus II., or 204, (14.), is (as above asserted) an *Ellipsoid*; its *centre* being at the origin (3.), and its *mean semiaxis* being $=\mathrm{T}\beta$; while $\mathrm{U}\beta$ has, by 204, (15.), the direction of the *axis* of a *circumscribed cylinder of revolution*, of which cylinder the *radius* is $\mathrm{T}\beta$; and α is, by the last cited sub-article, perpendicular to the plane of the *ellipse of contact*.

(10.) Those who are familiar with modern geometry, and who have caught the notations of quaternions, will easily see that this ellipsoid II., or IV., is a *deformation* of what may be called the *mean sphere* XIV., and is *homologous* thereto; the infinitely distant point in the direction of β being a *centre of homology*, and either of the two planes XI. or XII. being a *plane of homology* corresponding.

217. The recent form, X. or X′., of the quaternion equation of the ellipsoid, admits of being *interpreted*, in such a way as to conduct (comp. 216) to a simple *construction* of that surface; which we shall first investigate by calculation, and then illustrate by geometry.

(1.) Carrying on the Roman numerals from the sub-articles to 216, and observing that (by 190, &c.),

$$\frac{\rho}{\gamma} = \mathrm{K}\frac{\gamma}{\rho}.\mathrm{N}\frac{\rho}{\gamma}, \quad \text{and} \quad \mathrm{K}\frac{\rho}{\delta} = \frac{\delta}{\rho}.\mathrm{N}\frac{\rho}{\delta},$$

the equation X. takes the form,

$$\text{XV.} \ldots 1 = \mathrm{T}\left\{\left(\frac{\delta}{\mathrm{T}\delta^2} + \mathrm{K}\frac{\gamma}{\rho}.\frac{\rho}{\mathrm{T}\gamma^2}\right) : \frac{\rho}{\mathrm{T}\rho^2}\right\};$$

or

$$\text{XVI.} \ldots \frac{t^2}{\mathrm{T}\rho} = \mathrm{T}\left(\iota + \mathrm{K}\frac{\kappa}{\rho}.\rho\right),$$

if we make

$$\text{XVII.} \ldots \frac{\delta}{\mathrm{T}\delta^2} = \frac{\iota}{t^2} \quad \text{and} \quad \frac{\gamma}{\mathrm{T}\gamma^2} = \frac{\kappa}{t^2},$$

when ι and κ are two new constant vectors, and t is a new constant scalar, which we shall suppose to be positive, but of which the value may be chosen at pleasure.

(2.) The comparison of the forms X. and X'. shows that γ and δ may be interchanged, or that they enter symmetrically into the equation of the ellipsoid, although they may not at first seem to do so; it is therefore allowed to assume that

$$\text{XVIII.} \ldots \mathrm{T}\gamma > \mathrm{T}\delta, \quad \text{and therefore that} \quad \text{XVIII}'. \ldots \mathrm{T}\iota > \mathrm{T}\kappa;$$

for the supposition $\mathrm{T}\gamma = \mathrm{T}\delta$ would give, by VI.,

$$\mathrm{T}(\beta + \alpha) = \mathrm{T}(\beta - \alpha), \quad \text{and} \therefore \text{(by 186, (6.) \&c.)} \quad \beta \perp \alpha,$$

which latter case was excluded in 216, (1.).

(3.) We have thus,

$$\text{XIX.} \ldots \mathrm{U}\iota = \mathrm{U}\delta; \qquad \mathrm{U}\kappa = \mathrm{U}\gamma;$$

$$\text{XX.} \ldots \mathrm{T}\iota = \frac{t^2}{\mathrm{T}\delta}; \qquad \mathrm{T}\kappa = \frac{t^2}{\mathrm{T}\gamma};$$

$$\text{XXI.} \ldots \frac{\mathrm{T}\iota^2 - \mathrm{T}\kappa^2}{t^2} = \left(\frac{t}{\mathrm{T}\delta}\right)^2 - \left(\frac{t}{\mathrm{T}\gamma}\right)^2.$$

(4.) Let ABC be a plane triangle, such that

$$\text{XXII.} \ldots \mathrm{CB} = \iota, \quad \mathrm{CA} = \kappa;$$

let also

$$\mathrm{AE} = \rho.$$

Then if a sphere, which we shall call the *diacentric sphere*, be described round the point C as centre, with a radius $= \mathrm{T}\kappa$, and therefore so as to pass *through the centre* A (here written instead of O) of the ellipsoid, and if D be the point in which the line AE meets this sphere again, we shall have, by 213, (5.), (13.),

$$\text{XXIII.} \ldots \mathrm{CD} = -\mathrm{K}\frac{\kappa}{\rho}.\rho,$$

and therefore

$$\text{XXIII}'. \ldots \mathrm{DB} = \iota + \mathrm{K}\frac{\kappa}{\rho}.\rho;$$

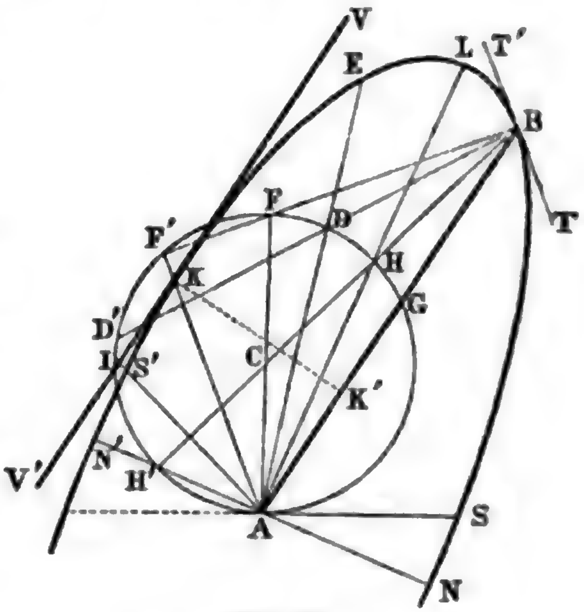

Fig. 53.

OA = α fig 51 p 215
OB = β
AC = γ-α

AE = ρ

so that the equation XVI. becomes,

$$\text{XXIV.}\ldots\ t^2 = \mathrm{T.AE.T.DB.}$$

(5.) The point B is *external* to the diacentric sphere (4.), by the assumption (2.); a real tangent (or rather cone of tangents) to this sphere can therefore be drawn from that point; and if we select the length of such a tangent as the value (1.) of the scalar t, that is to say, if we make each member of the formula XXI. equal to unity, and denote by D′ the second intersection of the right line BD with the sphere, as in Fig. 53, we shall have (by Euclid III.) the elementary relation,

$$\text{XXV.}\ldots\ t^2 = \mathrm{T.DB.T.BD'};$$

whence follows this *Geometrical Equation of the Ellipsoid*,

$$\text{XXVI.}\ldots\ \mathrm{T.AE} = \mathrm{T.BD'};$$

or in a somewhat more familiar notation,

$$\text{XXVII.}\ldots\ \overline{\mathrm{AE}} = \overline{\mathrm{BD'}};$$

where $\overline{\mathrm{AE}}$ denotes the *length* of the line AE, and similarly for $\overline{\mathrm{BD'}}$.

(6.) The following very simple *Rule of Construction* (comp. the recent Fig. 53) results therefore from our quaternion analysis:—

From a fixed point A, *on the surface of a given sphere, draw any chord* AD; *let* D′ *be the second point of intersection of the same spheric surface with the secant* BD, *drawn from a fixed external*[*] *point* B; *and take a radius vector* AE, *equal in length to the line* BD′, *and in direction either coincident with, or opposite to, the chord* AD: *the locus of the point* E *will be an ellipsoid, with* A *for its centre, and with* B *for a point of its surface.*

(7.) Or thus:—

If, of a plane but variable quadrilateral ABED′, *of which one side* AB *is given in length and in position, the two diagonals* AE, BD′ *be equal to each other in length, and if their intersection* D *be always situated upon the surface of a given sphere, whereof the side* AD′ *of the quadrilateral is a chord, then the opposite side* BE *is a chord of a given ellipsoid.*

218. From either of the two foregoing statements, of the *Rule of Construction for the Ellipsoid* to which quaternions have conducted, many *geometrical consequences* can easily be inferred, a few of which may be mentioned here, with their proofs by *calculation* annexed: the present Calculus being, of course, still employed.

(1.) That the corner B, of what may be called the *Generating Triangle* ABC, is in fact a point of the generated surface, with the construction 217, (6.), may be proved, by conceiving the variable chord AD of the given diacentric sphere to take the position AG; where G is the second intersection of the line AB with that spheric surface.

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[*] It is merely to fix the conceptions, that the point B is here supposed to be *external* (5.); the calculations and the construction would be almost the same, if we assumed B to be an *internal* point, or $\mathrm{T}\iota < \mathrm{T}\kappa$, $\mathrm{T}\gamma < \mathrm{T}\delta$.

(2.) If D be conceived to approach to A (instead of G), and therefore D′ to G (instead of A), the *direction* of AE (or of AD) then tends to become tangential to the sphere at A, while the *length* of AE (or of BD′) tends, by the construction, to become equal to the length of BG; the surface has therefore a *diametral* and *circular section*, in a plane which touches the diacentric sphere at A, and with a radius $= \overline{\text{BG}}$.

(3.) Conceive a circular section of the sphere through A, made by a plane perpendicular to BC; if D move along this circle, D′ will move along a parallel circle through G, and the length of BD′, or that of AE, will again be equal to $\overline{\text{BG}}$; such then is the radius of a *second* diametral and *circular section* of the *ellipsoid*, made by the lately mentioned plane.

[illegible]

(4.) The *construction* gives us thus *two cyclic planes* through A; the perpendiculars to which planes, or the *two cyclic normals* (216, (7.)) of the ellipsoid, are seen to have the directions of the *two sides*, CA, CB, of the *generating triangle* ABC (1.).

(5.) Again, since the rectangle

$$\overline{\text{BA}} \,.\, \overline{\text{BG}} = \overline{\text{BD}} \,.\, \overline{\text{BD}'} = \overline{\text{BD}} \,.\, \overline{\text{AE}} = \text{double area of triangle ABE} : \sin \text{BDE},$$

we have the equation,

$$\text{XXVIII.} \ldots \text{perpendicular distance of E from AB} = \overline{\text{BG}} \,.\, \sin \text{BDE};$$

the *third side*, AB, of the generating triangle (1.), is therefore the *axis of revolution* of a *cylinder*, which *envelopes* the ellipsoid, and of which the radius has the same length, BG, as the radius of each of the two diametral and circular sections.

(6.) For the points of contact of ellipsoid and cylinder, we have the geometrical relation,

$$\text{XXIX.} \ldots \text{BDE} = \text{a right angle}; \quad \text{or} \quad \text{XXIX}'. \ldots \text{ADB} = \text{a right angle};$$

the point D is therefore situated on a *second spheric surface*, which has the line AB for a diameter, and intersects the diacentric sphere in a *circle*, whereof the plane passes through A, and cuts the enveloping cylinder in an *ellipse of contact* (comp. 204, (15.), and 216, (9.)), of that cylinder with the ellipsoid.

(7.) Let AC meet the diacentric sphere again in F, and let BF meet it again in F′ (as in Fig. 53); the *common plane* of the last-mentioned circle and ellipse (6.) can then be easily proved to cut perpendicularly the plane of the generating triangle ABC in the line AF′; so that the line F′B is *normal to this plane of contact;* and therefore also (by conjugate diameters, &c.) *to the ellipsoid*, at B.

(8.) These *geometrical consequences of the construction* (217), to which many others might be added, can all be shown to be consistent with, and confirmed by, the *quaternion analysis* from which that construction itself was derived. Thus, the two *circular sections* (2.) (3.) had presented themselves in 216, (7.); and their two *cyclic normals* (4.), or the sides CA, CB of the triangle, being (by 217, (4.)) the two vectors κ, ι, have (by 217, (1.) or (3.)) the directions of the two former vectors γ, δ; which again agrees with 216, (7.).

(9.) Again, it will be found that the assumed relations between the *three pairs of constant vectors*, α, β; γ, δ; and ι, κ, any *one* of which *pairs* is sufficient to determine the ellipsoid, conduct to the following expressions (of which the investigation is left to the student, as an exercise):

$$\text{XXX.}\ldots\ \alpha = \frac{\delta}{\delta+\gamma}\gamma = \frac{\gamma}{\delta+\gamma}\delta = \frac{+t^2}{\mathrm{T}(\iota+\kappa)}\mathrm{U}(\iota+\kappa) = \mathrm{F'B};$$

$$\text{XXXI.}\ldots\ \beta = \frac{\delta}{\delta-\gamma}\gamma = \frac{\gamma}{\delta-\gamma}\delta = \frac{-t^2}{\mathrm{T}(\iota-\kappa)}\mathrm{U}(\iota-\kappa) = \mathrm{BG};$$

the letters B, F′, G referring here to Fig. 53, while $\alpha\beta\gamma\delta$ retain their former meanings (216), and are not interpreted as vectors of the points ABCD in that Figure. Hence the recent geometrical inferences, that AB (or BG) is the axis of revolution of an enveloping cylinder (5.), and that F′B is normal to the plane of the ellipse of contact (7.), agree with the former conclusions (216, (9.), or 204, (15.)), that β is such an axis, and that α is such a normal.

(10.) It is easy to prove, generally, that

$$\mathrm{S}\frac{q-1}{q+1} = \mathrm{S}\frac{(q-1)(\mathrm{K}q+1)}{(q+1)(\mathrm{K}q+1)} = \frac{\mathrm{N}q-1}{\mathrm{N}(q+1)}, \qquad \mathrm{S}\frac{q+1}{q-1} = \frac{\mathrm{N}q-1}{\mathrm{N}(q-1)};$$

whence

$$\text{XXXII.}\ldots\ \mathrm{S}\frac{\iota-\kappa}{\iota+\kappa} = \frac{\mathrm{T}\iota^2-\mathrm{T}\kappa^2}{\mathrm{T}(\iota+\kappa)^2}, \qquad \mathrm{S}\frac{\iota+\kappa}{\iota-\kappa} = \frac{\mathrm{T}\iota^2-\mathrm{T}\kappa^2}{\mathrm{T}(\iota-\kappa)^2},$$

whatever two vectors ι and κ may be. But we have here,

$$\text{XXXIII.}\ldots\ t^2 = \mathrm{T}\iota^2 - \mathrm{T}\kappa^2, \text{ by 217, (5.)};$$

the recent expressions (9.) for α and β become, therefore,

$$\text{XXXIV.}\ldots\ \alpha = +(\iota+\kappa)\,\mathrm{S}\frac{\iota-\kappa}{\iota+\kappa}; \qquad \beta = -(\iota-\kappa)\,\mathrm{S}\frac{\iota+\kappa}{\iota-\kappa}.$$

The last form 204, (14.), of the equation of the ellipsoid, may therefore be now thus written:

$$\text{XXXV.}\ldots\ \mathrm{T}\left(\mathrm{S}\frac{\rho}{\iota+\kappa} : \mathrm{S}\frac{\iota-\kappa}{\iota+\kappa} - \mathrm{V}\frac{\rho}{\iota-\kappa} : \mathrm{S}\frac{\iota+\kappa}{\iota-\kappa}\right) = 1;$$

in which the sign of the right part may be changed. And thus we verify by calculation the recent result (1.) of the construction, namely that B is a point of the surface; for we see that the last equation is satisfied, when we suppose

$$\text{XXXVI.}\ldots\ \rho = \mathrm{AB} = \iota - \kappa = \beta : \mathrm{S}\frac{\beta}{\alpha};$$

a value of ρ which evidently satisfies also the form 216, IV.

(11.) From the form 216, II., combined with the value XXXIV. of α, it is easy to infer that the plane,

$$\text{XXXVII.}\ldots\ \mathrm{S}\frac{\rho}{\alpha} = 1, \text{ or } \text{XXXVII}'.\ldots\ \mathrm{S}\frac{\rho}{\iota+\kappa} = \mathrm{S}\frac{\iota-\kappa}{\iota+\kappa},$$

which corresponds to the value $x = 1$ in 216, I., *touches the ellipsoid* at the point B, of which the vector ρ has been thus determined (10); the *normal to the surface*, at that point, has therefore the direction of $\iota + \kappa$, or of α, that is, of FB, or of F′B: so that the last geometrical inference (7.) is thus confirmed, by calculation with quaternions.

219. A few other consequences of the construction (217) may be here noted; especially as regards the geometrical determination

of the *three principal semiaxes* of the ellipsoid, and the major and minor semiaxes of any elliptic and *diametral section*; together with the assigning of a certain *system of spherical conics*, of which the *surface* may be considered to be the *locus*.

(1.) Let a, b, c denote the lengths of the greatest, the mean, and the least semiaxes of the ellipsoid, respectively; then if the side BC of the generating triangle cut the diacentric sphere in the points H and H′, the former lying (as in Fig. 53) between the points B and C, we have the values,

$$\text{XXXVIII.}\ldots\quad a = \overline{\mathrm{BH}'}; \qquad b = \overline{\mathrm{BG}}; \qquad c = \overline{\mathrm{BH}};$$

so that the lengths of the sides of the triangle ABC may be thus expressed, in terms of these semiaxes,

$$\text{XXXIX.}\ldots\quad \overline{\mathrm{BC}} = \mathrm{T}\iota = \frac{a+c}{2}; \qquad \overline{\mathrm{CA}} = \mathrm{T}\kappa = \frac{a-c}{2}; \qquad \overline{\mathrm{AB}} = \mathrm{T}(\iota - \kappa) = \frac{ac}{b};$$

and we may write,

$$\text{XL.}\ldots\quad a = \mathrm{T}\iota + \mathrm{T}\kappa; \qquad b = \frac{\mathrm{T}\iota^2 - \mathrm{T}\kappa^2}{\mathrm{T}(\iota - \kappa)}; \qquad c = \mathrm{T}\iota - \mathrm{T}\kappa.$$

(2.) If, in the respective directions of the two supplementary chords AH, AH′ of the sphere, or in the opposite directions, we set off lines AL, AN, with the lengths of BH′, BH, the points L, N, thus obtained, will be respectively a *major* and a *minor summit* of the surface. And if we erect, at the centre A of that surface, a perpendicular AM to the plane of the triangle, with a length $= \overline{\mathrm{BG}}$, the point M (which will be common to the two circular sections, and will be situated on the enveloping cylinder) will be a *mean summit* thereof.

(3.) Conceive that the sphere and ellipsoid are both cut by a plane through A, on which the points B′ and C′ shall be supposed to be the projections of B and C; then C′ will be the centre of the circular section of the sphere; and if the line B′C′ cut this new circle in the points D_1, D_2, of which D_1 may be supposed to be the nearer to B′, the two supplementary chords AD_1, AD_2 of the circle have the *directions* of the *major and minor semiaxes* of the *elliptic section* of the ellipsoid; while the *lengths* of those semiaxes are, respectively, $\overline{\mathrm{BA}}\,.\,\overline{\mathrm{BG}} : \overline{\mathrm{BD}_1}$, and $\overline{\mathrm{BA}}\,.\,\overline{\mathrm{BG}} : \overline{\mathrm{BD}_2}$; or $\overline{\mathrm{BD}'_1}$ and $\overline{\mathrm{BD}'_2}$, if the secants BD_1 and BD_2 meet the sphere again in D_1' and D_2'.

(4.) If these two semiaxes of the section be called $a_{,}$ and $c_{,}$, and if we still denote by t the tangent from B to the sphere, we have thus,

$$\text{XLI.}\ldots\quad \overline{\mathrm{BD}_1} = t^2 : a_{,} = aca_{,}^{-1}; \qquad \overline{\mathrm{BD}_2} = t^2 : c_{,} = acc_{,}^{-1};$$

but if we denote by p_1 and p_2 the inclinations of the plane of the section to the two cyclic planes of the ellipsoid, whereto CA and CB are perpendicular, so that the projections of these two sides of the triangle are

$$\text{XLII.}\ldots\quad \begin{cases} \overline{\mathrm{C'A}} = \overline{\mathrm{CA}}\,.\,\sin p_1 = \frac{1}{2}(a-c)\sin p_1, \\ \overline{\mathrm{C'B'}} = \overline{\mathrm{CB}}\,.\,\sin p_2 = \frac{1}{2}(a+c)\sin p_2, \end{cases}$$

we have

$$\text{XLIII.}\ldots\quad \overline{\mathrm{BD}_2}^2 - \overline{\mathrm{BD}_1}^2 = \overline{\mathrm{B'D}_2}^2 - \overline{\mathrm{B'D}_1}^2 = \overline{4\mathrm{B'C'}}\,.\,\overline{\mathrm{C'A}} = (a^2 - c^2)\sin p_1 \sin p_2;$$

whence follows the important formula,

$$\text{XLIV.}\ldots\quad c_{,}^{-2} - a_{,}^{-2} = (c^{-2} - a^{-2})\sin p_1 \sin p_2;$$

or in words, the known and useful theorem, that "*the difference of the inverse squares of the semiaxes, of a plane and diametral section of an ellipsoid, varies as the product of the sines of the inclinations of the cutting plane, to the two planes of circular section.*

(5.) As verifications, if the plane be that of the generating triangle ABC, we have

$$p_1 = p_2 = \frac{\pi}{2}, \quad \text{and} \quad a_{,} = a, \quad c_{,} = c;$$

but if the plane be perpendicular to either of the two sides, CA, CB, then either p_1 or $p_2 = 0$, and $c_{,} = a_{,}$.

(6.) If the ellipsoid be cut by any concentric sphere, *distinct* from the *mean sphere* XIV., so that

$$\text{XLV.} \ldots \overline{\mathrm{AE}} = \mathrm{T}\rho = r \lessgtr b, \text{ where } r \text{ is a given positive scalar;}$$

then

$$\text{XLVI.} \ldots \overline{\mathrm{BD}} = t^2 r^{-1} \lessgtr acb^{-1}, \quad \text{that is,} \quad \lessgtr \overline{\mathrm{BA}};$$

so that the *locus* of what may be called the *guide-point* D, through which, by the construction, the variable semidiameter AE of the ellipsoid (or one of its prolongations) passes, and which is still at a constant distance from the given external point B, is now again a *circle* of the diacentric sphere, but one of which the *plane* does *not pass* (as it did in 218, (3.)) *through the centre* A of the ellipsoid. The point E has therefore here, for one locus, the *cyclic cone* which has A for *vertex*, and rests on the last-mentioned circle as its *base;* and since it is also on the concentric *sphere* XLV., it must be on one or other of the *two spherical conics*, in which (comp. 196, (11.)) the cone and sphere last mentioned intersect.

(7.) The *intersection of an ellipsoid with a concentric sphere* is therefore, generally, a *system of two such conics*, varying with the value of the radius r, and becoming, as a *limit*, the *system of the two circular sections*, for the particular value $r = b$; and the ellipsoid itself may be considered as the *locus* of all such spherical conics, including those two circles.

(8.) And we see, by (6.), that the *two cyclic planes* (comp. 196, (17.), &c.) *of any one of the concentric cones*, which rest on any such conic, *coincide with the two cyclic planes of the ellipsoid:* all this resulting, with the greatest ease, from the *construction* (217) to which quaternions had conducted.

(9.) With respect to the Figure 53, which was designed to illustrate that construction, the signification of the letters ABCDD'EFF'GHH'LN has been already explained. But as regards the other letters we may here add, Ist, that N' is a second minor summit of the surface, so that AN' = NA; IInd, that K is a point in which the chord AF', of what we may here call the *diacentric circle* AGF, intersects what may be called the *principal ellipse*,* or the *section* NBLEN' of the ellipsoid, made by the *plane of the greatest and least axes*, that is by the plane of the *generating triangle* ABC, so that the lengths of AK and BF are equal; IIIrd, that the *tangent*, VKV', to this ellipse at this point, is *parallel to the side* AB *of the triangle*, or to the *axis of*

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* In the plane of what is called, by many modern geometers, the *focal hyperbola* of the ellipsoid.

revolution of the enveloping cylinder 218, (5.), being in fact *one side* (or *generatrix*) of that cylinder; IVth, that AK, AB are thus two *conjugate semidiameters* of the ellipse, and therefore the tangent TBT′, at the point B of that ellipse, is parallel to the line AKF′, or perpendicular to the line BFF′; Vth, that this latter line is thus the *normal* (comp. 218, (7.), (11.)) to the same elliptic section, and therefore also to the ellipsoid, at B; VIth, that the *least distance* KK′ between the parallels AB, KV, being = the radius b of the cylinder, is equal in length to the line BG, and also to each of the two semidiameters, AS, AS′, of the ellipse, which are *radii of the two circular sections* of the ellipsoid, in planes perpendicular to the plane of the Figure; VIIth, that AS touches the circle at A; and VIIIth, that the point S′ is on the chord AI of that circle, which is drawn at right angles to the side BC of the triangle.

220. The reader will easily conceive that the quaternion equation of the ellipsoid admits of being put under several other forms; among which, however, it may here suffice to mention one, and to assign its geometrical interpretation.

(1.) For any three vectors, ι, κ, ρ, we have the transformations,

$$\text{XLVII.} \ldots \mathrm{N}\left(\frac{\iota}{\rho}+\mathrm{K}\frac{\kappa}{\rho}\right)=\mathrm{N}\frac{\iota}{\rho}+\mathrm{N}\frac{\kappa}{\rho}+2\mathrm{S}\frac{\iota}{\rho}\frac{\kappa}{\rho}$$

$$=\mathrm{N}\frac{\iota}{\kappa}\,\mathrm{N}\frac{\kappa}{\rho}+\mathrm{N}\frac{\kappa}{\iota}\,\mathrm{N}\frac{\iota}{\rho}+2\mathrm{S}\frac{\iota}{\rho}\frac{\kappa}{\rho}\,\mathrm{T}\frac{\kappa}{\iota}\,\mathrm{T}\frac{\iota}{\kappa}$$

$$=\mathrm{N}\left(\frac{\iota}{\rho}\,\mathrm{T}\frac{\kappa}{\iota}+\mathrm{K}\frac{\kappa}{\rho}\,\mathrm{T}\frac{\iota}{\kappa}\right)=\mathrm{N}\left(\frac{\kappa}{\rho}\,\mathrm{T}\frac{\iota}{\kappa}+\mathrm{K}\frac{\iota}{\rho}\,\mathrm{T}\frac{\kappa}{\iota}\right)$$

$$=\mathrm{N}\left(\frac{\mathrm{U}\iota\,.\,\mathrm{T}\kappa}{\rho}+\mathrm{K}\frac{\mathrm{U}\kappa\,.\,\mathrm{T}\iota}{\rho}\right)=\mathrm{N}\left(\frac{\mathrm{U}\kappa\,.\,\mathrm{T}\iota}{\rho}+\mathrm{K}\frac{\mathrm{U}\iota\,.\,\mathrm{T}\kappa}{\rho}\right);$$

whence follows this other general transformation:

$$\text{XLVIII.} \ldots \mathrm{T}\left(\iota+\mathrm{K}\frac{\kappa}{\rho}\,.\,\rho\right)=\mathrm{T}\left(\mathrm{U}\kappa\,.\,\mathrm{T}\iota+\mathrm{K}\frac{\mathrm{U}\iota\,.\,\mathrm{T}\kappa}{\rho}\,.\,\rho\right).$$

(2.) If then we introduce two new auxiliary and constant vectors, ι' and κ', defined by the equations,

$$\text{XLIX.} \ldots \iota'=-\mathrm{U}\kappa\,.\,\mathrm{T}\iota, \qquad \kappa'=-\mathrm{U}\iota\,.\,\mathrm{T}\kappa,$$

which give,

$$\text{L.} \ldots \mathrm{T}\iota'=\mathrm{T}\iota, \qquad \mathrm{T}\kappa'=\mathrm{T}\kappa, \qquad \mathrm{T}(\iota'-\kappa')=\mathrm{T}(\iota-\kappa), \qquad \mathrm{T}\iota'^2-\mathrm{T}\kappa'^2=t^2,$$

we may write the equation XVI. (in 217) of the ellipsoid under the following precisely similar form:

$$\text{LI.} \ldots \frac{t^2}{\mathrm{T}\rho}=\mathrm{T}\left(\iota'+\mathrm{K}\frac{\kappa'}{\rho}\,.\,\rho\right);$$

in which ι' and κ' have simply taken the places of ι and κ.

(3.) Retaining then the *centre* A of the ellipsoid, construct a *new diacentric sphere*, with a new centre C′, and a *new generating triangle* AB′C′, where B′ is a *new fixed external point*, but the *lengths of the sides* are the *same*, by the conditions,

$$\text{LII.} \quad .\ \mathrm{AC'}=-\kappa', \qquad \mathrm{C'B'}=+\iota', \quad \text{and therefore} \quad \mathrm{AB'}=\iota'-\kappa';$$

draw any secant B′D″D‴ (instead of BDD′), and set off a line AE in the direction of

AD″, or in the opposite direction, with a length equal to that of BD‴; *the locus of the point* E *will be the same ellipsoid as before.*

(4.) The only inference which we shall here* draw from this new construction is, that there exists (as is known) a *second enveloping cylinder of revolution*, and that its axis is the side AB′ of the new triangle AB′C′; but that the *radius* of this second cylinder is equal to that of the first, namely to the *mean semiaxis*, b, of the ellipsoid; and that the *major semiaxis*, a, or the line AL in Fig. 58, *bisects the angle* BAB′, *between the two axes of revolution* of these two circumscribed cylinders: the *plane* of the new ellipse of contact being geometrically determined by a process exactly similar to that employed in 218, (7.); and being perpendicular to the new vector, $\iota' + \kappa'$, as the old plane of contact was (by 218, (11.)) to $\iota + \kappa$.

SECTION 14.—*On the Reduction of the General Quaternion to a Standard Quadrinomial Form; with a First Proof of the Associative Principle of Multiplication of Quaternions.*

221. Retaining the significations (181) of the three rectangular unit-lines OI, OJ, OK, as the *axes*, and therefore also the *indices* (159), of three given right versors i, j, k, in three mutually rectangular planes, we can express the *index* OQ of any *other* right quaternion, such as $\mathrm{V}q$, under the *trinomial form* (comp. 62),

$$\text{I.} \ldots \mathrm{IV}q = \mathrm{OQ} = x.\mathrm{OI} + y.\mathrm{OJ} + z.\mathrm{OK};$$

where xyz are some three scalar coefficients, namely, the three rectangular co-ordinates of the extremity Q of the index, with respect to the three axes OI, OJ, OK. Hence we may write also generally, by 206 and 126,

$$\text{II.} \ldots \mathrm{V}q = xi + yj + zk = ix + jy + kz;$$

and this last form, $ix + jy + kz$, may be said to be a *Standard Trinomial Form*, to which *every right quaternion*, or the *right part* $\mathrm{V}q$ of *any* proposed quaternion q, can be (as above) *reduced*. If then we denote by w the *scalar part*, $\mathrm{S}q$, of the same general quaternion q, we shall have, by 202, the following *General Reduction of a Quaternion to a* STANDARD QUADRINOMIAL FORM (183):

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* If room shall allow, a few additional remarks may be made, on the relations of the constant vectors ι, κ, &c., to the ellipsoid, and on some other constructions of that surface, when, in the following Book, its equation shall come to be put under the new form,

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2.$$

$$\text{III.} \ldots q = (\mathrm{S}q + \mathrm{V}q =) w + ix + jy + kz;$$

in which the *four scalars, wxyz*, may be said to be the *Four Constituents of the Quaternion*. And it is evident (comp. 202, (5.), and 133), that if we write in like manner,

$$\text{IV.} \ldots q' = w' + ix' + jy' + kz',$$

where *ijk* denote the same three given right versors (181) as before, then *the equation*

$$\text{V.} \ldots q' = q,$$

between these two quaternions, q and q', *includes the four following scalar equations between the constituents:*

$$\text{VI.} \ldots w' = w, \qquad x' = x, \qquad y' = y, \qquad z' = z;$$

which is a new justification (comp. 112, 116) of the *propriety* of *naming*, as we have done throughout the present Chapter, the *General Quotient of two Vectors* (101) a QUATERNION.

222. When the *Standard Quadrinomial Form* (221) is adopted, we have then not only

$$\text{I.} \ldots \mathrm{S}q = w, \quad \text{and} \quad \mathrm{V}q = ix + jy + kz,$$

as before, but also, by 204, XI.,

$$\text{II.} \ldots \mathrm{K}q = (\mathrm{S}q - \mathrm{V}q =) w - ix - jy - kz.$$

And because the *distributive property of multiplication* of quaternions (212), combined with the *laws of of the symbols ijk* (182), or with the *General and Fundamental Formula of this whole Calculus* (183), namely with the formula,

$$i^2 = j^2 = k^2 = ijk = -1, \tag{A}$$

gives the transformation,

$$\text{III.} \ldots (ix + jy + kz)^2 = -(x^2 + y^2 + z^2),$$

we have, by 204, &c., the following new expressions:

$$\text{IV.} \ldots \mathrm{NV}q = (\mathrm{TV}q)^2 = -\mathrm{V}q^2 = x^2 + y^2 + z^2;$$
$$\text{V.} \ldots \mathrm{TV}q = \sqrt{(x^2 + y^2 + z^2)};$$
$$\text{VI.} \ldots \mathrm{UV}q = (ix + jy + kz) : \sqrt{(x^2 + y^2 + z^2)};$$
$$\text{VII.} \ldots \mathrm{N}q = \mathrm{T}q^2 = \mathrm{S}q^2 - \mathrm{V}q^2 = w^2 + x^2 + y^2 + z^2;$$
$$\text{VIII.} \ldots \mathrm{T}q = \sqrt{(w^2 + x^2 + y^2 + z^2)};$$
$$\text{IX.} \ldots \mathrm{U}q = (w + ix + jy + kz) : \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\text{X.} \ldots \mathrm{SU}q = w : \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\text{XI.} \ldots \mathrm{VU}q = (ix + jy + kz) : \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\text{XII.} \ldots \mathrm{TVU}q = \sqrt{\frac{x^2 + y^2 + z^2}{w^2 + x^2 + y^2 + z^2}}.$$

(1.) To prove the recent formula III., we may arrange as follows the steps of the multiplication (comp. again 182):

$$\begin{array}{r}
\mathrm{V}q = ix + jy + kz, \\
\underline{\mathrm{V}q = ix + jy + kz}\,; \\
ix\,.\,\mathrm{V}q = -x^2 + kxy - jxz\,; \\
jy\,.\,\mathrm{V}q = -y^2 - kyx \qquad + iyz, \\
\underline{kz\,.\,\mathrm{V}q = -z^2 \qquad + jzx - izy}\,; \\
\mathrm{V}q^2 = \mathrm{V}q\,.\,\mathrm{V}q = -x^2 - y^2 - z^2.
\end{array}$$

(2.) We have, therefore,

$$\text{XIII.} \ldots (ix + jy + kz)^2 = -1, \quad \text{if} \quad x^2 + y^2 + z^2 = 1,$$

a result to which we have already alluded,* in connexion with the partial ***indeterminateness*** of signification, in the present calculus, of the symbol $\sqrt{-1}$, when considered as denoting a ***right radial*** (149), or a ***right versor*** (153), of which the ***plane*** or the ***axis*** is ***arbitrary.***

(3.) If $q'' = q'q$, then $\mathrm{N}q'' = \mathrm{N}q'\,.\,\mathrm{N}q$, by 191, (8.); but if $q = w + \&c.$, $q' = w' + \&c.$, $q'' = w'' + \&c.$, then

$$\text{XIV.} \ldots \left\{\begin{array}{l}
w'' = w'w - (x'x + y'y + z'z), \\
x'' = (w'x + x'w) + (y'z - z'y), \\
y'' = (w'y + y'w) + (z'x - x'z), \\
z'' = (w'z + z'w) + (x'y - y'x);
\end{array}\right.$$

and conversely these four scalar equations are jointly equivalent to, and may be summed up in, the quaternion formula,

$$\text{XV.} \ldots w'' + ix'' + jy'' + kz'' = (w' + ix' + jy' + kz')\,(w + ix + jy + kz);$$

we ought therefore, under these conditions XIV., to have the equation,

$$\text{XVI.} \ldots w''^2 + x''^2 + y''^2 + z''^2 = (w'^2 + x'^2 + y'^2 + z'^2)\,(w^2 + x^2 + y^2 + z^2);$$

which can in fact be verified by so easy an algebraical calculation, that its truth may be said to be obvious upon mere inspection, at least when the terms in the four quadrinomial expressions $w'' \ldots z''$ are arranged† as above.

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* Compare the first Note to page 131; and that to page 162.

† From having somewhat otherwise *arranged* those terms, the author had some little trouble at first, in verifying that the twenty-four *double products*, in the expansion of $w''^2 + \&c.$, destroy each other, leaving only the sixteen *products of squares*, or that XVI. follows from XIV., when he was led to anticipate that result through quaternions, in the year 1843. He believes, however, that the *algebraic theorem* XVI., as distinguished from the *quaternion formula* XV., with which it is here connected, had been discovered by the celebrated EULER.

223. The principal *use* which we shall here make of the standard quadrinomial form (221), is to prove by it the general *associative property of multiplication* of quaternions; which can now with great ease be done, the *distributive* property* (212) of such multiplication having been already proved. In fact, if we write, as in 222, (3.),

$$\text{I.} \ldots \begin{cases} q = w + ix + jy + kz, \\ q' = w' + ix' + jy' + kz', \\ q'' = w'' + ix'' + jy'' + kz'', \end{cases}$$

without now assuming that the relation $q'' = q'q$, or any other relation, exists between the three quaternions q, q', q'', and inquire whether it be true that the *associative formula*,

$$\text{II.} \ldots q''q' \cdot q = q'' \cdot q'q,$$

holds good, we see, by the distributive principle, that we have only to try whether this last formula is valid when the three quaternion factors q, q', q'' are replaced, in any one common order on both sides of the equation, and with or without repetition, by the three given right versors ijk; but this has already been proved, in Art. 183. We arrive then, thus, at the important conclusion, that *the General Multiplication of Quaternions is an Associative Operation*, as it had been previously seen (212) to be a *Distributive* one: although we had also found (168, 183, 191) that *such Multiplication is not* (in general) *Commutative:* or that *the two products*, $q'q$ and qq', are *generally unequal.* We may therefore *omit the point* (as in 183), and may denote each member of the equation II. by the symbol $q''q'q$.

(1.) Let $v = \text{V}q$, $v' = \text{V}q'$, $v'' = \text{V}q''$; so that v, v', v'' are any three right quaternions, and therefore, by 191, (2.), and 196, 204,

$$\text{K}v'v = vv', \qquad \text{S}v'v = \tfrac{1}{2}(v'v + vv'), \qquad \text{V}v'v = \tfrac{1}{2}(v'v - vv').$$

Let this last right quaternion be called $v_{,}$, and let $\text{S}v'v = s_{,}$, so that $v'v = s_{,} + v_{,}$; we shall then have the equations,

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* At a later stage, a sketch will be given of at least one proof of this *Associative Principle of Multiplication*, which will *not presuppose the Distributive Principle.*

$$2\mathrm{V}v''v_{,} = v''v_{,} - v_{,}v''\,; \qquad 0 = v''s_{,} - s_{,}v''\,;$$

whence, by addition,

$$\begin{aligned} 2\mathrm{V}v''v_{,} &= v''.v'v - v'v.v'' \\ &= (v''v' + v'v'')v - v'(v''v + vv'') \\ &= 2v\mathrm{S}v'v'' - 2v'\mathrm{S}v''v\,; \end{aligned}$$

and therefore generally, if v, v', v'' be still *right*, as above,

$$\text{III.} \ldots \mathrm{V}\,.\,v''\mathrm{V}v'v = v\mathrm{S}v'v'' - v'\mathrm{S}v''v\,;$$ + usef!

a formula with which *the student ought to make himself completely familiar*, on account of its extensive utility.

(2.) With the recent notations,

$$\mathrm{V}\,.\,v''\mathrm{S}v'v = \mathrm{V}v''s_{,} = v''s_{,} = v''\mathrm{S}vv'\,;$$

we have therefore this other very useful formula,

$$\text{IV.} \ldots \mathrm{V}\,.\,v''v'v = v\mathrm{S}v'v'' - v'\mathrm{S}v''v + v''\mathrm{S}vv',$$ ← comp.

where the point in the first member may often for simplicity be dispensed with; and in which it is still supposed that

$$\angle v = \angle v' = \angle v'' = \frac{\pi}{2}.$$

(3.) The formula III. gives (by 206),

$$\text{V.} \ldots \mathrm{IV}\,.\,v''\mathrm{V}v'v = \mathrm{I}v\,.\,\mathrm{S}v'v'' - \mathrm{I}v'\,.\,\mathrm{S}v''v\,;$$

hence this last vector, which is evidently *complanar with the two indices* Iv and Iv', is at the same time (by 208) *perpendicular to the third index* Iv'', and therefore (by (1.)) *complanar with the third quaternion* q''.

(4.) With the recent notations, the vector,

$$\text{VI.} \ldots \mathrm{I}v_{,} = \mathrm{IV}v'v = \mathrm{IV}(\mathrm{V}q'.\mathrm{V}q),$$

is (by 208, XXII.) a line perpendicular to both Iv and Iv'; or *common to the planes* of q and q'; being also such that the *rotation* round it from Iv' to Iv is *positive*: while *its length*,

$$\mathrm{TI}v_{,}, \quad \text{or} \quad \mathrm{T}v_{,}, \quad \text{or} \quad \mathrm{TV}\,.\,v'v, \quad \text{or} \quad \mathrm{TV}(\mathrm{V}q'.\mathrm{V}q),$$

bears to the unit of length the same ratio, as that which the parallelogram under the indices, Iv and Iv', *bears to the unit of area.*

(5.) To interpret (comp. IV.) the *scalar* expression,

$$\text{VII.} \ldots \mathrm{S}v''v'v = \mathrm{S}v''v_{,} = \mathrm{S}\,.\,v''\mathrm{V}v'v,$$

(because $\mathrm{S}v''s_{,} = 0$), we may employ the formula 208, V.; which gives the the transformation,

$$\text{VIII.} \ldots \mathrm{S}v''v'v = \mathrm{T}v''.\,\mathrm{T}v_{,}.\,\cos(\pi - x)\,;$$

where Tv'' denotes the *length* of the *line* Iv'', and T$v_{,}$ represents by (4.) the *area* (positively taken) of the *parallelogram* under Iv' and Iv; while x is (by 208), the *angle* between the two indices Iv'', I$v_{,}$. This angle will be *obtuse*, and therefore the cosine of its supplement will be *positive*, and equal to the *sine of the inclination of the line* Iv'' *to the plane of* Iv *and* Iv', if the *rotation* round Iv'' from Iv' to Iv be *negative*, that is, if the rotation round Iv from Iv' to Iv'' be *positive*; but that cosine will be equal the negative of this sine, if the direction of this rotation be reversed. We have therefore the important interpretation:

$$\text{IX.} \ldots \mathrm{S}v''v'v = \pm\ \textit{volume of parallelepiped under}\ \mathrm{I}v,\ \mathrm{I}v',\ \mathrm{I}v''\,;$$

the *upper* or the *lower sign* being taken, according as the rotation round $\mathrm{I}v$, from $\mathrm{I}v'$ to $\mathrm{I}v''$, is *positively* or *negatively directed.*

(6.) For example, we saw that the ternary products ijk and kji have scalar values, namely,

$$ijk = -1, \qquad kji = +1, \text{ by 183, (1,), (2.);}$$

and accordingly the *parallelepiped of indices* becomes, in this case, an *unit-cube;* while the rotation round the index of i, from that of j to that of k, is *positive* (181).

(7.) In general, for any three *right* quaternions $vv'v''$, we have the formula,

$$\text{X.} \ldots \mathrm{S}vv'v'' = -\mathrm{S}v''v'v;$$

and when the three indices are *complanar*, so that the *volume* mentioned in IX. *vanishes*, then each of these two last scalars becomes *zero;* so that we may write, as a new *Formula of Complanarity;*

$$\text{XI.} \ldots \mathrm{S}v''v'v = 0, \quad \text{if} \quad \mathrm{I}v'' \,|||\, \mathrm{I}v', \ \mathrm{I}v \ (123):$$

while, on the other hand, this scalar cannot vanish in any *other* case, if the quaternions (or their indices) be still supposed to be *actual* (1, 144); because it then represents an actual volume.

(8.) Hence also we may establish the following *Formula of Collinearity*, for *any three quaternions:*

$$\text{XII.} \ldots \mathrm{S}(\mathrm{V}q''.\mathrm{V}q'.\mathrm{V}q) = 0, \quad \text{if} \quad \mathrm{IV}q'' \,|||\, \mathrm{IV}q', \quad \mathrm{IV}q;$$

that is, by 209, if the *planes* of q, q', q'' have any *common line.*

(9.) In general, if we employ the *standard trinomial form* 221, II., namely,

$$v = \mathrm{V}q = ix + jy + kz, \qquad v' = ix' + \&\text{c.}, \qquad v'' = ix'' + \&\text{c.},$$

the laws (182, 183) of the symbols i, j, k give the transformation,

$$\text{XIII.} \ldots \mathrm{S}v''v'v = x''(z'y - y'z) + y''(x'z - z'x) + z''(y'x - x'y);$$

and accordingly this is the known expression for the volume (with a suitable sign) of the parallelepiped, which has the three lines OP, OP′, OP″ for three co-initial edges, if the rectangular co-ordinates* of the four corners, O, P, P′, P″ be 000, xyz, $x'y'z'$, $x''y''z''$.

(10.) Again, as another important consequence of the general associative property of multiplication, it may be here observed, that although products of *more than two* quaternions have *not* generally *equal scalars*, for *all* possible permutations of the factors, since we have just seen a case X. in which such a change of arrangement produces a change of *sign* in the result, yet *cyclical permutation is permitted, under the sign* S; or in symbols, that for *any three quaternions* (and the result is easily extended to *any greater number* of such factors) the following formula holds good:

$$\text{XIV.} \ldots \mathrm{S}q''q'q = \mathrm{S}qq''q'.$$

In fact, to prove this equality, we have only to write it thus,

$$\text{XIV}'. \ldots \mathrm{S}(q''q'.q) = \mathrm{S}\,(q.q''q'),$$

and to remember that the scalar of the product of any *two* quaternions remains unaltered (198, I.), when the order of those two factors is changed.

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* This result may serve as an example of the manner in which *quaternions*, although *not based* on any usual doctrine of *co-ordinates*, may yet be employed to *deduce*, or to *recover*, and often with great ease, important co-ordinate expressions.

(11.) In like manner, by 192, II., it may be inferred that

$$\text{XV.} \ldots \mathrm{K}q''q'q = \mathrm{K}(q''.q'q) = \mathrm{K}q'q\,.\,\mathrm{K}q'' = \mathrm{K}q\,.\,\mathrm{K}q'\,.\,\mathrm{K}q'',$$

with a corresponding result for any greater number of factors; whence by 192, I., if Πq and $\Pi' q$ denote the *products* of any one set of quaternions taken in two *opposite orders*, we may write,

$$\text{XVI.} \ldots \mathrm{K}\Pi q = \Pi'\mathrm{K}q; \qquad \text{XVII.} \ldots \mathrm{R}\Pi q = \Pi'\mathrm{R}q.$$

(12.) But if v be *right*, as above, then $\mathrm{K}v = -v$, by 144; hence,

$$\text{XVIII.} \ldots \mathrm{K}\Pi v = \pm \Pi' v; \quad \text{XIX.} \ldots \mathrm{S}\Pi v = \pm \mathrm{S}\Pi' v; \quad \text{XX.} \ldots \mathrm{V}\Pi v = \mp \mathrm{V}\Pi' v;$$

upper or *lower signs* being taken, according as the *number* of the right factors is *even* or *odd;* and under the same conditions,

$$\text{XXI.} \ldots \mathrm{S}\Pi v = \tfrac{1}{2}(\Pi v \pm \Pi' v); \qquad \text{XXII.} \ldots \mathrm{V}\Pi v = \tfrac{1}{2}(\Pi v \mp \Pi' v);$$

as was lately exemplified (1.), for the case where the number is *two.*

(13.) For the case where that number is *three*, the four last formulæ give,

$$\text{XXIII.} \ldots \mathrm{S}v''v'v = -\mathrm{S}vv'v'' = \tfrac{1}{2}(v''v'v - vv'v'');$$
$$\text{XXIV.} \ldots \mathrm{V}v''v'v = +\mathrm{V}vv'v'' = \tfrac{1}{2}(v''v'v + vv'v'');$$

results which obviously agree with X. and IV.

224. For the case of *Complanar Quaternions* (119), the power of reducing each (120) to the form of a fraction (101) which shall have, at pleasure, for its denominator or for its numerator, any arbitrary line in the given plane, furnishes some peculiar facilities for proving the *commutative* and *associative* properties of *Addition* (207), and the *distributive* and *associative* properties of *Multiplication* (212, 223); while, for this *case* of multiplication of quaternions, we have already seen (191, (1.)) that the *commutative* property *also* holds good, as it does in algebraic multiplication. It may therefore be not irrelevant nor useless to insert here a short Second Chapter on the subject of such *complanars:* in treating briefly of which, while assuming as proved the existence of all the foregoing properties, we shall have an opportunity to say something of Powers and Roots and Logarithms; and of the connexion of Quaternions with Plane Trigonometry, and with Algebraical Equations. After which, in the Third and last Chapter of this Second Book, we propose to resume, for a short time, the consideration of *Diplanar Quaternions;* and especially to show how the *Associative Principle of Multiplication* can be established, for them, *without*[*] *employing the Distributive Principle.*

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[*] Compare the Note to page 236.

CHAPTER II.

ON COMPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN ONE PLANE; AND ON POWERS, ROOTS, AND LOGARITHMS OF QUATERNIONS.

SECTION 1.—*On Complanar Proportion of Vectors; Fourth Proportional to Three, Third Proportional to Two, Mean Proportional, Square Root; General Reduction of a Quaternion in a given Plane, to a Standard Binomial Form.*

225. The Quaternions of the present Chapter shall all be supposed to be *complanar* (119); their *common plane* being assumed to coincide with that of the given right versor i (181). And *all the lines*, or vectors, such as α, β, γ, &c., or α_0, α_1, α_2, &c., to be here employed, shall be conceived to be *in that given plane* of i; so that we may write (by 123), for the purposes of this Chapter, the *formulæ of complanarity:*

$$q \mathbin{|||} q' \mathbin{|||} q'' \ldots \mathbin{|||} i; \qquad \alpha \mathbin{|||} i, \qquad \beta \mathbin{|||} i, \qquad \alpha_0 \mathbin{|||} i, \text{ \&c.}$$

226. Under these conditions, we can always (by 103, 117) interpret any symbol of the form $(\beta : \alpha) . \gamma$, as denoting a *line* δ in the given plane; which line may also be denoted (125) by the symbol $(\gamma : \alpha) . \beta$, but *not** (comp. 103) by either of the two apparently equivalent symbols, $(\beta . \gamma) : \alpha$, $(\gamma . \beta) : \alpha$; so that we may write,

$$\text{I.} \ldots \delta = \frac{\beta}{\alpha}\gamma = \frac{\gamma}{\alpha}\beta,$$

and may say that this line δ is the *Fourth Proportional* to the three lines α, β, γ; or to the three lines α, γ, β; the *two Means*, β and γ, of any such *Complanar Proportion of Four Vectors*, admitting thus of being *interchanged*, as in algebra. Under the same conditions we may write also (by 125),

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* In fact the symbols $\beta . \gamma$, $\gamma . \beta$, or $\beta\gamma$, $\gamma\beta$, have not as yet received with us *any* interpretation; and even when they shall come to be interpreted as representing certain quaternions, it will be found (comp. 168) that the two combinations, $\frac{\beta}{\alpha}\gamma$ and $\frac{\beta\gamma}{\alpha}$, have generally different significations.

$$\text{II.} \ldots \alpha = \frac{\beta}{\delta}\gamma = \frac{\gamma}{\delta}\beta; \qquad \beta = \frac{\alpha}{\gamma}\delta = \frac{\delta}{\gamma}\alpha; \qquad \gamma = \frac{\delta}{\beta}\alpha = \frac{\alpha}{\beta}\delta;$$

so that (still as in algebra) the *two Extremes*, α and δ, of any such proportion of four lines $\alpha, \beta, \gamma, \delta$, may likewise change places among themselves: while we may also make the *means* become the *extremes*, if we at the same time change the extremes to means. More generally, if $\alpha, \beta, \gamma, \delta, \varepsilon \ldots$ be *any odd number* of vectors in the given plane, we can always find *another* vector ρ in that plane, which shall satisfy the equation,

$$\text{III.} \ldots \quad \ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta}\alpha = \rho, \quad \text{or} \quad \text{III}'. \ldots \quad \ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta}\frac{\alpha}{\rho} = 1;$$

and when such a formula holds good, for any *one* arrangement of the *numerator-lines* $\alpha, \gamma, \varepsilon, \ldots$ and of the *denominator-lines* $\rho, \beta, \delta \ldots$ it can easily be proved to hold good also for any *other* arrangement of the numerators, and any other arrangement of the denominators. For example, whatever four (complanar) vectors may be denoted by $\beta\gamma\delta\varepsilon$, we have the transformations,

$$\text{IV.} \ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta} = \frac{\varepsilon}{\delta}\gamma : \beta = \frac{\gamma}{\delta}\varepsilon : \beta = \frac{\gamma}{\delta}\frac{\varepsilon}{\beta},$$

the two numerators being thus interchanged. Again,

$$\text{IV}'. \ldots \frac{\varepsilon}{\delta}\frac{\gamma}{\beta} = \frac{\gamma}{\beta}\frac{\varepsilon}{\delta} = \frac{\varepsilon}{\beta}\frac{\gamma}{\delta}, \text{ by IV.};$$

so that the two denominators also may change places.

227. An interesting *case* of such *proportion* (226) is that in which the *means coincide;* so that only *three distinct lines*, such as α, β, γ, are involved: and that we have (comp. Art. 149, and Fig. 42) an equation of the form,

$$\text{I.} \ldots \gamma = \frac{\beta}{\alpha}\beta, \quad \text{or} \quad \alpha = \frac{\beta}{\gamma}\beta,$$

but *not*[*] $\gamma = \beta\beta : \alpha$, nor $\alpha = \beta\beta : \gamma$. In this case, it is said that the *three lines* $\alpha\beta\gamma$ form a *Continued Proportion;* of which α and γ are now the *Extremes*, and β is the *Mean:* this line β being also said to be *a*[†] *Mean Proportional* between the two others, α and γ; while γ is *the Third Proportional* to the two lines α and β; and α is, at the same time, the third proportional to γ and β. Under the same conditions, we have

$$\text{II.} \ldots \beta = \frac{\alpha}{\beta}\gamma = \frac{\gamma}{\beta}\alpha;$$

so that this *mean*, β, between α and γ, is also the *fourth proportional* (226) *to itself, as first*, and to those *two other* lines. We have also (comp. again 149),

$$\text{III.} \ldots \left(\frac{\beta}{\alpha}\right)^2 = \frac{\gamma}{\alpha}, \qquad \left(\frac{\beta}{\gamma}\right)^2 = \frac{\alpha}{\gamma};$$

whence it is natural to write,

$$\text{IV.} \ldots \frac{\beta}{\alpha} = \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}}, \qquad \frac{\beta}{\gamma} = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}},$$

and therefore (by 103),

$$\text{V.} \ldots \beta = \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}}\alpha, \qquad \beta = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}\gamma;$$

although we are *not* here to write $\beta = (\gamma\alpha)^{\frac{1}{2}}$, nor $\beta = (\alpha\gamma)^{\frac{1}{2}}$. But because we have always, as in algebra (comp. 199, (3.)), the equation or identity, $(-q)^2 = q^2$, we are equally well entitled to write,

$$\text{VI.} \ldots \left(\frac{-\beta}{\alpha}\right)^2 = \frac{\gamma}{\alpha}, \qquad \left(\frac{-\beta}{\gamma}\right)^2 = \frac{\alpha}{\gamma}, \qquad -\beta = \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}}\alpha = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}\gamma;$$

the symbol $q^{\frac{1}{2}}$ denoting thus, in general, *either of two opposite quaternions*, whereof however one, namely that one of which the *angle* is *acute*, has been already *selected* in 199, (1.), as that which shall be called by us *the Square Root* of the quaternion

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[*] Compare the Note to the foregoing Article.

[†] We say, *a mean proportional;* because we shall shortly see that the *opposite line*, $-\beta$, is in the same sense *another mean;* although a *rule* will presently be given, for distinguishing between them, and for *selecting one*, as that which may be called, by eminence, *the mean proportional.*

q, and denoted by $\sqrt{q}$. We may therefore establish the formula,

$$\text{VII.} \ldots \beta = \pm\sqrt{\left(\frac{\gamma}{\alpha}\right)}.\alpha = \pm\sqrt{\left(\frac{\alpha}{\gamma}\right)}.\gamma,$$

if α, β, γ form, as above, a continued proportion; the *upper signs* being taken when (as in Fig. 42) the *angle* AOC, between the extreme lines α, γ, is *bisected* by the line OB, or β, *itself;* but the *lower signs*, when that angle is bisected by the *opposite line*, $-\beta$, or when β bisects the *vertically opposite angle* (comp. again 199, (3.)): but the *proportion of tensors*,

$$\text{VIII.} \ldots \mathrm{T}\gamma : \mathrm{T}\beta = \mathrm{T}\beta : \mathrm{T}\alpha,$$

and the resulting formulæ,

$$\text{IX.} \ldots \mathrm{T}\beta^2 = \mathrm{T}\alpha . \mathrm{T}\gamma, \qquad \mathrm{T}\beta = \sqrt{(\mathrm{T}\alpha . \mathrm{T}\gamma)},$$

in *each* case holding good. And when we shall speak simply of *the Mean Proportional between two vectors*, α and γ, which make any acute, or right, or obtuse angle with each other, we shall always henceforth understand the *former* of these two bisectors; namely, the *bisector* OB of that *angle* AOC *itself*, and *not* that of the *opposite* angle: thus taking upper signs, in the recent formula VII.

(1.) At the limit when the angle AOC *vanishes*, so that $\mathrm{U}\gamma = \mathrm{U}\alpha$, then $\mathrm{U}\beta =$ each of these two unit-lines; and the mean proportional β has the *same* common *direction* as each of the two given extremes. This comes to our agreeing to write,

$$\text{X.} \ldots \sqrt{1} = +1, \quad \text{and generally,} \quad \text{X}'. \ldots \sqrt{(a^2)} = +a,$$

if a be any positive scalar.

(2.) At the *other* limit, when $\text{AOC} = \pi$, or $\mathrm{U}\gamma = -\mathrm{U}\alpha$, the *length* of the mean proportional β is still determined by IX., as the *geometric mean* (in the usual sense) between the lengths of the two given extremes (comp. the two Figures 41); but, even with the supposed restriction (225) on the *plane* in which all the lines are situated, an *ambiguity* arises in this case, from the doubt *which* of the two *opposite perpendiculars* at O, to the line AOC, is to be taken as the direction of the *mean vector*. To remove this ambiguity, we shall suppose that the *rotation* round the axis of i (to which *axis all* the lines considered in this Chapter are, by 225, perpendicular), from the first line OA to the second line OB, is in this case *positive;* which supposition is equivalent to writing, for present purposes,

XI.[*] . . $\sqrt{-1} = +i$; and XI′. . . $\sqrt{(-a^2)} = ia$, if $a > 0$.

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[*] It is to be carefully observed that *this square root of negative unity* is *not*, in any sense, *imaginary*, nor even *ambiguous*, in its geometrical *interpretation*, but denotes a *real and given right versor* (181).

And thus the *mean proportional* between two vectors (in the given plane) becomes, in *all* cases, *determined:* at least if their *order* (as first and third) be given.

(3.) If the *restriction* (225) on the *common plane* of the lines, were *removed*, we might *then*, on the recent plan (227), fix *definitely* the *direction*, as well as the *length*, of the *mean* OB, in *every* case *but one:* this excepted case being that in which, as in (2.), the two *given extremes*, OA, OC, have exactly opposite directions; so that the *angle* ($\text{AOC} = \pi$) between them has *no one definite bisector*. In this case, the sought point B would have *no one determined position*, but only a *locus:* namely the *circumference of a circle*, with O for centre, and with a radius equal to the geometric mean between $\overline{\text{OA}}$, $\overline{\text{OC}}$, while its plane would be perpendicular to the given right line AOC. (Comp. again the Figures 41; and the remarks in 148, 149, 153, 154, on the *square* of a *right radial*, or versor, and on the partially indeterminate character of the *square root* of a *negative scalar*, when interpreted, on the plan of this Calculus, as a *real* in geometry.)

228. The quotient of any two complanar and right quaternions has been seen (191, (6.)) to be a scalar; since then we here suppose (225) that $q \parallel\!| i$, we are at liberty to write,

$$\text{I.} \ldots \mathrm{S}q = x; \qquad \mathrm{V}q : i = y; \qquad \mathrm{V}q = yi = iy;$$

and consequently may establish the following *Reduction of a Quaternion in the given Plane* (of i) to a *Standard Binomial Form** (comp. 221):

$$\text{II.} \ldots q = x + iy, \quad \text{if} \quad q \parallel\!| i;$$

x and y being some *two scalars*, which may be called the *two constituents* (comp. again 221) of this binomial. And then an *equation* between *two* quaternions, considered as binomials of this form, such as the equation,

$$\text{III.} \ldots q' = q, \quad \text{or} \quad \text{III}'. \ldots x' + iy' = x + iy,$$

breaks up (by 202, (5.)) into *two scalar equations* between their respective *constituents*, namely,

$$\text{IV.} \ldots x' = x, \qquad y' = y,$$

notwithstanding the *geometrical reality* of the right versor, i.

(1.) On comparing the recent equations II., III., IV., with those marked as III., V., VI., in 221, we see that, in thus passing from *general* to *complanar* quaternions, we have merely *suppressed the coefficients of j and k*, as being for our present purpose, *null;* and have then written x and y, instead of w and x.

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* It is *permitted*, by 227, XI., to write this expression as $x + y\sqrt{-1}$; but the form $x + iy$ is shorter, and perhaps less liable to any ambiguity of interpretation.

(2.) As the word "binomial" has other meanings in algebra, it may be convenient to call the form II. a COUPLE; and the two constituent scalars x and y, of which the values serve to distinguish one such couple from another, may not unnaturally be said to be the *Co-ordinates* of that *Couple*, for a reason which it may be useful to state.

(3.) Conceive, then, that the plane of Fig. 50 coincides with that of i, and that positive rotation round Ax. i is, in that Figure, directed towards the *left-hand;* which may be reconciled with our general convention (127), by imagining that this *axis* of i is directed from O towards the *back* of the Figure; or *below** it, if horizontal. This being assumed, and perpendiculars BB′, BB″ being let fall (as in the Figure) on the indefinite line OA itself, and on a normal to that line at O, which normal we may call OA′, and may suppose it to have a length equal to that of OA, with a left-handed rotation AOA′, so that

$$\text{V.} \ldots \mathrm{OA}' = i . \mathrm{OA}, \quad \text{or briefly,} \quad \text{V}'. \ldots \alpha' = i\alpha,$$

while $\beta' = \mathrm{OB}'$, and $\beta'' = \mathrm{OB}''$, as in 201, and $q = \beta : \alpha$, as in 202;

then, on whichever side of the indefinite right line OA the point B may be situated, a comparison of the quaternion q with the binomial form II. will give the two equations,

$$\text{VI.} \ldots x\,(= \mathrm{S}q) = \beta' : \alpha; \qquad y\,(= \mathrm{V}q : i = \beta'' : i\alpha) = \beta'' : \alpha';$$

so that *these two scalars*, x and y, are precisely the *two rectangular co-ordinates of the point* B, *referred to the two lines* OA and OA′, *as two rectangular unit-axes*, of the *ordinary* (or Cartesian) kind. And since *every other* quaternion, $q' = x' + iy'$, in the given plane, can be reduced to the form $\gamma : \alpha$, or OC : OA, where C is a point in that plane, which can be projected into C′ and C″ in the same way (comp. 197, 205), we see that the two *new scalars*, or constituents, x' and y', are simply (for the same reason) the *co-ordinates of the new point* C, referred to the same pair of axes.

(4.) It is evident (from the principles of the foregoing Chapter), that if we thus express as *couples* (2.) *any two* complanar quaternions, q and q', we shall have the following general transformations for their *sum*, *difference*, and *product*:

$$\text{VII.} \ldots q' \pm q = (x' \pm x) + i\,(y' \pm y);$$

$$\text{VIII.} \ldots q' . q = (x'x - y'y) + i\,(x'y + y'x).$$

(5.) Again, for any *one* such couple, q, we have (comp. 222) not only $\mathrm{S}q = x$, and $\mathrm{V}q = iy$, as above, but also,

$$\text{IX.} \ldots \mathrm{K}q = x - iy; \qquad \text{X.} \ldots \mathrm{N}q = x^2 + y^2; \qquad \text{XI.} \ldots \mathrm{T}q = \sqrt{(x^2 + y^2)};$$

$$\text{XII.} \ldots \mathrm{U}q = \frac{x + iy}{\sqrt{(x^2 + y^2)}}; \qquad \text{XIII.} \ldots \frac{1}{q} = \frac{x - iy}{x^2 + y^2}; \text{ \&c.}$$

(6.) Hence, for the *quotient* of any two such couples, we have,

$$\text{XIV.} \ldots \begin{cases} \dfrac{q'}{q} = \dfrac{x' + iy'}{x + iy} = \dfrac{x'' + iy''}{x^2 + y^2}, \qquad x'' + iy'' = q'\mathrm{K}q, \\ \qquad x'' = x'x + y'y, \qquad y'' = y'x - x'y. \end{cases}$$

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* Compare the second Note to page 108.

(7.) The *law of the norms* (191, (8.)), or the formula, $Nq'q = Nq' . Nq$, is expressed here (comp. 222, (8.)) by the well-known algebraic equation, or identity,

$$\text{XV.} \ldots (x'^2 + y'^2)(x^2 + y^2) = (x'x - y'y)^2 + (x'y + y'x)^2 ;$$

in which $xyx'y'$ may be *any four scalars*.

SECTION 2.—*On Continued Proportion of Four or more Vectors; Whole Powers and Roots of Quaternions; and Roots of Unity.*

229. The conception of *continued proportion* (227) may easily be extended from the case of *three* to that of *four* or more (complanar) vectors; and thus a theory may be formed of *cubes* and *higher whole powers of quaternions*, with a correspondingly extended theory of *roots* of quaternions, including roots of *scalars*, and in particular of *unity*. Thus if we suppose that the *four* vectors $\alpha\beta\gamma\delta$ form a continued proportion, expressed by the formulæ,

$$\text{I.} \ldots \frac{\delta}{\gamma} = \frac{\gamma}{\beta} = \frac{\beta}{\alpha}, \quad \text{whence} \quad \text{II.} \ldots \frac{\delta}{\alpha} = \frac{\delta}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha} = \left(\frac{\beta}{\alpha}\right)^3,$$

(by an obvious extension of usual algebraic notation,) we may say that the quaternion $\delta : \alpha$ is *the cube*, or the *third power*, of $\beta : \alpha$; and that the latter quaternion is, conversely, *a cube-root* (or *third root*) of the former; which last relation may naturally be denoted by writing,

$$\text{III.} \ldots \frac{\beta}{\alpha} = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}}, \quad \text{or} \quad \text{III}'. \ldots \beta = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}}\alpha \,(\text{comp. } 227, \text{IV., V.}).$$

230. But it is important to observe that as the equation $q^2 = Q$, in which q is a sought and Q is a given quaternion, was found to be satisfied by *two* opposite quaternions q, of the form $\pm \sqrt{Q}$ (comp. 227, VII.), so the slightly less simple equation $q^3 = Q$ is satisfied by *three* distinct and real quaternions, if Q be actual and real; whereof *each*, divided by *either* of the other two, gives for *quotient* a *real* quaternion, which is equal to *one of the cube-roots of positive unity*. In fact, if we conceive (comp. the annexed Fig. 54) that β' and β'' are two other but equally long vectors in the given plane, ob-

tained from β by two successive and positive rotations, each through the third part of a circumference, so that

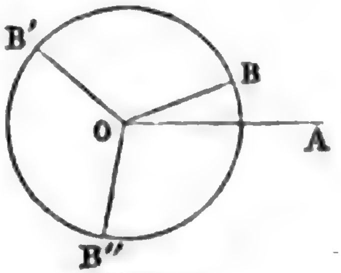

Fig. 54.

$$\text{IV.} \ldots \frac{\beta}{\beta''} = \frac{\beta''}{\beta'} = \frac{\beta'}{\beta},$$

or

$$\text{IV}'. \ldots \frac{\beta}{\beta'} = \frac{\beta'}{\beta''} = \frac{\beta''}{\beta},$$

and therefore

$$\text{V.} \ldots \left(\frac{\beta'}{\beta}\right)^3 = \left(\frac{\beta''}{\beta}\right)^3 = 1, \text{ while V}'. \ldots \frac{\beta''}{\beta} = \left(\frac{\beta'}{\beta}\right)^2, \ \frac{\beta'}{\beta} = \left(\frac{\beta''}{\beta}\right)^2,$$

we shall have

$$\text{VI.} \ldots \left(\frac{\beta'}{\alpha}\right)^3 = \left(\frac{\beta'}{\beta}\right)^3 \left(\frac{\beta}{\alpha}\right)^3 = \frac{\delta}{\alpha}, \quad \text{and} \quad \text{VI}'. \ldots \left(\frac{\beta''}{\alpha}\right)^3 = \frac{\delta}{\alpha};$$

so that we are equally entitled, at this stage, to write, instead of III. or III'., these other equations:

$$\text{VII.} \ldots \frac{\beta'}{\alpha} = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}}, \quad \beta' = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}} \alpha;$$

or

$$\text{VII}'. \ldots \frac{\beta''}{\alpha} = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}}, \quad \beta'' = \left(\frac{\delta}{\alpha}\right)^{\frac{1}{3}} \alpha.$$

231. A (real and actual) quaternion Q may thus be said to have *three* (real, actual, and) *distinct cube-roots;* of which however only *one* can have an *angle less than sixty degrees;* while *none* can have an angle *equal* to sixty degrees, unless the proposed quaternion Q degenerates into a *negative scalar*. In every *other* case, *one of the three* cube-roots of Q, or one of the three values of the symbol $Q^{\frac{1}{3}}$, may be considered as *simpler* than either of the other two, because it has a *smaller angle* (comp. 199, (1.)); and if we, for the present, *denote this one*, which we shall call the *Principal Cube-Root* of the quaternion Q, by the *symbol* $\sqrt[3]{Q}$, we shall thus be enabled to establish the formula of inequality,

$$\text{VIII.} \ldots \angle \sqrt[3]{Q} < \frac{\pi}{3}, \quad \text{if} \quad \angle Q < \pi.$$

232. At the limit, when Q degenerates, as above, into a negative scalar, *one* of its cube-roots is *itself* a negative scalar, and has there-

fore its angle $=\pi$; while *each* of the two other roots has its angle $=\frac{\pi}{3}$. In *this* case, among these two roots of which the angles are equal to each other, and are less than that of the third, we shall consider as *simpler*, and therefore as *principal*, the one which answers (comp. 227, (2.)) to a *positive rotation* through sixty degrees; and so shall be led to write,

$$\text{IX.} \ldots \sqrt[3]{-1} = \frac{1+i\sqrt{3}}{2}; \quad \text{and} \quad \text{X.} \ldots \angle \sqrt[3]{-1} = \frac{\pi}{3};$$

using thus the *positive sign* for the radical $\sqrt{3}$, by which i is multiplied in the expression IX. for $2\sqrt[3]{-1}$; with the connected formula,

$$\text{IX}' \ldots \sqrt[3]{(-a^3)} = \frac{a}{2}(1+i\sqrt{3}), \quad \text{if} \quad a>0;$$

although it might at first have seemed more natural to adopt as *principal* the *scalar* value, and to write thus,

$$\sqrt[3]{-1} = -1;$$

which latter is in fact *one value* of the symbol, $(-1)^{\frac{1}{3}}$.

(1.) We have, however, on the present plan, as in arithmetic,

$$\text{XI.} \ldots \sqrt[3]{1} = 1; \quad \text{and} \quad \text{XI}' \ldots \sqrt[3]{(a^3)} = a, \text{ if } a>0.$$

(2.) The equations,

$$\text{XII.} \ldots \left(\frac{1+i\sqrt{3}}{2}\right)^3 = -1, \qquad \text{and} \quad \text{XIII.} \ldots \left(\frac{-1+i\sqrt{3}}{2}\right)^3 = +1,$$

can be verified in *calculation*, by actual *cubing*, exactly as in algebra; the only difference being, as regards the *conception* of the subject, that although i satisfies the equation $i^2 = -1$, it is regarded *here* as altogether *real*; namely, as a real *right versor*[*] (181).

233. There is no difficulty in conceiving how the same general principles may be extended (comp. 229) to a *continued proportion* of $n+1$ complanar vectors,

$$\text{I.} \ldots \alpha, \alpha_1, \alpha_2, \ldots \alpha_n,$$

---

[*] *This conception differs fundamentally* from one which had occurred to several able writers, *before* the invention of the quaternions; and according to which the symbols 1 and $\sqrt{-1}$ were interpreted as representing *a pair of equally long and mutually rectangular right lines, in a given plane.* In *Quaternions, no line* is represented by the *number*, ONE, except as regards its *length*; the *reason* being, mainly, that we require, in the present Calculus, to be able to deal with *all possible planes*; and that *no one right line is common to all such.*

when n is a whole number greater than three; nor in interpreting, in connexion therewith, the equations,

$$\text{II.}\ .\ .\ \frac{a_n}{a} = \left(\frac{a_1}{a}\right)^n; \quad \text{III.}\ .\ .\ \frac{a_1}{a} = \left(\frac{a_n}{a}\right)^{\frac{1}{n}}; \quad \text{IV.}\ .\ .\ a_1 = \left(\frac{a_n}{a}\right)^{\frac{1}{n}} a.$$

Denoting, for the moment, what we shall call the *principal* n^{th} *root* of a quaternion Q by the symbol $\sqrt[n]{Q}$, we have, on this plan (comp. 231, VIII.),

$$\text{V.}\ .\ .\ \angle \sqrt[n]{Q} < \frac{\pi}{n}, \quad \text{if} \quad \angle Q < \pi;$$

$$\text{VI.}\ .\ .\ \angle (\sqrt[n]{-1}) = \frac{\pi}{n}; \qquad \text{VII.}\ .\ .\ \mathrm{V}(\sqrt[n]{-1}) : i > 0;$$

this last condition, namely that there shall be a *positive* (scalar) *coefficient* y of i, in the *binomial* (or *couple*) *form* $x + iy$ (228), for the quaternion $\sqrt[n]{-1}$, thus serving to complete the determination of that *principal* n^{th} root of *negative unity*; or of any *other* negative scalar, since -1 may be changed to $-a$, if $a > 0$, in each of the two last formulæ. And as to the *general* n^{th} *root of a quaternion*, we may write, on the same principles,

$$\text{VIII.}\ .\ .\ Q^{\frac{1}{n}} = 1^{\frac{1}{n}} . \sqrt[n]{Q};$$

where the factor $1^{\frac{1}{n}}$, representing the *general* n^{th} *root of positive unity*, has n *different values*, depending on the division of the circumference of a circle into n equal parts, in the way lately illustrated, for the case $n = 3$, by Figure 54; and only differing from ordinary algebra by the *reality* here attributed to i. In fact, *each* of these n^{th} *roots of unity* is with us a *real versor;* namely the *quotient of two radii of a circle*, which make with each other an *angle*, equal to the nth *part of some whole number of circumferences.*

(1.) We propose, however, to interpret the particular symbol $i^{\frac{1}{n}}$, as always denoting the *principal value* of the nth root of i; thus writing,

$$\text{IX.}\ .\ .\ i^{\frac{1}{n}} = \sqrt[n]{i};$$

whence it will follow that when this root is expressed under the form of a *couple* (228), the two constituents x and y shall both be positive, and the quotient $y : x$ shall have a smaller value than for any other couple $x + iy$ (with constituents thus positive), of which the nth power equals i.

(2.) For example, although the equation

$$q^2 = (x + iy)^2 = i,$$

is satisfied by the *two* values, $\pm (1 + i) : \sqrt{2}$, we shall write *definitely*,

$$\text{X.} \ldots i^{\frac{1}{2}} = +\sqrt{i} = \frac{1+i}{\sqrt{2}}.$$

(3.) And although the equation,

$$q^3 = (x + iy)^3 = i,$$

is satisfied by the *three* distinct and real couples, $(i \pm \sqrt{3}) : 2$, and $-i$, we shall adopt only the *one* value,

$$\text{XI.} \ldots i^{\frac{1}{3}} = \sqrt[3]{i} = \frac{i+\sqrt{3}}{2}.$$

(4.) In general, we shall thus have the expression,

$$\text{XII.} \ldots i^{\frac{1}{n}} = \cos\frac{\pi}{2n} + i\sin\frac{\pi}{2n};$$

which we shall occasionally *abridge* to the following:

$$\text{XII}'. \ldots i^{\frac{1}{n}} = \operatorname{cis}\frac{\pi}{2n};$$

and this *root*, $i^{\frac{1}{n}}$, thus interpreted, denotes a *versor*, which *turns* any line on which it operates, through an angle equal to the nth *part of a right angle*, in the positive direction of rotation, round the given axis of i.

234. If m and n be *any two positive whole numbers*, and q any quaternion, the definition contained in the formula 233, II., of the *whole power*, q^n, enables us to write, as in algebra, the two equations:

$$\text{I.} \ldots q^m q^n = q^{m+n}; \qquad \text{II.} \ldots (q^n)^m = q^{mn};$$

and we propose to extend the former to the case of *null* and *negative* whole *exponents*, writing therefore,

$$\text{III.} \ldots q^0 = 1; \qquad \text{IV.} \ldots q^{m-n} = q^m : q^n;$$

and in particular,

$$\text{V.} \ldots q^{-1} = 1 : q = \frac{1}{q} = \textit{reciprocal}^{*}\ (134) \text{ of } q.$$

We shall also extend the formula II., by writing

$$\text{VI.} \ldots (q^{\frac{1}{n}})^m = q^{\frac{m}{n}},$$

whether m be positive or negative; so that this last symbol, if m and n be still *whole* numbers, whereof n may be supposed to be *positive*, has *as many distinct values* as there are *units in the denominator* of its *fractional exponent*, when *reduced to its*

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* Compare the Note to page 121.

least terms; among which values of $q^{\frac{m}{n}}$, we shall naturally consider as the *principal* one, that which is the mth *power* of the principal n^{th} *root* (233) of q.

(1.) For example, the symbol $q^{\frac{2}{3}}$ denotes, on this plan, the *square of any cube-root* of q; it has therefore *three* distinct values, namely, the three values of the *cube-root of the square* of the same quaternion q; but among these we regard as principal, the *square of the principal cube-root* (231) of that proposed quaternion.

(2.) Again, the symbol $q^{\frac{2}{4}}$ is interpreted, on the same plan, as denoting the *square of any fourth root* of q; but because $(1^{\frac{1}{4}})^2 = 1^{\frac{1}{2}} = \pm 1$, this *square* has only *two* distinct values, namely those of the *square root* $q^{\frac{1}{2}}$, the fractional exponent $\frac{2}{4}$ being thus reduced to its least terms; and among these the *principal* value is the *square of the principal fourth root*, which square is, at the same time, the *principal square root* (199, (1.), or 227) of the quaternion q.

(3.) The symbol $q^{-\frac{1}{2}}$ denotes, as in algebra, the reciprocal of a square-root of q; while q^{-2} denotes the reciprocal of the square, &c.

(4.) If the exponent t, in a symbol of the form q^t, be still a *scalar*, but a *surd* (or incommensurable), we may consider this *surd exponent*, t, as a *limit*, towards which a *variable fraction tends:* and the symbol itself may then be interpreted as the corresponding limit of a *fractional power* of a quaternion, which has however (in this case) *indefinitely many values*, and can therefore be of little or no *use*, until a *selection* shall have been made, of *one* value of this *surd power* as *principal*, according to a law which will be best understood by the introduction of the conception of the *amplitude* of a quaternion, to which in the next Section we shall proceed.

(5.) Meanwhile (comp. 233), (4.)), we may already *definitely interpret* the symbol i^t as denoting a *versor*, which *turns* any line in the given plane, *through t right angles*, round Ax. i, in the positive or negative direction, according as this *scalar exponent*, t, whether rational or irrational, is itself positive or negative; and thus may establish the formula,

$$\text{VII.} \ldots \ i^t = \cos\frac{t\pi}{2} + i\sin\frac{t\pi}{2};$$

or briefly (comp. 233, XII′.),

$$\text{VIII.} \ldots \ i^t = \operatorname{cis}\frac{t\pi}{2}.$$

SECTION 3.—*On the Amplitudes of Quaternions in a given Plane; and on Trigonometric Expressions for such Quaternions, and for their Powers.*

235. Using the binomial or *couple form* (228) for a quaternion in the plane of i (225), if we introduce two new and real scalars, r and z, whereof the former shall be supposed to be positive, and which are connected with the two former scalars x and y by the equations,

$$\text{I.} \ldots \ x = r\cos z, \qquad y = r\sin z, \qquad r > 0,$$

we shall then evidently have the formulæ (comp. 228, (5.)) :

$$\text{II.} \ldots \mathrm{T}q = \mathrm{T}(x + iy) = r;$$

$$\text{III.} \ldots \mathrm{U}q = \mathrm{U}(x + iy) = \cos z + i \sin z;$$

which last expression may be conveniently abridged (comp. 233, XII'., and 234, VIII.) to the following:

$$\text{IV.} \ldots \mathrm{U}q = \operatorname{cis} z; \quad \text{so that} \quad \text{V.} \ldots q = r \operatorname{cis} z.$$

And the arcual or angular quantity, z, may be called the *Amplitude** of the quaternion q; this name being here preferred by us to "*Angle*," because we have already appropriated the latter name, and the corresponding symbol $\angle q$, to denote (130) an *angle of the Euclidean kind*, or at least one not exceeding, in either direction, the *limits* 0 and π; whereas the *amplitude*, z, considered as obliged only to satisfy the equations I., may have *any real and scalar value*. We shall *denote* this amplitude, at least for the present, by the *symbol*,† $\mathrm{am}\,.\,q$, or simply, $\mathrm{am}\, q$; and thus shall have the following formula, of *connexion between amplitude and angle*,

$$\text{VI.} \ldots (z =) \, \mathrm{am}\,.\,q = 2n\pi \pm \angle q;$$

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* Compare the Note to Art. 130.

† The symbol V was spoken of, in 202, as completing the *system of notations* peculiar to the present Calculus; and in fact, besides the *three letters*, i, j, k, of which the laws are expressed by the *fundamental formula* (A) of Art. 183, and which were originally (namely in the year 1843, and in the two following years) the *only peculiar symbols of quaternions* (see Note to page 160), that Calculus does not *habitually* employ, with peculiar significations, any more than the *five characteristics of operation*, K, S, T, U, V, for *conjugate*, *scalar*, *tensor*, *versor*, and *vector* (or *right part*): although perhaps the mark N for *norm*, which in the present work has been adopted from the *Theory of Numbers*, will gradually come more into use than it has yet done, in connexion with quaternions also. As to the marks, $\angle$, Ax., I, R, and now am. (or am_n), for *angle*, *axis*, *index*, *reciprocal*, and *amplitude*, they are to be considered as chiefly available for the present *exposition* of the system, and as not often wanted, nor employed, in the subsequent *practice* thereof; and the same remark applies to the recent *abridgment* cis, for cos + i sin; to some notations in the present Section for *powers* and *roots*, serving to express the conception of one n^{th} root, &c., as *distinguished* from another; and to the characteristic P, of what we shall call in the next section the *ponential* of a quaternion, though not requiring that notation afterwards. No apology need be made for employing the purely *geometrical signs*, $\perp$, $\|$, $|||$, for *perpendicularity*, *parallelism*, and *complanarity*: although the *last* of them was perhaps first introduced by the present writer, who has found it frequently useful.

the upper or the lower *sign* being taken, according as $\mathrm{Ax}.\, q = \pm \mathrm{Ax}.\, i$; and n being *any whole number*, positive or negative or null. We may then write also (for any quaternion $q \parallel\!| \, i$) the general transformations following:

$$\text{VII.} \ldots \mathrm{U}q = \mathrm{cis\,am}\, q; \qquad \text{VIII.} \ldots q = \mathrm{T}q \,.\, \mathrm{cis\,am}\, q.$$

(1.) Writing $q = \beta : \alpha$, the amplitude $\mathrm{am}.\,q$, or $\mathrm{am}\,(\beta : \alpha)$, is thus a scalar quantity, expressing (*with its proper sign*) the *amount of rotation*, round Ax. i, from the line α to the line β; and admitting, *in general*, of being increased or diminished by *any whole number of circumferences*, or of *entire revolutions*, when only the *directions of the two lines*, α and β, in the given plane of i, are given.

(2.) But the *particular* quaternion, or right versor, i *itself*, shall be considered as having *definitely*, for *its* amplitude, *one right angle;* so that we shall establish the particular formula,

$$\text{IX.} \ldots \mathrm{am}.\,i = \angle\, i = \frac{\pi}{2}.$$

(3.) When, for any *other* given quaternion q, the generally *arbitrary integer* n in VI. receives any one *determined value*, the corresponding value of the amplitude may be denoted by either of the two following temporary symbols,* which we here treat as equivalent to each other,

$$\mathrm{am}_n.\,q, \quad \text{or} \quad \angle_n\, q;$$

so that (with the same rule of signs as before) we may write, as a more *definite* formula than VI., the equation:

$$\text{X.} \ldots \mathrm{am}_n.\,q = \angle_n\, q = 2n\pi \pm \angle\, q;$$

and may say that this last quantity is the nth *value of the amplitude* of q; while the *zero-value*, $\mathrm{am}_0\, q$, may be called the *principal amplitude* (or the *principal value* of the amplitude).

(4.) With these notations, and with the convention, $\mathrm{am}_0\,(-1) = +\pi$, we may write,

$$\text{XI.} \ldots \mathrm{am}_0\, q = \angle_0\, q = \pm \angle\, q;$$

$$\text{XII.} \ldots \mathrm{am}_n\, a = \mathrm{am}_n\, 1 = \angle_n\, 1 = 2n\pi, \quad \text{if} \quad a > 0;$$

and

$$\text{XIII.} \ldots \mathrm{am}_n(-a) = \mathrm{am}_n(-1) = \angle_n(-1) = (2n+1)\pi,$$

if a be still a positive scalar.

236. From the foregoing definition of amplitude, and from the formerly established connexion of *multiplication of versors* with *composition of rotations* (207), it is obvious that (within the given *plane*, and with abstraction made of *tensors*) *multiplication and division of quaternions* answer respectively to

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* Compare the recent Note, respecting the *notations* employed.

(algebraical) *addition and subtraction of amplitudes:* so that, if the symbol $\mathrm{am}.q$ be interpreted in the *general* (or indefinite) sense of the equation 235, VI., we may write:

I. . . $\mathrm{am}(q'.q) = \mathrm{am}\,q' + \mathrm{am}\,q$; II. . . $\mathrm{am}(q':q) = \mathrm{am}\,q' - \mathrm{am}\,q$;

implying hereby that, in each formula, *one* of the values of the first member is *among* the values of the second member; but not here specifying *which* value. With the same generality of signification, it follows evidently that, for a *product* of *any number* of (complanar) quaternions, and for a *whole power* of any one quaternion, we have the analogous formulæ:

III. . . $\mathrm{am}\,\Pi q = \Sigma\,\mathrm{am}\,q$; IV. . . $\mathrm{am}.q^p = p.\mathrm{am}\,q$;

where the exponent p may be any positive or negative integer, or zero.

(1.) It was proved, in 191, II., that for *any two* quaternions, the formula $\mathrm{U}q'q = \mathrm{U}q'.\mathrm{U}q$ holds good; a result which, by the associative principle of multiplication (223), is easily extended to *any number* of quaternion factors (complanar or diplanar), with an analogous result for tensors: so that we may write, generally,

V. . . $\mathrm{U}\Pi q = \Pi\mathrm{U}q$; VI. . . $\mathrm{T}\Pi q = \Pi\mathrm{T}q$.

(2.) Confining ourselves to the first of these two equations, and combining it with III., and with 235, VII., we arrive at the important formula:

VII. . . $\Pi\,\mathrm{cis}\,\mathrm{am}\,q\ (= \Pi\mathrm{U}q = \mathrm{U}\Pi q = \mathrm{cis}\,\mathrm{am}\,\Pi q) = \mathrm{cis}\,\Sigma\,\mathrm{am}\,q$;

whence in particular (comp. IV.),

VIII. . . $(\mathrm{cis}\,\mathrm{am}\,q)^p = \mathrm{cis}(p\,.\,\mathrm{am}\,q)$,

at least if the exponent p be still any whole number.

(3.) In these last formulæ, the amplitudes $\mathrm{am}.q$, $\mathrm{am}.q'$, &c., may represent *any angular quantities*, z, z', &c.; we may therefore write them thus,

IX. . . $\Pi\,\mathrm{cis}\,z = \mathrm{cis}\,\Sigma z$; X. . . $(\mathrm{cis}\,z)^p = \mathrm{cis}\,pz$;

including thus, under *abridged forms*, some known and useful theorems, respecting *cosines and sines of sums and multiples* of arcs.

(4.) For example, if the number of factors of the form $\mathrm{cis}\,z$ be *two*, we have thus,

IX'. . . $\mathrm{cis}\,z'.\mathrm{cis}\,z = \mathrm{cis}(z'+z)$; X'. . . $(\mathrm{cis}\,z)^2 = \mathrm{cis}\,2z$;

whence

$$\cos(z'+z) = \mathrm{S}(\mathrm{cis}\,z'.\mathrm{cis}\,z) = \cos z' \cos z - \sin z' \sin z;$$
$$\sin(z'+z) = i^{-1}\mathrm{V}(\mathrm{cis}\,z'.\mathrm{cis}\,z) = \cos z' \sin z + \sin z' \cos z;$$
$$\cos 2z = (\cos z)^2 - (\sin z)^2; \qquad \sin 2z = 2\cos z \sin z;$$

with similar results for more factors than two.

(5.) Without expressly introducing the conception, or at least the notation of *amplitude*, we may derive the recent formulæ IX. and X., from the consideration of the *power* i^t (234), as follows. That *power of* i, with a *scalar exponent*, t, has been interpreted in 234, (5.), as a symbol satisfying an equation which may be written thus:

$$\text{XI.} \ldots i^t = \text{cis}\, z, \quad \text{if} \quad z = \tfrac{1}{2} t\pi;$$

or geometrically as a *versor*, which *turns a line through t right angles*, where t may be *any scalar*. We see then at once, *from this interpretation*, that if t' be either the same or any *other* scalar, the formula,

$$\text{XII.} \ldots i^t i^{t'} = i^{t+t'}, \quad \text{or} \quad \text{XIII.} \ldots \Pi . i^t = i^{\Sigma t},$$

must hold good, as in algebra. And because the number of the factors i^t is easily seen to be arbitrary in this last formula, we may write also,

$$\text{XIV.} \ldots (i^t)^p = i^{pt},$$

if p be any whole* number. But the two last formulæ may be changed by XI., to the equations IX. and X., which are therefore thus *again* obtained; although the later *forms*, namely XIII. and XIV., are perhaps somewhat *simpler*: having indeed the appearance of being mere *algebraical identities*, although we see that their *geometrical interpretations*, as given above, are important.

(6.) In connexion with the same interpretation XI. of the same useful symbol i^t, it may be noticed here that

$$\text{XV.} \ldots \text{K} . i^t = i^{-t};$$

and that therefore,

$$\text{XVI.} \ldots \cos \frac{t\pi}{2} = \text{S} . i^t = \tfrac{1}{2}(i^t + i^{-t});$$

$$\text{XVII.} \ldots \sin \frac{t\pi}{2} = i^{-1} \text{V} . i^t = \tfrac{1}{2} i^{-1} (i^t - i^{-t}).$$

(7.) Hence, by raising the double of each member of XVI. to any positive whole power p, halving, and substituting z for $\frac{1}{2}t\pi$, we get the equation,

$$\begin{aligned}\text{XVIII.} \ldots 2^{p-1}(\cos z)^p = \tfrac{1}{2}(i^t + i^{-t})^p &= \tfrac{1}{2}(i^{pt} + i^{-pt}) + \tfrac{1}{2}p\,(i^{(p-2)t} + i^{(2-p)t}) + \&\text{c.} \\ &= \cos pz + p \cos (p-2) z + \frac{p(p-1)}{2} \cos (p-4) z + \&\text{c.},\end{aligned}$$

with the usual rule for halving the coefficient of $\cos 0z$, if p be an even integer; and with analogous processes for obtaining the known expansions of $2^{p-1}(\sin z)^p$, for any positive whole value, even or odd, of p; and many other known results of the same kind.

237. If p be still a whole number, we have thus the transformation,

$$\text{I.} \ldots q^p = (r \,\text{cis}\, z)^p = r^p \,\text{cis}\, pz = (\text{T}q)^p \,\text{cis}\, (p . \text{am}\, q);$$

in which (comp. 190, 161) the two factors, of the tensor and versor kinds, may be thus written:

$$\text{II.} \ldots \text{T}(q)^p = (\text{T}q)^p = \text{T}q^p; \qquad \text{III.} \ldots \text{U}(q^p) = (\text{U}q)^p = \text{U}q^p;$$

and *any value* (235) of the amplitude $\text{am} . q$ may be taken, since *all* will conduct to one *common* value of this *whole power* q^p. And if, for I., we substitute this slightly different formula (comp. 235, (3.)),

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* It will soon be seen that there is a sense, although one not quite so *definite*, in which this formula holds good, even when the exponent p is fractional, or surd; namely, that the second member is then *one of the values* of the first.

$$\text{IV.} \ldots (q^p)_n = \mathrm{T}q^p . \operatorname{cis}(p . \operatorname{am}_n q), \text{ with } p = \frac{m'}{n'},\ n' > 0,$$

m', n', n being whole numbers whereof the first is supposed to be *prime* to the second, so that the *exponent* p is here a *fraction in its least terms*, with a *positive denominator* n', while the factor $\mathrm{T}q^p$ is interpreted as a *positive scalar* (of which the positive or negative logarithm, in any given system, is equal to $p \times$ the logarithm of $\mathrm{T}q$), then the expression in the second member admits of n' *distinct values*, answering to different values of n; which are precisely the n' values (comp. 234) of the *fractional power* q^p, on principles already established: the *principal* value of that power corresponding to the value $n = 0$.

(1.) For *any* value of the integer n, we may say that the symbol $(q^p)_n$, defined by the formula IV., represents the nth *value of the power* q^p; such values, however, *recurring* periodically, when p is, as above, a *fraction*.

(2.) Abridging $(1^p)_n$ to $1^p{}_n$, we have thus, *generally*, by 235, XII.,

$$\text{V.} \ldots 1^p{}_n = \operatorname{cis} 2pn\pi, \text{ if } p \text{ be any fraction,}$$

a restriction which however we shall soon remove; and in particular,

$$\text{VI.} \ldots \textit{Principal value of } 1^p = 1^p{}_0 = 1.$$

(3.) Thus, making successively $p = \frac{1}{2}$, $p = \frac{1}{3}$, we have

$$\text{VII.} \ldots 1^{\frac{1}{2}}{}_n = \operatorname{cis} n\pi, \quad 1^{\frac{1}{2}}{}_0 = +1, \quad 1^{\frac{1}{2}}{}_1 = -1, \quad 1^{\frac{1}{2}}{}_2 = +1, \ \&\text{c.};$$

$$\text{VIII.} \ldots 1^{\frac{1}{3}}{}_n = \operatorname{cis} \frac{2n\pi}{3}, \quad 1^{\frac{1}{3}}{}_0 = 1, \quad 1^{\frac{1}{3}}{}_1 = \frac{-1 + i\sqrt{3}}{2}, \quad 1^{\frac{1}{3}}{}_2 = \frac{-1 - i\sqrt{3}}{2}, \quad 1^{\frac{1}{3}}{}_3 = 1, \ \&\text{c.}$$

(4.) Denoting in like manner the nth value of $(-1)^p$ by the abridged symbol $(-1)^p{}_n$, we have, on the same plan (comp. 235, XIII.), for any fractional* value of p,

$$\text{IX.} \ldots (-1)^p{}_n = \operatorname{cis} p(2n+1)\pi; \text{ whence (comp. 232),}$$

$$\text{X.} \ldots (-1)^{\frac{1}{2}}{}_0 = \operatorname{cis}\frac{\pi}{2} = +i, \quad (-1)^{\frac{1}{2}}{}_1 = \operatorname{cis}\frac{3\pi}{2} = -i, \quad (-1)^{\frac{1}{2}}{}_2 = +i, \ \&\text{c.};$$

and

$$\text{XI.} \ldots (-1)^{\frac{1}{3}}{}_0 = \frac{1 + i\sqrt{3}}{2}, \quad (-1)^{\frac{1}{3}}{}_1 = -1, \quad (-1)^{\frac{1}{3}}{}_2 = \frac{1 - i\sqrt{3}}{2}, \ \&\text{c.},$$

these three values of $(-1)^{\frac{1}{3}}$ recurring periodically.

(5.) The formula IV. gives, generally, by V., the transformation,

$$\text{XII.} \ldots (q^p)_n = (q^p)_0 \operatorname{cis} 2pn\pi = 1^p{}_n (q^p)_0;$$

so that the nth *value* of q^p is equal to the *principal value* of that power of q, multiplied by the ***corresponding value*** of the ***same power of positive unity;*** and it may be remarked, that if the ***base*** a be any ***positive scalar***, the ***principal*** p^{th} ***power***, $(a^p)_0$, is simply, by our definitions, the *arithmetical value* of a^p.

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* As before, this restriction is only a temporary one.

(6.) The nth value of the pth power of any ***negative scalar***, $-a$, is in like manner equal to the ***arithmetical*** p^{th} ***power*** of the positive opposite, $+a$, multiplied by the corresponding value of the same power of ***negative unity;*** or in symbols,

$$\text{XIII.} \ldots (-a)^{p}{}_{n} = (-1)^{p}{}_{n}\,(a^p)_0 = (a^p)_0 \operatorname{cis} p\,(2n+1)\,\pi.$$

(7.) The formula IV., with its consequences V. VI. IX. XII. XIII., may be ***extended*** so as to include, ***as a limit***, the case when the ***exponent*** p being still ***scalar***, becomes ***incommensurable***, or surd; and although the ***number of values*** of the power q^p becomes thus ***unlimited*** (comp. 234, (4.)), yet we can still consider ***one*** of them as the ***principal value*** of this (now) ***surd power:*** namely the value,

$$\text{XIV.} \ldots (q^p)_0 = \mathrm{T}q^p \,.\, \operatorname{cis}(p \operatorname{am}_0 q),$$

which answers to the ***principal amplitude*** (235, (3.)) of the proposed quaternion q.

238. We may therefore consider the *symbol*,

$$q^p,$$

in which the *base*, q, is *any quaternion*, while the *exponent*, p, is *any scalar*, as being now fully *interpreted;* but no interpretation has been as yet assigned to this *other* symbol of the same kind,

$$q^{q'},$$

in which *both* the base q, and the exponent q', are supposed to be (generally) *quaternions*, although for the purposes of this Chapter *complanar* (225). To do this, in a way which shall be completely *consistent* with the foregoing conventions and conclusions, or rather which shall *include* and *reproduce* them, for the *case* where the new *quaternion exponent*, q', *degenerates* (131) *into a scalar*, will be one main object of the following Section: which however will also contain a theory of *logarithms of quaternions*, and of the *connexion* of both *logarithms* and *powers* with the properties of a certain function, which we shall call the *ponential* of a quaternion, and to consider which we next proceed.

SECTION 4.—*On the Ponential and Logarithm of a Quaternion; and on Powers of Quaternions, with Quaternions for their Exponents.*

239. If we consider the polynomial function,

$$\text{I.} \ldots \mathrm{P}(q,\, m) = 1 + q_1 + q_2 + \ldots q_m,$$

in which q is any quaternion, and m is any positive whole number, while it is supposed (for conciseness) that

$$\text{II.}\ \ldots\ q_m = \frac{q^m}{1.2.3..m} \left(= \frac{q^m}{\Gamma(m+1)}\right),$$

then it is not difficult to prove that *however great*, but *finite* and *given*, the *tensor* Tq may be, a *finite number* m can be assigned, for which the inequality

$$\text{III.}\ \ldots\ \mathrm{T}(\mathrm{P}(q,\ m+n) - \mathrm{P}(q,\ m)) < a, \quad \text{if} \quad a > 0,$$

shall be satisfied, *however large* the (positive whole) number n may be, and *however small* the (positive) scalar a, provided that this last is *given*. In other words, if we write (comp. 228),

$$\text{IV.}\ \ldots\ q = x + iy, \qquad \mathrm{P}(q,\ m) = X_m + iY_m,$$

a finite value of the number m can always be assigned, such that the following inequality,

$$\text{V.}\ \ldots\ (X_{m+n} - X_m)^2 + (Y_{m+n} - Y_m)^2 < a^2,$$

shall hold good, however large the number n, and however small (but given and >0) the scalar a may be. It follows evidently that *each of the two scalar series*, or succession of scalar functions,

$$\text{VI.}\ \ldots\ X_0 = 1, \qquad X_1 = 1 + x, \qquad X_2 = 1 + x + \frac{x^2 - y^2}{2}, \ldots \qquad X_m, \ldots$$

$$\text{VII.}\ \ldots\ Y_0 = 0, \qquad Y_1 = y, \qquad Y_2 = y + xy, \ldots \qquad Y_m, \ldots$$

converges ultimately to a fixed and finite limit, whereof the one may be called X_∞, or simply X, and the latter Y_∞, or Y, and of which each is a certain *function of the two scalars*, x and y. Writing then

$$\text{VIII.}\ \ldots\ Q = X_\infty + iY_\infty = X + iY,$$

we must consider this *quaternion* Q (namely the *limit* to which the following *series of quaternions*,

$$\text{IX.}\ldots \mathrm{P}(q, 0) = 1,\ \ \mathrm{P}(q, 1) = 1 + q,\ \ \mathrm{P}(q, 2) = 1 + q + \frac{q^2}{2}, \ldots\ \mathrm{P}(q, m), \ldots$$

converges ultimately) as being in like manner a certain *function*, which we shall call the *ponential function*, or simply *the Ponential of* q, in consequence of its possessing certain *exponential properties*; and which may be denoted by any one of the three symbols,

$$\mathrm{P}(q,\ \infty), \quad \text{or} \quad \mathrm{P}(q), \quad \text{or simply} \quad \mathrm{P}q.$$

We have therefore the equation,

$$\text{X.}\ \ldots\ \textit{Ponential of } q = Q = \mathrm{P}q = 1 + q_1 + q_2 + \ldots + q_\infty,$$

with the signification II. of the term q_m.

(1.) In connexion with the *convergence* of this *ponential series*, or with the inequality III., it may be remarked that if we write (comp. 235) $r = \mathrm{T}q$, and $r_m = \mathrm{T}q_m$, we shall have, by 212, (2.),

$$\text{XI.} \ldots \mathrm{T}(\mathrm{P}(q, m+n) - \mathrm{P}(q, m)) \leqq \mathrm{P}(r, m+n) - \mathrm{P}(r, m);$$

it is sufficient then to prove that this last difference, or the sum of the n positive terms, $r_{m+1}, \ldots r_{m+n}$, can be made $< a$. Now if we take a number $p > 2r - 1$, we shall have $r_{p+1} < \frac{1}{2} r_p$, $r_{p+2} < \frac{1}{2} r_{p+1}$, &c., so that a finite number $m > p > 2r - 1$ can be assigned, such that $r_m < a$; and then,

$$\text{XII.} \ldots \mathrm{P}(r, m+n) - \mathrm{P}(r, m) < a(2^{-1} + 2^{-2} + \ldots + 2^{-n}) < a;$$

the asserted inequality is therefore proved to exist.

(2.) In general, if an ascending series with positive coefficients, such as

$$\text{XIII.} \ldots \mathrm{A}_0 + \mathrm{A}_1 q + \mathrm{A}_2 q^2 + \&\text{c.}, \quad \text{where} \quad \mathrm{A}_0 > 0,\ \mathrm{A}_1 > 0,\ \&\text{c.},$$

be *convergent* when q is changed to a *positive scalar*, it will *à fortiori* converge, when q is a *quaternion*.

240. Let q and q' be any two complanar quaternions, and let q'' be their sum, so that

$$\text{I.} \ldots q'' = q' + q, \qquad q'' \parallel\!| q' \parallel\!| q;$$

then, as in algebra, with the signification 239, II. of q_m, and with corresponding significations of q'_m and q''_m, we have

$$\text{II.} \ldots q_m'' = \frac{(q'+q)^m}{1.2.3..m} = q'_m q_0 + q'_{m-1} q_1 + q'_{m-2} q_2 + \ldots + q'_0 q_m,$$

where $q_0 = q'_0 = 1$. Hence, writing again $r = \mathrm{T}q$, $r_m = \mathrm{T}q_m$, and in like manner $r' = \mathrm{T}q'$, $r'' = \mathrm{T}q''$, &c., the two differences,

$$\text{III.} \ldots \mathrm{P}(r', m).\mathrm{P}(r, m) - \mathrm{P}(r'', m),$$

and

$$\text{IV.} \ldots \mathrm{P}(r'', 2m) - \mathrm{P}(r', m).\mathrm{P}(r, m),$$

can be expanded as sums of positive terms of the form $r'_{p'}.r_p$ (one sum containing $\frac{1}{2}m(m+1)$, and the other containing $m(m+1)$ such terms); but, by 239, III., the *sum* of these two positive differences can be made less than any given small positive scalar a, since

$$\text{V.} \ldots \mathrm{P}(r'', 2m) - \mathrm{P}(r'', m) < a, \quad \text{if} \quad a > 0,$$

provided that the number m is taken large enough; *each* difference, therefore, separately *tends* to 0, as m tends to ∞; a tendency which must exist *à fortiori*, when the *tensors*, r, r', r'', are replaced by the *quaternions*, q, q', q''. The *function* $\mathrm{P}q$ is therefore subject to the *Exponential Law*,

$$\text{VI.} \ldots \mathrm{P}(q'+q) = \mathrm{P}q'.\mathrm{P}q = \mathrm{P}q.\mathrm{P}q', \quad \text{if} \quad q' \parallel\!| q.$$

(1.) *If* we write (comp. 237, (5.)),

$$\text{VII.} \ldots \mathrm{P}1 = \epsilon, \quad \text{then} \quad \text{VIII.} \ldots \mathrm{P}x = (\epsilon^x)_0 = \textit{arithmetical value of } \epsilon^x;$$

where ϵ is the known base of the natural system of logarithms, and x is any scalar. We shall henceforth write simply ϵ^x to denote this *principal* (or arithmetical) value of the xth power of ϵ, and so shall have the simplified equation,

$$\text{VIII}'. \ldots \mathrm{P}x = \epsilon^x.$$

(2.) Already we have thus a motive for writing, *generally*,

$$\text{IX.} \ldots \mathrm{P}q = \epsilon^q;$$

but this formula is *here* to be considered merely as a *definition* of the sense in which we *interpret* this *exponential symbol*, ϵ^q; namely as what we have lately called the *ponential function*, Pq, considered as the sum of the infinite but converging *series*, 239, X. It will however be soon seen to be *included* in a *more general definition* (comp. 238) of the symbol $q^{q'}$.

(3.) For any scalar x, we have by VIII. the transformation:

$$\text{X.} \ldots x = \mathrm{lP}x = \textit{natural logarithm of ponential of } x.$$

241. The exponential law (240) gives the following general *decomposition of a ponential into factors*,

$$\text{I.} \ldots \mathrm{P}q = \mathrm{P}(x + iy) = \mathrm{P}x . \mathrm{P}iy;$$

in which we have just seen that the factor Px is a positive scalar. The other factor, Piy, is easily proved to be a versor, and therefore to be *the versor of* Pq, while Px is the *tensor* of the same ponential; because we have in general,

$$\text{II.} \ldots \mathrm{P}q . \mathrm{P}(-q) = \mathrm{P}0 = 1, \quad \text{and} \quad \text{III.} \ldots \mathrm{PK}q = \mathrm{KP}q,$$

since $\text{IV.} \ldots (\mathrm{K}q)^m = \mathrm{K}(q^m) = (\text{say}) \ \mathrm{K}q^m$ (comp. 199, IX.);

and therefore, in particular (comp. 150, 158), ∴ $Nq = qKq$

$$\text{V.} \ldots 1 : \mathrm{P}iy = \mathrm{P}(-iy) = \mathrm{KP}iy, \quad \text{or} \quad \text{VI.} \ldots \mathrm{NP}iy = 1.$$ — Hence? [illegible]

We may therefore write (comp. 240, IX., X.),

$$\text{VII.} \ldots \mathrm{TP}q = \mathrm{PS}q = \mathrm{P}x = \epsilon^x; \qquad \text{VIII.} \ldots x = \mathrm{S}q = \mathrm{lTP}q;$$

$$\text{IX.} \ldots \mathrm{UP}q = \mathrm{PV}q = \mathrm{P}iy = \epsilon^{iy} = \text{cis}\, y \ \text{(comp. 235, IV.)};$$

this last transformation being obtained from the two series,

$$\text{X.} \ldots \mathrm{SP}iy = 1 - \frac{y^2}{2} + \&\text{c.} = \cos y;$$

$$\text{XI.} \ldots i^{-1}\,\mathrm{VP}iy = y - \frac{y^3}{2 . 3} + \&\text{c.} = \sin y.$$

Hence the ponential Pq may be thus transformed:

$$\text{XII.} \ldots \mathrm{P}q = \mathrm{P}(x + iy) = \epsilon^x \ \text{cis}\, y.$$

(1) If we had not chosen to *assume as known* the *series* for *cosine* and *sine*, nor to *select* (at first) any one *unit of angle*, such as that known one on which their validity depends, we might then have proceeded as follows. Writing

$$\text{XIII.}\ldots\ \mathrm{P}iy = fy + i\phi y, \qquad f(-y) = +fy, \qquad \phi(-y) = -\phi y,$$

we should have, by the exponential law (240),

$$\text{XIV.}\ldots\ f(y+y') = \mathrm{S}(\mathrm{P}iy\,.\,\mathrm{P}iy') = fy\,.\,fy' - \phi y\,.\,\phi y';$$

$$\text{XV.}\ldots\ f(y-y') = \qquad fy\,.\,fy' + \phi y\,.\,\phi y';$$

and then the *functional equation*, which results, namely,

$$\text{XVI.}\ldots\ f(y+y') + f(y-y') = 2fy\,.\,fy',$$

would show that

$$\text{XVII.}\ldots\ fy = \cos\left(\frac{y}{c} \times \textit{a right angle}\right),$$

whatever unit of angle may be adopted, provided that we determine the *constant* c by the condition,

$$\text{XVIII.}\ldots\ c = \textit{least positive root of the equation } fy\,(= \mathrm{SP}iy) = 0\,;$$

or nearly,

$$\text{XVIII}'.\ldots\ c = 1{\cdot}5708, \text{ as the study of the series* would show.}$$

(2.) A *motive* would thus arise for *representing a right angle by this numerical constant, c;* or for *so selecting* the *angular unit*, as to have the equation (π still denoting two right angles),

$$\text{XIX.}\ldots\ \pi = 2c = \textit{least positive root of the equation } fy = -1\,;$$

giving nearly,

$$\text{XIX}'.\ldots\ \pi = 3{\cdot}14159, \text{ as usual;}$$

for thus we should reduce XVII. to the simpler form,

$$\text{XX.}\ldots\ fy = \cos y.$$

(3.) As to the function ϕy, since

$$\text{XXI.}\ldots\ (fy)^2 + (\phi y)^2 = \mathrm{P}iy\,.\,\mathrm{P}(-iy) = 1,$$

it is evident that $\phi y = \pm \sin y$; and it is easy to prove that the upper sign is to be taken. In fact, it can be shown (without supposing any previous knowledge of cosines or sines) that ϕc is positive, and therefore that

$$\text{XXII.}\ldots\ \phi c = +1, \quad \text{or} \quad \text{XXIII.}\ldots\ \mathrm{P}ic = i\,;$$

whence

$$\text{XXIV.}\ldots\ \phi y = \mathrm{S}\,.\,i^{-1}\mathrm{P}iy = \mathrm{SP}i(y-c) = f(y-c),$$

and

$$\text{XXV.}\ldots\ \mathrm{P}iy = fy + if(y-c).$$

If then we replace c by $\frac{\pi}{2}$, we have

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* In fact, the value of the constant c may be obtained to *this* degree of accuracy, by simple interpolation between the two approximate values of the function f,

$$f(1{\cdot}5) = +\,0{\cdot}070737, \qquad f(1{\cdot}6) = -\,0{\cdot}029200\,;$$

and of course there are artifices, not necessary to be mentioned here, by which a far more accurate value can be found.

$$\text{XXVI.} \ldots \phi y = \cos\left(y - \frac{\pi}{2}\right) = \sin y; \quad \text{and} \quad \text{XXVII.} \ldots \mathrm{P}iy = \operatorname{cis} y, \text{ as in IX.}$$

(4.) The series X. XI. for cosine and sine might thus be *deduced*, instead of being *assumed* as known: and since we have the limiting value,

$$\text{XXIX.} \ldots \lim_{y=0} .\, y^{-1} \sin y = \lim_{y=0} .\, y^{-1} i^{-1}\, \mathrm{VP}iy = 1,$$

it follows that the *unit of angle*, which thus gives $\mathrm{P}iy = \operatorname{cis} y$, is (as usual) the angle subtended at the centre by the *arc equal to radius*; or that the *number* π (or $2c$) is to 1, as the *circumference* is to the *diameter* of a circle.

(5.) If any *other angular unit* had been, for any reason, chosen, then a *right angle* would of course be represented by a *different number*, and not by 1·5708 nearly; but we should *still* have the *transformation*,

$$\text{XXX.} \ldots \mathrm{P}iy = \operatorname{cis}\left(\frac{y}{c} \times \text{a right angle}\right),$$

though *not* the same *series* as before, for $\cos y$ and $\sin y$.

242. The usual unit being retained, we see, by 241, XII., that

$$\text{I.} \ldots \mathrm{P}.\, 2in\pi = 1, \quad \text{and} \quad \text{II.} \ldots \mathrm{P}(q + 2in\pi) = \mathrm{P}q,$$

if n be any whole number; it follows, then, that the *inverse ponential function*, $\mathrm{P}^{-1}q$, or what we may call the *Imponential*, of a given quaternion q, has *indefinitely many values*, which may all be represented by the formula,

$$\text{III.} \ldots \mathrm{P}_n^{-1}q = \mathrm{lT}q + i\, \mathrm{am}_n\, q;$$

and of which *each* satisfies the equation,

$\therefore \mathrm{P}\mathrm{P}_n^{-1}q = \mathrm{P}(x + iy)$

$$\text{IV.} \ldots \mathrm{P}\mathrm{P}_n^{-1}q = q;$$

$= \epsilon^{x} \operatorname{cis} y$

while the one which corresponds to $n = 0$ may be called the *Principal Imponential*. It will be found that when the *exponent* p is *any scalar*, the definition already given (237, IV., XII.) for the nth *value* of the pth *power* of q enables us to establish the formula,

$$\text{V.} \ldots (q^p)_n = \mathrm{P}(p\mathrm{P}_n^{-1}q);$$

and we now propose to *extend* this last formula, by a *new definition*, to the *more general case* (238), when the *exponent* is a *quaternion* q': thus writing generally, for *any two complanar quaternions*, q and q', the *General Exponential Formula*,

$$\text{VI.} \ldots (q^{q'})_n = \mathrm{P}(q'\mathrm{P}_n^{-1}q);$$

the *principal value* of $q^{q'}$ being still conceived to correspond to $n = 0$, or to the *principal amplitude* of q (comp. 235, (3.)).

(1.) For example,

$$\text{VII.}\ldots(\epsilon^q)_0 = \mathrm{P}(q\mathrm{P}_0^{-1}\epsilon) = \mathrm{P}q, \quad \text{because} \quad \mathrm{P}_0^{-1}\epsilon = l\epsilon = 1;$$

the *ponential* Pq, which we agreed, in 240, (2.), to denote simply by ϵ^q, is therefore now seen to be in fact, by our general definition, the ***principal value of that power***, or exponential.

(2.) With the same notations,

$$\text{VIII.}\ldots\epsilon^{iy} = \operatorname{cis} y, \qquad \cos y = \tfrac{1}{2}(\epsilon^{iy} + \epsilon^{-iy}), \qquad \sin y = \frac{1}{2i}(\epsilon^{iy} - \epsilon^{-iy});$$

these two last only differing from the usual imaginary expressions for cosine and sine, by the geometrical *reality*[*] of the versor i.

(3.) The ***cosine and sine of a quaternion*** (in the given plane) may now be ***defined*** by the equations:

$$\text{IX.}\ldots\cos q = \tfrac{1}{2}(\epsilon^{iq} + \epsilon^{-iq}); \qquad \text{X.}\ldots\sin q = \frac{1}{2i}(\epsilon^{iq} - \epsilon^{-iq});$$

and we may write (comp. 241, IX.),

$$\text{XI.}\ldots\operatorname{cis} q = \epsilon^{iq} = \mathrm{P}iq.$$

(4.) With this interpretation of cis q, the exponential properties, 236, IX., X., continue to hold good; and we may write,

$$\text{XII.}\ldots(q^{q'})_n = \mathrm{P}(q'l\mathrm{T}q).\mathrm{P}(iq'\operatorname{am}_n q) = (\mathrm{T}q)_0{}^{q'}\operatorname{cis}(q'\operatorname{am}_n q);$$

a formula which evidently includes the corresponding one, 237, IV., for the nth value of the pth power of q, when p is scalar.

(5.) The definitions III. and VI., combined with 235, XII., give generally,

$$\text{XIII.}\ldots 1_n{}^{q'} = (1^{q'})_n = \mathrm{P}.2in\pi q'; \qquad \text{XIV.}\ldots(q^{q'})_n = 1_n{}^{q'}.(q^{q'})_0;$$

this last equation including the formula 237, XII.

(6.) The same definitions give,

$$\text{XV.}\ldots\mathrm{P}_0^{-1}i = \frac{i\pi}{2}; \qquad \text{XVI.}\ldots(i^i)_0 = \epsilon^{-\frac{\pi}{2}};$$

which last equation agrees with a known interpretation of the symbol,

$$\sqrt{-1}^{\sqrt{-1}},$$

considered as denoting in algebra a real quantity.

(7.) The formula VI. may even be extended to the case where the *exponent* q' is a *quaternion*, which is ***not in the given plane of*** i, and therefore ***not complanar with the base*** q; thus we may write,

$$\text{XVII.}\ldots(i^j)_0 = \mathrm{P}(j\mathrm{P}_0^{-1}i) = \mathrm{P}\left(-\frac{k\pi}{2}\right) = -k;$$

but it would be foreign (225) to the plan of this Chapter to enter into any further details, on the subject of the interpretation of the exponential symbol $q^{q'}$, for this case of *diplanar quaternions*, though we see that there would be no difficulty in treating it, after what has been shown respecting *complanars*.

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[*] Compare 232, (2.), and the Notes to pages 243, 248.

243. As regards the *general logarithm* q' *of a quaternion* q (in the given plane), we may regard it as any quaternion which satisfies the equation,

$$\text{I.} \ldots \epsilon^{q'} = \mathrm{P}q' = q;$$

and in this view it is simply the *Imponential* $\mathrm{P}^{-1}q$, of which the nth value is expressed by the formula 242, III. But the *principal imponential*, which answers (as above) to $n = 0$, may be said to be the *principal logarithm*, or simply *the Logarithm*, of the quaternion q, and may be denoted by the symbol,

$$\mathrm{l}q;$$

so that we may write,

$$\text{I.} \ldots \mathrm{l}q = \mathrm{P}_0^{-1}q = \mathrm{lT}q + i\,\mathrm{am}_0\,q;$$

or still more simply,

$$\text{II.} \ldots \mathrm{l}q = \mathrm{l}(\mathrm{T}q\,.\,\mathrm{U}q) = \mathrm{lT}q + \mathrm{lU}q,$$

because $\mathrm{lTU}q = \mathrm{l}1 = 0$, and therefore,

$$\text{III.} \ldots \mathrm{lU}q = i\,\mathrm{am}_0\,q.$$

We have thus the two general equations,

$$\text{IV.} \ldots \mathrm{Sl}q = \mathrm{lT}q; \qquad \text{V.} \ldots \mathrm{Vl}q = \mathrm{lU}q;$$

in which $\mathrm{lT}q$ is still the scalar and natural logarithm of the positive scalar $\mathrm{T}q$.

(1.) As examples (comp. 235, (2.) and (4.)),

$$\text{VI.} \ldots \mathrm{l}i = \tfrac{1}{2}i\pi; \qquad \text{VII.} \ldots \mathrm{l}(-1) = i\pi.$$

(2.) The *general logarithm* of q may be denoted by any one of the symbols,

$$\log . q, \text{ or } \log q, \text{ or } (\log q)_n,$$

this last denoting the nth *value;* and then we shall have,

$$\text{VIII.} \ldots (\log q)_n = \mathrm{l}q + 2in\pi.$$

(3.) The formula,

$$\text{IX.} \ldots \log . q'q = \log q' + \log q, \quad \text{if} \quad q' \parallel\!| \, q,$$

holds good, in the sense that *every* value of the first member is *one* of the values of the second (comp. 236).

(4.) *Principal value of* $q^{q'} = \epsilon^{q'\mathrm{l}q}$; and *one* value of $\log . q^{q'} = q'\mathrm{l}q$.

(5.) The *quotient* of *two* general logarithms,

$$\text{X.} \ldots (\log q')_{n'} : (\log q)_n = \frac{\mathrm{l}q' + 2in'\pi}{\mathrm{l}q + 2in\pi},$$

may be said to be the *general logarithm of the quaternion, q', to the complanar quaternion base, q*; and we see that its expression involves[*] *two arbitrary and independent integers*, while its *principal value* may be defined to be $\mathrm{l}q' : \mathrm{l}q$.

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[*] As the corresponding expression in algebra, according to Graves and Ohm.

SECTION 5.—*On Finite* (or Polynomial) Equations of Algebraic Form, involving Complanar Quaternions; and on the Existence of* n *Real Quaternion Roots, of any such Equation of the* nth *Degree.*

244. We have seen (233) that an equation of the form,

$$\text{I.} \ldots q^n - Q = 0,$$

where n is any given positive integer, and Q is any† given, real, and actual quaternion (144), has always *n real, actual, and unequal quaternion roots*, q, complanar with Q; namely, the n distinct and real values of the symbol $Q^{\frac{1}{n}}$ (233, VIII.), determined on a plan lately laid down. This result is, however, included in a much more general *Theorem*, respecting *Quaternion Equations of Algebraic Form*; namely, that *if q_1, q_2, .. q_n be any n given, real, and complanar quaternions,* then *the equation,*

$$\text{II.} \ldots q^n + q_1 q^{n-1} + q_2 q^{n-2} + \ldots + q_n = 0,$$

has always n real quaternion roots, q', q'', .. $q^{(n)}$, and no more in the given plane; of which roots it is possible however that some, or all may become *equal*, in consequence of certain *relations* existing between the n given *coefficients*.

245. As another statement of the same Theorem, if we write,

$$\text{I.} \ldots \mathrm{F}_n q = q^n + q_1 q^{n-1} + \ldots + q_n,$$

the coefficients $q_1 \ldots q_n$ being as before, we may say that *every such polynomial function, $\mathrm{F}_n q$, is equal to a product of n real, complanar, and linear* (or *binomial*) *factors*, of the form $q - q'$; or that an equation of the form,

$$\text{II.} \ldots \mathrm{F}_n q = (q - q')(q - q'') \ldots (q - q^{(n)}),$$

can be *proved* in all cases to *exist:* although we may not be able, with our present methods, to *assign expressions for the roots*, $q', \dots q^{(n)}$, *in terms of the coefficients* $q_1, \dots q_n$.

---

* By saying *finite equations*, we merely intend to exclude here equations with *infinitely many terms*, such as $\mathrm{P}q = 1$, which has been seen (242) to have *infinitely many roots*, represented by the expression $q = 2in\pi$, where n may be any whole number.

† It is true that we have supposed $Q \mathrel{|||} i$ (225); but nothing hinders us, in any other case, from substituting for i the versor $\mathrm{UV}Q$, and then proceeding as before.

246. Or we may say that there is always a certain *system of n real quaternions*, q', &c., $|||\, i$, which satisfies the *system of equations*, of known algebraic form,

$$\text{III} \;..\; \begin{cases} q' + q'' + .. + q^{(n)} = -q_1; \\ q'q'' + q'q''' + q''q''' + .. = +q_2; \\ q'q''q''' + .. = -q_3; \ \&\text{c.} \end{cases}$$

247. Or because the difference $\mathrm{F}_n q - \mathrm{F}_n q'$ is *divisible* by $q - q'$, as in algebra, under the supposed conditions of *complanarity* (224), it is sufficient to say that *at least one real quaternion q' always exists* (whether we can *assign* it or not), *which satisfies the equation*,

$$\text{IV.} \;..\; \mathrm{F}_n q' = 0,$$

with the foregoing form (245, I.) of the polynomial function F.

248. Or finally, because the theorem is evidently true for the case $n = 1$, while the case 244, I., has been considered, and the case $q_n = 0$ is satisfied by the supposition $q = 0$, we may, without essential loss of generality, reduce the enunciation to the following:

Every equation of the form,*

$$\text{I.} \;..\; q(q - q')(q - q'')\,..\,(q - q^{(n-1)}) = Q,$$

in which $q', q'', ..$ and Q are any n real and given quaternions in the given plane, whereof at least Q and q' may be supposed *actual* (144), *is satisfied by at least one real, actual, and complanar quaternion*, q.

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* The corresponding *form*, of the *algebraical equation of the* n^{th} *degree*, was proposed by Mourey, in his very ingenious and original little work, entitled *La vraie théorie des Quantités Négatives, et des Quantités prétendues Imaginaires* (Paris, 1828). *Suggestions* also, towards the *geometrical proof* of the theorem in the text have been taken from the same work; in which, however, the *curve* here called (in 251) an *oval* is not perhaps defined with sufficient precision: the *inequality*, here numbered as 251, XII., being not employed. It is to be observed that Mourey's book contains *no hint* of the *present calculus*, being *confined*, like the *Double Algebra* of Prof. De Morgan (London, 1849), and like the earlier work of Mr. Warren (Cambridge, 1828), to questions *within the plane:* whereas the very *conception* of the *Quaternion* involves, as we have seen, a reference to *Tridimensional* SPACE.

249. Supposing that the $m-1$ last of the $n-1$ given quaternions $q' \,.\,. q^{(n-1)}$ vanish, but that the $n-m$ first of them are actual, where m may be any whole number from 1 to $n-1$, and introducing a new real, known, complanar, and actual quaternion q_0, which satisfies the condition,

$$\text{II.} \ldots q_0{}^m = \frac{Q}{q'q'' \,.\,. q^{(n-m)}},$$

we may write thus the recent equation I.,

$$\text{III.} \,.\,. fq = \left(\frac{q}{q_0}\right)^m \left(\frac{q}{q'} - 1\right)\left(\frac{q}{q''} - 1\right) \,.\,. \left(\frac{q}{q^{(n-m)}} - 1\right) = 1;$$

and may (by 187, 159, 235) decompose it into the two following:

$$\text{IV.} \ldots \mathrm{T}fq = 1; \quad \text{and} \quad \text{V.} \ldots \mathrm{U}fq = 1, \quad \text{or} \quad \text{VI.} \ldots \mathrm{am}\, fq = 2p\pi;$$

in which p is some whole number (negatives and zero included).

250. To give a more *geometrical form* to the equation, let λ be any given or assumed line $|||\, i$, and let it be supposed that α, β, .. and ρ, σ, or OA, OB, ... and OP, OS, are $n-m+2$ other lines in the same planes, and that $\phi\rho$ is a known scalar function of ρ, such that

$$\text{VII.} \ldots \alpha = q'\lambda, \qquad \beta = q''\lambda, \,.\,. \qquad \rho = q\lambda, \qquad \sigma = q_0\lambda,$$

and

$$\text{VIII.} \ldots \phi\rho = fq = \left(\frac{\rho}{\sigma}\right)^m \cdot \frac{\rho - \alpha}{\alpha} \cdot \frac{\rho - \beta}{\beta} \cdots = \left(\frac{\mathrm{OP}}{\mathrm{OS}}\right)^m \cdot \frac{\mathrm{AP}}{\mathrm{OA}} \cdot \frac{\mathrm{BP}}{\mathrm{OB}} \cdots;$$

the theorem to be proved may then be said to be, that *whatever system of real points*, O, A, B, .. and S, in a given plane, and *whatever positive whole number m*, *may be assumed*, or given, *there is always at least one real point* P, in the same plane, *which satisfies the two conditions:*

$$\text{IX.} \ldots \mathrm{T}\phi\rho = 1; \qquad \text{X.} \ldots \mathrm{am}\, \phi\rho = 2p\pi.$$

251. Whatever value $\iota \,|||\, i$ we may assume for the *versor* (or unit-vector) $\mathrm{U}\rho$, there always exists *at least one* value of the *tensor* $\mathrm{T}\rho$, which satisfies the condition IX.; because the function $\mathrm{T}\phi\rho$ vanishes with $\mathrm{T}\rho$, and becomes infinite when $\mathrm{T}\rho = \infty$, having varied continuously (although perhaps with fluctuations) in the interval. Attending then only to the *least value* (if there be more than one) of $\mathrm{T}\rho$, which thus renders $\mathrm{T}\phi\rho$ equal to unity, we can conceive a real, unambiguous, and scalar function $\psi\iota$, which shall have the two following properties:

$$\text{XI.} \ldots \mathrm{T}\phi(\iota\psi\iota) = 1; \qquad \text{XII.} \ldots \mathrm{T}\phi(x\iota\psi\iota) < 1, \text{ if } x > 0, < 1.$$

And in this way the equation, or system of equations,

$$\text{XIII.} \ldots \rho = \iota\psi\iota, \quad \text{or} \quad \text{XIV.} \ldots \mathrm{U}\rho = \iota, \quad \mathrm{T}\rho = \psi\iota,$$

may be conceived to *determine a real, finite, and plane closed curve*, which we shall call generally an *Oval*, and which shall have the two following properties: Ist, *every right line*, or *ray*, drawn *from the origin* O, in any arbitrary direction within the plane, *meets the curve once*, but *once only;* and IInd, *no one* of the $n - m$ *other given points* A, B, .. *is on the oval*, because $\phi\alpha = \phi\beta = .. = 0$.

252. This being laid down, let us conceive a point P to perform *one circuit* of the oval, moving in the *positive* direction relatively to the given interior point O; so that, whatever the given direction of the line OS may be, the *amplitude* $\mathrm{am}(\rho : \sigma)$, if supposed to vary *continuously*,* will have *increased by four right angles*, or by 2π, in the course of this *one positive circuit;* and consequently, the amplitude of the left-hand factor $(\rho : \sigma)^m$, of $\phi\rho$, will have increased, at the same time, by $2m\pi$. Then, if the point A be also *interior* to the oval, so that the line OA must be prolonged to meet that curve, the ray AP will have likewise made *one* positive revolution, and the amplitude of the factor $(\rho - \alpha) : \alpha$ will have increased by 2π. But if A be an *exterior* point, so that the *finite line* OA *intersects* the curve in a point M, and therefore *never meets it again* if prolonged, although the prolongation of the *opposite line* AO must meet it *once* in some point N, then while the *point* P performs first what we may call the positive *half-circuit* from M to N, and afterwards the *other* positive half-circuit from N to M *again*, the *ray* AP has only *oscillated* about its initial and final direction, namely that of the line AO, without ever attaining the *opposite direction;* in *this* case, therefore, the amplitude $\mathrm{am}(\mathrm{AP} : \mathrm{OA})$, if still supposed to vary *continuously*, has only *fluctuated* in its value, and has (upon the whole) undergone *no change* at all. And since precisely similar remarks apply to the other given points, B, &c., it follows that the amplitude, $\mathrm{am}\,\phi\rho$, of the *product* (VIII.) of all these factors, has (by 236) received a *total increment* $= 2(m + t)\pi$, if t be the *number* (perhaps zero) *of given internal points*, A, B, ..; while the number m is (by 249) *at least* $= 1$. Thus, while P performs (as above) *one positive circuit*, *the amplitude* $\mathrm{am}\,\phi\rho$ *has passed at least m times*, and therefore *at least once, through a value of the form* $2p\pi$; and consequently the *condition* X. has been *at least once satisfied.* But the *other* condition, IX., is satisfied *throughout*, by the

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* That is, so as not to receive any *sudden* increment, or decrement, of one or more whole circumferences (comp. 235, (1.)).

supposed *construction* of the oval: there is therefore *at least one real position* P, upon that curve, for which $\phi\rho$ or $fq = 1$; so that, *for this position* of that point, the equation 249, III., and therefore also the equation 248, I., is satisfied. The *theorem* of Art. 248, and consequently also, by 247, the theorem of 244, with its transformations 245 and 246, is therefore in this manner *proved.*

253. This conclusion is so important, that it may be useful to illustrate the general reasoning, by applying it to the case of a *quadratic equation*, of the form,

$$\text{I.} \ldots fq = \frac{q}{q_0}\left(\frac{q}{q'} - 1\right) = 1; \text{ or II.} \ldots \phi\rho = \frac{\rho}{\sigma}\left(\frac{\rho}{\alpha} - 1\right) = \frac{\text{OP}}{\text{OS}} \cdot \frac{\text{AP}}{\text{OA}} = 1.$$

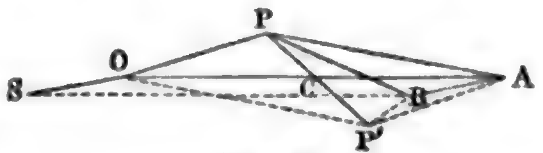

Fig. 55.

We have now to prove (comp. 250, VIII.) that a (real) point P exists, which renders the fourth proportional (226) to the three lines OA, OP, AP equal to a given line OS, or AB, if this latter be drawn = OS; or which satisfies the following condition of similarity of triangles (118),

$$\text{III.} \ldots \Delta\,\text{AOP} \propto \text{PAB};$$

which *includes* the equation of rectangles,

$$\text{IV.} \ldots \overline{\text{OP}}\cdot\overline{\text{AP}} = \overline{\text{OA}}\cdot\overline{\text{AB}}.$$

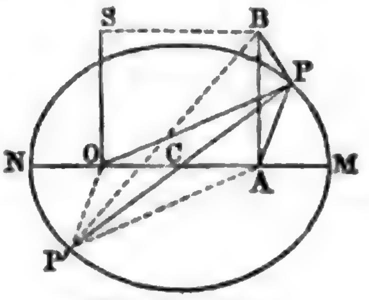

Fig. 55, *bis.*

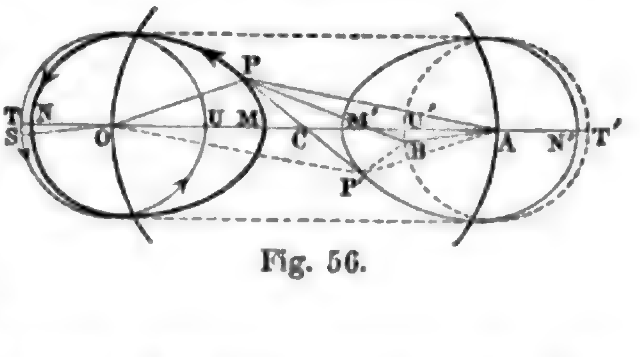

Fig. 56.

(Compare the annexed Figures, 55, and 55, *bis.*) Conceive, then, that a continuous curve* is described as a *locus* (or as *part* of the locus) of P, by means of this equality IV., with the additional condition when necessary, that O shall be *within* it; in such a manner that when (as in Fig. 56) a right line from O meets the general or total *locus* in *several* points, M,

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* This curve of the fourth degree is the well-known *Cassinian;* but when it breaks up, as in Fig. 56, into *two separate ovals*, we *here retain*, as *the oval of the proof*, *only* the one round O, *rejecting* for the present that round A.

M′, N′, we *reject* all but the point M which is *nearest* to O, as not belonging (comp. 251, XII.) to the *oval here considered.* Then while P moves upon *that* oval, in the positive direction relatively to O, from M to N, and from N to M again, so that the ray OP performs one positive revolution, and the amplitude of the factor OP : OS increases continuously by 2π, the ray AP performs in like manner one positive revolution, or (on the whole) does not revolve at all, and the amplitude of the factor AP : OA increases by 2π or by 0, according as the point A is *interior* or *exterior* to the oval. In the one case, therefore, the amplitude $\mathrm{am}\,\phi\rho$ of the *product* increases by 4π (as in Fig. 55, *bis*); and in the other case, it increases by 2π (as in Fig. 56); so that in *each* case, it passes *at least once* through a value of the *form* $2p\pi$, whatever its initial value may have been. Hence, *for at least one real position*, P, upon the oval, we have

$$\text{V.} \ldots \mathrm{am}\,\phi\rho = 1, \quad \text{and therefore} \quad \text{VI.} \ldots \mathrm{U}\phi\rho = 1;$$

but

$$\text{VII.} \ldots \mathrm{T}\phi\rho = 1,$$

throughout, by the *construction*, or by the equation of the *locus* IV.; the *geometrical condition* $\phi\rho = 1$ (II.) is therefore satisfied by *at least one real vector* ρ; and consequently the *quadratic equation* $fq = 1$ (I.) is satisfied by *at least one real quaternion root*, $q = \rho : \lambda$ (250, VII.). But the recent form I. has the same generality as the earlier form,

$$\text{VIII.} \ldots \mathrm{F}_2 q = q^2 + q_1 q + q_2 = 0 \ \text{(comp. 245)},$$

where q_1 and q_2 are any two given, real, actual, and complanar quaternions; thus there is always a real quaternion q' in the given plane, which satisfies the equation,

$$\text{VIII}'. \ldots \mathrm{F}_2 q' = q'^2 + q_1 q' + q_2 = 0 \ \text{(comp. 247)};$$

subtracting, therefore, and dividing by $q - q'$, as in algebra (comp. 224), we obtain the following *depressed* or *linear* equation q,

$$\text{IX.} \ldots q + q' + q_1 = 0, \quad \text{or} \quad \text{IX}'. \ldots q = q'' = -q' - q_1 \ \text{(comp. 246)}.$$

The quadratic VIII. has therefore a *second real quaternion root*, q'', *related* in this manner to the *first;* and because the *quadratic function* $\mathrm{F}_2 q$ (comp. again 245) is thus decomposable into *two linear factors*, or can be put under the form,

$$\text{X.} \ldots \mathrm{F}_2 q = (q - q')\,(q - q''),$$

it *cannot vanish* for *any third real quaternion*, q; so that (comp. 244) the quadratic equation has *no more than two* such *real roots*.

(1.) The *cubic equation* may therefore be put under the *form* (comp. 248),

$$\text{X.} \ldots \mathrm{F}_3 q = q^3 + q_1 q^2 + q_2 q + q_3 = q\,(q - q')\,(q - q'') + q_3 = 0\,;$$

it has therefore *one real root*, say q', by the *general proof* (252), which has been above illustrated by the case of the *quadratic* equation; subtracting therefore (compare 247) the equation $\mathrm{F}_3 q' = 0$, and dividing by $q - q'$, we can *depress* the cubic to a quadratic, which will have *two* new real roots, q'' and q'''; and thus the *cubic function* may be put under the form,

$$\text{XI.} \ldots \mathrm{F}_3 q = (q - q')\,(q - q'')\,(q - q'''),$$

which cannot vanish for any *fourth* real value of q; the cubic equation X. has therefore *no more than three real quaternion roots* (comp. 244): and similarly for equations of *higher degrees*.

(2.) The existence of *two* real roots q of the *quadratic* I., or of *two real vectors*, ρ and ρ', which satisfy the equation II., might have been *geometrically anticipated*, from the recently proved increase $= 4\pi$ of amplitude $\phi\rho$, in the course of one circuit, for the case of Fig. 55, *bis*, in consequence of which there must be *two real positions*, P and P′, *on the one oval* of *that* Figure, of which *each* satisfies the condition of similarity III.; and for the case of Fig. 56, from the consideration that the *second* (or *lighter*) *oval*, which in this case exists, although not employed above, is related to A exactly as the *first* (or *dark*) oval of the Figure is related to O; so that, to the real position P on the first, there must correspond *another* real position P′, upon the second.

(3.) As regards the *law* of this *correspondence*, if the equation II. be put under the form,

$$\text{XII.} \ldots \left(\frac{\rho}{\alpha}\right)^2 - \left(\frac{\rho}{\alpha}\right)^1 - \frac{\sigma}{\alpha} = 0,$$

and if we now write

$$\text{XIII.} \ldots \rho = q\alpha, \quad \text{we may write} \quad \text{XIV.} \ldots q_1 = -1, \qquad q_2 = -\sigma : \alpha,$$

for comparison with the form VIII.; and then the recent relation IX′. (or 246) between the two *roots* will take the form of the following relation between *vectors*,

$$\text{XV.} \ldots \rho + \rho' = \alpha\,; \quad \text{or} \quad \text{XV}'. \ldots \mathrm{OP}' = \rho' = \alpha - \rho = \mathrm{PA}\,;$$

so that the point P′ completes (as in the cited Figures) the parallelogram OPAP′, and the line PP′ is bisected by the middle point C of OA. Accordingly, with this position of P′, we have (comp. III.) the similarity, and (comp. II. and 226) the equation,

$$\text{XVI.} \ldots \Delta\,\mathrm{AOP}' \propto \mathrm{P'AB}\,; \qquad \text{XVII.} \ldots \phi\rho' = \phi(\alpha - \rho) = \phi\rho = 1.$$

(4.) The *other* relation between the two roots of the quadratic VIII., namely (comp. 246),

$$\text{XVIII.} \ldots q'q'' = q_2, \quad \text{gives} \quad \text{XIX.} \ldots \frac{\rho}{\alpha}\,\rho' = -\sigma\,;$$

and accordingly, the line σ, or OS, is a fourth proportional to the three lines OA, OP, and AP, or α, ρ, and $-\rho'$.

(5.) The *actual solution*, by *calculation*, of the *quadratic equation* VIII. in *complanar quaternions*, is performed *exactly as in algebra* ; the *formula* being,

$$\text{XX.} \ldots q = -\tfrac{1}{2}q_1 \pm \surd(\tfrac{1}{4}q_1^2 - q_2),$$

in which, however, the *square root* is to be interpreted as a *real quaternion*, on principles already laid down.

(6.) *Cubic* and *biquadratic* equations, with quaternion coefficients of the kind considered in 244, are in like manner *resolved* by the known *formulæ* of algebra; but we have now (as has been proved) *three real* (quaternion) *roots* for the former, and *four* such real roots for the latter.

254. The following is another mode of presenting the geometrical reasonings of the foregoing Article, without expressly introducing the notation or conception of *amplitude*. The equation $\phi\rho = 1$ of 253 being written as follows,

$$\text{I.} \ldots \sigma = \chi\rho = \frac{\rho}{\alpha}(\rho - \alpha), \text{ or II.} \ldots \mathrm{T}\sigma = \mathrm{T}\chi\rho, \text{ and III.} \ldots \mathrm{U}\sigma = \mathrm{U}\chi\rho,$$

we may thus regard the *vector* σ as a *known function of the vector* ρ, or the *point* S as a *function of the point* P; in the sense that, while O and A are *fixed*, P and S *vary together* : although it may (and does) happen, that S may *return* to a former position without P having similarly returned. Now the essential property of the *oval* (253) may be said to be this: that it is *the locus of the points* P *nearest to* O, *for which the tensor* $\mathrm{T}\chi\rho$ *has a given value*, say b; namely the *given value of* $\mathrm{T}\sigma$, or of $\overline{\text{OS}}$, when the *point* S, like O and A, is *given*. If then we conceive the point P to *move*, as before, *along the oval*, and the point S *also* to *move*, according to the *law* expressed by the recent formula I., this *latter* point must move (by II.) *on the circumference of a given circle* (comp. again Fig. 56), with the given origin O for *centre ;* and the *theorem* is, that *in so moving*, S will *pass*, at least *once*, through *every position on that circle*, while P performs *one circuit* of the oval. And *this* may be proved by observing that (by III.) the *angular motion of the radius* OS is equal to the *sum of the angular motions* of the *two rays*, OP and AP; but this latter *sum* amounts to *eight right angles* for the case of Fig. 55, *bis*,, and to *four right angles* for the case of Fig. 56; the radius OS, and the point S, must therefore have *revolved twice* in the first case, and *once* in the second case, which proves the theorem in question.

(1.) In the first of these two cases, namely when A is an interior point, *each* of the three angular velocities is *positive* throughout, and the *mean angular velocity of* *the radius* OS is *double* of that of *each* of the *two rays* OP, AP. But in the second case, when A is exterior, the *mean* angular velocity of the ray AP is *zero;* and we might for a moment doubt, whether the *sometimes negative* velocity of *that* ray might not, for parts of the circuit, *exceed* the *always positive* velocity of the ray OP, and so cause the radius OS to move *backwards*, for a while. This cannot be, however; for if we conceive P to describe, like P′, a circuit of the *other* (or lighter) *oval*, in Fig. 56, the point S (if still dependent on it by the law I.) would *again* traverse the whole of the same circumference as before; if then it could ever *fluctuate* in its motion, it would pass *more than twice* through some given series of real positions on that circle, during the successive description of the *two ovals* by P; and thus, within certain limiting values of the coefficients, the quadratic equation would have *more than two real roots:* a result which has been proved to be impossible.

(2.) While S thus describes a circle round O, we may conceive the *connected point* B to describe an *equal circle* round A; and in the case at least of Fig. 56, it is easy to prove *geometrically*, from the constant equality (253, IV.) of the rectangles $\overline{\text{OP}}\cdot\overline{\text{AP}}$ and $\overline{\text{OA}}\cdot\overline{\text{AB}}$, that these *two circles* (with T′U and T′U′ as *diameters*), and the *two ovals* (with MN and M′N′ as *axes*), have *two common tangents*, parallel to the line OA, which connects what we may call the *two given foci* (or *focal points*), O and A: the new or *third circle*, which is described on this *focal interval* OA as diameter, passing *through the four points of contact on the ovals*, as the Figure may serve to exhibit.

(3.) To prove the same things *by quaternions*, we shall find it convenient to *change the origin* (18), for the sake of symmetry, to the *central point* C; and thus to denote *now* CP by ρ, and CA by α, writing also $\overline{\text{CA}} = \text{T}\alpha = a$, and representing still the radius of each of the two equal circles by b. We shall then have, as the *joint equation* of the system of the *two ovals*, the following:

$$\text{IV.}\ldots\ \text{T}(\rho+\alpha).\text{T}(\rho-\alpha) = 2ab;$$

or

$$\text{V.}\ldots\ \text{T}(q^2-1) = 2c, \quad \text{if} \quad q = \frac{\rho}{\alpha} \quad \text{and} \quad c = \frac{b}{a}.$$

But because we have *generally* (by 199, 204, &c.) the transformations,

$$\text{VI.}\ldots\ \text{S}.q^2 = 2\text{S}q^2 - \text{T}q^2 = \text{T}q^2 + 2\text{V}q^2 = 2\text{NS}q - \text{N}q = \text{N}q - 2\text{NV}q,$$

the square of the equation V. may (by 210, (8.)) be written under either of the two following forms:

$$\text{VII.}\ldots\ (\text{N}q-1)^2 + 4\text{NV}q = 4c^2; \qquad \text{VIII.}\ldots\ (\text{N}q+1)^2 - 4\text{NS}q = 4c^2;$$

whereof the first shows that the *maximum* value of $\text{TV}q$ is c, at least if $2c < 1$, as happens for this case of Fig. 56; and that this maximum corresponds to the value $\text{T}q = 1$, or $\text{T}\rho = a$: results which, when interpreted, reproduce those of the preceding sub-article.

(4.) When $2c > 1$, it is permitted to suppose $\text{S}q = 0$, $\text{NV}q = \text{N}q = 2c - 1$; and then we have only *one* continuous oval, as in the case of Fig. 55, *bis;* but if $c < 1$, though $> \frac{1}{2}$, there exists a certain *undulation* in the form of the curve (not represented in that Figure), $\text{TV}q$ being a *minimum* for $\text{S}q = 0$, or for $\rho \perp \alpha$, but becoming (as before) a *maximum* when $\text{T}q = 1$, and *vanishing* when $\text{S}q^2 = 2c + 1$, namely at the two *summits* M, N, where the oval meets the axis.

(5.) In the intermediate case, when $2c = 1$, the *Cassinian curve* IV. becomes (as is known) a *lemniscata;* of which the *quaternion equation* may, by V., be written (comp. 200, (8.)) under any one of the following forms:

$$\text{IX.} \ldots \mathrm{T}(q^2 - 1) = 1; \quad \text{or} \quad \text{X.} \ldots \mathrm{N}q^2 = 2\mathrm{S}.q^2; \quad \text{or} \quad \text{XI.} \ldots \mathrm{T}q^2 = 2\mathrm{SU}.q^2;$$

or finally,

$$\text{XII.} \ldots \mathrm{T}\rho^2 = 2\mathrm{T}\alpha^2 \cos 2 \angle \frac{\rho}{\alpha};$$

which last, when written as

$$\text{XII}'. \ldots \overline{\mathrm{CP}}^2 = 2\overline{\mathrm{CA}}^2 . \cos 2\mathrm{ACP},$$

agrees evidently with known results.

(6.) This corresponds to the case when

$$\text{XIII.} \ldots \sigma = \frac{-\alpha}{4}, \quad \text{and} \quad \text{XIV.} \ldots \rho = \rho' = +\frac{\alpha}{2}, \text{ in 253, XII.},$$

that *quadratic equation* having thus its roots *equal;* and in general, for *all degrees*, cases of *equal roots* answer to some interesting *peculiarities of form* of the *ovals*, on which we cannot here delay.

(7.) It may, however, be remarked, in passing, that if we *remove the restriction* that the vector ρ, or CP, shall be *in a given plane* (225), drawn through the line which connects the *two foci*, O and A, the recent equation V. will then represent the *surface* (or *surfaces*) generated by the *revolution* of the *oval* (or ovals), or lemniscata, about that line OA as an *axis*.

255. If we look back, for a moment, on the formula of *similarity*, 253, III., we shall see that it involves not merely an *equality of rectangles*, 253, IV., but also an *equality of angles*, AOP and PAB; so that the angle OAB represents (in the Figures 55) a *given difference of the base angles* AOP, PAO of the triangle OAP: but to *construct* a *triangle*, by means of *such* a given *difference*, combined with a *given base*, and a *given rectangle of sides*, is a known problem of elementary geometry. To solve it briefly, as an exercise, *by quaternions*, let the given base be the line AA′, with O for its middle point, as in the annexed Figure 57; let BAA′ represent the given difference of base angles, PAA′ − AA′P; and let $\overline{\mathrm{OA}}.\overline{\mathrm{AB}}$ be equal to the given rectangle of sides, $\overline{\mathrm{AP}}.\overline{\mathrm{A'P}}$. We shall then have the similarity and equation,

$$\text{I.} \ldots \Delta\, \mathrm{OA'P} \propto \mathrm{PAB}; \qquad \text{II.} \ldots \frac{\rho + \alpha}{\alpha} = \frac{\beta - \alpha}{\rho - \alpha};$$

whence it follows by the simplest calculations, that

$$\text{III.} \ldots \left(\frac{\rho}{\alpha}\right)^2 = \left(\frac{\rho}{\alpha}+1\right)\left(\frac{\rho}{\alpha}-1\right) + 1 = \frac{\beta - \alpha}{\alpha} + 1 = \frac{\beta}{\alpha};$$

or that ρ is a mean proportional (227) between α and β. Draw, therefore, a line OP, which shall be in length a *geometric mean* between the two given lines, OA, OB, and shall also *bisect their angle*

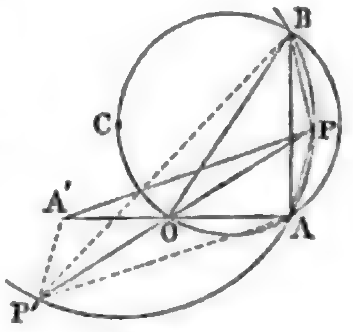

Fig. 57.

AOB; its extremity will be the required *vertex*, P, of the sought triangle AA′P: a result of the *quaternion analysis*, which *geometrical synthesis*[*] easily confirms.

(1.) The equation III. is however satisfied also (comp. 227) by the *opposite* vector, OP′ = PO, or $\rho' = -\rho$; and because $\beta = (\rho : \alpha) . \rho$, we have

$$\text{IV.} \ldots \frac{\rho+\beta}{\rho+\alpha} = \frac{\rho}{\alpha} = \frac{\beta}{\rho} = \frac{\rho'}{\alpha'}, \quad \text{or} \quad \text{IV}'. \ldots \frac{\mathrm{P'B}}{\mathrm{P'A}} = \frac{\mathrm{OP}}{\mathrm{OA}} = \frac{\mathrm{OB}}{\mathrm{OP}} = \frac{\mathrm{OP'}}{\mathrm{OA'}};$$

so that the *four* following triangles are *similar* (the *two first* of them indeed being *equal*):

$$\text{V.} \ldots \Delta\, \mathrm{A'OP'} \propto \mathrm{AOP} \propto \mathrm{POB} \propto \mathrm{AP'B};$$

as geometry again would confirm.

(2.) The angles AP′B, BPA, are therefore *supplementary*, their sum being equal to the sum of the angles in the triangle OAP; whence it follows that *the four points* A, P, B, P′ *are concircular*:[†] or in other words, *the quadrilateral* APBP′ is *inscriptible in a circle*, which (we may add) passes *through the centre* C *of the circle* OAB (see again Fig. 57), because the angle AOB is *double* of the angle AP′B, by what has been already proved.

(3.) Quadratic equations in quaternions may also be employed in the solution of many other geometrical problems; for example, to decompose a given vector into two others, which shall have a given geometrical mean, &c.

SECTION 6.—*On the* n² − n *Imaginary (or Symbolical) Roots of a Quaternion Equation of the* nᵗʰ *Degree, with Coefficients of the kind considered in the foregoing Section.*

256. The polynomial function $F_n q$ (245), like the quaternions $q, q_1, \ldots q_n$ on which it depends, may always be reduced to the form of a couple (228); and thus we may establish the transformation (comp. 239),

$$\text{I.} \ldots F_n q = F_n(x+iy) = X_n + iY_n = G_n(x,y) + iH_n(x,y),$$

X_n and Y_n, or G_n and H_n, being *two* known, real, finite, and scalar *functions* of the *two sought scalars*, x and y; which functions, rela-

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[*] In fact, the two triangles I. are similar, as required, because their angles at O and P are equal, and the sides about them are proportional.

[†] Geometrically, the construction gives at once the similarity,

$$\Delta\, \mathrm{AOP} \propto \mathrm{POB}, \quad \text{whence} \quad \angle\, \mathrm{BPA} = \mathrm{OPA} + \mathrm{PAO} = \mathrm{POA'};$$

and if we complete the parallelogram APA′P′, the new similarity,

$$\Delta\, \mathrm{OA'P} \propto \mathrm{OP'B}, \quad \text{gives} \quad \angle\, \mathrm{AP'B} = \mathrm{OA'P} + \mathrm{A'PO} = \mathrm{AOP};$$

thus the opposite angles BPA, AP′B are supplementary, and the quadrilateral APBP′ is inscriptible. It will be shown, in a shortly subsequent Section, that these four points, A, P, B, P′, form a *harmonic group* upon their common circle.

tively to *them*, are each of the nth dimension, but which involve also, though only in the *first* dimension, the $2n$ *given* and *real scalars*, $x_1, y_1, \ldots x_n, y_n$. And since the *one quaternion* (or *couple*) *equation*, $F_n q = 0$, is equivalent (by 228, IV.) to the *system* of the *two scalar equations*,

$$\text{II.} \ldots X_n = 0, \quad Y_n = 0, \quad \text{or} \quad \text{III.} \ldots G_n(x, y) = 0, \quad H_n(x, y) = 0,$$

we see (by what has been stated in 244, and proved in 252) that *such a system*, of *two* equations of the nth dimension, *can always be satisfied by n systems* (or *pairs*) *of real scalars*, and by *not more than n*, such as

$$\text{IV.} \ldots x', y'; \quad x'', y''; \ldots \quad x^{(n)}, y^{(n)};$$

although it may happen that *two* or *more* of these systems shall *coincide* with (or become *equal* to) each other.

(1.) If x and y be treated as *co-ordinates* (comp. 228, (3.)), the two equations II. or III. represent a *system of two curves*, in the given plane; and then the *theorem* is, that these two curves *intersect each other* (*generally**) *in n real points*, and in *no more*: although two or more of these n points may happen to *coincide* with each other.

(2.) Let h denote, as a temporary abridgment, the *old* or *ordinary imaginary*, $\sqrt{-1}$, of *algebra*, considered as an *uninterpreted symbol*, and as *not* equal to *any real versor*, such as i (comp. 181, and 214, (3.)), but as following the *rules of scalars*, especially as regards the *commutative property* of multiplication (126); so that

$$\text{V.} \ldots h^2 + 1 = 0, \quad \text{and} \quad \text{VI.} \ldots hi = ih, \quad \text{but} \quad \text{VII.} \ldots h \text{ not} = \pm i.$$

(3.) Let q denote still a *real quaternion*, or *real couple*, $x + iy$; and with the meaning just now proposed of h, let $[q]$ denote the connected but *imaginary algebraic quantity*, or *bi-scalar* (214, (7.)), $x + hy$; so that

$$\text{VIII.}\ q = x + iy, \quad \text{but} \quad \text{IX.} \ldots [q] = x + hy;$$

and let any *biquaternion* (214), (8.), or (as we may *here* call it) BI-COUPLE, of the *form* $[q'] + i[q'']$, be said to be *complanar* with i; with the old *notation* (123) of complanarity.

(4.) Then, for the *polynomial equation in real and complanar quaternions*, $F_n q = 0$ (244, 245), we may be led to *substitute* the following *connected algebraical equation*, of the *same degree*, n, and *involving real scalars similarly*:

$$\text{X.} \ldots [F_n q] = [q]^n + [q_1][q]^{n-1} + \ldots + [q_n] = 0;$$

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* Cases of *equal roots* may cause points of intersection, which are *generally imaginary*, to *become real*, but *coincident* with each other, and with *former* real roots: for instance the *hyperbola*, $x^2 - y^2 = a$, is intersected in *two real* and *distinct points*, by the *pair of right lines* $xy = 0$, if the scalar $a >$ or < 0; but for the case $a = 0$, the *two pairs of lines*, $x^2 - y^2 = 0$ and $xy = 0$, may be considered to have *four coincident intersections* at the origin.

which, after the reductions depending on the substitution V. of -1 for h^2, receives the form,

$$\text{XI.} \ldots [F_n q] = X_n + hY_n = 0\,;$$

where X_n and Y_n are the *same real and scalar functions* as in I.

(5.) But we have seen in II., that *these two real functions* can be made to *vanish together*, by selecting *any one of n real pairs* IV. of *scalar values, x and y*; *the General Algebraical Equation* X., *of the* n^{th} *Degree, has therefore n Real or Imaginary Roots*,* *of the Form* $x + y\sqrt{-1}$; and it has *no more than n* such roots.

(6.) *Elimination* of y, between the two equations II. or III., conducts generally to an algebraic equation in x, of the degree n^2; which equation has therefore n^2 *algebraic roots* (5.), real *or* imaginary; namely, by what has been lately proved, n *real* and *scalar roots*, $x', \ldots x^{(n)}$, with *real* and *scalar values* $y', \ldots y^{(n)}$ (comp. IV.) of y to *correspond*; and $n(n-1)$ *other roots*, with the same number of corresponding values of y, which may be thus denoted,

$$\text{XII.} \ldots [x^{(n+1)}, \ldots [x^{(n^2)}]\,; \qquad \text{XIII.} \ldots [y^{(n+1)}], \ldots [y^{(n^2)}]\,;$$

and which are either themselves *imaginary* (or *bi-scalar*, 214, (7.)), or at least *correspond*, by the supposed elimination, to *imaginary* or *bi-scalar values* of y; since if $x^{(n+1)}$ and $y^{(n+1)}$, for example, could *both* be *real*, the *quaternion equation* $F_n q = 0$ would then have an $(n+1)$st *real root*, of the form, $q^{(n+1)} = x^{(n+1)} + iy^{(n+1)}$, contrary to what has been proved (252).

257. On the whole, then, it results that the equation $F_n q = 0$ in complanar quaternions, of the nth degree, with real coefficients, while it admits of only *n real quaternion roots*,

$$\text{I.} \ldots q', q'', \ldots q^{(n)} \text{ (244, \&c.)},$$

is *symbolically satisfied also* (comp. 214, (3.)) *by* $n(n-1)$ *imaginary quaternion roots*, or by $n^2 - n$ *bi-quaternions* (214, (8.)), or *bi-couples* (256, (3.)), which may be thus denoted,

$$\text{II.} \ldots [q^{(n+1)}], \ldots [q^{(n^2)}]\,;$$

and of which the first, for example, has the *form*,

$$\text{III.} \ldots [q^{(n+1)}] = [x^{(n+1)}] + i[y^{(n+1)}] = x_{\prime}^{(n+1)} + hx_{\prime\prime}^{(n+1)} + i(y_{\prime}^{(n+1)} + hy_{\prime\prime}^{(n+1)})\,;$$

where $x_{\prime}^{(n+1)}$, $x_{\prime\prime}^{(n+1)}$, $y_{\prime}^{(n+1)}$, and $y_{\prime\prime}^{(n+1)}$ are *four real scalars*, but h is the *imaginary of algebra* (256, (2.)).

(1.) There must, for instance, be $n(n-1)$ *imaginary* n^{th} *roots of unity, in the given plane* of i (comp. 256, (3.)), besides the *n real roots* already determined (233,

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* This celebrated *Theorem of Algebra* has long been known, and has been proved in other ways; but it seemed necessary, or at least useful, for the purpose of the present work, to prove it anew, in connexion with *Quaternions*: or rather to establish the theorem (244, 252), to which in the present Calculus it *corresponds*. Compare the Note to page 266.

237); and accordingly in the case $n = 2$, we have the *four* following *square-roots of* $1 \; ||| \; i$, two real and two imaginary:

$$\text{IV.} \ldots +1, \quad -1; \qquad +hi, \quad -hi;$$

for, by 256, (2.), we have

$$\text{V.} \ldots (\pm hi)^2 = h^2 i^2 = (-1)(-1) = +1.$$

And the *two imaginary roots* of the *quadratic* equation $F_2 q = 0$, which *generally* exist, at least as *symbols* (214, (3.)), may be obtained by *multiplying the square-root* in the formula 253, XX. by hi; so that in the particular *case*, when that *radical vanishes*, the *four roots* of the equation become *real* and *equal*: *zero* having thus *only itself* for a *square-root*.

(2.) Again, if we write (comp. 237, (3.)),

$$\text{VI.} \ldots q = 1^{\frac{1}{3}}_1 = \frac{-1 + i\sqrt{3}}{2}, \qquad q^2 = 1^{\frac{1}{3}}_2 = \frac{-1 - i\sqrt{3}}{2},$$

so that $1, q, q^2$ are the *three real cube-roots of positive unity*, in the given plane; and if we write also,

$$\text{VII.} \ldots \theta = [q] = \frac{-1 + h\sqrt{3}}{2}, \qquad \theta^2 = [q]^2 = \frac{-1 - h\sqrt{3}}{2},$$

so that θ and θ^2 are (as usual) the *two ordinary* (or *algebraical*) *imaginary cube-roots of unity*; then the *nine cube-roots of* $1 \, (||| \, i)$ are the following:

$$\text{VIII.} \ldots 1; \quad q, q^2; \quad \theta, \theta^2; \quad \theta q, \theta^2 q; \quad \theta^2 q, \theta^2 q^2;$$

whereof the first is a *real scalar*; the two next are *real couples*, or *quaternions* $||| \, i$; the two following are *imaginary scalars*, or *biscalars*; and the four that remain are *imaginary couples*, or *bi-couples*, or *biquaternions*.

(3.) The *sixteen fourth roots of unity* $(||| \, i)$ are:

$$\text{IX.} \ldots \pm 1; \quad \pm i; \quad \pm h; \quad \pm hi; \quad \pm \tfrac{1}{2}(1 \pm h)(1 \pm i);$$

the three ambiguous signs in the last expression being all independent of each other.

(4.) *Imaginary roots*, of *this* sort, are sometimes *useful*, or rather *necessary*, in calculations respecting *ideal intersections*,* and *ideal contacts*, in *geometry*: although in what remains of the present Volume, we shall have little or no occasion to employ them.

(5.) We may, however, here observe, that when the *restriction* (225) on the *plane* of the quaternion q is *removed*, the *General Quaternion Equation of the* n^{th} *Degree* admits, by the foregoing principles, no fewer than n^4 *Roots*, *real or imaginary*: because, when that general equation is reduced, by 221, to the *Standard Quadrinomial Form*,

$$\text{X.} \ldots F_n q = W_n + iX_n + jY_n + kZ_n = 0,$$

it breaks up (comp. 221, VI.) into a *System of Four Scalar Equations*, *each* (generally) of the nth *dimension*, in w, x, y, z; namely,

$$\text{XI.} \ldots W_n = 0, \qquad X_n = 0, \qquad Y_n = 0, \qquad Z_n = 0;$$

and if x, y, z be eliminated between these four, the result is (generally) *a scalar* (or algebraical) *equation of the degree* n^4, relatively to the remaining *constituent*, w;

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* Comp. Art. 214, and the Notes there referred to.

which therefore has n^4 (algebraical) *values*, real or imaginary: and similarly for the three other constituents, x, y, z, of the sought quaternion q.

(6.) It may even happen, *when no plane is given*, that the *number of roots* (or solutions) of a *finite* equation in quaternions* shall become *infinite;* as has been seen to be the case for the equation $q^2 = -1$ (149, 154), *even* when we confine ourselves to what we have considered as *real* roots. If *imaginary* roots be admitted, we may write, *still more generally*, besides the *two biscalar values*, $\pm h$, the expression,

$$\text{XII.} \ldots (-1)^{\frac{1}{2}} = v + hv', \qquad \mathrm{S}v = \mathrm{S}v' = \mathrm{S}vv' = 0, \qquad \mathrm{N}v - \mathrm{N}v' = 1;$$

v and v' being thus *any two real and right quaternions, in rectangular planes*, provided that the *norm* of the *first exceeds* that of the *second* by *unity*.

(7.) And in like manner, besides the *two real* and *scalar values*, ± 1, we have this general symbolical expression for a square root of positive unity, with merely the difference of the norms reversed:

$$\text{XIII.} \ldots 1^{\frac{1}{2}} = v + hv', \qquad \mathrm{S}v = \mathrm{S}v' = \mathrm{S}vv' = 0, \qquad \mathrm{N}v' - \mathrm{N}v = 1.$$

SECTION 7.—*On the Reciprocal of a Vector, and on Harmonic Means of Vectors; with Remarks on the Anharmonic Quaternion of a Group of Four Points, and on Conditions of Concircularity.*

258. When two vectors, α and α', are so related that

$$\text{I.} \ldots \alpha' = -\mathrm{U}\alpha : \mathrm{T}\alpha, \quad \text{and therefore} \quad \text{II.} \ldots \alpha = -\mathrm{U}\alpha' : \mathrm{T}\alpha,$$

or that

$$\text{III.} \ldots \mathrm{T}\alpha \,.\, \mathrm{T}\alpha' = 1, \quad \text{and} \quad \text{IV.} \ldots \mathrm{U}\alpha + \mathrm{U}\alpha' = 0,$$

we shall say that each of these two vectors is the *Reciprocal*† of the other; and shall (at least for the present) *denote* this relation between them, by writing

$$\text{V.} \ldots \alpha' = \mathrm{R}\alpha, \quad \text{or} \quad \text{VI.} \ldots \alpha = \mathrm{R}\alpha';$$

so that for *every vector* α, and *every right quotient* v,

$$\text{VII.} \ldots \mathrm{R}\alpha = -\mathrm{U}\alpha : \mathrm{T}\alpha; \qquad \text{VIII.} \ldots \mathrm{R}^2\alpha = \mathrm{RR}\alpha = \alpha;$$

and

$$\text{IX.} \ldots \mathrm{RI}v = \mathrm{IR}v \text{ (comp. 161, (3.), and 204, XXXV'.).}$$

259. One of the most important properties of such reciprocals is contained in the following theorem:

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* Compare the Note to page 265.

† Accordingly, under these conditions, we shall afterwards *denote* this *reciprocal* of a vector α by the *symbol* α^{-1}; but we postpone the use of this notation, until we shall be prepared to connect it with a general theory of *products and powers of vectors*. Compare 234, V., and the Note to page 121. And as regards the temporary use of the characteristic R, compare the second Note to page 252.

If any two vectors OA, OB, have OA′, OB′ for their *reciprocals*, then (comp. Fig. 58) the *right line* A′B′ is *parallel to the tangent* OD, *at the origin* O, *to the circle* OAB; and the *two triangles*, OAB, OB′A′, are *inversely similar* (118). Or in symbols,

$$\text{I.}\ldots \text{ if } \text{OA}' = \text{R}.\text{OA}, \quad \text{and} \quad \text{OB}' = \text{R}.\text{OB},$$

then

$$\Delta\,\text{OAB} \propto' \text{OB}'\text{A}'.$$

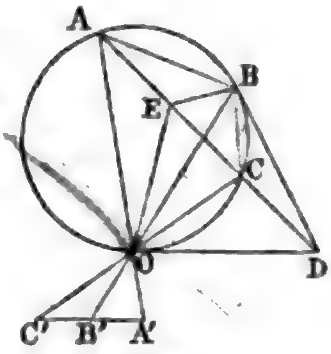

Fig. 58.

(1.) Of course, under the same conditions, the tangent at O to the circle OA′B′ is parallel to the line AB.

(2.) The angles BAO and OB′A′ or BOD being equal, the fourth proportional (226) to AB, AO, and OB, or to BA, OA, and OB, has the direction of OD, or the *direction opposite* to that of A′B′; and its *length* is easily proved to be the *reciprocal* (or *inverse*) of the length of the same line A′B′, because the similar triangles give,

$$\text{II.}\ldots (\overline{\text{OA}} : \overline{\text{BA}}).\overline{\text{OB}} = (\overline{\text{OB}'} : \overline{\text{A}'\text{B}'}).\overline{\text{OB}} = 1 : \overline{\text{A}'\text{B}'},$$

it being remembered that

$$\text{III.}\ldots \overline{\text{OA}}.\overline{\text{OA}'} = \overline{\text{OB}}.\overline{\text{OB}'} = 1;$$

we may therefore write,

$$\text{IV.}\ldots (\text{OA} : \text{BA}).\text{OB} = \text{R}.\text{A}'\text{B}', \quad \text{or} \quad \text{V.}\ldots \frac{\alpha}{\alpha - \beta}\beta = \text{R}(\text{R}\beta - \text{R}\alpha),$$

whatever two vectors α and β may be.

(3.) Changing α and β to their reciprocals, the last formula becomes,

$$\text{VI.}\ldots \text{R}(\beta - \alpha) = \frac{\text{R}\alpha}{\text{R}\alpha - \text{R}\beta}.\text{R}\beta; \quad \text{or} \quad \text{VII.}\ldots (\text{OA}' : \text{B}'\text{A}').\text{OB}' = \text{R}.\text{AB}.$$

(4.) The inverse similarity I. gives also, generally, the relation,

$$\text{VIII.}\ldots \text{K}\frac{\beta}{\alpha} = \frac{\text{R}\alpha}{\text{R}\beta}.$$

(5.) Since, then, by 195, II., or 207, (2.),

$$\text{IX.}\ldots \text{K}\frac{\beta}{\alpha} \pm 1 = \text{K}\frac{\beta \pm \alpha}{\alpha}, \quad \text{we have} \quad \text{X.}\ldots \frac{\text{R}\alpha \pm \text{R}\beta}{\text{R}\beta} = \frac{\text{R}\alpha}{\text{R}(\beta \pm \alpha)};$$

the lower signs agreeing with VI.

(6.) In general, the *reciprocals* of *opposite* vectors are themselves *opposite*; or in symbols,

$$\text{XI.}\ldots \text{R}(-\alpha) = -\text{R}\alpha.$$

(7.) More generally,

$$\text{XII.}\ldots \text{R}x\alpha = x^{-1}\text{R}\alpha,$$

if x be any scalar.

(8.) Taking lower signs in X., changing α to γ, dividing, and taking conjugates, we find for *any three vectors* α, β, γ (*complanar or diplanar*) the formula:

$$\text{XIII.}\ldots \text{K}\frac{\text{R}\gamma - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = \text{K}\left(\frac{\text{R}\gamma}{\text{R}(\beta - \gamma)}.\frac{\text{R}(\beta - \alpha)}{\text{R}\alpha}\right) = \frac{\alpha}{\beta - \alpha}.\frac{\gamma - \beta}{-\gamma} = \frac{\text{OA}}{\text{AB}}.\frac{\text{BC}}{\text{CO}},$$

if $\alpha = \text{OA}$, $\beta = \text{OB}$, and $\gamma = \text{OC}$, as usual.

(9.) If then we *extend*, to *any four points of space*, the *notation* (25),

$$\text{XIV.} \ldots (\text{ABCD}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DA}},$$

interpreting each of these two *factor-quotients* as a *quaternion*, and *defining* that their *product* (in *this order*) is the *anharmonic quaternion function*, or simply *the Anharmonic, of the Group of four points* A, B, C, D, or of the (*plane or gauche*) *Quadrilateral* ABCD, we shall have the following general and useful *formula of transformation :*

$$\text{XV.} \ldots \quad (\text{OABC}) = \text{K}\,\frac{\text{R}\gamma - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = \text{K}\,\frac{\text{B}'\text{C}'}{\text{B}'\text{A}'},$$

where OA′, OB′, OB′ are supposed to be reciprocals of OA, OB, OC.

(10.) With this notation XIV., we have *generally*, and not merely for *collinear groups* (35), the relations:

$$\text{XVI.} \ldots (\text{ABCD}) + (\text{ACBD}) = 1\,; \qquad \text{XVII.} \ldots (\text{ABCD}).(\text{ADCB}) = 1.$$

(11.) Let O, A, B, C, D be *any five points*, and OA′, .. OD′ the reciprocals of OA, .. OD; we shall then have, by XV.,

$$\text{XVIII.} \ldots \frac{\text{B}'\text{A}'}{\text{B}'\text{C}'} = \text{K}\,(\text{OCBA}), \qquad \frac{\text{D}'\text{C}'}{\text{D}'\text{A}'} = \text{K}\,(\text{OADC})\,;$$

and therefore,

$$\text{XIX.} \ldots \text{K}\,(\text{A}'\text{B}'\text{C}'\text{D}') = (\text{OADC})\,(\text{OCBA}) = -\,(\text{OADCBA}),$$

if we agree to write generally, for *any six points*, the formula,*

$$\text{XX.} \ldots (\text{ABCDEF}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DE}} \cdot \frac{\text{EF}}{\text{FA}}.$$

(12.) If then the five points O .. D be *complanar* (225), we have, by 226, and by XIV.,

$$\text{XXI.} \ldots \text{K}(\text{A}'\text{B}'\text{C}'\text{D}') = (\text{ABCD}), \quad \text{or} \quad \text{XXI}'. \ldots (\text{A}'\text{B}'\text{C}'\text{D}') = \text{K}\,(\text{ABCD})\,;$$

the *anharmonic quaternion* (ABCD) being thus changed to its *conjugate*, when the *four rays* OA, .. OD are changed to their *reciprocals*.

260. Another very important consequence from the definition (258) of reciprocals of vectors, or from the recent theorem (259), may be expressed as follows:

If any three coinitial vectors, OA, OB, OC, *be chords of one common circle*, then (see again Fig. 58) *their three coinitial re-*

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* There is a convenience in calling, generally, this *product of three quotients*, (ABCDEF), the *evolutionary quaternion*, or simply *the Evolutionary*, of the *Group of Six Points*, A .. F, or (if they be not collinear) of the plane or gauche *Hexagon* ABCDEF : because the equation,

$$(\text{ABCA}'\text{B}'\text{C}') = -\,1,$$

expresses either Ist, that the *three pairs of points*, AA′, BB′, CC′, form a *collinear involution* (26) of a well-known kind; or IInd, that those three *pairs*, or the three corresponding *diagonals* of the hexagon, compose a *complanar* or a *homospheric Involution*, of a *new kind suggested by quaternions* (comp. 261, (11.)).

ciprocals, OA′, OB′, OC′, *are termino-collinear* (24): or, in other words, if the *four points* O, A, B, C be *concircular*, then the *three points* A′, B′, C′ are situated *on one right line*.

And conversely, *if three coinitial vectors*, OA′, OB′, OC′, thus terminate on one *right line*, then their three coinitial *reciprocals*, OA, OB, OC, are *chords* of one *circle;* the *tangent* to which circle, at the origin, is *parallel* to the *right line;* while the *anharmonic function* (259, (9.)), of the *inscribed quadrilateral* OABC, reduces itself to a *scalar quotient of segments* of that line (which therefore is its own conjugate, by 139): namely,

$$\text{I.}\ldots(\text{OABC}) = \text{B}'\text{C}' : \text{B}'\text{A}' = (\infty\,\text{A}'\text{B}'\text{C}') = (\text{O}.\text{OABC}),$$

if the symbol ∞ be used *here* to denote the *point at infinity* on the right line A′B′C′; and if, in thus employing the notation (35) for the *anharmonic of a plane pencil*, we consider the *null chord*, OO, as having the *direction** of the *tangent*, OD.

(1.) If $\rho = \text{OP}$ be the *variable vector* of a point P upon the circle OAB, the *quaternion equation* of that circle may be thus written:

$$\text{II.}\ldots \text{R}\rho = \text{R}\beta + x(\text{R}\alpha - \text{R}\beta), \quad \text{where} \quad \text{III.}\ldots x = (\text{OABP});$$

the coefficient x being thus a *variable scalar* (comp. 99, I.), which depends on the *variable position* of the point P on the circumference.

(2.) Or we may write,

$$\text{IV.}\ldots \text{R}\rho = \frac{t\text{R}\alpha + u\text{R}\beta}{t+u},$$

as another form of the equation of the same *circle* OAB; with which may usefully be contrasted the earlier form (comp. 25), of the equation of the *line* AB,

$$\text{V.}\ldots \rho = \frac{t\alpha + u\beta}{t+u}.$$

(3.) Or, dividing the second member of IV. by the first, and taking conjugates, we have for the circle,

$$\text{VI.}\ldots \frac{t\rho}{\alpha} + \frac{u\rho}{\beta} = t + u; \quad \text{while} \quad \text{VII.}\ldots \frac{t\alpha}{\rho} + \frac{u\beta}{\rho} = t + u,$$

for the right line.

(4.) Or we may write, by II.,

$$\text{VIII.}\ldots \text{V}\frac{\text{R}\rho - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = 0; \quad \text{or} \quad \text{VIII}'.\ldots \frac{\text{R}\rho - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = \text{V}^{-1}0;$$

this latter symbol, by 204, (18.), denoting *any scalar*.

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* Compare the remarks in the second Note to page 139, respecting the possible determinateness of signification of the symbol U0, when the *zero* denotes a *line*, which *vanishes* according to a *law*.

(5.) Or still more briefly,

$$\text{IX.}\,.\,.\; \mathrm{V}(\mathrm{OABP}) = 0\,; \quad \text{or} \quad \text{IX}'.\,.\,.\; (\mathrm{OABP}) = \mathrm{V}^{-1}\,0.$$

(6.) If the *four* points O, A, B, C be still *concircular*, and if P be *any fifth point in their plane*, while PO_1, .. PC_1 are the reciprocals of PO, .. PC, then by 259, XXI., we have the relation,

$$\text{X.}\,.\,.\; (\mathrm{O}_1\mathrm{A}_1\mathrm{B}_1\mathrm{C}_1) = \mathrm{K}(\mathrm{OABC}) = (\mathrm{OABC}) = \mathrm{V}^{-1}0\,;$$

the *four new points* O_1 .. C_1 are therefore *generally concircular.*

(7.) If, however, the point P be again placed *on the circle* OABC, those four new points are (by the present Article) *collinear;* being the *intersections* of the *pencil* P.OABC with a *parallel to the tangent* at P. In this case, therefore, we have the equation,

$$\text{XI.}\,.\,.\; (\mathrm{P.OABC}) = (\mathrm{O}_1\mathrm{A}_1\mathrm{B}_1\mathrm{C}_1) = (\mathrm{OABC})\,;$$

so that the *constant anharmonic of the pencil* (35) is thus seen to be equal to what we have defined (259, (9.)) to be the *anharmonic of the group.*

(8.) And because the *anharmonic* of a *circular group* is a *scalar*, it is equal (by 187, (8.)) to its own *tensor*, either positively or negatively taken: we may therefore write, for any *inscribed quadrilateral* OABC, the formula,

$$\text{XII.}\,.\,.\; (\mathrm{OABC}) = \mp\mathrm{T}(\mathrm{OABC}) = \mp(\overline{\mathrm{OA}}\cdot\overline{\mathrm{BC}}) : (\overline{\mathrm{AB}}\cdot\overline{\mathrm{CO}}),$$

$=\mp$ a *quotient of rectangles of opposite sides;* the upper or the lower *sign* being taken, according as the point B′ falls, or does not fall, *between* the points A′ and C′: that is, according as the quadrilateral OABC is an *uncrossed* or a *crossed* one.

(9.) Hence it is easy to infer that *for any circular group* O, A, B, C, we have the equation,

$$\text{XIII.}\,.\,.\; \mathrm{U}\frac{\mathrm{OA}}{\mathrm{AB}} = \pm\,\mathrm{U}\frac{\mathrm{CO}}{\mathrm{CB}}\,;$$

the *upper* sign being taken when the *succession* OABC is a *direct* one, that is, when the *quadrilateral* OABC is *uncrossed;* and the lower sign, in the contrary case, namely, when the succession is (what may be called) *indirect*, or when the quadrilateral is *crossed:* while conversely this equation XIII. is sufficient to prove, whenever it occurs, that the anharmonic (OABC) is a negative or a positive *scalar*, and therefore by (5.) that the *group* is *circular* (if not *linear*), as above.

(10.) If A, B, C, D, E be any *five homospheric points* (or points upon the surface of *one sphere*), and if O be *any sixth point of space*, while OA′, .. OE′ are the reciprocals of OA, .. OE, then the *five new points* A′ .. E′ are *generally homospheric* (with each other); but if O happens to be *on the sphere* ABCDE, then A′ .. E′ are *complanar*, their common plane being *parallel to the tangent plane* to the given sphere at O: with resulting anharmonic relations, on which we cannot here delay.

261. An interesting *case* of the foregoing theory is that when the generally *scalar* anharmonic of a *circular* group becomes equal to *negative unity :* in which case (comp. 26), the group is said to be *harmonic.* A few remarks upon such *circular and harmonic groups* may here be briefly made: the stu-

dent being left to fill up hints for himself, as what must be now to him an easy exercise of calculation.

(1.) For such a group (comp. again Fig. 58), we have thus the equation,

$$\text{I.} \ldots (\text{OABC}) = -1; \quad \text{and therefore} \quad \text{II.} \ldots \text{A}'\text{B}' = \text{B}'\text{C}';$$

or

$$\text{III.} \ldots \text{R}\beta = \tfrac{1}{2}(\text{R}\alpha + \text{R}\gamma);$$

and under this condition, we shall say (comp. 216, (5.) that the *Vector* β is the *Harmonic Mean* between the *two* vectors, α and γ.

(2.) Dividing, and taking conjugates (comp. 260, (3.), and 216, (5.)), we thus obtain the equation,

$$\text{IV.} \ldots \frac{\beta}{\alpha} + \frac{\beta}{\gamma} = 2; \quad \text{or} \quad \text{V.} \ldots \beta = \frac{2\alpha}{\gamma + \alpha}\gamma = \frac{2\gamma}{\gamma + \alpha}\alpha;$$

or

$$\text{VI.} \ldots \beta = \frac{\alpha}{\iota}\gamma = \frac{\gamma}{\iota}\alpha, \quad \text{if} \quad \text{VII.} \ldots \iota = \tfrac{1}{2}(\gamma + \alpha);$$

ι thus denoting here the vector OE (Fig. 58) of the middle point of the chord AC. We may then say that the *harmonic mean* between any two lines is (as in algebra) the *fourth proportional to their semisum, and to themselves*.

(3.) Geometrically, we have thus the similar triangles,

$$\text{VIII.} \ldots \Delta\,\text{AOB} \propto \text{EOC}; \qquad \text{VIII}'. \ldots \Delta\,\text{AOE} \propto \text{BOC};$$

whence, either because the angles OBA and OCA, or because the angles OAC and OBC are equal, we may infer (comp. 260, (5.)) that, when the equation I. is satisfied, the four points O, A, B, C, if not *collinear*, are *concircular*.

(4.) We have also the similarities,

$$\text{IX.} \ldots \Delta\,\text{OEC} \propto \text{CEB}, \quad \text{and} \quad \text{IX}'. \ldots \Delta\,\text{OEA} \propto \text{AEB};$$

or the equations,

$$\text{X.} \ldots \frac{\beta - \iota}{\gamma - \iota} = \frac{\gamma - \iota}{-\iota}, \quad \text{and} \quad \text{X}'. \ldots \frac{\beta - \iota}{\alpha - \iota} = \frac{\alpha - \iota}{-\iota};$$

in fact we have, by VI. and VII.,

$$\text{XI.} \ldots \frac{\alpha}{\iota} + \frac{\gamma}{\iota} = 2; \qquad \text{XII.} \ldots \frac{\beta - \iota}{-\iota}\left(= 1 - \frac{\beta}{\alpha}\frac{\alpha}{\iota} = 1 - \frac{\gamma}{\iota}\alpha\right) = \left(1 - \frac{\alpha}{\iota}\right)^2.$$

(5.) Hence the line EC, in Fig. 58, is the *mean proportional* (227) between the lines EO and EB; or in words, the *semisum* (OE), the *semidifference* (EC), and the *excess* (BE) *of the semisum over the harmonic mean* (OB), form (as in algebra) a *continued proportion* (227).

(6.) Conversely, if any three coinitial vectors, EO, EC, EB, form thus a continued proportion, and if we take EA = CE, then the four points OABC will compose a circular and harmonic group; for example, the points APBP′ of Fig. 57 are arranged so as to form such a group.*

(7.) It is easy to prove that, for the *inscribed quadrilateral* OABC of Fig. 58, the *rectangles under opposite sides* are each equal to *half* of the rectangle under the

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* Compare the Note to 255, (2.). In that sub-article, the text should have run thus: of which (we may add) the centre C is on the circle OAB, &c. In Fig. 58, the centre of the circle OABC is concircular with the three points O, E, B.

diagonals; which geometrical relation answers to either of the two anharmonic equations (comp. 259, (10.)):

$$\text{XIII.}\ldots(\text{OBAC}) = +2; \qquad \text{XIII}'.\ldots(\text{OCAB}) = +\tfrac{1}{2}.$$

(8.) Hence, or in other ways, it may be inferred that these diagonals, OB, AC, are *conjugate chords* of the circle to which they belong: in the sense that *each* passes *through the pole* of the *other*, and that thus the line DB is the *second tangent* from the point D, in which the chord AC prolonged intersects the tangent at O.

(9.) Under the same conditions, it is easy to prove, either by quaternions or by geometry, that we have the harmonic equations:

$$\text{XIV.}\ldots(\text{ABCO}) = (\text{BCOA}) = (\text{COAB}) = -1;$$

so that AC is the harmonic mean between AB and AO; BO is such a mean between BC and BA; and CA between CO and CB.

(10.) In any such group, *any two opposite points* (or opposite corners of the quadrilateral), as for example O and B, may be said to be *harmonically conjugate* to each other, *with respect to the two other points*, A and C; and we see that when these *two* points A and C are *given*, then to *every third point* O (whether in a given plane, or in space) there always *corresponds* a *fourth* point B, which is in this sense *conjugate* to that third point: this fourth point being always *complanar* with the three points A, C, O, and being even *concircular* with them, unless they happen to be *collinear* with each other; in which extreme (or *limiting*) *case*, the *fourth point* B is *still determined*, but is now collinear with the others (as in 26, &c.).

(11.) When, after thus *selecting two* points*, A and C, or treating them as *given* or *fixed*, we determine (10.) the harmonic *conjugates* B, B′, B″, with respect to *them*, of *any three assumed points*, O, O′, O″, then the *three pairs of points*, O, B; O′, B′; O″, B″, may be said to form an *Involution*,† either *on the right line* AC, (in which case it will only be one of an already well-known kind), or *in a plane* through that line, or even generally *in space:* and the two points A, C may in all these cases be said to be the two *Double Points* (or *Foci*) of this Involution. But the field thus opened, for geometrical investigation by Quaternions, is far too extensive to be more than *mentioned* here.

(12.) We shall therefore only at present add, that the conception of the *harmonic mean* between *two vectors* may easily be extended to *any number* of such, and need not be limited to the *plane:* since we may define that η is the harmonic mean of the n arbitrary vectors $\alpha_1, \ldots \alpha_n$, when it satisfies the equation,

$$\text{XV.}\ldots \mathrm{R}\eta = \frac{1}{n}(\mathrm{R}\alpha_1 + \ldots + \mathrm{R}\alpha_n); \quad \text{or} \quad \text{XVI.}\ldots n\mathrm{R}\eta = \Sigma\mathrm{R}\alpha.$$

(13.) Finally, as regards the *notation* $\mathrm{R}\alpha$, and the *definition* (258) of the *reciprocal of a vector*, it may be observed that if we had chosen to define reciprocal vectors as having *similar* (instead of *opposite*) *directions*, we should indeed have had the positive sign in the equation 258, VII.; but should have been obliged to write, instead of 258, IX., the much less simple formula,

$$\mathrm{RI}v = -\,\mathrm{IR}v.$$

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* There is a sense in which the geometrical process here spoken of can be applied, even when the two fixed points, or *foci*, are *imaginary*. Compare the *Géométrie Supérieure* of M. Chasles, page 136.

† Compare the Note to 259, (11.).

CHAPTER III.

ON DIPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN SPACE: AND ESPECIALLY ON THE ASSOCIATIVE PRINCIPLE OF MULTIPLICATION OF SUCH QUATERNIONS.

SECTION 1.—*On some Enunciations of the Associative Property, or Principle, of Multiplication of Diplanar Quaternions.*

262. In the preceding Chapter we have confined ourselves almost entirely, as had been proposed (224, 225), to the consideration of quaternions *in a given plane* (that of i); alluding only, in some instances, to possible extensions* of results so obtained. But we must now return to consider, as in the First Chapter of this Second Book, the subject of *General Quotients of Vectors:* and especially their *Associative Multiplication* (223), which has hitherto been only proved in connexion with the *Distributive Principle* (212), and with the *Laws of the Symbols i, j, k* (183). And first we shall give a few *geometrical enunciations* of that associative principle, which shall be independent of the distributive one, and in which it will be sufficient to consider (comp. 191) the *multiplication of versors;* because the multiplication of *tensors* is *evidently* an associative operation, as corresponding simply to *arithmetical* multiplication, or to the *composition of ratios* in geometry.† We shall therefore suppose, throughout the present Chapter, that q, r, s are some *three given but arbitrary versors*, in *three given and distinct planes;*‡ and our object will be to throw

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* As in 227, (3.); 242, (7.); 254, (7.); 257, (6.) and (7.); 259, (8.), (9.), (10.), (11.); 260, (10.); and 261, (11.) and (12.).

† Or, more generally, for any three pairs of magnitudes, each pair separately being homogeneous.

‡ If the factors q, r, s were *complanar*, we could always (by 120) put them

some additional light, by new enunciations in this Section, and by new demonstrations in the next, on the very important, although very simple, *Associative Formula* (223, II.), which may be written thus:

$$\text{I.} \ldots sr.q = s.rq;$$

or thus, more fully,

$$\text{II.} \ldots q'q = t, \quad \text{if} \quad q' = sr, \quad s' = rq, \quad \text{and} \quad t = ss';$$

q', s', and t being here *three new and derived versors*, in *three new and derived planes*.

263. Already we may see that this *Associative Theorem of Multiplication*, in all its forms, has an essential reference to a *System of Six Planes*, namely the planes of these *six versors*,

$$\text{IV.} \ldots q, r, s, rq, sr, srq, \quad \text{or} \quad \text{IV}'. \ldots q, r, s, s', q', t;$$

on the judicious selection and arrangement of which, the clearness and elegance of every geometrical statement or proof of the theorem must very much depend: while the *versor character* of the factors (in the only part of the theorem for which *proof* is required) suggests a reference to a *Sphere*, namely to what we have called the *unit-sphere* (128). And the *three* following *arrangements* of the six planes appear to be the most natural and simple that can be considered: namely, Ist, the arrangement in which the planes all pass *through the centre* of the sphere; IInd, that in which they all *touch* its surface; and IIIrd, that in which they are the six *faces of an inscribed solid*. We proceed to consider successively these three arrangements.

264. When the *first* arrangement (263) is adopted, it is natural to employ *arcs of great circles*, as *representatives* of the *versors*, on the

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under the forms,

$$q = \frac{\beta}{\alpha}, \qquad r = \frac{\gamma}{\beta}, \qquad s = \frac{\delta}{\gamma};$$

and then should have (comp. 183, (1.)) the *two equal ternary products*,

$$sr.q = \frac{\delta}{\beta}\frac{\beta}{\alpha} = \frac{\delta}{\alpha} = \frac{\delta}{\gamma}\frac{\gamma}{\alpha} = s.rq;$$

so that in *this* case (comp. 224) the associative property would be proved without any difficulty.

plan of Art. 162. Representing thus the factor q by the arc AB, and r by the successive arc BC, we represent (167) their *product* rq, or s', by AC; or by any *equal arc* (165), such as DE, in Fig. 59, may be supposed to be. Again, representing s by EF, we shall have DF as the representative of the *ternary product* $s.rq$, or ss', or t, taken in *one order of association*. To represent the *other* ternary product, $sr.q$, or $q'q$, we may first determine three new points, G, H, I, by arcual equations (165), between GH, BC, and between HI, EF, so that BC, EF intersect in H, as the arcs representing s' and s had intersected in E; and then, after thus finding an arc GI which represents sr, or q', may determine three other points, K, L, M, by equations between KL, AB, and between LM, GI, so that these two new arcs, KL, LM, represent q and q', and that AB, GI intersect in L; for in this way we shall have an arc, namely KM, which represents $q'q$ as required. And the *theorem* then is, that this last arc KM is *equal* to the former arc DF, *in the full sense* of Art. 165; or that *when* (as under the foregoing conditions of construction) *the five arcual equations*,

q r

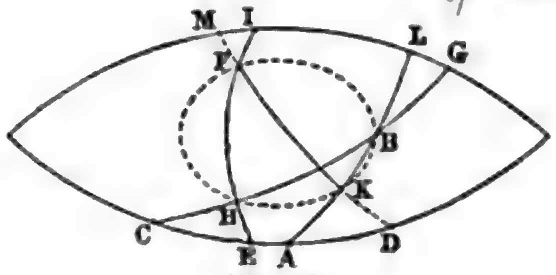

Fig. 59.

$$\text{I.} \ldots \quad \frown AB = \frown KL, \quad \frown BC = \frown GH, \quad \frown EF = \frown HI, \quad \frown AC = \frown DE, \quad \frown GI = \frown LM,$$

exist, then this sixth equation of the same kind is satisfied also,

$$\text{II.} \ldots \quad \frown DF = \frown KM:$$

the *two points*, K and M, being both *on the same great circle* as the two previously determined points, D and F; or D and M being on the great circle through F and K: and the *two arcs*, DF and KM, of that great circle, or the two *dotted* arcs, DK, FM in the Figure, being *equally long*, and *similarly directed* (165).

(1.) Or, after determining the *nine points* A . . I so as to satisfy the three middle equations I., we might determine the *three other* points, K, L, M, without any other arcual *equations*, as *intersections* of the three pairs of arcs AB, DF; AB, GI; DF, GI; and then the theorem would be, that (if these three last points be suitably distinguished from their own opposites upon the sphere) the two extreme equations I., and the equation II., are satisfied.

(2.) The same geometrical theorem may also be thus enunciated: *If the first, third, and fifth sides* (KL, GH, ED) *of a spherical hexagon* KLGHED *be respectively and arcually equal* (165) *to the first, second, and third sides* (AB, BC, CA) *of a spherical triangle* ABC, *then the second, fourth, and sixth sides* (LG, HE, DK) *of the same hexagon are equal to the three successive sides* (MI, IF, FM) *of another spherical triangle*, MIF.

(3.) It may also be said, that *if five successive sides* (KL, .. ED) of *one* spherical *hexagon* be respectively and arcually *equal* to the *five successive diagonals* (AB, MI, BC, IF, CA) of *another* such hexagon (AMBICF), then the *sixth side* (DK) of the *first* is equal to the *sixth diagonal* (FM) of the *second.*

(4.) Or, if we adopt the conception mentioned in 180, (3.), of an *arcual sum*, and *denote* such a sum by inserting + between the symbols of the two summands, that of the *added arc* being written to the *left-hand*, we may state the theorem, in connexion with the recent Fig. 59, by the formula:

$$\text{III.} \ldots \ \frown\mathrm{DF} + \frown\mathrm{BA} = \frown\mathrm{EF} + \frown\mathrm{BC}, \quad \text{if} \quad \frown\mathrm{DA} = \frown\mathrm{EC};$$

where B and F may denote *any two points* upon the sphere.

(5.) We may also express* the same principle, although somewhat less simply, as follows (see again Fig. 59, and compare sub-art. (2.)):

$$\text{IV.} \ldots \text{if} \ \frown\mathrm{ED} + \frown\mathrm{GH} + \frown\mathrm{KL} = 0, \quad \text{then} \quad \frown\mathrm{DK} + \frown\mathrm{HE} + \frown\mathrm{LG} = 0.$$

(6.) If, for a moment, we agree to write (comp. Art. 1),

$$\text{V.} \ldots \ \frown\mathrm{AB} = \overparen{\mathrm{B} - \mathrm{A}},$$

we may then express the recent statement IV. a little more lucidly thus:

$$\text{VI.} \ldots \text{if} \ \overparen{\mathrm{D} - \mathrm{E}} + \overparen{\mathrm{H} - \mathrm{G}} + \overparen{\mathrm{L} - \mathrm{K}} = 0, \quad \text{then} \quad \overparen{\mathrm{K} - \mathrm{D}} + \overparen{\mathrm{E} - \mathrm{H}} + \overparen{\mathrm{G} - \mathrm{L}} = 0.$$

(7.) Or still more simply, if $\frown$, $\frown'$, $\frown''$ be supposed to denote *any three diplanar arcs*, which are to be *added* according to the *rule* (180, (3.)) above referred to, the *theorem* may be said to be, that

$$\text{VII.} \ldots (\frown'' + \frown') + \frown = \frown'' + (\frown' + \frown);$$

or in words, that *Addition of Arcs on a Sphere is an Associative Operation.*

(8.) Conversely, if any independent demonstration be given, of the truth of any one of the foregoing statements, considered as expressing a *theorem of spherical geometry*,† a *new proof* will thereby be furnished, of the associative property of *multiplication of quaternions.*

265. In the *second* arrangement (263) of the *six planes*, instead of representing the three given versors, and their partial or total products, by *arcs*, it is natural to represent them (174, II.) by *angles* on the sphere. Conceive then that the two versors, q and r, are represented, in Fig. 60, by the two spherical angles, EAB and ABE; and therefore (175) that their product, rq or s', is represented by the external vertical angle at E, of the triangle ABE. Let the second versor r be also represented by the angle FBC, and the third versor s by BCF; then the other binary product, sr or q', will be represented by the external angle at F, of the new triangle BCF. Again, to represent the *first* ternary product, $t = ss' = s.rq$, we have only to take the external angle at D of the triangle ECD, if D be a point determined by the two conditions, that the angle ECD shall be *equal* to BCF, and DEC *supplementary* to BEA. On the other hand, if we conceive a point D′ determined by the conditions that D′AF shall be equal to EAB, and AFD′ supplementary to CFB, then the external angle at D′, of the triangle AFD′, will represent the *second* ternary product, $q'q = sr.q$, which (by the associative principle) must be *equal to the first.* Conceiving then that ED is prolonged to G, and FD′ to H, the *two spherical angles*, GDC and AD′H, must be *equal in all respects;* their *vertices* D and D′ *coinciding*, and the *rotations* (174, 177) which they represent being not only *equal in amount*, but also *similarly directed.* Or, to express the same thing otherwise, we may *enunciate* (262) *the Associative Principle* by saying, that *when the three angular equations,*

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* Some of these formulæ and figures, in connexion with the associative principle, are taken, though for the most part with modifications, from the author's Sixth Lecture on Quaternions, in which that whole subject is very fully treated. Comp. the Note to page 160.

† Such a demonstration, namely a deduction of the equation II. from the five equations I., by known properties of *spherical conics*, will be briefly given in the ensuing Section.

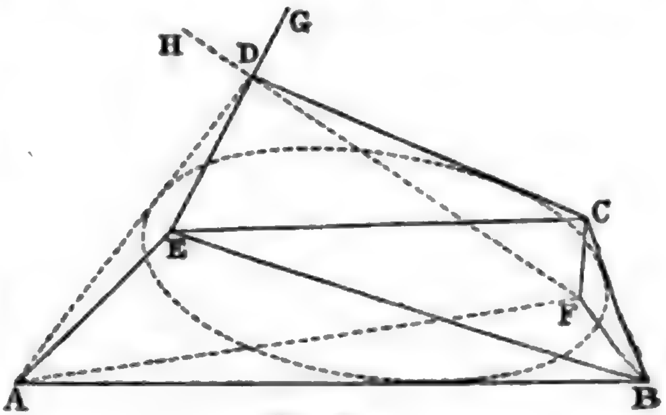

Fig. 60.

I. . . ABE = FBC, BCF = ECD, DEC = π − BEA,

are satisfied, then these three other equations,

II. . . DAF = EAB, FDA = CDE, AFD = π − CFB,

are satisfied also. For not only is this *theorem of spherical geometry* a *consequence* of the associative principle of *multiplication of quaternions*, but conversely any independent demonstration* of the theorem is, at the same time, a proof of the principle.

266. The *third arrangement* (263) of the six planes may be illustrated by conceiving a *gauche hexagon*, AB′CA′BC′, to be *inscribed in a sphere*, in such a manner that the intersection D of the three planes, C′AB′, B′CA′, A′BC′, is on the surface; and therefore that the *three small circles*, denoted by these three last triliteral symbols, *concur*

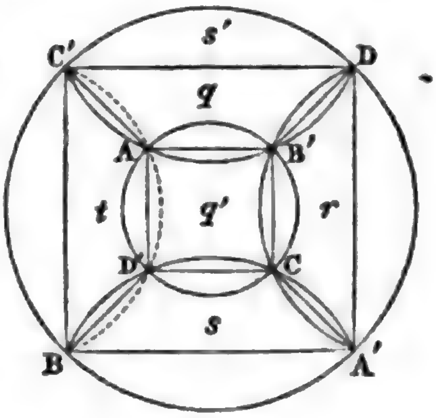

Fig. 61.

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* Such as we shall sketch, in the following Section, with the help of the known properties of the *spherical conics*. Compare the Note to the foregoing Article.

in one point D; while the second intersection of the two other small circles, AB′C, CA′B, may be denoted by the letter D′, as in the annexed Fig. 61. Let it be also for simplicity at first supposed, that (as in the Figure) the *five circular successions*,

I. . . C′AB′D, AB′CD′, B′CA′D, CA′BD′, A′BC′D,

are all *direct*; or that the *five inscribed quadrilaterals*, denoted by these symbols I., are all *uncrossed* ones. Then (by 260, (9.)) it is allowed to introduce *three versors*, q, r, s, each having *two expressions*, as follows:

$$\text{II.}\ldots\ q = \mathrm{U}\frac{\mathrm{B'D}}{\mathrm{DC'}} = +\mathrm{U}\frac{\mathrm{AB'}}{\mathrm{AC'}};\quad r = \mathrm{U}\frac{\mathrm{DA'}}{\mathrm{B'D}} = +\mathrm{U}\frac{\mathrm{CA'}}{\mathrm{CB'}};$$

$$s = \mathrm{U}\frac{\mathrm{CD'}}{\mathrm{CA'}} = +\mathrm{U}\frac{\mathrm{BD'}}{\mathrm{A'B}};$$

although (by the cited sub-article) the last members of these three formulæ should receive the *negative* sign, if the first, third, and fourth of the successions I. were to become *indirect*, or if the corresponding quadrilaterals were *crossed* ones. We have thus (by 191) the derived expressions,

$$\text{III.}\ldots\ s' = rq = \mathrm{U}\frac{\mathrm{DA'}}{\mathrm{DC'}} = \mathrm{U}\frac{\mathrm{A'B}}{\mathrm{BC'}};\qquad q' = sr = \mathrm{U}\frac{\mathrm{CD'}}{\mathrm{CB'}} = \mathrm{U}\frac{\mathrm{D'A}}{\mathrm{AB'}};$$

whereof, however, the two versors in the first formula would differ in their signs, if the fifth succession I. were indirect; and those in the second formula, if the second succession were such. Hence,

$$\text{IV.}\ldots\ t = ss' = s\,.\,rq = \mathrm{U}\frac{\mathrm{BD'}}{\mathrm{BC'}};\qquad q'q = sr\,.\,q = \mathrm{U}\frac{\mathrm{D'A}}{\mathrm{AC'}};$$

and since, by the associative principle, these two last versors are to be *equal*, it follows that, under the supposed conditions of construction, the *four points*, B, C′, A, D′, compose a *circular* and *direct succession*; or that the *quadrilateral*, BC′AD′, is *plane*, *inscriptible*,* and *uncrossed*.

267. It is easy, by suitable changes of sign, to adapt the recent reasoning to the case where some or all of the successions I. are indirect; and thus to *infer*, from the associative principle, this *theorem of spherical geometry*: *If* AB′CA′BC′

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* Of course, since the four points BC′AD′ are known to be *homospheric* (comp. 260. (10.)), the *inscriptibility* of the quadrilateral in a *circle* would *follow* from its being *plane*, if the latter were otherwise proved: but it is *here* deduced from the *equality* of the *two versors* IV., on the plan of 260, (9.).

be a spherical hexagon, such that the three small circles C′AB′, B′CA′, A′BC′ *concur in one point* D, *then*, Ist, *the three other small circles*, AB′C, CA′B, BC′A, *concur in another point*, D′; and IInd, *of the six circular successions*, 266, I., and BC′AD′, *the number of those which are indirect is always even* (including *zero*). And conversely, any independent demonstration* of this geometrical theorem will be a *new proof* of the associative principle.

268. The same fertile principle of *associative multiplication* may be enunciated in other ways, without limiting the factors to be *versors*, and without introducing the conception of a *sphere*. Thus we may say (comp. 264, (2.)), that if O.ABCDEF (comp. 35) be any *pencil of six rays in space*, and O.A′B′C′ any *pencil of three rays*, and if the three angles AOB, COD, EOF of the first pencil be respectively equal to the angles B′OC′, C′OA′, A′OB′ of the second, then *another* pencil of *three* rays, O.A″B″C″, can be assigned, such that the three other angles BOC, DOE, FOA of the *first* pencil shall be equal to the angles B″OC″, C″OA″, A″OB″ of the *third*: *equality of angles* (with one vertex) being *here* understood (comp. 165) to *include complanarity*, and *similarity of direction of rotations.*

(1.) Again (comp. 264, (4.)), we may establish the following formula, in which the four vectors $\alpha\beta\gamma\delta$ form a complanar proportion (226), but ϵ and ζ are any two lines in space:

$$\text{I.}\,\ldots\;\frac{\zeta}{\gamma}\frac{\delta}{\epsilon}=\frac{\zeta}{\alpha}\frac{\beta}{\epsilon},\quad\text{if}\quad\frac{\delta}{\gamma}=\frac{\beta}{\alpha};$$

for, under this last condition, we have (comp. 125),

$$\text{II.}\,\ldots\;\frac{\zeta}{\gamma}\frac{\delta}{\epsilon}=\frac{\zeta}{\alpha}\frac{\alpha}{\gamma}.\frac{\delta}{\epsilon}=\frac{\zeta}{\alpha}.\frac{\beta}{\delta}\frac{\delta}{\epsilon}.$$

(2.) Another enunciation of the associative principle is the following:

$$\text{III.}\,\ldots\;\text{if}\quad\frac{\delta}{\gamma}\frac{\beta}{\alpha}=\frac{\zeta}{\epsilon},\quad\text{then}\quad\frac{\epsilon}{\alpha}\frac{\beta}{\gamma}=\frac{\zeta}{\delta};$$

for if we determine (120) six new vectors, $\eta\theta\iota$, and $\kappa\lambda\mu$, so that

$$\text{IV.}\,\ldots\left\{\begin{array}{l}\dfrac{\theta}{\eta}=\dfrac{\delta}{\gamma},\quad\dfrac{\eta}{\iota}=\dfrac{\beta}{\alpha},\quad\text{whence}\quad\dfrac{\theta}{\iota}=\dfrac{\zeta}{\epsilon},\\ \qquad\qquad\text{and}\\ \qquad\dfrac{\lambda}{\kappa}=\dfrac{\epsilon}{\alpha},\quad\dfrac{\kappa}{\mu}=\dfrac{\beta}{\gamma},\end{array}\right.$$

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* An elementary proof, by *stereographic projection*, will be proposed in the following Section.

we shall have the transformations,

$$\text{V.} \ldots \frac{\lambda}{\zeta} = \frac{\lambda}{\varepsilon}\frac{\iota}{\theta} = \frac{\lambda}{\varepsilon} \cdot \frac{\iota}{\eta}\frac{\eta}{\theta} = \frac{\lambda}{\varepsilon}\frac{\iota}{\eta} \cdot \frac{\eta}{\theta} = \frac{\kappa}{\beta}\frac{\gamma}{\delta} = \frac{\mu}{\delta}, \quad \text{or} \quad \text{VI.} \ldots \frac{\lambda}{\mu} = \frac{\zeta}{\delta}.$$

(3.) Conversely, the assertion that this last equation or proportion VI. is true, whenever the twelve vectors $\alpha \ldots \mu$ are connected by the five proportions IV., is a form of enunciation of the associative principle; for it conducts (comp. IV. and V.) to the equation,

$$\text{VII.} \ldots \frac{\lambda}{\varepsilon} \cdot \frac{\iota}{\eta}\frac{\eta}{\theta} = \frac{\lambda}{\varepsilon}\frac{\iota}{\eta} \cdot \frac{\eta}{\theta}, \quad \text{at least if} \quad \varepsilon \parallel\!| \iota, \theta;$$

but, even with this last restriction, the three factor-quotients in VII. may represent *any three quaternions*.

SECTION 2.—*On some Geometrical Proofs of the Associative Property of Multiplication of Quaternions, which are independent of the Distributive* Principle.*

269. We propose, in this Section, to furnish *three* geometrical Demonstrations of the Associative Principle, in connexion with the three Figures (59–61) which were employed in the last Section for its Enunciation; and with the *three arrangements* of *six planes*, which were described in Art. 263. The two first of these proofs will suppose the knowledge of a few properties of *spherical conics* (196, (11.)); but the third will only employ the doctrine of *stereographic projection*, and will therefore be of a more strictly *elementary character*. The Principle itself is, however, of such great importance in this Calculus, that its nature and its evidence can scarcely be put in too many different points of view.

270. The only properties of a spherical conic, which we shall in this Article assume as known,† are the three following: Ist, that *through any three given points* on a given sphere, which are not on a great circle, a *conic* can be described (consisting generally of *two opposite ovals*), which shall have a *given great circle* for *one* of its *two cyclic arcs;* IInd, that if a *transversal* arc cut *both* these arcs, and the conic, the *intercepts* (suitably measured) on this transversal are *equal;* and IIIrd, that if the *vertex* of a spherical angle *move along the conic,* while its *legs* pass always *through two fixed points* thereof, those legs

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* Compare 224 and 262; and the Note to page 236.

† The reader may consult the Translation (Dublin, 1841, pp. 46, 50, 55) by the present Dean Graves, of two Memoirs by M. Chasles, on *Cones of the Second Degree, and Spherical Conics.*

intercept a *constant interval*, upon *each* cyclic arc, separately taken. Admitting these three properties, we see that if, in Fig. 59, we conceive a spherical conic to be described, so as to pass through the three points B, F, H, and to have the great circle DAEC for *one* cyclic arc, the second and third equations I. of 264 will prove that the arc GLIM is the *other* cyclic arc for this conic; the first equation I. proves next that the conic passes through K; and if the arcual chord FK be drawn and prolonged, the two remaining equations prove that it meets the cyclic arcs in D and M; after which, the equation II. of the same Art. 264 immediately results, at least with the arrangement* adopted in the Figure.

(1.) The Ist property is easily seen to correspond to the possibility of circumscribing a circle about a given plane triangle, namely that of which the corners are the intersections of a plane parallel to the plane of the given cyclic arc, with the three radii drawn to the three given points upon the sphere: but it may be worth while, as an exercise, to prove here the IInd property *by quaternions*.

(2.) Take then the equation of a *cyclic cone*, 196, (8.), which may (by 196, XII.) be written thus:

$$\text{I.} \ldots \mathrm{S}\frac{\rho}{\alpha}\,\mathrm{S}\frac{\rho}{\beta}=\mathrm{N}\frac{\rho}{\beta};\quad\text{and let}\quad \text{II.} \ldots \mathrm{S}\frac{\rho'}{\alpha}\,\mathrm{S}\frac{\rho'}{\beta}=\mathrm{N}\frac{\rho'}{\beta},$$

ρ and ρ' being thus *two rays* (or *sides*) of the cone, which may also be considered to be the vectors of two points P and P′ of a *spherical conic*, by supposing that their lengths are each unity. Let τ and τ' be the vectors of the two points T and T′ on the two cyclic arcs, in which the arcual chord PP′ of the conic cuts them; so that

$$\text{III.} \ldots \mathrm{S}\frac{\tau}{\alpha}=0,\quad \mathrm{S}\frac{\tau'}{\beta}=0,\quad\text{and}\quad \text{IV.} \ldots \mathrm{T}\tau=\mathrm{T}\tau'=1.$$

The theorem may then be stated thus: that

$$\text{V.} \ldots \text{if } \rho=x\tau+x'\tau',\quad\text{then}\quad \text{VI.} \ldots \rho'=x'\tau+x\tau';$$

or that this expression VI. satisfies II., if the equations I. III. IV. V. be satisfied. Now, by III. V. VI., we have

$$\text{VII.} \ldots \mathrm{S}\frac{\rho}{\alpha}=x'\mathrm{S}\frac{\tau'}{\alpha}=\frac{x'}{x}\,\mathrm{S}\frac{\rho'}{\alpha},\qquad \mathrm{S}\frac{\rho}{\beta}=x\mathrm{S}\frac{\tau}{\beta}=\frac{x}{x'}\,\mathrm{S}\frac{\rho'}{\beta};$$

whence it follows that the first members of I. and II. are equal, and it only remains to prove that their second members are equal also, or that $\mathrm{T}\rho'=\mathrm{T}\rho$, if $\mathrm{T}\tau'=\mathrm{T}\tau$. Accordingly we have, by V. and VI.,

$$\text{VIII.} \ldots \frac{\rho'-\rho}{\rho'+\rho}=\frac{x'-x}{x'+x}\cdot\frac{\tau-\tau'}{\tau+\tau'}=\mathrm{S}^{-1}0,\ \text{by 200, (11.), and 204, (19.)};$$

and the property in question is proved.

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* Modifications of that arrangement may be conceived, to which however it would be easy to adapt the reasoning.

271. To prove the associative principle, with the help of Fig. 60, three other properties of a spherical conic shall be supposed known:[*] Ist, that for every such curve *two focal points* exist, possessing several important relations to it, one of which is, that if these *two foci* and *one tangent arc* be *given*, the conic can be constructed; IInd, that if, from any point upon the sphere, *two tangents* be drawn to the conic, and also *two arcs to the foci*, then *one focal arc* makes *with one tangent* the *same angle* as the *other* focal arc with the *other* tangent; and IIIrd, that if a spherical quadrilateral be circumscribed to such a conic (supposed here for simplicity to be a spherical *ellipse*, or the *opposite* ellipse being neglected), *opposite sides subtend supplementary angles*, at *either* of the two (interior) foci. Admitting these known properties, and supposing the arrangement to be as in Fig. 60, we may conceive a conic described, which shall have E and F for its two focal points, and shall touch the arc BC; and then the two first of the equations I., in 265, will prove that it touches also the arcs AB and CD, while the third of those equations proves that it touches AD, so that ABCD is a circumscribed[†] quadrilateral: after which the three equations II., of the same article, are consequences of the same properties of the curve.

272. Finally, to prove the same important Principle in a more completely elementary way, by means of the arrangement represented in Fig. 61, or to prove the theorem of spherical geometry enunciated in Art. 267, we may assume the point D as the *pole* of a *stereographic projection*, in which the three small circles through that point shall be represented by *right lines*, but the three others by *circles*, all being in one *common plane*. And then (interchanging accents) the theorem comes to be thus stated:

If A′, B′, C′ *be any three points* (comp. Fig. 62) *on the sides* BC, CA, AB *of any plane triangle*, or on those sides prolonged, *then*, Ist, *the three circles*,

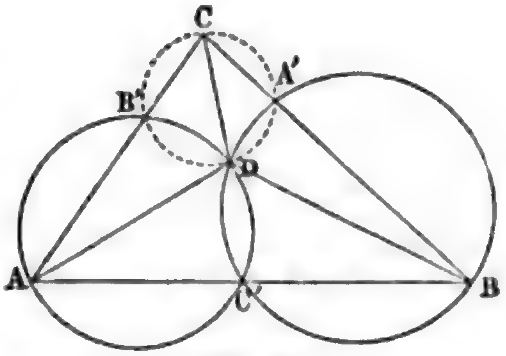

Fig. 62.

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[*] The reader may again consult pages 46 and 50 of the Translation lately cited. In strictness, there are of course *four foci*, opposite two by two.

[†] The writer has elsewhere proposed the *notation*, EF(. .)ABCD, to denote the relation of the focal points E, F to this circumscribed quadrilateral.

$$\text{I.} \ldots C'AB', \quad A'BC', \quad B'CA',$$

will *meet in one point* D; and IInd, *an even number* (if any) *of the six* (linear or circular) *successions*,

$$\text{II.} \ldots AB'C, \quad BC'A, \quad CA'B, \quad \text{and} \quad \text{II}'. \ldots C'AB'D, \quad A'BC'D, \quad B'CA'D,$$

will be *direct;* an *even* number therefore *also* (if any) being *indirect*. But, under this *form*,* the theorem can be proved by very elementary considerations, and still without any employment of the *distributive principle* (224, 262).

(1.) The *first part* of the theorem, as thus stated, is evident from the Third Book of Euclid; but to prove *both parts* together, it may be useful to proceed as follows, admitting the conception (235) of *amplitudes*, or of angles as representing *rotations*, which may have *any values*, positive or negative, and are to be *added* with attention to their *signs*.

(2.) We may thus write the three *equations*,

$$\text{III.} \ldots AB'C = n\pi, \quad BC'A = n'\pi, \quad CA'B = n''\pi,$$

to express the three *collineations*, AB'C, &c. of Fig. 62; the *integer*, n, being *odd* or *even*, according as the point B' is on the finite line AC, or on a prolongation of that line; or in other words, according as the first *succession* II. is *direct* or *indirect*: and similarly for the two other coefficients, n' and n''.

(3.) Again, if OPQR be *any four points* in one plane, we may establish the formula,

$$\text{IV.} \ldots POQ + QOR = POR + 2m\pi,$$

with the same conception of addition of amplitudes; if then D be *any point in the plane* of the triangle ABC, we may write,

$$\text{V.} \ldots AB'D + DB'C = n\pi, \quad BC'D + DC'A = n'\pi, \quad CA'D + DA'B = n''\pi;$$

and therefore,

$$\text{VI.} \ldots (AB'D + DC'A) + (BC'D + DA'B) + (CA'D + DB'C) = (n + n' + n'')\pi.$$

(4.) Again, if any four points OPQR be not merely *complanar* but *concircular*, we have the general formula,

$$\text{VII.} \ldots OPQ + QRO = p\pi,$$

the integer p being *odd* or *even*, according as the succession OPQR is *direct* or *indi-*

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* The Associative Principle of Multiplication was stated nearly under this *form*, and was illustrated by the same simple *diagram*, in paragraph XXII. of a communication by the present author, which was entitled *Letters on Quaternions*, and has been printed in the First and Second Editions of the late Dr. Nichol's *Cyclopædia of the Physical Sciences* (London and Glasgow, 1857 and 1860). The same communication contained other illustrations and consequences of the same principle, which it has not been thought necessary here to reproduce (compare however Note C); and others may be found in the Sixth of the author's already cited *Lectures on Quaternions* (Dublin, 1853), from which (as already observed) some of the formulæ and figures of this Chapter have been taken.

rect; if then we denote by D the *second intersection* of the first and second circles I., whereof C′ is a *first* intersection, we shall have

$$\text{VIII.} \ldots \mathrm{AB'D} + \mathrm{DC'A} = p\pi, \qquad \mathrm{BC'D} + \mathrm{DA'B} = p'\pi,$$

p and p' being *odd*, when the two first successions II′. are direct, but *even* in the contrary case.

(5.) Hence, by VI., we have,

$$\text{IX.} \ldots \mathrm{CA'D} + \mathrm{DB'C} = p''\pi, \quad \text{where X.} \ldots p + p' + p'' = n + n' + n'';$$

the *third succession* II′. is therefore *always circular*, or the *third circle* I. passes *through the intersection* D of the *two first;* and it is *direct* or *indirect*, that is to say, p'' is *odd* or *even*, according as the *number of even coefficients*, among the *five* previously considered, is itself *even* or *odd;* or in other words, according as the *number of indirect successions*, among the *five* previously considered, is *even* (including *zero*), or *odd*.

(6.) In every case, therefore, the *total number* of successions of *each kind* is *even*, and *both* parts of the theorem are proved: the importance of the *second part* of it (respecting the *even partition*, if any, of the *six successions* II. II′.) arising from the necessity of proving that we have *always*, as in algebra,

$$\text{XI} \ldots sr \,.\, q = + s \,.\, rq, \quad \text{and } never \quad \text{XII.} \ldots sr \,.\, q = - s \,.\, rq,$$

if q, r, s be *any three actual quaternions*.

(7.) The *associative principle* of multiplication may also be proved, without the *distributive* principle, by certain considerations of *rotations of a system*, on which we cannot enter here.

SECTION 3.—*On some Additional Formulæ.*

273. Before concluding the Second Book, a few additional remarks may be made, as regards some of the notations and transformations which have already occurred, or others analogous to them. And first as to *notation*, although we have reserved for the Third Book the *interpretation* of such expressions as $\beta\alpha$, or α^2, yet we have agreed, in 210, (9.), to *abridge* the frequently occurring *symbol* $(\mathrm{T}\alpha)^2$ to $\mathrm{T}\alpha^2$; and we *now* propose to abridge it still further to $\mathrm{N}\alpha$, and to *call* this *square of the tensor* (or of the *length*) *of a vector*, α, *the Norm of that Vector:* as we had (in 190, &c.), the equation $\mathrm{T}q^2 = \mathrm{N}q$, and called $\mathrm{N}q$ the *norm* of the *quaternion* q (in 145, (11.)). We shall therefore now write generally, *for any vector* α, the formula,

$$\text{I.} \ldots (\mathrm{T}\alpha)^2 = \mathrm{T}\alpha^2 = \mathrm{N}\alpha.$$

(1.) The equations (comp. 186, (1.) (2.) (3.) (4.)),

$$\text{II.} \ldots \mathrm{N}\rho = 1; \qquad \text{III.} \ldots \mathrm{N}\rho = \mathrm{N}\alpha; \qquad \text{IV.} \ldots \mathrm{N}(\rho - \alpha) = \mathrm{N}\alpha;$$
$$\text{V.} \ldots \mathrm{N}(\rho - \alpha) = \mathrm{N}(\beta - \alpha),$$

represent, respectively, the *unit-sphere;* the sphere through A, with O for centre; the sphere through O, with A for centre; and the sphere through B, with the same centre A.

(2.) The equations (comp. 186, (6.) (7.)),

$$\text{VI.} \ldots \mathrm{N}(\rho+\alpha)=\mathrm{N}(\rho-\alpha); \qquad \text{VII.} \ldots \mathrm{N}(\rho-\beta)=\mathrm{N}(\rho-\alpha),$$

represent, respectively, the *plane* through O, perpendicular to the line OA; and the plane which *perpendicularly bisects* the line AB.

274. As regards *transformations*, the few following may here be added, which relate partly to the *quaternion forms* (204, 216, &c.) of the *Equation*[*] *of the Ellipsoid.*

(1.) Changing $\mathrm{K}(\kappa:\rho)$ to $\mathrm{R}\rho:\mathrm{R}\kappa$, by 259, VIII., in the equation 217, XVI. of the ellipsoid, and observing that the three vectors ρ, $\mathrm{R}\rho$, and $\mathrm{R}\kappa$ are complanar, while $1:\mathrm{T}\rho=\mathrm{TR}\rho$ by 258, that equation becomes, when divided by $\mathrm{TR}\rho$, and when the value 217, (5.) for t^2 is taken, and the notation 273 is employed:

$$\text{I.} \ldots \mathrm{T}\left(\frac{\iota}{\mathrm{R}\rho}+\frac{\rho}{\mathrm{R}\kappa}\right)=\mathrm{N}\iota-\mathrm{N}\kappa;$$

of which the first member will soon be seen to admit of being written[†] as $\mathrm{T}(\iota\rho+\rho\kappa)$, and the second member as $\kappa^2-\iota^2$.

(2.) If, in connexion with the earlier forms (204, 216) of the equation of the same surface, we introduce a *new auxiliary vector*, σ or OS, such that (comp. 216, VIII.)

$$\text{II.} \ldots \sigma=\left(\mathrm{S}\frac{\rho}{\alpha}+\mathrm{V}\frac{\rho}{\beta}\right)\beta=\rho+2\beta\mathrm{S}\frac{\rho}{\delta},$$

the equation may, by 204, (14.), be reduced to the following extremely simple form:

$$\text{III.} \ldots \mathrm{T}\sigma=\mathrm{T}\beta;$$

which expresses that the *locus* of the *new auxiliary point* S is what we have called the *mean sphere*, 216, XIV.; while the *line* PS, or $\sigma-\rho$, which *connects* any two *corresponding points*, P and S, on the ellipsoid and sphere, is seen to be *parallel to the fixed line* β; which is *one element of the homology*, mentioned in 216, (10.).

(3.) It is easy to prove that

$$\text{IV.} \ldots \mathrm{S}\frac{\sigma}{\delta}=\mathrm{S}\frac{\beta}{\alpha}\,\mathrm{S}\frac{\rho}{\delta}, \quad \text{and therefore} \quad \text{V.} \ldots \mathrm{S}\frac{\sigma'}{\delta}:\mathrm{S}\frac{\sigma}{\delta}=\mathrm{S}\frac{\rho'}{\delta}:\mathrm{S}\frac{\rho}{\delta},$$

if ρ' and σ' be the vectors of two new but corresponding points, P′ and S′, on the ellipsoid and sphere; whence it is easy to infer this *other element of the homology*, that any two *corresponding chords*, PP′ and SS′, of the two surfaces, *intersect each other on the cyclic plane* which has δ for its *cyclic normal* (comp. 216, (7.)): in fact, they intersect in the point T of which the vector is,

$$\text{VI.} \ldots \tau=\frac{x\rho+x'\rho'}{x+x'}=\frac{x\sigma+x'\sigma'}{x+x'}, \quad \text{if} \quad x=\mathrm{S}\frac{\rho'}{\delta}, \quad \text{and} \quad x'=-\mathrm{S}\frac{\rho}{\delta};$$

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[*] In the verification 216, (2.) of the equation 216, (1.), considered as representing a *surface of the second order*, $\mathrm{V}\frac{\lambda}{\beta}$ and $\mathrm{V}\frac{\mu}{\beta}$ ought to have been printed, instead of $\mathrm{V}\frac{\lambda}{\alpha}$ and $\mathrm{V}\frac{\mu}{\alpha}$; but this does not affect the reasoning.

[†] Compare the Note to page 283.

and this point is on the plane just mentioned (comp. 216, XI.), because

$$\text{VII.}\ldots \mathrm{S}\frac{\tau}{\delta} = 0.$$

(4.) Quite similar results would have followed, if we had assumed

$$\text{VIII.}\ldots \sigma = \left(-\mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\beta}\right)\beta = \rho - 2\beta\mathrm{S}\frac{\rho}{\gamma},$$

which would have given again, as in III.,

$$\text{IX.}\ldots \mathrm{T}\sigma = \mathrm{T}\beta, \quad \text{but with} \quad \text{X.}\ldots \mathrm{S}\frac{\sigma}{\gamma} = -\mathrm{S}\frac{\beta}{\alpha}\,\mathrm{S}\frac{\rho}{\gamma};$$

the *other cyclic plane*, with γ instead of δ for its *normal*, might therefore have been taken (as asserted in 216, (10.)), as *another plane of homology* of ellipsoid and sphere, with the *same centre of homology* as before: namely, the *point at infinity on the line* β, or on the *axis* (204, (15.)) of one of the two *circumscribed cylinders of revolution* (comp. 220, (4.)).

(5.) The same ellipsoid is, *in two other ways*, homologous to the same mean sphere, with the same two cyclic planes as *planes* of homology, but with a *new centre* of homology, which is the infinitely distant point on the axis of the *second circumscribed cylinder* (or on the line AB′ of the sub-article last cited).

(6.) Although not specially connected with the *ellipsoid*, the following general transformations may be noted here (comp. 199, XII., and 204, XXXIV′.):

$$\text{XI.}\ldots \mathrm{TV}\sqrt{q} = \sqrt{\{\tfrac{1}{2}(\mathrm{T}q - \mathrm{S}q)\}}; \quad \text{XII.}\ldots \tan\tfrac{1}{2}\angle q = (\mathrm{TV}:\mathrm{S})\sqrt{q} = \sqrt{\frac{\mathrm{T}q - \mathrm{S}q}{\mathrm{T}q + \mathrm{S}q}}.$$

(7.) The equations 204, XVI. and XXXV., give easily,

$$\text{XIII.}\ldots \mathrm{UV}q = \mathrm{UVU}q; \qquad \text{XIV.}\ldots \mathrm{UIV}q = \mathrm{Ax}.\,q; \qquad \text{XV.}\ldots \mathrm{TIV}q = \mathrm{TV}q;$$

or the more symbolical forms,

$$\text{XIII}'.\ldots \mathrm{UVU} = \mathrm{UV}; \qquad \text{XIV}'.\ldots \mathrm{UIV} = \mathrm{Ax}.; \qquad \text{XV}'.\ldots \mathrm{TIV} = \mathrm{TV};$$

and the identity 200, IX. becomes more evident, when we observe that

$$\text{XVI.}\ldots q - \mathrm{N}q = q(1 - \mathrm{K}q).$$

(8.) We have also generally (comp. 200, (10.) and 218, (10.)),

$$\text{XVII.}\ldots \frac{q-1}{q+1} = \frac{(q-1)(\mathrm{K}q+1)}{(q+1)(\mathrm{K}q+1)} = \frac{\mathrm{N}q - 1 + 2\mathrm{V}q}{\mathrm{N}q + 1 + 2\mathrm{S}q}.$$

(9.) The formula,*

$$\text{XVIII.}\ldots \mathrm{U}(rq + \mathrm{K}qr) = \mathrm{U}(\mathrm{S}r.\mathrm{S}q + \mathrm{V}r.\mathrm{V}q) = r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1},$$

in which q and r may be *any two quaternions*, is not perhaps of any great importance in itself, but will be found to furnish a student with several useful exercises in transformation.

(10.) When it was said, in 257, (1.), that *zero* had *only itself* for a *square-root*, the meaning was (comp. 225), that *no binomial expression of the form* $x + iy$ (228) *could satisfy the equation*,

$$\text{XIX.}\ldots 0 = q^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy,$$

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* This formula was given, but in like manner without proof, in page 587 of the author's Lectures on Quaternions.

for any real or imaginary values of the two scalar coefficients x and y, different from zero;* for if *biquaternions* (214, (8.)) be admitted, and if h again denote, as in 256, (2.), the *imaginary of algebra*, then (comp. 257, (6.) and (7.)) we may write, generally, *besides the real value* $0^{\frac{1}{2}} = 0$, the *imaginary expression*,

$$\text{XX.} \ldots \quad 0^{\frac{1}{2}} = v + hv', \quad \text{if} \quad \mathrm{S}v = \mathrm{S}v' = \mathrm{S}vv' = \mathrm{N}v' - \mathrm{N}v = 0\,;$$

v and v' being thus *any two real right quaternions, with equal norms* (or with *equal tensors*), in planes perperpendicular to each other.

(11.) For example, by 256, (2.) and by the laws (183) of ijk, we have the transformations,

$$\text{XXI.} \ldots \quad (i + hj)^2 = i^2 - j^2 + h(ij + ji) = 0 + h0 = 0\,;$$

so that the bi-quaternion $i + hj$ is one of the imaginary values of the symbol $0^{\frac{1}{2}}$.

(12.) In general, when *bi-quaternions* are admitted into calculation, not only the *square* of *one*, but the *product* of *two* such factors may *vanish*, without *either* of them *separately* vanishing: a circumstance which may throw some light on the existence of those *imaginary* (or *symbolical*) *roots of equations*, which were treated of in 257.

(13.) For example, although the equation

$$\text{XXII.} \ldots \quad q^2 - 1 = (q - 1)(q + 1) = 0$$

has *no real roots except* ± 1, and therefore cannot be verified by the *substitution* of *any other real scalar*, or *real quaternion*, for q, yet if we substitute for q the *bi-quaternion*† $v + hv'$, with the conditions 257, XIII., this equation XXII. is verified.

(14.) It will be found, however, that when *two imaginary* but *non-evanescent factors* give thus a *null product, the norm of each is zero;* provided that we agree to *extend to bi-quaternions* the formula $\mathrm{N}q = \mathrm{S}q^2 - \mathrm{V}q^2$ (204, XXII.); or to *define* that the *Norm of a Biquaternion* (like that of an ordinary or real quaternion) is equal to the *Square of the Scalar Part, minus the Square of the Right Part: each* of these two *parts* being generally *imaginary*, and the *former* being what we have called a *Bi-scalar*.

(15.) With this definition, if q and q' be any two *real quaternions*, and if h be, as above, the ordinary imaginary of *algebra*, we may establish the formula:

$$\text{XXIII.} \ldots \quad \mathrm{N}(q + hq') = (\mathrm{S}q + h\mathrm{S}q')^2 - (\mathrm{V}q + h\mathrm{V}q')^2\,;$$

or (comp. 200, VII., and 210, XX.),

$$\text{XXIV.} \ldots \quad \mathrm{N}(q + hq') = \mathrm{N}q - \mathrm{N}q' + 2h\mathrm{S}\,.\,q\mathrm{K}q'.$$

(16.) As regards the *norm of the sum* of any two *real* quaternions, or real vectors (273), the following transformations are occasionally useful (comp. 220, (2.)):

$$\text{XXV.} \ldots \quad \mathrm{N}(q' + q) = \mathrm{N}(\mathrm{T}q'.\,\mathrm{U}q + \mathrm{T}q\,.\,\mathrm{U}q')\,;$$

$$\text{XXVI.} \ldots \quad \mathrm{N}(\beta + \alpha) = \mathrm{N}(\mathrm{T}\beta\,.\,\mathrm{U}\alpha + \mathrm{T}\alpha\,.\,\mathrm{U}\beta)\,;$$

in each of which it is permitted to change the *norms* to the *tensors* of which they are the *squares*, or to write T for N.

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* Compare the Note to page 276.

† This *includes* the expression ± hi, of 257, (1.), for a *symbolical square-root* of *positive unity*. Other such roots are ± hj, and ± hk.

BOOK III.

ON QUATERNIONS, CONSIDERED AS PRODUCTS OR POWERS OF VECTORS; AND ON SOME APPLICATIONS OF QUATERNIONS.

CHAPTER I.

ON THE INTERPRETATION OF A PRODUCT OF VECTORS, OR POWER OF A VECTOR, AS A QUATERNION.

SECTION 1.—*On a First Method of interpreting a Product of Two Vectors as a Quaternion.*

ART. 275. In the First Book of these *Elements* we interpreted, Ist, the *difference* of any two directed right lines in space (4); IInd, the *sum* of two or more such lines (5–9); IIIrd, the *product* of one such line, multiplied by or into a positive or negative *number* (15); IVth, the *quotient* of such a line, divided by such a number (16), or by what we have called generally a SCALAR (17); and Vth, the sum of a *system* of such lines, each affected (97) with a scalar *coefficient* (99), as being in each case *itself* (generally) a *Directed Line** *in Space*, or what we have called a VECTOR (1).

276. In the Second Book, the fundamental principle or pervading conception has been, that the *Quotient of two such Vectors* is, generally, a QUATERNION (112, 116). It is however to be remembered, that we have *included* under this general conception, which *usually* relates to what may be called an *Oblique Quotient*, or the quotient of two lines in space making either an *acute* or an *obtuse angle* with each other

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* The *Fourth Proportional* to any *three complanar lines* has also been since interpreted (226), as being *another line* in the *same plane*.

(130), the *three* following particular *cases:* Ist, the *limiting case*, when the angle becomes *null*, or when the two lines are *similarly directed*, in which case the quotient *degenerates* (131) into a *positive scalar;* IInd, the *other* limiting case, when the angle is equal to *two right angles*, or when the lines are *oppositely directed*, and when in consequence the quotient *again* degenerates, but now into a *negative* scalar; and IIIrd, the *intermediate* case, when the angle is *right*, or when the two lines are *perpendicular* (132), instead of being *parallel* (15), and when therefore their quotient becomes what we have called (132) a *Right Quotient*, or a RIGHT QUATERNION: which has been seen to be a case not less important than the two former ones.

277. But *no Interpretation* has been assigned, in either of the two foregoing Books, for a PRODUCT *of two or more Vectors;* or for the SQUARE, or other POWER *of a Vector:* so that the *Symbols*,

$$\text{I.} \ldots \beta\alpha,\ \gamma\beta\alpha, \ldots \quad \text{and} \quad \text{II.} \ldots \alpha^2,\ \alpha^3, \ldots \alpha^{-1}, \ldots \alpha^t,$$

in which α, β, γ .. denote *vectors*, but t denotes a *scalar*, remain as yet entirely *uninterpreted;* and we are therefore *free* to assign, at this stage, *any meanings* to these *new symbols*, or *new combinations* of symbols, which shall *not contradict each other*, and shall appear to be consistent with *convenience* and *analogy*. And to do so will be the chief object of this First Chapter of the Third (and last) Book of these *Elements:* which is designed to be a much shorter one than either of the foregoing.

278. As a commencement of such *Interpretation* we shall here *define*, that *a vector α is multiplied by another vector β*, or that the *latter* vector is multiplied *into** the *former*, or that the *product* $\beta\alpha$ is obtained, *when the multiplier-line β is divided by the reciprocal* Rα (258) *of the multiplicand-line α*; as we had *proved* (136) that one *quaternion* is *multiplied into another*, when it is *divided by* the *reciprocal* thereof. In symbols, we shall therefore write, *as a first definition*, the formula:

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* Compare the Notes to pages 146, 159.

$$\text{I.} \ldots \beta\alpha = \beta : \mathrm{R}\alpha; \quad \text{where} \quad \text{II.} \ldots \mathrm{R}\alpha = -\,\mathrm{U}\alpha : \mathrm{T}\alpha \ (258, \text{VII.}).$$

And we proceed to consider, in the following Section, some of the general *consequences* of this definition, or interpretation, of a *Product of two Vectors*, as being *equal* to a certain *Quotient*, or *Quaternion*.

SECTION 2.—*On some Consequences of the foregoing Interpretation.*

279. The definition (278) gives the formula:

$$\text{I.} \ldots \beta\alpha = \frac{\beta}{\mathrm{R}\alpha}; \quad \text{and similarly,} \quad \text{I}'. \ldots \alpha\beta = \frac{\alpha}{\mathrm{R}\beta};$$

it gives therefore, by 259, VIII., the general relation,

$$\text{II.} \ldots \beta\alpha = \mathrm{K}\alpha\beta; \quad \text{or} \quad \text{II}'. \ldots \alpha\beta = \mathrm{K}\beta\alpha.$$

The *Products of two Vectors*, taken in two *opposite orders*, are therefore *Conjugate Quaternions;* and the *Multiplication of Vectors*, like that of Quaternions (168), is (generally) a *Non-Commutative Operation*.

(1.) It follows from II. (by 196, comp. 223, (1.)), that

$$\text{III.} \ldots \mathrm{S}\beta\alpha = +\,\mathrm{S}\alpha\beta = \tfrac{1}{2}(\beta\alpha + \alpha\beta).$$

(2.) It follows also (by 204, comp. again 223, (1.)), that

$$\text{IV.} \ldots \mathrm{V}\beta\alpha = -\,\mathrm{V}\alpha\beta = \tfrac{1}{2}(\beta\alpha - \alpha\beta).$$

280. Again, by the same general formula 259, VIII., we have the transformations,

$$\text{I.} \ldots \frac{\beta}{\mathrm{R}(\alpha + \alpha')} = \mathrm{K}\frac{\alpha + \alpha'}{\mathrm{R}\beta} = \mathrm{K}\frac{\alpha}{\mathrm{R}\beta} + \mathrm{K}\frac{\alpha'}{\mathrm{R}\beta} = \frac{\beta}{\mathrm{R}\alpha} + \frac{\beta}{\mathrm{R}\alpha'};$$

it follows, then, from the definition (278), that

$$\text{II.} \ldots \beta(\alpha + \alpha') = \beta\alpha + \beta\alpha';$$

whence also, by taking conjugates (279), we have this other general equation,

$$\text{III.} \ldots (\alpha + \alpha')\beta = \alpha\beta + \alpha'\beta.$$

Multiplication of Vectors is, therefore, like that of *Quaternions* (212), a *Doubly Distributive Operation*.

281. As we have not yet assigned any signification for a *ternary product of vectors*, such as $\gamma\beta\alpha$, we are not yet pre-

pared to pronounce, whether the *Associative Principle* (223) of *Multiplication of Quaternions* does or does not extend to *Vector-Multiplication*. But we can already derive several other consequences from the definition (278) of a *binary product*, $\beta\alpha$; among which, attention may be called to the *Scalar character* of a *Product of two Parallel Vectors;* and to the *Right character* of a *Product of two Perpendicular Vectors*, or of two lines at right angles with each other.

(1.) The definition (278) may be thus written,

$$\text{I.} \ldots \beta\alpha = -\mathrm{T}\beta \,.\, \mathrm{T}\alpha \,.\, \mathrm{U}(\beta : \alpha);$$

it gives, therefore,

$$\text{II.} \ldots \mathrm{T}\beta\alpha = \mathrm{T}\beta \,.\, \mathrm{T}\alpha; \qquad \text{III.} \ldots \mathrm{U}\beta\alpha = -\mathrm{U}(\beta : \alpha) = \mathrm{U}\beta \,.\, \mathrm{U}\alpha;$$

the *tensor* and *versor* of the *product* of two *vectors* being thus *equal* (as for quaternions, 191) to the *product of the tensors*, and to the *product of the versors*, respectively.

(2.) Writing for abridgment (comp. 208),

$$\text{IV.} \ldots a = \mathrm{T}\alpha, \qquad b = \mathrm{T}\beta, \qquad \gamma = \mathrm{Ax}.(\beta : \alpha), \qquad x = \angle(\beta : \alpha),$$

we have thus,

$$\text{V.} \ldots \mathrm{T}\beta\alpha = ba; \qquad \text{VI.} \ldots \mathrm{S}\beta\alpha = \mathrm{S}\alpha\beta = -ba\cos x;$$

$$\text{VII.} \ldots \mathrm{SU}\beta\alpha = \mathrm{SU}\alpha\beta = -\cos x; \qquad \text{VIII.} \ldots \angle\beta\alpha = \pi - x;$$

so that (comp. 198) the *angle of the product* of any two vectors is the *supplement of the angle of the quotient*.

(3.) We have next the transformations (comp. again 208),

$$\text{IX.} \ldots \mathrm{TV}\beta\alpha = \mathrm{TV}\alpha\beta = ba\sin x; \qquad \text{X.} \ldots \mathrm{TVU}\beta\alpha = \mathrm{TVU}\alpha\beta = \sin x;$$

$$\text{XI.} \ldots \mathrm{IV}\beta\alpha = -\gamma ba\sin x; \qquad \text{XI}'. \ldots \mathrm{IV}\alpha\beta = +\gamma ab\sin x;$$

$$\text{XII.} \ldots \mathrm{IUV}\beta\alpha = \mathrm{Ax}.\,\beta\alpha = -\gamma; \qquad \text{XII}'. \ldots \mathrm{IUV}\alpha\beta = \mathrm{Ax}.\,\alpha\beta = +\gamma;$$

so that *the rotation round the axis of a product of two vectors, from the multiplier to the multiplicand, is positive.*

(4.) It follows also, by IX., that the *tensor of the right part* of such a *product*, $\beta\alpha$, is equal to the *parallelogram under the factors;* or to the *double of the area of the triangle* OAB, whereof those two factors α, β, or OA, OB, are two coinitial *sides:* so that if we denote here this last-mentioned *area* by the symbol

$$\Delta\,\text{OAB},$$

we may write the equation,

$$\text{XIII.} \ldots \mathrm{TV}\beta\alpha = \textit{parallelogram under } \alpha,\ \beta, = 2\Delta\,\text{OAB};$$

and the *index*, $\mathrm{IV}\beta\alpha$, is a *right line perpendicular to the plane* of this *parallelogram*, of which line the *length represents* its *area*, in the sense that they bear *equal ratios* to their respective *units* (of length and of area).

(5.) Hence, by 279, IV.,

$$\text{XIV.} \ldots \mathrm{T}(\beta\alpha - \alpha\beta) = 2 \times \text{parallelogram} = 4\,\Delta\,\text{OAB}.$$

(6.) For any two vectors, α, β,

$$\text{XV.}\ldots\ \mathrm{S}\beta\alpha = -\,\mathrm{N}\alpha\,.\,\mathrm{S}(\beta:\alpha)\,; \qquad \text{XVI.}\ldots\ \mathrm{V}\beta\alpha = -\,\mathrm{N}\alpha\,.\,\mathrm{V}(\beta:\alpha)\,;$$

or briefly,*

$$\text{XVII.}\ldots\ \beta\alpha = -\,\mathrm{N}\alpha\,.(\beta:\alpha),$$

with the signification (273) of $\mathrm{N}\alpha$, as denoting $(\mathrm{T}\alpha)^2$.

(7.) If the two factor-lines be *perpendicular* to each other, so that x is a *right angle*, then the *parallelogram* (4.) becomes a *rectangle*, and the *product* $\beta\alpha$ becomes a *right quaternion* (132); so that we may write,

$$\text{XVIII.}\ldots\ \mathrm{S}\beta\alpha = \mathrm{S}\alpha\beta = 0, \quad \text{if} \quad \beta \perp \alpha, \text{ and reciprocally.}$$

(8.) Under the same condition of perpendicularity,

$$\text{XIX.}\ldots\ \angle\,\beta\alpha = \angle\,\alpha\beta = \frac{\pi}{2}\,; \quad \text{XX.}\ldots\ \mathrm{I}\beta\alpha = -\,\gamma ba\,; \quad \text{XXI.}\ldots\ \mathrm{I}\alpha\beta = +\,\gamma ab.$$

(9.) On the other hand, if the two factor-lines be *parallel*, the *right part* of their product vanishes, or that product reduces itself to a *scalar*, which is *negative* or *positive* according as the two vectors multiplied have *similar* or *opposite directions;* for we may establish the formula,

$$\text{XXII.}\ldots\ \text{if } \beta \parallel \alpha, \quad \text{then} \quad \mathrm{V}\beta\alpha = 0, \quad \mathrm{V}\alpha\beta = 0\,;$$

and, under the same condition of *parallelism*,

$$\text{XXIII.}\ldots\ \beta\alpha = \alpha\beta = \mathrm{S}\beta\alpha = \mathrm{S}\alpha\beta = \mp\, ba,$$

the *upper* or the *lower sign* being taken, according as $x = 0$, or $= \pi$.

(10.) We may also write (by 279, (1.) and (2.)) the following *formula of perpendicularity*, and *formula of parallelism :*

$$\text{XXIV.}\ldots\ \text{if } \beta \perp \alpha, \quad \text{then} \quad \beta\alpha = -\,\alpha\beta, \text{ and reciprocally}\,;$$
$$\text{XXV.}\ldots\ \text{if } \beta \parallel \alpha, \quad \text{then} \quad \beta\alpha = +\,\alpha\beta, \text{ with the converse.}$$

(11.) If α, β, γ be *any three unit-lines*, considered as vectors of the corners A, B, C of a *spherical triangle*, with *sides* equal to three new positive scalars, a, b, c, then because, by XVII., $\beta\alpha = -\,\beta:\alpha$, and $\gamma\beta = -\,\gamma:\beta$, the sub-articles to 208 allow us to write,

$$\text{XXVI.}\ldots\ \mathrm{S}(\mathrm{V}\gamma\beta\,.\,\mathrm{V}\beta\alpha) = \sin a \sin c \cos \mathrm{B}\,;$$
$$\text{XXVII.}\ldots\ \mathrm{IV}(\mathrm{V}\gamma\beta\,.\,\mathrm{V}\beta\alpha) = \pm\,\beta \sin a \sin c \sin \mathrm{B}\,;$$
$$\text{XXVIII.}\ldots\ (\mathrm{IV}:\mathrm{S})\,(\mathrm{V}\gamma\beta\,.\,\mathrm{V}\beta\alpha) = \pm\,\beta \tan \mathrm{B}\,;$$

upper or lower *signs* being taken, in the two last formulæ, according as the *rotation* round β from α to γ, or that round B from A to C, is positive or negative.

(12.) The equation 274, I., of the *Ellipsoid*, may now be written thus:

$$\text{XXIX.}\ldots\ \mathrm{T}(\iota\rho + \rho\kappa) = \mathrm{T}\iota^2 - \mathrm{T}\kappa^2\,; \quad \text{or} \quad \text{XXX.}\ldots\ \mathrm{T}(\iota\rho + \rho\kappa) = \mathrm{N}\iota - \mathrm{N}\kappa.$$

282. Under the general head of a *product* of two *parallel vectors*, two interesting *cases* occur, which furnish two first examples of *Powers of Vectors :* namely, Ist, the case when

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* *All* the consequences of the interpretation (278), of the product $\beta\alpha$ of two vectors, might be deduced from this formula XVII.; which, however, it would not have been so natural to have assumed for a *definition* of that symbol, as it was to assume the formula 278, I.

the two factors are *equal*, which gives this remarkable result, that *the Square of a Vector is always equal to a Negative Scalar;* and IInd, the case when the factors are (in the sense already defined, 258) *reciprocal* to each other, in which case it follows from the definition (278) that their *product* is equal to *Positive Unity:* so that *each* may, in this case, be considered as equal to *unity divided by the other*, or to the *Power* of that other which has *Negative Unity* for its *Exponent*.

(1.) When $\beta = \alpha$, the product $\beta\alpha$ reduces itself to what we may call the *square* of α, and may denote by α^2; and thus we may write, as a particular but important *case* of 281, XXIII., the formula (comp. 278),

$$\text{I.} \ldots \alpha^2 = -\alpha^2 = -(\mathrm{T}\alpha)^2 = -\mathrm{N}\alpha;$$

so that the *square of any vector* α is equal to the *negative of the norm* (273) of that vector; or to the *negative of the square of the number* $\mathrm{T}\alpha$, which expresses (185) the *length* of the same vector.

(2.) More immediately, the definition (278) gives,

$$\text{II.} \ldots \alpha^2 = \alpha\alpha = \alpha : \mathrm{R}\alpha = -(\mathrm{T}\alpha)^2 = -\mathrm{N}\alpha, \text{ as before.}$$

(3.) Hence (compare the notations 161, 190, 199, 204),

$$\text{III.} \ldots \mathrm{S}.\alpha^2 = -\mathrm{N}\alpha; \qquad \text{IV.} \ldots \mathrm{V}.\alpha^2 = 0;$$

and

$$\text{V.} \ldots \mathrm{T}.\alpha^2 = \mathrm{T}(\alpha^2) = +\mathrm{N}\alpha = (\mathrm{T}\alpha)^2 = \mathrm{T}\alpha^2;$$

the omission of the *parentheses*, or of the *point*, in this last symbol of a tensor,* for the square of a *vector*, as well as for the square of a *quaternion* (190), being thus justified: and in like manner we may write,

$$\text{VI.} \ldots \mathrm{U}.\alpha^2 = \mathrm{U}(\alpha^2) = -1 = (\mathrm{U}\alpha)^2 = \mathrm{U}\alpha^2;$$

the *square of an unit-vector* (129) being always equal to *negative unity*, and parentheses (or points) being *again* omitted.

(4.) The equation

$$\text{VII.} \ldots \rho^2 = \alpha^2, \quad \text{gives} \quad \text{VII}' \ldots \mathrm{N}\rho = \mathrm{N}\alpha, \quad \text{or} \quad \text{VII}'' \ldots \mathrm{T}\rho = \mathrm{T}\alpha;$$

it represents therefore, by 186, (2.), the *sphere* with O for centre, which passes through the point A.

(5.) The more general equation,

$$\text{VIII.} \ldots (\rho - \alpha)^2 = (\beta - \alpha)^2, \qquad (\text{comp.}\dagger\ 186,\ (4.),)$$

represents the sphere with A for centre, which passes through the point B.

(6.) For example, the equation,

$$\text{IX.} \ldots (\rho - \alpha)^2 = \alpha^2, \qquad (\text{comp. } 186,\ (3.),)$$

represents the sphere with A for centre, which passes through the origin O.

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* Compare the Note to page 210.

† Compare also the sub-articles to 278.

(7.) The equations (comp. 186, (6.), (7.)),

$$\text{X.} \ldots (\rho + \alpha)^2 = (\rho - \alpha)^2; \qquad \text{XI.} \ldots (\rho - \beta)^2 = (\rho - \alpha)^2,$$

represent, respectively, the *plane* through O, perpendicular to the line OA; and the plane which perpendicularly bisects the line AB.

(8.) The *distributive principle* of *vector-multiplication* (280), and the formula 279, III., enable us to establish generally (comp. 210, (9.)) the formula,

$$\text{XII.} \ldots (\beta \pm \alpha)^2 = \beta^2 \pm 2\text{S}\beta\alpha + \alpha^2;$$

the recent equations IX. and X. may therefore be thus transformed:

$$\text{IX}'. \ldots \rho^2 = 2\text{S}\alpha\rho; \quad \text{and} \quad \text{X}'. \ldots \text{S}\alpha\rho = 0.$$

(9.) The equations,

$$\text{XIII.} \ldots \rho^2 + a^2 = 0; \qquad \text{XIV.} \ldots \rho^2 + 1 = 0,$$

represent the spheres with O for centre, which have *a* and 1 for their respective radii; so that this very simple formula, $\rho^2 + 1 = 0$, is (comp. 186, (1.)) *a form of the Equation of the Unit-Sphere* (128), and is, as such, of great importance in the present Calculus.

(10.) The equation,

$$\text{XV.} \ldots \rho^2 - 2\text{S}\alpha\rho + c = 0,$$

may be transformed to the following,

$$\text{XVI.} \ldots \text{N}(\rho - \alpha) = -(\rho - \alpha)^2 = c - \alpha^2 = c + \text{N}\alpha;$$

or

$$\text{XVI}'. \ldots \text{T}(\rho - \alpha) = \sqrt{(c - \alpha^2)} = \sqrt{(c + \text{N}\alpha)};$$

it represents therefore a (real or imaginary) sphere, with A for *centre*, and with this last radical (if real) for *radius*.

(11.) This *sphere* is therefore necessarily *real*, if *c* be a *positive* scalar; or if this scalar constant, *c*, *though negative*, be (algebraically) *greater than* α^2, or than $-\text{N}\alpha$: but it becomes *imaginary*, if $c + \text{N}\alpha < 0$.

(12.) The *radical plane* of the *two spheres*,

$$\text{XVII.} \ldots \rho^2 - 2\text{S}\alpha\rho + c = 0, \qquad \rho^2 - 2\text{S}\alpha'\rho + c' = 0,$$

has for equation,

$$\text{XVIII.} \ldots 2\text{S}(\alpha' - \alpha)\rho = c' - c;$$

it is therefore *always real*, if the given *vectors* α, α' and the given *scalars* c, c' be such, even if one or both of the *spheres themselves* be *imaginary*.

(13.) The equation 281, XXIX., or XXX., of the *Central Ellipsoid* (or of the ellipsoid with its *centre* taken for the *origin* of vectors), may now be still further simplified,* as follows:

$$\text{XIX.} \ldots \text{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2.$$

(14.) The definition (278) gives also,

$$\text{XX.} \ldots \alpha\text{R}\alpha = \alpha : \alpha = 1; \quad \text{or} \quad \text{XX}'. \ldots \text{R}\alpha . \alpha = \text{R}\alpha : \text{R}\alpha = 1;$$

whence it is natural to write,†

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* Compare the Note to page 238.

† Compare the second Note to page 279.

$$\text{XXI.} \ldots \mathrm{R}\alpha = 1 : \frac{1}{\alpha} = \alpha^{-1},$$

if we so far anticipate here the general theory of *powers of vectors*, above alluded to (277), as to use this last symbol to denote the *quotient*, of *unity divided by the vector* α; so as to have *identically*, or for *every* vector, the equation,

$$\text{XXII.} \ldots \alpha . \alpha^{-1} = \alpha^{-1} . \alpha = 1.$$

(15.) It follows, by 258, VII., that

$$\text{XXIII.} \ldots \alpha^{-1} = -\mathrm{U}\alpha : \mathrm{T}\alpha ; \quad \text{and} \quad \text{XXIV.} \ldots \beta\alpha = \beta : \alpha^{-1}.$$

(16.) If we had adopted the equation XXIII. *as a definition** of the *symbol* α^{-1}, then the formula XXIV. might have been used, as a *formula of interpretation* for the symbol $\beta\alpha$. But we proceed to consider an entirely *different method*, of arriving at the *same* (or an *equivalent*) *Interpretation* of this latter symbol: or of a *Binary Product of Vectors*, considered as equal to a *Quaternion*.

SECTION 3.—*On a Second Method of arriving at the same Interpretation, of a Binary Product of Vectors.*

283. It cannot fail to have been observed by any attentive reader of the Second Book, how close and intimate a *connexion*† has been found to exist, between a *Right Quaternion* (132), and its *Index*, or *Index-Vector* (133). Thus, if v and v' denote (as in 223, (1.), &c.) any two *right* quaternions, and if $\mathrm{I}v$, $\mathrm{I}v'$ denote, as usual, their *indices*, we have already seen that

$$\text{I.} \ldots \mathrm{I}v' = \mathrm{I}v, \quad \text{if} \quad v' = v, \text{ and conversely (133);}$$

$$\text{II.} \ldots \mathrm{I}(v' \pm v) = \mathrm{I}v' \pm \mathrm{I}v \text{ (206);}$$

$$\text{III.} \ldots \mathrm{I}v' : \mathrm{I}v = v' : v \text{ (193);}$$

to which may be added the more recent formula,

$$\text{IV.} \ldots \mathrm{RI}v = \mathrm{IR}v \text{ (258, IX.).}$$

284. It could not therefore have appeared strange, if we had proposed to establish this new formula of the same kind,

$$\text{I.} \ldots \mathrm{I}v' . \mathrm{I}v = v' . v = v'v,$$

as a definition (supposing that the recent definition 278 had not occurred to us), whereby to *interpret the product of any two indices of right quaternions*, as being equal to the *product of those two quaternions themselves*. And then, *to interpret the product* $\beta\alpha$, of *any two given vectors*, taken *in a given order*, we should only have had to conceive (as we always may), that the two proposed *factors*, α and β, are the *indices* of *two right quaternions*, v and v', and to *multiply these latter*, in the *same order*. For thus we should have been led to establish the formula,

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* Compare the Note to page 305.

† Compare the Note to page 174.

$$\text{II.} \ldots \beta\alpha = v'v, \quad \text{if} \quad \alpha = \mathrm{I}v, \quad \text{and} \quad \beta = \mathrm{I}v';$$

or we should have this slightly more *symbolical* equation,

$$\text{III.} \ldots \beta\alpha = \beta \,.\, \alpha = \mathrm{I}^{-1}\beta \,.\, \mathrm{I}^{-1}\alpha\,;$$

in which the symbols,

$$\mathrm{I}^{-1}\alpha \text{ and } \mathrm{I}^{-1}\beta,$$

are understood to denote the two right quaternions, whereof the two lines α and β are the indices.

(1.) To establish now the substantial *identity of these two interpretations*, 278 and 284, of a *binary product of vectors* $\beta\alpha$, notwithstanding *the difference of form* of the *definitional equations* by which they have been expressed, we have only to observe that it has been found, as a *theorem* (194), that

$$\text{IV.} \ldots v'v = \mathrm{I}v' : \mathrm{I}(1 : v) = \mathrm{I}v' : \mathrm{IR}v\,;$$

but the definition (258) of $\mathrm{R}\alpha$ gave us the lately cited equation, $\mathrm{RI}v = \mathrm{IR}v$; we have therefore, by the recent formula II., the equation,

$$\text{V.} \ldots \mathrm{I}v' . \mathrm{I}v = \mathrm{I}v' : \mathrm{RI}v; \quad \text{or} \quad \text{VI.} \ldots \beta \,.\, \alpha = \beta : \mathrm{R}\alpha,$$

as in 278, I.; α and β still denoting *any two vectors*. The *two interpretations* therefore *coincide*, at least in their *results*, although they have been obtained by *different processes*, or *suggestions*, and are expressed by two different *formulæ*.

(2.) The result 279, II., respecting *conjugate products of vectors*, corresponds thus to the result 191, (2.), or to the first formula of 223, (1.).

(3.) The two formulæ of 279, (1.) and (2.), respecting the *scalar and right parts* of the product $\beta\alpha$, answer to the two other formulæ of the same sub-article, 223, (1.), respecting the corresponding parts of $v'v$.

(4.) The *doubly distributive property* (280), of *vector-multiplication*, is on this plan seen to be *included* in the corresponding but more general property (212), of *multiplication of quaternions*.

(5.) By changing $\mathrm{IV}q$, $\mathrm{IV}q'$, t, t', and ∂, to α, β, a, b, and γ, in those formulæ of Art. 208 which are previous to its sub-articles, we should obtain, *with the recent definition* (or *interpretation*) II. of $\beta\alpha$, several of the consequences lately given (in sub-arts. to 281), as resulting from the former definition, 278, I. Thus, the equations,

VI., VII., VIII,, IX., X., XI., XII., XXII., and XXIII.,

of 281, correspond to, and may (with our last definition) be deduced from, the formulæ,

V., VI., VIII., XI., XII., XXII., XX., XIV., and XVI., XVIII.,

of 208. (Some of the consequences from the sub-articles to 208 have been already considered, in 281, (11.))

(6.) The *geometrical properties of the line* $\mathrm{IV}\beta\alpha$, deduced from the *first definition* (278) of $\beta\alpha$ in 281, (3.) and (4.), (namely, the *positive rotation* round that line, from β to α; its *perpendicularity to their plane*; and the *representation* by the same line of the *paralellogram under those two factors*, regard being had to *units* of *length* and of *area*,) might also have been deduced from 223, (4.), by means of the *second definition* (284), of the *same product*, $\beta\alpha$.

SECTION 4.—*On the Symbolical Identification of a Right Quaternion with its own Index: and on the Construction of a Product of Two Rectangular Lines, by a Third Line, rectangular to both.*

285. It has been seen, then, that the recent formula 284, II. or III., *may replace the formula* 278, I., as a *second definition* of a *product of two vectors*, which conducts to the *same consequences*, and therefore ultimately to the same *interpretation* of such a product, as the *first*. Now, in the *second* formula, we have *interpreted that product*, $\beta\alpha$, by *changing* the *two factor-lines*, α and β, to the *two right quaternions*, v and v', or $\mathrm{I}^{-1}\alpha$ and $\mathrm{I}^{-1}\beta$, of which they are the *indices*; and by then *defining* that the sought product $\beta\alpha$ is *equal* to the product $v'v$, of those two right quaternions. It becomes, therefore, important to inquire, at this stage, *how far such substitution*, of $\mathrm{I}^{-1}\alpha$ for α, or of v for $\mathrm{I}v$, together with the *converse* substitution, is permitted in this Calculus, consistently with principles already established. For it is evident that if such substitutions can be shown to be generally legitimate, or allowable, we shall thereby be enabled to *enlarge* greatly the existing field of *interpretation*: and to treat, in *all* cases, *Functions of Vectors*, as being, at the same time, *Functions of Right Quaternions*.

286. We have first, by 133 (comp. 283, I.), the *equality*,

$$\mathrm{I.}\ \ldots\ \mathrm{I}^{-1}\beta = \mathrm{I}^{-1}\alpha, \quad \text{if} \quad \beta = \alpha.$$

In the next place, by 206 (comp. 283, II.), we have the formula of *addition or subtraction*,

$$\mathrm{II.}\ \ldots\ \mathrm{I}^{-1}(\beta \pm \alpha) = \mathrm{I}^{-1}\beta \pm \mathrm{I}^{-1}\alpha;$$

with these more general results of the same kind (comp. 207 and 99),

$$\mathrm{III.}\ \ldots\ \mathrm{I}^{-1}\Sigma\alpha = \Sigma\mathrm{I}^{-1}\alpha; \qquad \mathrm{IV.}\ \ldots\ \mathrm{I}^{-1}\Sigma x\alpha = \Sigma x\mathrm{I}^{-1}\alpha.$$

In the third place, by 193 (comp. 283, III.), we have, for *division*, the formula,

$$\text{V.} \ldots \mathrm{I}^{-1}\beta : \mathrm{I}^{-1}\alpha = \beta : \alpha;$$

while the *second definition* (284) of *multiplication of vectors*, which has been proved to be *consistent* with the *first* definition (278), has given us the analogous equation,

$$\text{VI.} \ldots \mathrm{I}^{-1}\beta \,.\, \mathrm{I}^{-1}\alpha = \beta \,.\, \alpha = \beta\alpha.$$

It would seem, then, that we might at once proceed to *define*, for the purpose of *interpreting any proposed Function of Vectors as a Quaternion*, that the following general *Equation* exists:

$$\text{VII.} \ldots \mathrm{I}^{-1}\alpha = \alpha; \quad \text{or} \quad \text{VIII.} \ldots \mathrm{I}v = v, \quad \text{if} \quad v = \frac{\pi}{2};$$

or still more briefly and *symbolically*, if it be understood that the *subject* of the *operation* I is always a *right* quaternion,

$$\text{IX.} \ldots \mathrm{I} = 1.$$

But, before finally adopting this conclusion, there is a *case* (or rather a *class* of cases), which it is necessary to examine, in order to be certain that *no contradiction to former results* can ever be thereby caused.

287. The *most general form of a vector-function*, or of a vector regarded as a function of other vectors and of scalars, which was considered in the First Book, was the form (99, comp. 275),

$$\text{I.} \ldots \rho = \Sigma x\alpha;$$

and we have seen that *if we change, in this form*, each *vector* α to the corresponding *right quaternion* $\mathrm{I}^{-1}\alpha$, and then take the *index* of the new right quaternion which *results*, we shall thus be conducted to precisely the *same vector* ρ, as that which had been otherwise obtained before; or in symbols, that

$$\text{II.} \ldots \Sigma x\alpha = \mathrm{I}\Sigma x\mathrm{I}^{-1}\alpha \text{ (comp. 286, IV.).}$$

But *another form of a vector-function* has been considered in the Second Book; namely, the form,

$$\text{III.} \ldots \rho = \ldots \frac{\epsilon}{\delta}\frac{\gamma}{\beta}\alpha \text{ (226, III.);}$$

in which α, β, γ, δ, ϵ ... are *any odd number of complanar vectors*. And before we accept, *as general*, the *equation* VII. or VIII. or IX. of 286, we must inquire whether we are at liberty to write, under the same *conditions of complanarity*, and with the *same signification* of the *vector* ρ, the equation,

$$\text{IV.} \ldots \rho = \mathrm{I}\left(\ldots \frac{\mathrm{I}^{-1}\epsilon}{\mathrm{I}^{-1}\delta} \cdot \frac{\mathrm{I}^{-1}\gamma}{\mathrm{I}^{-1}\beta} \cdot \mathrm{I}^{-1}a\right).$$

288. To examine this, let there be at first only *three given* complanar *vectors*, $\gamma \parallel\!| a, \beta$; in which case there will always be (by 226) a *fourth vector* ρ, in the same plane, which will represent or construct the function $(\gamma : \beta) . a$; namely, the *fourth proportional* to β, γ, a. Taking then what we may call the *Inverse Index-Functions*, or *operating* on these four vectors a, β, γ, ρ by the *characteristic* I^{-1}, we obtain *four collinear and right quaternions* (209), which may be denoted by v, v', v'', v'''; and we shall have the equation,

$$\text{V.} \ldots v''' : v = (\rho : a = \gamma : \beta =) v'' : v';$$

or

$$\text{VI.} \ldots v''' = (v'' : v') . v;$$

which proves what was required. Or, more symbolically,

$$\text{VII.} \ldots \frac{\mathrm{I}^{-1}\rho}{\mathrm{I}^{-1}a} = \frac{\rho}{a} = \frac{\gamma}{\beta} = \frac{\mathrm{I}^{-1}\gamma}{\mathrm{I}^{-1}\beta};$$

$$\text{VIII.} \ldots \frac{\gamma}{\beta} . a = \rho = \mathrm{I}(\mathrm{I}^{-1}\rho) = \mathrm{I}\left(\frac{\mathrm{I}^{-1}\gamma}{\mathrm{I}^{-1}\beta} . \mathrm{I}^{-1}a\right).$$

And it is so easy to extend this reasoning to the case of any greater odd number of given vectors in one plane, that we may now consider the recent formula IV. as proved.

289. We shall therefore *adopt*, as *general*, the *symbolical equations* VII. VIII. IX. of 286; and shall thus be enabled, in a shortly subsequent Section, to *interpret ternary* (and other) *products of vectors*, as well as *powers* and other *Functions of Vectors*, as being *generally Quaternions;* although they may, in particular *cases*, *degenerate* (131) into *scalars*, or may become *right* quaternions (132): in which latter event they may, in virtue of the same principle, be *represented by*, and *equated* to, *their own indices* (133), and so be treated as *vectors*. In symbols, we shall write *generally*, for *any set of vectors* $a, \beta, \gamma, \ldots$ and *any function* f, the equation,

$$\text{I.} \ldots f(a, \beta, \gamma, \ldots) = f(\mathrm{I}^{-1}a, \mathrm{I}^{-1}\beta, \mathrm{I}^{-1}\gamma, \ldots) = q,$$

q being some *quaternion;* while in the particular *case* when this quaternion is *right*, or when

$$q = v = \mathrm{S}^{-1}0 = \mathrm{I}^{-1}\rho,$$

we shall write *also*, and usually *by preference* (for *that* case), the formula,

$$\text{II.} \ldots f(\alpha, \beta, \gamma, \ldots) = \mathrm{I}f(\mathrm{I}^{-1}\alpha, \mathrm{I}^{-1}\beta, \mathrm{I}^{-1}\gamma, \ldots) = \rho,$$

ρ being a *vector*.

290. For example, instead of saying (as in 281) that the *Product* of any *two Rectangular Vectors* is a *Right Quaternion*, with certain properties of its *Index*, already pointed out (284, (6.)), we may *now* say that *such* a *product* is *equal* to that *index*. And hence will follow the important consequence, that *the Product of any Two Rectangular Lines in Space is equal to* (or may be *constructed by*) *a Third Line, rectangular to both; the Rotation round this Product-Line, from the Multiplier-Line to the Multiplicand-Line*, being *Positive:* and the *Length of the Product* being equal to the *Product of the Lengths of the Factors*, or *representing* (with a suitable reference to *units*) the *Area of the Rectangle* under them. And generally we may now, for all purposes of *calculation* and *expression*, *identify*[*] a *Right Quaternion with its own Index.*

SECTION 5.—*On some Simplifications of Notation, or of Expression, resulting from this Identification; and on the Conception of an Unit-Line as a Right Versor.*

291. An immediate consequence of the symbolical equation 286, IX., is that we may now *suppress the Characteristic* I, *of the Index of a Right Quaternion*, in all the formulæ into which it has entered; and so may *simplify the Notation*. Thus, instead of writing,

$$\mathrm{Ax.}\,q = \mathrm{IUV}q, \quad \text{or} \quad \mathrm{Ax.} = \mathrm{IUV}, \quad \text{as in 204, (23.),}$$

or

$$\mathrm{Ax.}\,q = \mathrm{UIV}q, \quad \mathrm{Ax.} = \mathrm{UIV}, \quad \text{as in 274, (7.),}$$

we may *now* write simply[†],

$$\text{I.} \ldots \mathrm{Ax.}\,q = \mathrm{U\,V}q; \quad \text{or} \quad \text{II.} \ldots \mathrm{Ax.} = \mathrm{UV}.$$

The Characteristic Ax., *of the Operation of taking the Axis of a Quaternion* (132, (6.)), *may therefore henceforth be replaced* whenever we may think fit to *dispense with it*, by this *combination of two other characteristics*, U and V, which are of greater and more *general* utility, and indeed *cannot*[*] be dispensed with, in the *practice* of the present Calculus.

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[*] Compare the Notes to pages 119, 136, 174, 191, 200.

[†] Compare the first Note to page 118, and the second Note to page 200.

292. We are now enabled also to *diminish*, to some extent, the *number of technical terms*, which have been employed in the foregoing Book. Thus, whereas we defined, in 202, that the right quaternion Vq was the *Right Part* of the Quaternion q, or of the *sum* Sq + Vq, we may now, by 290, *identify* that *part* with its own *index-vector* IVq, and so may be led to *call* it *the vector part*, or simply *the* VECTOR,[†] of that Quaternion q, without henceforth *speaking* of the *right part:* although the *plan of exposition*, adopted in the Second Book, required that we should do so for some time. And thus an *enunciation*, which was put forward at an early stage of the present work, namely, at the end of the First Chapter of the First Book, or the assertion (17) that

"*Scalar plus Vector equals Quaternion,*"

becomes entirely intelligible, and acquires a perfectly *definite* signification. For we are in this manner led to conceive a *Number* (positive or negative) as being *added to a Line,*[‡] *when it is added* (according to rules already established) *to that right quotient* (132), of which the *line* is the *Index*. In symbols, we are thus led to establish the formula,

$$\text{I.} \ldots q = a + \alpha, \quad \text{when} \quad \text{II.} \ldots q = a + \mathrm{I}^{-1}\alpha;$$

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[*] Of course, any one who chooses may *invent new symbols*, to *denote the same operations on quaternions*, as those which are denoted in these *Elements*, and in the elsewhere cited *Lectures*, by the letters U and V; but, under *some form*, such symbols *must* be *used:* and it appears to have been hitherto thought expedient, by other writers, not hastily to innovate on notations which have been already employed in several published researches, and have been found to answer their purpose. As to the *type* used for these, and for the analogous characteristics K, S, T, that must evidently be a mere affair of taste and convenience: and in fact they have all been printed as small italic capitals, in some examination-papers by the author.

[†] Compare the Note to page 191.

[‡] On account of this possibility of conceiving a quaternion to be the *sum* of a number and a line, it was at one time suggested by the present author, that a *Quaternion* might *also* be called a *Grammarithm*, by a combination of the two Greek words, γραμμή and ἀριθμός, which signify respectively a *Line* and a *Number*.

whatever scalar, and *whatever vector*, may be denoted by a and α. And because *either* of these *two parts*, or *summands*, may *vanish* separately, we are entitled to say, that *both Scalars and Vectors*, or *Numbers and Lines*, are *included in the Conception of a Quaternion*, as now enlarged or modified.

293. Again, the same *symbolical identification* of $\mathrm{I}v$ with v (286, VIII.) leads to the forming of a *new conception of an Unit-Line*, or *Unit-Vector* (129), as being also a *Right Versor* (153); or an *Operator*, of which the *effect* is to *turn a line*, in a plane perpendicular to itself, through a *positive quadrant of rotation:* and thereby to oblige the *Operand-Line* to take a *new direction*, *at right angles* to its *old* direction, but *without* any *change of length*. And then the remarks (154) on the equation $q^2 = -1$, where q was a *right versor* in the *former sense* (which is still a permitted one) of its being a *right radial quotient* (147), or the *quotient of two equally long but mutually rectangular lines*, become immediately applicable to the *interpretation of the equation*,

$$\rho^2 = -1, \quad \text{or} \quad \rho^2 + 1 = 0 \ (282, \text{XIV.});$$

where ρ is still an *unit-vector*.

(1.) Thus (comp. Fig. 41), if α be any line perpendicular to *such* a vector ρ, we have the equations,

$$\text{I.} \ldots \rho\alpha = \beta; \qquad \text{II.} \ldots \rho^2\alpha = \rho\beta = \alpha' = -\alpha;$$

β being *another* line perpendicular to ρ, which is, at the same time, at right angles to α, and of the same length with it; and from which a *third line* α', or $-\alpha$, *opposite* to the line α, but still *equally long*, is formed by a *repetition* of the *operation*, denoted by (what we may here call) the *characteristic* ρ; or having that *unit-vector* ρ for the *operator*, or *instrument employed*, as a sort of *handle*, or *axis** *of rotation*.

(2.) More generally (comp. 290), if α, β, γ be any three lines at right angles to each other, and if the *length* of γ be numerically equal to the *product of the lengths* of α and β, then (by what precedes) the *line* γ *represents*, or *constructs*, or is *equal* to, the *product of the two other lines*, at least if a certain *order* of the *factors* (comp. 279) be observed: so that we may write the equation (comp. 281, XXI.),

$$\text{III.} \ldots \alpha\beta = \gamma, \quad \text{if} \quad \text{IV.} \ldots \beta \perp \alpha,\ \gamma \perp \alpha,\ \gamma \perp \beta, \quad \text{and} \quad \text{V.} \ldots \mathrm{T}\alpha \,.\, \mathrm{T}\beta = \mathrm{T}\gamma,$$

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* Compare the first Note to page 136.

provided that the *rotation round* α, *from* β *to* γ, or that round γ from α to β, &c., has the direction taken as the *positive* one.

(3.) In this more general case, we may *still* conceive that the *multiplier-line* α has *operated* on the *multiplicand-line* β, so as to *produce* (or *generate*) the *product-line* γ; but *not now* by an operation of *version alone*, since the *tensor* of β is (generally) *multiplied* by that of α, in order to form, by V., the *tensor of the product*, γ.

(4.) And if (comp. Fig. 41, *bis*, in which α was first changed to β, and then to α') we *repeat this compound operation*, of *tension and version combined* (comp. 189), or if we *multiply again by* α, we obtain a *fourth line* β', *in the plane* of β, γ, but with a *direction opposite* to that of β, and with a *length generally different*: namely the line,

$$\text{VI.}\ \ldots\ \alpha\gamma = \alpha\alpha\beta = \alpha^2\beta = \beta' = -a^2\beta, \quad \text{if} \quad a = \text{T}\alpha.$$

(5.) The *operator* α^2, or $\alpha\alpha$, is therefore *equivalent*, in its *effect on* β, to the *negative scalar*, $-a^2$, or $-(\text{T}\alpha)^2$, or $-\text{N}\alpha$, considered as a *coefficient*, or as a (scalar) *multiplier* (15): whence the *equation*,

$$\text{VII.}\ \ldots\ \alpha^2 = -\text{N}\alpha \ (282,\ \text{I.}),$$

may be *again deduced*, but now with a *new interpretation*, which is, however, as we see, completely *consistent*, in all its *consequences*, with the one first proposed (282).

SECTION 6.—*On the Interpretation of a Product of Three or more Vectors, as a Quaternion.*

294. There is now no difficulty in interpreting a *ternary product of vectors* (comp. 277, I.), or a product of *more vectors than three*, taken always in some *given order*; namely, as the result (289, I.) of the *substitution* of the corresponding *right quaternions* in that product: which *result* is *generally* what we have lately called (276) an *Oblique Quotient*, or a Quaternion with either an *acute* or an *obtuse angle* (130); but may *degenerate* (131) into a *scalar*, or may become *itself* a *right* quaternion (132), and so be *constructed* (289, II.) by a *new vector*. It follows (comp. 281), that *Multiplication of Vectors*, like that of *Quaternions* (223), in which indeed we now see that it is *included*, is an *Associative Operation*: or that we may write generally (comp. 223, II.), for *any three vectors*, α, β, γ, the *Formula*,

$$\text{I.}\ \ldots\ \gamma\beta\cdot\alpha = \gamma\cdot\beta\alpha.$$

(1.) The formulæ 223, III. and IV., are now replaced by the following:

$$\text{II.}\ \ldots\ \text{V}.\gamma\text{V}\beta\alpha = \alpha\text{S}\beta\gamma - \beta\text{S}\gamma\alpha;$$

$$\text{III.}\ \ldots\ \text{V}\gamma\beta\alpha = \alpha\text{S}\beta\gamma - \beta\text{S}\gamma\alpha + \gamma\text{S}\alpha\beta;$$

in which $V\gamma\beta\alpha$ is written, for simplicity, instead of $V(\gamma\beta\alpha)$, or $V.\gamma\beta\alpha$; and with which, as with the earlier equations referred to, a student of this Calculus will find it useful to render himself *very familiar*.

(2.) Another useful form of the equation II. is the following:

$$\text{IV.}\ldots\ V(V\alpha\beta.\gamma) = \alpha S\beta\gamma - \beta S\gamma\alpha.$$

(3.) The equations IX. X. XIV. of 223 enable us now to write, for *any three vectors*, the formula:

$$\begin{aligned}\text{V.}\ldots\ S\gamma\beta\alpha &= -S\alpha\beta\gamma = S\alpha\gamma\beta = -S\beta\gamma\alpha = S\beta\alpha\gamma = -S\gamma\alpha\beta \\ &= \pm \textit{ volume of parallelepiped under } \alpha, \beta, \gamma, \\ &= \pm 6 \times \textit{volume of pyramid } \text{OABC};\end{aligned}$$

upper or lower *signs* being taken, according as the *rotation* round α from β to γ is positive or negative: or in other words, the *scalar* $S\gamma\beta\alpha$, of the *ternary product of vectors* $\gamma\beta\alpha$, being *positive* in the first case, but *negative* in the second.

(4.) The *condition of complanarity* of *three vectors*, α, β, γ, is therefore expressed by the equation (comp. 223, XI.):

$$\text{VI.}\ldots\ S\gamma\beta\alpha = 0; \quad \text{or} \quad \text{VI}'.\ldots\ S\alpha\beta\gamma = 0; \ \&\text{c.}$$

(5.) If α, β, γ be *any three vectors*, complanar or diplanar, the expression,

$$\text{VII.}\ldots\ \delta = \alpha S\beta\gamma - \beta S\gamma\alpha,$$

gives

$$\text{VIII.}\ldots\ S\gamma\delta = 0, \quad \text{and} \quad \text{IX.}\ldots\ S\alpha\beta\delta = 0;$$

it represents therefore (comp. II. and IV.) a *fourth vector* δ, which is *perpendicular* to γ, but *complanar* with α and β: or in symbols,

$$\text{X.}\ldots\ \delta \perp \gamma, \quad \text{and} \quad \text{XI.}\ldots\ \delta \mathbin{|||} \alpha, \beta.$$

(Compare the notations 123, 129.)

(6.) For *any four vectors*, we have by II. and IV. the transformations,

$$\begin{aligned}\text{XII.}\ldots\ V(V\alpha\beta.V\gamma\delta) &= \delta S\alpha\beta\gamma - \gamma S\alpha\beta\delta; \\ \text{XIII.}\ldots\ V(V\alpha\beta.V\gamma\delta) &= \alpha S\beta\gamma\delta - \beta S\alpha\gamma\delta;\end{aligned}$$

and each of these three equivalent expressions represents a *fifth vector* ϵ, which is at once complanar with α, β, and with γ, δ; or a *line* OE, which is in the *intersection of the two planes*, OAB and OCD.

(7.) Comparing them, we see that any *arbitrary vector* ρ may be expressed as a *linear function* of any *three given diplanar vectors*, α, β, γ, by the formula:

$$\text{XIV.}\ldots\ \rho S\alpha\beta\gamma = \alpha S\beta\gamma\rho + \beta S\gamma\alpha\rho + \gamma S\alpha\beta\rho;$$

which is found to be one of extensive utility.

(8.) Another very useful formula, of the same kind, is the following:

$$\text{XV.}\ldots\ \rho S\alpha\beta\gamma = V\beta\gamma.S\alpha\rho + V\gamma\alpha.S\beta\rho + V\alpha\beta.S\gamma\rho;$$

in the second member of which, the points may be omitted.

(9.) One mode of proving the correctness of this last formula XV., is to *operate* on both members of it, by the *three symbols*, or *characteristics of operation*,

$$\text{XVI.}\ldots\ S.\alpha, \quad S.\beta, \quad S.\gamma;$$

the common results on both sides being respectively the three scalar products,

$$\text{XVII.}\ldots\ S\alpha\rho.S\alpha\beta\gamma, \quad S\beta\rho.S\alpha\beta\gamma, \quad S\gamma\rho.S\alpha\beta\gamma;$$

where again the points may be omitted.

(10.) We here employ the principle, that *if the three vectors* α, β, γ *be actual and diplanar*, then *no actual vector* λ *can satisfy at once the three scalar equations*,

$$\text{XVIII.} \ldots \mathrm{S}\alpha\lambda = 0, \quad \mathrm{S}\beta\lambda = 0, \quad \mathrm{S}\gamma\lambda = 0;$$

because it *cannot be perpendicular at once to those three diplanar vectors.*

(11.) If, then, in any investigation with quaternions, we meet a system of this form XVIII., we can at once infer that

$$\text{XIX.} \ldots \lambda = 0, \quad \text{if} \quad \text{XX.} \ldots \mathrm{S}\alpha\beta\gamma \gtrless 0;$$

while, conversely, if λ *be* an *actual* vector, then α, β, γ must be *complanar* vectors, or $\mathrm{S}\alpha\beta\gamma = 0$, as in VI'.

(12.) Hence also, under the same condition XX., the three scalar equations,

$$\text{XXI.} \ldots \mathrm{S}\alpha\lambda = \mathrm{S}\alpha\mu, \quad \mathrm{S}\beta\lambda = \mathrm{S}\beta\mu, \quad \mathrm{S}\gamma\lambda = \mathrm{S}\gamma\mu,$$

give

$$\text{XXII.} \ldots \lambda = \mu.$$

(13.) *Operating* (comp. (9.)) on the equation XV. by the symbol, or *characteristic*, $\mathrm{S}.\delta$, in which δ is any new vector, we find a result which may be written thus (with or without the points):

$$\text{XXIII.} \ldots 0 = \mathrm{S}\alpha\rho.\mathrm{S}\beta\gamma\delta - \mathrm{S}\beta\rho.\mathrm{S}\gamma\delta\alpha + \mathrm{S}\gamma\rho.\mathrm{S}\delta\alpha\beta - \mathrm{S}\delta\rho.\mathrm{S}\alpha\beta\gamma;$$

where α, β, γ, δ, ρ may denote *any five vectors.*

(14.) In drawing this last inference, we assume that the equation XV. holds good, *even* when the three vectors α, β, γ are *complanar:* which in fact must be true, *as a limit*, since the equation has been proved, by (9.) and (12.), to be valid, if γ be *ever so little out of the plane* of α and β.

(15.) We have therefore this new formula:

$$\text{XXIV.} \ldots \mathrm{V}\beta\gamma\mathrm{S}\alpha\rho + \mathrm{V}\gamma\alpha\mathrm{S}\beta\rho + \mathrm{V}\alpha\beta\mathrm{S}\gamma\rho = 0, \quad \text{if} \quad \mathrm{S}\alpha\beta\gamma = 0;$$

in which ρ may denote *any fourth vector*, whether *in*, or *out of*, the *common plane* of α, β, γ.

(16.) If ρ be *perpendicular to that plane*, the last formula is *evidently* true, each term of the first member vanishing separately, by 281, (7.); and if we change ρ to a vector δ *in the plane* of α, β, γ, we are conducted to the following equation, *as an interpretation* of the same formula XXIV., which expresses a known theorem of plane trigonometry, including several others under it:

$$\text{XXV.} \ldots \sin \mathrm{BOC}.\cos \mathrm{AOD} + \sin \mathrm{COA}.\cos \mathrm{BOD} + \sin \mathrm{AOB}.\cos \mathrm{COD} = 0,$$

for *any four complanar and co-initial lines*, OA, OB, OC, OD.

(17.) By passing from OD to a line perpendicular thereto, but in their common plane, we have this other known* equation:

$$\text{XXVI.} \ldots \sin \mathrm{BOC} \sin \mathrm{AOD} + \sin \mathrm{COA} \sin \mathrm{BOD} + \sin \mathrm{AOB} \sin \mathrm{COD} = 0;$$

which, like the former, admits of many transformations, but is only mentioned here as offering itself naturally to our notice, when we seek to *interpret the formula* XXIV. obtained as above by quaternions.

(18.) Operating on that formula by $\mathrm{S}.\delta$, and changing ρ to ϵ, we have this new equation:

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* Compare page 20 of the *Géométrie Supérieure* of M. Chasles.

$$\text{XXVII.}\ldots\ 0 = \mathrm{S}\alpha\iota\mathrm{S}\beta\gamma\delta + \mathrm{S}\beta\iota\mathrm{S}\gamma\alpha\delta + \mathrm{S}\gamma\iota\mathrm{S}\alpha\beta\delta, \quad \text{if} \quad \mathrm{S}\alpha\beta\gamma = 0;$$

which might indeed have been at once deduced from XXIII.

(19.) The equation XIV., as well as XV., must hold good at the *limit*, when α, β, γ are *complanar;* hence

$$\text{XXVIII.}\ldots\ \alpha\mathrm{S}\beta\gamma\rho + \beta\mathrm{S}\gamma\alpha\rho + \gamma\mathrm{S}\alpha\beta\rho = 0, \quad \text{if} \quad \mathrm{S}\alpha\beta\gamma = 0.$$

(20.) This last formula is evidently true, by (4.), if ρ be *in* the common plane of the three other vectors; and if we suppose it to be *perpendicular* to that plane, so that

$$\text{XXIX.}\ldots\ \rho \parallel \mathrm{V}\beta\gamma \parallel \mathrm{V}\gamma\alpha \parallel \mathrm{V}\alpha\beta,$$

and therefore, by 281, (9.), since $\mathrm{S}(\mathrm{S}\beta\gamma.\rho) = 0$,

$$\text{XXX.}\ldots\ \mathrm{S}\beta\gamma\rho = \mathrm{S}(\mathrm{V}\beta\gamma.\rho) = \mathrm{V}\beta\gamma.\rho, \text{ \&c.},$$

we may *divide each term by* ρ, and so obtain this other formula,

$$\text{XXXI.}\ldots\ \alpha\mathrm{V}\beta\gamma + \beta\mathrm{V}\gamma\alpha + \gamma\mathrm{V}\alpha\beta = 0, \quad \text{if} \quad \mathrm{S}\alpha\beta\gamma = 0.$$

(21.) In general, the *vector* (292) of this last expression *vanishes* by II.; the expression is therefore equal to its own *scalar*, and we may write,

$$\text{XXXII.}\ldots\ \alpha\mathrm{V}\beta\gamma + \beta\mathrm{V}\gamma\alpha + \gamma\mathrm{V}\alpha\beta = 3\mathrm{S}\alpha\beta\gamma,$$

whatever three vectors may be denoted by α, β, γ.

(22.) For the *case of complanarity*, if we suppose that the three vectors are *equally long*, we have the proportion,

$$\text{XXXIII.}\ldots\ \mathrm{V}\beta\gamma : \mathrm{V}\gamma\alpha : \mathrm{V}\alpha\beta = \sin \mathrm{BOC} : \sin \mathrm{COA} : \sin \mathrm{AOB};$$

and the formula XXXI. becomes thus,

$$\text{XXXIV.}\ldots\ \mathrm{OA}.\sin \mathrm{BOC} + \mathrm{OB}.\sin \mathrm{COA} + \mathrm{OC}.\sin \mathrm{AOB} = 0;$$

where OA, OB, OC are *any three radii of one circle*, and the equation is interpreted as in Articles 10, 11, &c.

(23.) The equation XXIII. might have been deduced from XIV., instead of XV., by first operating with $\mathrm{S}.\delta$, and then interchanging δ and ρ.

(24.) A *vector* ρ may in general be considered (221) as *depending on three scalars* (the *co-ordinates* of its *term*); it cannot then be *determined* by *fewer than three scalar equations;* nor can it be *eliminated* between *fewer than four*.

(25.) As an example of such *determination* of a vector, let α, β, γ be again any *three given* and *diplanar vectors;* and let the three given *equations* be,

$$\text{XXXV.}\ldots\ \mathrm{S}\alpha\rho = a, \quad \mathrm{S}\beta\rho = b, \quad \mathrm{S}\gamma\rho = c;$$

in which a, b, c are supposed to denote *three given scalars*. Then the *sought vector* ρ has for its expression, by XV.,

$$\text{XXXVI.}\ldots\ \rho = e^{-1}(a\mathrm{V}\beta\gamma + b\mathrm{V}\gamma\alpha + c\mathrm{V}\alpha\beta), \quad \text{if} \quad \text{XXXVII.}\ldots\ e = \mathrm{S}\alpha\beta\gamma.$$

(26.) As another example, let the three equations be,

$$\text{XXXVIII.}\ldots\ \mathrm{S}\beta\gamma\rho = a', \quad \mathrm{S}\gamma\alpha\rho = b', \quad \mathrm{S}\alpha\beta\rho = c';$$

then, with the same signification of the scalar e, we have, by XIV.,

$$\text{XXXIX.}\ldots\ \rho = e^{-1}(a'\alpha + b'\beta + c'\gamma).$$

(27.) As an example of *elimination of a vector*, let there be the *four* scalar equations,

$$\text{XL.}\ldots\ \mathrm{S}\alpha\rho = a, \quad \mathrm{S}\beta\rho = b, \quad \mathrm{S}\gamma\rho = c, \quad \mathrm{S}\delta\rho = d;$$

then, by XXIII., we have this *resulting equation*, *into which* ρ *does not enter*, but only the *four vectors*, $\alpha \, . \, . \, \delta$, and the *four scalars*, $a \, . \, . \, d$:

$$\text{XLI.} \ldots a \, . \, S\beta\gamma\delta - b \, . \, S\gamma\delta\alpha + c \, . \, S\delta\alpha\beta - d \, . \, S\alpha\beta\gamma = 0.$$

(28.) This last equation may therefore be considered as the *condition of concurrence of the four planes*, represented by the *four scalar equations* XL., in one *common point;* for, although it has not been expressly stated before, it follows evidently from the *definition* 278 of a *binary product of vectors*, combined with 196, (5.), that *every scalar equation of the linear form* (comp. 282, XVIII.),

$$\text{XLII.} \ldots S\alpha\rho = a, \quad \text{or} \quad S\rho\alpha = a,$$

in which $\alpha = \mathrm{OA}$, and $\rho = \mathrm{OP}$, as usual, *represents a plane locus of the point* P; the *vector of the foot* S, of the *perpendicular on that plane from the origin*, being

$$\text{XLIII.} \ldots \mathrm{OS} = \sigma = a\mathrm{R}\alpha = a\alpha^{-1} \text{ (282, XXI.)}.$$

(29.) If we conceive a *pyramidal volume* (68) as having an *algebraical* (or *scalar*) *character*, so as to be capable of bearing either a *positive* or a *negative ratio* to the volume of a *given pyramid*, with a *given order of its points*, we may then *omit the ambiguous sign*, in the last expression (3.) for the *scalar of a ternary product of vectors:* and so may write, generally, OABC denoting such a *volume*, the formula,

$$\text{XLIV.} \ldots S\alpha\beta\gamma = 6 \, . \, \mathrm{OABC},$$

= a *positive* or a *negative scalar*, according as the *rotation* round OA from OB to OC is *negative* or *positive*.

(30.) More generally, changing O to D, and OA or α to $\alpha - \delta$, &c., we have thus the formula:

$$\text{XLV.} \ldots 6 \, . \, \mathrm{DABC} = S(\alpha - \delta)\,(\beta - \delta)\,(\gamma - \delta) = S\alpha\beta\gamma - S\beta\gamma\delta + S\gamma\delta\alpha - S\delta\alpha\beta;$$

in which it may be observed, that the expression is *changed to its own opposite*, or *negative*, or is *multiplied by* -1, *when any two of the four vectors*, $\alpha, \beta, \gamma, \delta$, or when *any two of the four points*, A, B, C, D, *change places with each other;* and therefore is *restored* to its former value, by a *second* such *binary interchange*.

(31.) Denoting then the *new origin* of $\alpha, \beta, \gamma, \delta$ by E, we have first, by XLIV., XLV., the equation,

$$\text{XLVI.} \ldots \mathrm{DABC} = \mathrm{EABC} - \mathrm{EBCD} + \mathrm{ECDA} - \mathrm{EDAB};$$

and may then write the result (comp. 68) under the more *symmetric form* (because $-\mathrm{EBCD} = \mathrm{BECD} =$ &c.):

$$\text{XLVII.} \ldots \mathrm{BCDE} + \mathrm{CDEA} + \mathrm{DEAB} + \mathrm{EABC} + \mathrm{ABCD} = 0;$$

in which A, B, C, D, E may denote *any five points of space*.

(32.) And an analogous formula (69, III.) of the First Book, for *any six points* OABCDE, namely the equation (comp. 65, 70),

$$\text{XLVIII.} \ldots \mathrm{OA} \, . \, \mathrm{BCDE} + \mathrm{OB} \, . \, \mathrm{CDEA} + \mathrm{OC} \, . \, \mathrm{DEAB} + \mathrm{OD} \, . \, \mathrm{EABC} + \mathrm{OE} \, . \, \mathrm{ABCD} = 0,$$

in which the *additions* are performed according to the rules of *vectors*, the *volumes* being treated as scalar *coefficients*, is easily recovered from the foregoing principles and results. In fact, by XLVII., this last formula may be written as

$$\text{XLIX.} \ldots \mathrm{ED} \, . \, \mathrm{EABC} = \mathrm{EA} \, . \, \mathrm{EBCD} + \mathrm{EB} \, . \, \mathrm{ECAD} + \mathrm{EC} \, . \, \mathrm{EABD};$$

or, substituting $\alpha, \beta, \gamma, \delta$ for EA, EB, EC, ED, as

$$\text{L.} \ldots \delta S\alpha\beta\gamma = \alpha S\beta\gamma\delta + \beta S\gamma\alpha\delta + \gamma S\alpha\beta\delta;$$

which is only another form of XIV., and ought to be *familiar* to the student.

(33.) The formula 69, II. may be deduced from XXXI., by observing that, when the *three vectors* α, β, γ are *complanar*, we have the proportion,

$$\text{LI.} \ldots V\beta\gamma : V\gamma\alpha : V\alpha\beta : V(\beta\gamma + \gamma\alpha + \alpha\beta) = \text{OBC} : \text{OCA} : \text{OAB} : \text{ABC},$$

if *signs* (or *algebraic* or *scalar ratios*) *of areas* be attended to (28, 63); and the formula 69, I., for the case of *three collinear points* A, B, C, may now be written as follows:

$$\text{LII.} \ldots \alpha(\beta - \gamma) + \beta(\gamma - \alpha) + \gamma(\alpha - \beta) = 2V(\beta\gamma + \gamma\alpha + \alpha\beta)$$
$$= 2V(\beta - \alpha)(\gamma - \alpha) = 0,$$

if the *three coinitial vectors* α, β, γ be *termino-collinear* (24).

(34.) The case when *four coinitial vectors* α, β, γ, δ are *termino-complanar* (64), or when they terminate in *four complanar points* A, B, C, D, is expressed by equating to zero the second or the third member of the formula XLV.

(35.) Finally, for *ternary products of vectors* in general, we have the formula:

$$\text{LIII.} \ldots \alpha^2\beta^2\gamma^2 + (S\alpha\beta\gamma)^2 = (V\alpha\beta\gamma)^2 = (\alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta)^2$$
$$= \alpha^2(S\beta\gamma)^2 + \beta^2(S\gamma\alpha)^2 + \gamma^2(S\alpha\beta)^2 - 2S\beta\gamma S\gamma\alpha S\alpha\beta.$$

295. The *identity* (290) of a *right quaternion* with its *index*, and the *conception* (293) of an *unit-line* as a *right versor*, allow us now to treat the three important versors, i, j, k, as *constructed by*, and even as (in our present view) *identical with*, their own *axes;* or with the *three lines* OI, OJ, OK of 181, considered as being each a certain *instrument*, or *operator*, or *agent in a right rotation* (293, (1.)), which *causes any line*, in a plane perpendicular to itself, *to turn in that plane*, through a *positive quadrant*, without any change of its *length*. With this conception, or construction, the *Laws of the Symbols ijk* are *still included in the Fundamental Formula* of 183, namely,

$$i^2 = j^2 = k^2 = ijk = -1; \qquad \text{(A)}$$

and if we now, in conformity with the same conception, *transfer* the *Standard Trinomial Form* (221) from *Right Quaternions* to *Vectors*, so as to write generally an expression of the form,

$$\text{I.} \ldots \rho = ix + jy + kz, \quad \text{or} \quad \text{I}'. \ldots \alpha = ia + jb + kc, \ \&c.,$$

where xyz and abc are *scalars* (namely, *rectangular co-ordinates*), we can *recover* many of the foregoing results with ease: and can, if we think fit, *connect* them with *co-ordinates*.

(1.) As to the *laws* (182), included in the *Fundamental Formula* A, the law $i^2 = -1$, &c., may be interpreted on the plan of 293, (1.), as representing the *reversal* which results from *two successive quadrantal rotations*.

(2.) The *two contrasted laws*, or formulæ,

$$ij = +k, \qquad ji = -k, \qquad \text{(182, II. and III.)}$$

may now be interpreted as expressing, that although a *positive rotation through a right angle, round the line i as an axis, brings a revolving line from the position j to the position k*, or $+k$, yet, on the contrary, a *positive quadrantal rotation round the line j*, as a *new axis*, brings a *new revolving line* from a *new initial position*, i, to a *new final position*, denoted by $-k$, or *opposite*[*] to the *old* final position, $+k$.

(3.) Finally, the law $ijk = -1$ (183) may be interpreted by conceiving, that we *operate on a line* α, which has at first the direction of $+j$, *by the three lines*, k, j, i, *in succession;* which gives *three new* but equally long *lines*, β, γ, δ, in the directions of $-i$, $+k$, $-j$, and so conducts at last to a line $-\alpha$, which has a *direction opposite* to *the initial one*.

(4.) The foregoing *laws of ijk*, which are all (as has been said) *included* (184) in the Formula A, when combined with the recent expression I. for ρ, give (comp. 222, (1.)) for the *square* of that *vector* the value:

$$\text{II.} \ldots \rho^2 = (ix + jy + kz)^2 = -(x^2 + y^2 + z^2);$$

this *square of the line* ρ is therefore equal to the *negative of the square of its length* $\mathrm{T}\rho$ (185), or to the *negative of its norm* $\mathrm{N}\rho$ (273), which agrees with the former result[†] 282, (1.) or (2.).

(5.) The *condition of perpendicularity* of the *two lines* ρ and α, when they are represented by the two trinomials I. and I'., may be expressed (281, XVIII.) by the formula,

$$\text{III.} \ldots 0 = \mathrm{S}\alpha\rho = -(ax + by + cz);$$

which agrees with a well-known theorem of rectangular co-ordinates.

(6.) The *condition of complanarity* of *three* lines, ρ, ρ', ρ'', represented by the trinomial forms,

$$\text{IV.} \ldots \rho = ix + jy + kz, \quad \rho' = ix' + \&\text{c.}, \quad \rho'' = ix'' + \&\text{c.},$$

is (by 294, VI.) expressed by the formula (comp. 223, XIII.),

$$\text{V.} \ldots 0 = \mathrm{S}\rho''\rho'\rho = x''(z'y - y'z) + y''(x'z - z'x) + z''(y'x - x'y);$$

agreeing again with known results.

(7.) When the three lines ρ, ρ', ρ'', or OP, OP', OP'', are *not in one plane*, the recent expression for $\mathrm{S}\rho''\rho'\rho$ gives, by 294, (3.), the *volume of the parallelepiped*

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[*] In the *Lectures*, the *three rectangular unit-lines*, i, j, k, were supposed (in order to fix the conceptions, and with a reference to northern latitudes) to be directed, respectively, towards the *south*, the *west*, and the *zenith;* and then the *contrast* of the two formulæ, $ij = +k$, $ji = -k$, came to be illustrated by conceiving, that we at *one* time *turn a moveable line*, which is at first directed *westward*, round an *axis* (or *handle*) directed towards the *south*, with a *right-handed* (or *screwing*) *motion*, through a *right angle*, which causes the line to take an *upward position*, as its *final* one; and that at *another* time we operate, in a precisely similar manner, on a line directed at first *southward*, with an *axis* directed to the *west*, which obliges this new line to take finally a *downward* (instead of, as before, an *upward*) *direction*.

[†] Compare also 222, IV.

(comp. 223, (9.)) of which they are *edges;* and this *volume*, thus expressed, is a *positive* or a *negative scalar*, according as the *rotation* round ρ from ρ' to ρ'' is itself *positive* or *negative:* that is, according as it has the *same direction* as that round $+x$ from $+y$ to $+z$ (or round i from j to k), or the direction *opposite* thereto.

(8.) It may be noticed here (comp. 223, (13.)), that if α, β, γ be *any three vectors*, then (by 294, III. and V.) we have:

$$\text{VI.}\ \ldots\ \mathrm{S}\alpha\beta\gamma = -\mathrm{S}\gamma\beta\alpha = \tfrac{1}{2}(\alpha\beta\gamma - \gamma\beta\alpha);$$
$$\text{VII.}\ \ldots\ \mathrm{V}\alpha\beta\gamma = +\mathrm{V}\gamma\beta\alpha = \tfrac{1}{2}(\alpha\beta\gamma + \gamma\beta\alpha).$$

(9.) More generally (comp. 223, (12.)), since a *vector*, considered as representing a *right quaternion* (290), is always (by 144) the *opposite of its own conjugate*, so that we have the important formula,*

$$\text{VIII.}\ \ldots\ \mathrm{K}\alpha = -\alpha, \quad \text{and therefore} \quad \text{IX.}\ \ldots\ \mathrm{K}\Pi\alpha = \pm\Pi'\alpha,$$

we may write for *any number of vectors*, the transformations,

$$\text{X.}\ \ldots\ \mathrm{S}\Pi\alpha = \pm\mathrm{S}\Pi'\alpha = \tfrac{1}{2}(\Pi\alpha \pm \Pi'\alpha),$$
$$\text{XI.}\ \ldots\ \mathrm{V}\Pi\alpha = \mp\mathrm{V}\Pi'\alpha = \tfrac{1}{2}(\Pi\alpha \mp \Pi'\alpha),$$

upper or *lower* signs being taken, according as that number is *even* or *odd:* it being understood that

$$\text{XII.}\ \ldots\ \Pi'\alpha = \ldots\gamma\beta\alpha, \quad \text{if} \quad \Pi\alpha = \alpha\beta\gamma\ldots$$

(10.) The relations of *rectangularity*,

$$\text{XIII.}\ \ldots\ \mathrm{Ax}.\,i \perp \mathrm{Ax}.\,j; \quad \mathrm{Ax}.\,j \perp \mathrm{Ax}.\,k; \quad \mathrm{Ax}.\,k \perp \mathrm{Ax}.\,i,$$

which result at once from the definitions (181), may now be written more briefly, as follows:

$$\text{XIV.}\ \ldots\ i \perp j; \qquad j \perp k, \qquad k \perp i;$$

and similarly in other cases, where the *axes*, or the *planes*, of any two right quaternions are at *right angles* to each other.

(11.) But, with the notations of the Second Book, we might *also* have writtten, by 123, 181, such formulæ of *complanarity* as the following, $\mathrm{Ax}.\,j \mathbin{|||} i$, to express (comp. 225) that the *axis* of j was a *line* in the *plane* of i; and it might cause some confusion, if we were now to *abridge* that formula to $j \mathbin{|||} i$. In general, it seems convenient that we should not henceforth employ the *sign* $|||$, except as connecting either *symbols of three lines*, considered still as *complanar;* or else *symbols of three right quaternions*, considered as being *collinear* (209), because their *indices* (or *axes*) are *complanar:* or finally, *any two complanar quaternions* (123).

(12.) On the other hand, no inconvenience will result, if we now insert the *sign of parallelism*, between the symbols of two right quaternions which are, in the former sense (123), *complanar:* for example, we may write, on our present plan,

$$\text{XV.}\ \ldots\ xi \parallel i, \quad yj \parallel j, \quad zk \parallel k,$$

if xyz be any three scalars.

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* If, in like manner, we interpret, on our present plan, the symbols $\mathrm{U}\alpha$, $\mathrm{T}\alpha$, $\mathrm{N}\alpha$ as equivalent to $\mathrm{UI}^{-1}\alpha$, $\mathrm{TI}^{-1}\alpha$, $\mathrm{NI}^{-1}\alpha$, we are reconducted (compare the Notes to page 136) to the same significations of those symbols as before (155, 185, 273); and it is evident that on the same plan we have now,

$$\mathrm{S}\alpha = 0, \quad \mathrm{V}\alpha = \alpha.$$

296. There are a few particular but remarkable *cases*, of *ternary* and other *products of vectors*, which it may be well to mention here, and of which some may be worth a student's while to remember: especially as regards the *products of successive sides of closed polygons, inscribed in circles*, or in *spheres*.

(1.) If A, B, C, D be any four *concircular points*, we know, by the sub-articles to 260, that their *anharmonic function* (ABCD), as defined in 259, (9.), is *scalar*; being also *positive* or *negative*, according to a law of *arrangement* of those four points, which has been already stated.

(2.) But, by that definition, and by the *scalar* (though *negative*) character of the *square of a vector* (282), we have generally, for *any plane or gauche quadrilateral* ABCD, the formula:

$$\text{I.} \ldots e^2(\mathrm{ABCD}) = \mathrm{AB}.\mathrm{BC}.\mathrm{CD}.\mathrm{DA} = \textit{the continued product of the four sides};$$

in which the coefficient e^2 is a *positive scalar*, namely the product of two negative or of two positive squares, as follows:

$$\text{II.} \ldots e^2 = \mathrm{BC}^2.\mathrm{DA}^2 = \overline{\mathrm{BC}}^2.\overline{\mathrm{DA}}^2 > 0.$$

(3.) If then ABCD be a *plane* and *inscribed* quadrilateral, we have, by 260, (8.), the formula,

$$\text{III.} \ldots \mathrm{AB}.\mathrm{BC}.\mathrm{CD}.\mathrm{DA} = \textit{a positive or negative scalar},$$

according as this *quadrilateral in a circle* is a *crossed* or an *uncrossed* one.

(4.) The *product* $\alpha\beta\gamma$ of *any three complanar vectors* is a *vector*, because its *scalar part* $S\alpha\beta\gamma$ *vanishes*, by 294, (3.) and (4.); and if the factors be three *successive sides* AB, BC, CD of a quadrilateral thus *inscribed* in a circle, their product has either the *direction* of the *fourth successive side*, DA, or else the *opposite* direction, or in symbols,

$$\text{IV.} \ldots \mathrm{AB}.\mathrm{BC}.\mathrm{CD} : \mathrm{DA} > \text{or} < 0,$$

according as the *quadrilateral* ABCD is an *uncrossed* or a *crossed* one.

(5.) By conceiving the *fourth point* D to *approach*, continuously and indefinitely, *to the first point* A, we find that the *product of the three successive sides of any plane triangle*, ABC, is given by an equation of the form:

$$\text{V.} \ldots \mathrm{AB}.\mathrm{BC}.\mathrm{CA} = \mathrm{AT};$$

AT being a *line* (comp. Fig. 63) *which touches the circumscribed circle*, or (more fully) which *touches the segment* ABC of that circle, at the point A; or *represents the initial direction of motion, along the circumference, from* A *through* B *to* C: while the *length* of this *tangential product-line*, AT, is *equal* to, or *represents*, with the usual reference to an *unit* of length, the *product of the lengths of the three sides*, of the same inscribed triangle ABC.

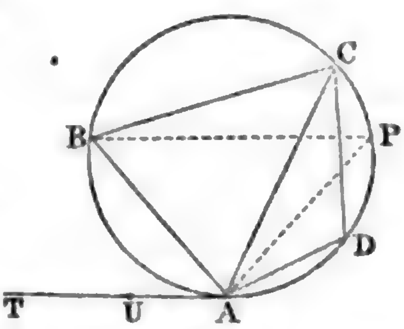

Fig. 63.

(6.) Conversely, if this theorem respecting the product of the sides of an *inscribed triangle* be supposed to have been *otherwise proved*, and if it be *remembered*, then since it will give in like manner the equation,

$$\text{VI.} \ldots \text{AC}.\text{CD}.\text{DA} = \text{AU},$$

if D be *any fourth point, concircular with* A, B, C, while AU is, as in the annexed Figures 63, a *tangent to the new segment* ACD, we can *recover* easily the theorem (3.), respecting the *product of the sides of an inscribed quadrilateral;* and thence can *return* to the corresponding theorem (260, (8.)), respecting the *anharmonic function* of any such figure ABCD: for we shall thus have, by V. and VI., the equation,

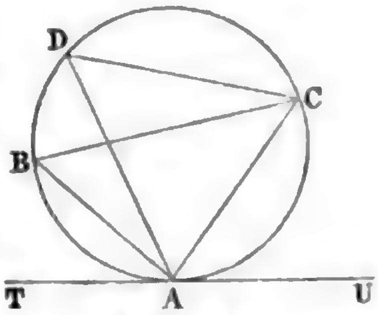

Fig. 63, *bis*.

$$\text{VII.} \ldots \text{AB}.\text{BC}.\text{CD}.\text{DA} = (\text{AT}.\text{AU}) : (\text{CA}.\text{AC}),$$

in which the *divisor* CA.AC or N.AC, or $\overline{\text{AC}}^2$, is *always positive* (282, (1.)), but the *dividend* AT.AU is *negative* (281, (9.)) for the case of an *uncrossed* quadrilateral (Fig. 63), being on the contrary *positive* for the other case of a *crossed* one (Fig. 63, *bis*).

(7.) If P be any point on the circle through a given point A, which touches at a given origin O a given line $\text{OT} = \tau$, as represented in Fig. 64, we shall then have by (5.) an equation of the form,

$$\text{VIII.} \ldots \text{OA}.\text{AP}.\text{PO} = x.\text{OT},$$

in which x is some scalar coefficient, which varies with the position of P. Making then $\text{OA} = \alpha$, and $\text{OP} = \rho$, as usual, we shall have

$$\text{IX.} \ldots \alpha(\rho - \alpha)\rho = -x\tau,$$

or

$$\text{IX}'. \ldots \rho^{-1} - \alpha^{-1} = x\tau : \alpha^2\rho^2,$$

or

$$\text{IX}''. \ldots \text{V}\tau\rho^{-1} = \text{V}\tau\alpha^{-1};$$

and any one of these may be considered as *a form of the equation of the circle*, determined by the given conditions.

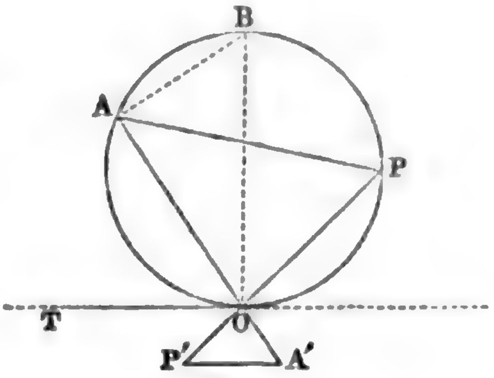

Fig. 64.

(8.) Geometrically, the last formula IX.″ expresses, that *the line* $\rho^{-1} - \alpha^{-1}$, or $\text{R}\rho - \text{R}\alpha$, or A′P′ (see again Fig. 64), if $\text{OA}' = \alpha^{-1} = \text{R}\alpha = \text{R}.\text{OA}$, and $\text{OP}' = \rho^{-1} = \text{R}.\text{OP}$, *is parallel to the given tangent* τ *at* O; which agrees with Fig. 58, and with Art. 260.

(9.) If B be the point *opposite* to O upon the circle, then the *diameter* OB, or β, as being $\perp \tau$, so that $\tau\beta^{-1}$ is a vector, is given by the formula,

$$\text{X.} \ldots \tau\beta^{-1} = \text{V}\tau\alpha^{-1}; \quad \text{or} \quad \text{X}'. \ldots \beta = -\tau : \text{V}\tau\alpha^{-1};$$

in which the tangent τ admits, as it ought to do, of being multiplied by any scalar, without the value of β being changed.

(10.) As another verification, the last formula gives,

$$\text{XI.} \ldots \overline{\text{OB}} = \text{T}\beta = \text{T}\alpha : \text{TVU}\tau\alpha^{-1} = \overline{\text{OA}} : \sin \text{AOT}.$$

(11.) If a quadrilateral OABC be *not inscriptible* in a circle, then, whether it be *plane* or *gauche*, we can always circumscribe (as in Fig. 65) *two circles*, OAB and OBC, about the *two triangles*, formed by drawing the *diagonal* OB; and then, on the plan of (6.), we can draw *two tangents* OT, OU, to the *two segments* OAB, OBC, so as to represent the *two ternary products*, OA . AB . BO, and OB . BC . CO;

after which we shall have the *quaternary product*,

XII. . . OA . AB . BC . CO = OT . OU : OB²;

where the *divisor*, $\overline{\text{OB}}^2$, or BO . OB, or N . OB, is a *positive scalar*, but the *dividend* OT . OU, and therefore also the *quotient* in the *second* member, or the *product* in the *first* member, is a *quaternion*.

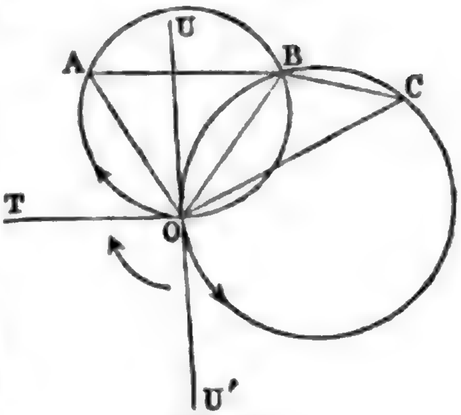

Fig. 65.

(12.) The *axis* of this quaternion is *perpendicular to the plane* TOU of the *two tangents;* and therefore to the *plane itself* of the quadrilateral OABC, if that be a *plane figure;* but if it be *gauche*, then the axis is *normal to the circumscribed sphere at the point* O: being also in all cases such, that the *rotation* round it, from OT to OU, is *positive*.

(13.) The *angle* of the same quaternion is the *supplement* of the angle TOU between the two tangents above mentioned; it is therefore *equal* to the angle U′OT, if OU′ touch the *new segment* OCB, or proceed in a new and *opposite direction* from O (see again Fig. 65); it may therefore be said to be the *angle between the two arcs*, OAB and OCB, *along which a point should move*, in order to go *from* O, on the two circumferences, *to* the *opposite corner* B of the *quadrilateral* OABC, *through* the *two other corners*, A and C, respectively: or the angle between the arcs OCB, OAB.

(14.) These results, respecting the *axis* and *angle* of the *product of the four successive sides*, of any quadrilateral OABC, or ABCD, apply without any modification to the *anharmonic quaternion* (259, (9.)) of the same quadrilateral; and although, for the *case* of a quadrilateral *in a circle*, the *axis* becomes indeterminate, because the quaternary product and the anharmonic function degenerate together into *scalars*, or because the figure may then be conceived to be *inscribed in indefinitely many spheres*, yet the *angle* may still be determined by the *same rule* as in the *general* case: this angle being $= \pi$, for the inscribed and *uncrossed* quadrilateral (Fig. 63); but $= 0$, for the inscribed and *crossed* one (Fig. 63, *bis*).

(15.) For the *gauche* quadrilateral OABC, which may always be conceived to be inscribed *in a determined sphere*, we may say, by (13.), that the *angle of the quaternion product*, ∠ (OA . AB . BC . CO), is equal to the *angle of the lunule*, bounded (generally) by the two *arcs of small circles* OAB, OCB; with the *same construction* for the equal *angle of the anharmonic*,

∠ (OABC), or ∠ (OA : AB . BC : CO).

(16.) It is evident that the general principle 223, (10.), of the permissibility of *cyclical permutation* of quaternion factors *under the sign* S, must hold good for the *case* when those quaternions *degenerate* (294) into *vectors;* and it is still more obvious, that *every* permutation of factors is allowed, under the sign T: whence *cyclical* permutation is *again* allowed, *under this other sign* SU; and consequently also (comp. 196, XVI.) under the sign ∠.

(17.) Hence generally, for *any four vectors*, we have the three equations,

$$\text{XIII.} \ldots \mathrm{S}\alpha\beta\gamma\delta = \mathrm{S}\beta\gamma\delta\alpha; \qquad \text{XIV.} \ldots \mathrm{SU}\alpha\beta\gamma\delta = \mathrm{SU}\beta\gamma\delta\alpha;$$

$$\text{XV.} \ldots \angle\, \alpha\beta\gamma\delta = \angle\, \beta\gamma\delta\alpha;$$

and in particular, for the *successive sides* of any plane or gauche *quadrilateral* ABCD, we have the *four equal angles*,

$$\text{XVI.} \ldots \angle(\mathrm{AB}.\mathrm{BC}.\mathrm{CD}.\mathrm{DA}) = \angle(\mathrm{BC}.\mathrm{CD}.\mathrm{DA}.\mathrm{AB}) = \&c.;$$

with the corresponding equality of the *angles of the four anharmonics*,

$$\text{XVII.} \ldots \angle(\mathrm{ABCD}) = \angle(\mathrm{BCDA}) = \angle(\mathrm{CDAB}) = \angle(\mathrm{DABC});$$

or of those of the four *reciprocal* anharmonics (259, XVII.),

$$\text{XVII}'. \ldots \angle(\mathrm{ADCB}) = \angle(\mathrm{BADC}) = \angle(\mathrm{CBAD}) = \angle(\mathrm{DCBA}).$$

(18.) Interpreting now, by (13.) and (15.), these last equations, we derive from them the following *theorem*, for the *plane*, or for *space* :—

Let ABCD be *any four points*, connected by *four circles*, each passing through *three* of the points: then, not only is the *angle at* A, between the *arcs* ABC, ADC, *equal* to the *angle at* C, between CDA and CBA, but also it is equal (comp. Fig. 66) to the *angle at* B, between the *two other arcs* BCD and BAD, and to the *angle at* D, between the arcs DAB, DCB.

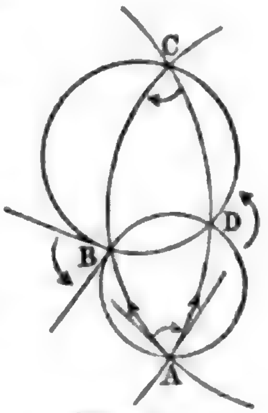

Fig. 66.

(19.) Again, let ABCDE be any *pentagon*, *inscribed in a sphere;* and conceive that the *two diagonals* AC, AD are drawn. We shall then have three equations, of the forms,

$$\text{XVIII.} \ldots \mathrm{AB}.\mathrm{BC}.\mathrm{CA} = \mathrm{AT}; \quad \mathrm{AC}.\mathrm{CD}.\mathrm{DA} = \mathrm{AU}; \quad \mathrm{AD}.\mathrm{DE}.\mathrm{EA} = \mathrm{AV};$$

where AT, AU, AV are three tangents to the sphere at A, so that their product is a fourth tangent at that point. But the equations XVIII. give

$$\text{XIX.} \ldots \mathrm{AB}.\mathrm{BC}.\mathrm{CD}.\mathrm{DE}.\mathrm{EA} = (\mathrm{AT}.\mathrm{AU}.\mathrm{AV}) : (\overline{\mathrm{AC}}^2.\overline{\mathrm{AD}}^2)$$
$$= \mathrm{AW} = \textit{a new vector, which touches the sphere at } \mathrm{A}.$$

We have therefore this *Theorem*, which includes several others under it :—

"*The product of the five successive sides, of any (generally gauche) pentagon inscribed in a sphere, is equal to a tangential vector, drawn from the point at which the pentagon begins and ends.*"

(20.) Let then P be a point on the sphere which passes through O, and through three given points A, B, C; we shall have the equation,

$$\text{XX.} \ldots 0 = \mathrm{S}(\mathrm{OA}.\mathrm{AB}.\mathrm{BC}.\mathrm{CP}.\mathrm{PO}) = \mathrm{S}\alpha(\beta-\alpha)(\gamma-\beta)(\rho-\gamma)(-\rho)$$
$$= \alpha^2\mathrm{S}\beta\gamma\rho + \beta^2\mathrm{S}\gamma\alpha\rho + \gamma^2\mathrm{S}\alpha\beta\rho - \rho^2\mathrm{S}\alpha\beta\gamma.$$

(21.) Comparing with 294, XIV., we see that the *condition* for the four co-initial vectors $\alpha, \beta, \gamma, \rho$ thus *terminating on one spheric surface*, which passes *through their common origin* O, may be thus expressed:

$$\text{XXI.} \ldots \text{if } \rho = x\alpha + y\beta + z\gamma, \quad \text{then} \quad \rho^2 = x\alpha^2 + y\beta^2 + z\gamma^2.$$

(22.) If then we *project* (comp. 62) the variable point P into points A', B', C' on the *three given chords* OA, OB, OC, by *three planes* through that point P, respectively *parallel* to the planes BOC, COA, AOB, we shall have the equation:

$$\text{XXII.} \ldots \mathrm{OP}^2 = \mathrm{OA}.\mathrm{OA}' + \mathrm{OB}.\mathrm{OB}' + \mathrm{OC}.\mathrm{OC}'.$$

(23.) That the equation XX. does in fact represent a *spheric locus* for the point P, is evident from its mere *form* (comp. 282, (10.)); and that this sphere passes *through the four given points*, O, A, B, C, may be proved by observing that the equation is satisfied, when we change ρ to any one of the four vectors, 0, α, β, γ.

(24.) Introducing an *auxiliary vector*, OD or δ, determined by the equation,

$$\text{XXIII.}\ldots\ \delta S\alpha\beta\gamma = \alpha^2 V\beta\gamma + \beta^2 V\gamma\alpha + \gamma^2 V\alpha\beta,$$

or by the system of the three scalar equations (comp. 294, (25.)),

$$\text{XXIV.}\ldots\ \alpha^2 = S\delta\alpha, \quad \beta^2 = S\delta\beta, \quad \gamma^2 = S\delta\gamma,$$

or

$$\text{XXIV}'.\ldots\ S\delta\alpha^{-1} = S\delta\beta^{-1} = S\delta\gamma^{-1} = 1,$$

the equation XX. of the sphere becomes simply,

$$\text{XXV.}\ldots\ \rho^2 = S\delta\rho, \quad \text{or} \quad \text{XXV}'.\ldots\ S\delta\rho^{-1} = 1;$$

so that D is the point of the sphere *opposite* to O, and δ is a *diameter* (comp. 282, IX′.; and 196, (6.)).

(25.) The formula XXIII., which determines this diameter, may be written in this other way:

$$\text{XXVI.}\ldots\ \delta S\alpha\beta\gamma = V\alpha(\beta-\alpha)(\gamma-\beta)\gamma;$$

or

$$\text{XXVI}'.\ldots\ 6.\text{OABC}.\text{OD} = -V(\text{OA}.\text{AB}.\text{BC}.\text{CO});$$

where the symbol OABC, considered as a *coefficient*, is interpreted as in 294, XLIV.; namely, as denoting the *volume* of the *pyramid* OABC, which is here an *inscribed* one.

(26.) This result of calculation, so far as it regards the *direction* of the *axis of the quaternion* OA . AB . BC . CO, agrees with, and may be used to confirm, the theorem (12.), respecting the *product of the successive sides of a gauche quadrilateral*, OABC; including the *rule of rotation*, which *distinguishes* that axis from its *opposite*.

(27.) The formula XXIII. for the diameter δ may also be thus written:

$$\text{XXVII.}\ldots\ \delta . S\alpha^{-1}\beta^{-1}\gamma^{-1} = V(\beta^{-1}\gamma^{-1} + \gamma^{-1}\alpha^{-1} + \alpha^{-1}\beta^{-1})$$
$$= V(\beta^{-1} - \alpha^{-1})(\gamma^{-1} - \alpha^{-1});$$

and the equation XX. of the sphere may be transformed to the following:

$$\text{XXVIII.}\ldots\ 0 = S(\beta^{-1} - \alpha^{-1})(\gamma^{-1} - \alpha^{-1})(\rho^{-1} - \alpha^{-1});$$

which expresses (by 294, (34.), comp. 260, (10.)), that the *four reciprocal vectors*,

$$\text{XXIX.}\ldots\ \text{OA}' = \alpha' = \alpha^{-1}, \quad \text{OB}' = \beta' = \beta^{-1}, \quad \text{OC}' = \gamma' = \gamma^{-1}, \quad \text{OP}' = \rho' = \rho^{-1},$$

are *termino-complanar* (64); the plane A′B′C′P′, in which they all terminate, being *parallel to the tangent plane to the sphere at* O: because the perpendicular let fall on this plane from O is

$$\text{XXX.}\ldots\ \delta' = \delta^{-1},$$

as appears from the three scalar equations,

$$\text{XXXI.}\ldots\ S\alpha'\delta = S\beta'\delta = S\gamma'\delta = 1.$$

(28.) In general, if D be the *foot of the perpendicular from* O, *on the plane* ABC, then

$$\text{XXXII.}\ldots\ \delta = S\alpha\beta\gamma : V(\beta\gamma + \gamma\alpha + \alpha\beta);$$

because this expression satisfies, and may be deduced from, the three equations,

$$\text{XXXIII.}\ldots\ S\alpha\delta^{-1} = S\beta\delta^{-1} = S\gamma\delta^{-1} = 1.$$

As a verification, the formula shows that the *length* $T\delta$, of this perpendicular, or *altitude*, OD, is equal to *the sextuple volume of the pyramid* OABC, *divided by the double area of the triangular base* ABC. (Compare 281, (4.), and 294, (8.), (33.).)

(29.) The equation XX., of the *sphere* OABC, might have been obtained by the *elimination of the vector* δ, between the *four scalar equations* XXIV. and XXV., on the plan of 294, (27.).

(30.) And another form of equation of the same sphere, answering to the development of XXVIII., may be obtained by the analogous elimination of the same vector δ, between the four other equations, XXIV′. and XXV′.

(31.) The *product* of *any even number* of *complanar vectors* is *generally a quaternion* with an *axis perpendicular to their plane;* but the product of the *successive sides of a hexagon* ABCDEF, or any other *even-sided figure, inscribed in a circle*, is a *scalar:* because by drawing *diagonals* AC, AD, AE from the *first* (or *last*) *point* A of the polygon, we find as in (6.) that it differs only by a scalar coefficient, or divisor, from the product of an *even number of tangents*, at the first point.

(32.) On the other hand, the *product* of *any odd number of complanar vectors* is *always a line*, in the same plane; and in particular (comp. (19.)), the product of the *successive sides* of a *pentagon*, or *heptagon*, &c., inscribed *in a circle*, is equal to a *tangential vector*, drawn from the *first point* of that *inscribed* and *odd-sided polygon:* because it differs only by a scalar coefficient from the product of an *odd number* of such *tangents*.

(33.) The *product* of *any number* of *lines in space* is *generally* a *quaternion* (289); and if they be the *successive sides* of a *hexagon*, or other *even-sided polygon*, inscribed *in a sphere*, the *axis* of this quaternion (comp. (12.)) is *normal to that sphere, at the initial* (or final) *point* of the polygon.

(34.) But the product of the successive sides of a *heptagon*, or other *odd-sided polygon in a sphere*, is equal (comp. (19.)) to a *vector*, which *touches the sphere* at the initial or final point; because it bears a scalar ratio to the product of an *odd number* of vectors, in the *tangent plane* at that point.

(35.) The equation XX., or its transformation XXVIII., may be called the condition or *equation of homosphericity* (comp. 260, (10.)) of the *five points* O, A, B, C, P; and the analogous equation for the five points ABCDE, with vectors $\alpha\beta\gamma\delta\epsilon$ from any arbitrary origin O, may be written thus:

$$\text{XXXIV.}\ldots\ 0 = \mathrm{S}(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\epsilon)(\epsilon-\alpha);$$

or thus,

$$\text{XXXV.}\ldots\ 0 = a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\epsilon^2,$$

six times the second member of this last formula being found to be equal to the second member of the one preceding it, if

$$\text{XXXVI.}\ldots\ a = \mathrm{BCDE},\quad b = \mathrm{CDEA},\quad c = \mathrm{DEAB},\quad d = \mathrm{EABC},\quad e = \mathrm{ABCD},$$

or more fully,

$$\text{XXXVII.}\ldots\ 6a = \mathrm{S}(\gamma-\beta)(\delta-\beta)(\epsilon-\beta) = \mathrm{S}(\gamma\delta\epsilon - \delta\epsilon\beta + \epsilon\beta\gamma - \beta\gamma\delta),\ \&\text{c.};$$

so that, by 294, XLVIII. and XLVII., we have also (comp. 65, 70) the equation,

$$\text{XXXVIII.}\ldots\ 0 = a\alpha + b\beta + c\gamma + d\delta + e\epsilon,$$

with the relation between the coefficients,

$$\text{XXXIX.}\ldots\ 0 = a + b + c + d + e,$$

which allows (as above) the *origin* of vectors to be *arbitrary*.

(36.) The equation or condition XXXV. may be obtained as the result of an *elimination* (294, (27.)), of a *vector* κ, and of a *scalar* g, between *five scalar equations* of the *form* 282, (10.), namely the five following,

$$\text{XL.} \ldots \alpha^2 - 2\mathrm{S}\kappa\alpha + g = 0, \quad \beta^2 - 2\mathrm{S}\kappa\beta + g = 0, \ldots \quad \epsilon^2 - 2\mathrm{S}\kappa\epsilon + g = 0;$$

κ being the *vector of the centre* K of the *sphere* ABCD, of which the equation may be written as

$$\text{XLI.} \ldots \rho^2 - 2\mathrm{S}\kappa\rho + g = 0,$$

g being some scalar constant; and *on which*, by the *condition* referred to, the *fifth point* E is situated.

(37.) By treating this fifth point, or its vector ϵ, as *arbitrary*, we recover the condition or *equation of concircularity* (3.), of the *four points* A, B, C, D; or the formula,

$$\text{XLII.} \ldots 0 = \mathrm{V}(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \alpha).$$

(38.) The *equation of the circle* ABC, and the *equation of the sphere* ABCD, may in general be written thus:

$$\text{XLIII.} \ldots 0 = \mathrm{V}(\alpha - \beta)(\beta - \gamma)(\gamma - \rho)(\rho - \alpha);$$
$$\text{XLIV.} \ldots 0 = \mathrm{S}(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \rho)(\rho - \alpha);$$

ρ being as usual the vector of a *variable point* P, on the one or the other *locus*.

(39.) The equations of the *tangent to the circle* ABC, and of the *tangent plane to the sphere* ABCD, at the point A, are respectively,

$$\text{XLV.} \ldots 0 = \mathrm{V}(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\rho - \alpha),$$

and

$$\text{XLVI.} \ldots 0 = \mathrm{S}(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \alpha)(\rho - \alpha).$$

(40.) Accordingly, whether we combine the two equations XLIII. and XLV., or XLIV. and XLVI., we find in each case the equation,

$$\text{XLVII.} \ldots (\rho - \alpha)^2 = 0, \quad \text{giving} \quad \rho = \alpha, \quad \text{or} \quad \mathrm{P} = \mathrm{A}\ (20);$$

it being supposed that the *three* points A, B, C are *not collinear*, and that the *four points*, A, B, C, D are *not complanar*.

(41.) If the *centre* of the *sphere* ABCD be taken for the *origin* O, so that

$$\text{XLVIII.} \ldots \alpha^2 = \beta^2 = \gamma^2 = \delta^2 = -r^2, \quad \text{or} \quad \text{XLIX.} \ldots \mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma = \mathrm{T}\delta = r,$$

the positive scalar r denoting the *radius*, then after some reductions we obtain the transformation,

$$\text{L.} \ldots \mathrm{V}(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \alpha) = 2\alpha\mathrm{S}(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha).$$

(42.) Hence, generally, if K be, as in (36.), the centre of the sphere, we have the equation (comp. XXVI'.),

$$\text{LI.} \ldots \mathrm{V}(\mathrm{AB}.\mathrm{BC}.\mathrm{CD}.\mathrm{DA}) = 12\,\mathrm{KA}.\mathrm{ABCD}.$$

(43.) We may therefore ennunciate this *theorem*:—

"*The vector part of the product of four successive sides, of a gauche quadrilateral inscribed in a sphere, is equal to the diameter drawn to the initial point of the polygon, multiplied by the sextuple volume of the pyramid, which its four points determine.*"

(44.) In effecting the *reductions* (41.), the following *general formulæ* of transformation have been employed, which may be useful on other occasions:

$$\text{LII.} \ldots \alpha q + q\alpha = 2(\alpha\mathrm{S}q + \mathrm{S}q\alpha); \qquad \text{LII'.} \ldots \alpha q\alpha = \alpha^2\mathrm{K}q + 2\alpha\mathrm{S}q\alpha;$$

where α may be *any vector*, and q may be *any quaternion*.

SECTION 7.—*On the Fourth Proportional to Three Diplanar Vectors.*

297. In general, when any *four quaternions*, q, q', q'', q''', satisfy the *equation of quotients*,

$$\text{I.} \ldots q''' : q'' = q' : q,$$

or the equivalent formula,

$$\text{II.} \ldots q''' = (q' : q) \cdot q'' = q' q^{-1} q'',$$

we shall say that they form a *Proportion;* and that the *fourth*, namely q''', is the *Fourth Proportional* to the *first*, *second*, and *third* quaternions, namely to q, q', and q'', taken in this given *order*. This definition will *include* (by 288) the one which was assigned in 226, for the fourth proportional to *three complanar vectors*, α, β, γ, namely that *fourth vector in the same plane*, $\delta = \beta\alpha^{-1}\gamma$, which has been already considered; and it will enable us to *interpret* (comp. 289) the *symbol*

$$\text{III.} \ldots \beta\alpha^{-1}\gamma, \text{ when } \gamma \textit{ not } |||\ \alpha, \beta,$$

as denoting *not* indeed a *Vector*, in this new case, *but* at least a *Quaternion*, which may be called (on the present general plan) *the Fourth Proportional to these Three Diplanar Vectors*, α, β, γ. *Such fourth proportionals* possess some interesting properties, especially with reference to their *vector parts*, which it will be useful briefly to consider, and to illustrate by showing their connexion with *spherical trigonometry*, and generally with *spherical geometry*.

(1.) Let α, β, γ be (as in 208, (1.), &c.) the vectors of the corners of a *triangle* ABC on the *unit-sphere*, whereof the sides are a, b, c; and let us write,

$$\text{IV.} \ldots \begin{cases} l = \cos a = \mathrm{S}\gamma\beta^{-1} = -\,\mathrm{S}\beta\gamma, \\ m = \cos b = \mathrm{S}\alpha\gamma^{-1} = -\,\mathrm{S}\gamma\alpha, \\ n = \cos c = \mathrm{S}\beta\alpha^{-1} = -\,\mathrm{S}\alpha\beta; \end{cases}$$

where it is understood that

$$\text{V.} \ldots \alpha^2 = \beta^2 = \gamma^2 = -1, \quad \text{or} \quad \text{VI.} \ldots \mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma = 1;$$

it being also at first supposed, for the sake of fixing the conceptions, that each of these three cosines, l, m, n, is greater than zero, or that each side of the triangle ABC is less than a quadrant.

(2.) Then, introducing three new vectors, δ, ϵ, ζ, defined by the equations,

$$\text{VII.} \ldots \begin{cases} \delta = \mathrm{V}\beta\alpha^{-1}\gamma = \mathrm{V}\gamma\alpha^{-1}\beta = m\beta + n\gamma - l\alpha, \\ \epsilon = \mathrm{V}\gamma\beta^{-1}\alpha = \mathrm{V}\alpha\beta^{-1}\gamma = n\gamma + l\alpha - m\beta, \\ \zeta = \mathrm{V}\alpha\gamma^{-1}\beta = \mathrm{V}\beta\gamma^{-1}\alpha = l\alpha + m\beta - n\gamma, \end{cases}$$

we find that these *three derived vectors* have all one *common length*, say r, because they have one *common norm*; namely,

$$\text{VIII.} \ldots \mathrm{N}\delta = \mathrm{N}\varepsilon = \mathrm{N}\zeta = l^2 + m^2 + n^2 - 2lmn = r^2;$$

so that

$$\text{IX.} \ldots \mathrm{T}\delta = \mathrm{T}\varepsilon = \mathrm{T}\zeta = r = \sqrt{(l^2 + m^2 + n^2 - 2lmn)}.$$

(3.) This common length, r, is *less than unity*; for if we write,

$$\text{X.} \ldots \mathrm{S}\alpha\beta\gamma = \mathrm{S}\beta\alpha^{-1}\gamma = e,$$

we shall have the relation,

$$\text{XI.} \ldots e^2 + r^2 = \mathrm{N}\beta\alpha^{-1}\gamma = 1;$$

and the scalar e is different from zero, because the vectors α, β, γ are diplanar.

(4.) *Dividing* the three lines δ, ε, ζ by their *length*, r, we change them to their *versors* (155, 156); and so obtain a *new triangle*, DEF, *on the unit-sphere*, of which the corners are determined by the *three new unit-vectors*,

$$\text{XII.} \ldots \mathrm{OD} = \mathrm{U}\delta = r^{-1}\delta; \quad \mathrm{OE} = \mathrm{U}\varepsilon = r^{-1}\varepsilon;$$
$$\mathrm{OF} = \mathrm{U}\zeta = r^{-1}\zeta.$$

(5.) The *sides* opposite to D, E, F, in this *new* or *derived* triangle, are *bisected*, as in Fig. 67, by the *corners* A, B, C of the *old* or *given* triangle; because we have the three equations,

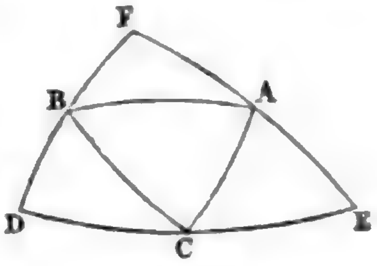

Fig. 67.

$$\text{XIII.} \ldots \varepsilon + \zeta = 2l\alpha; \quad \zeta + \delta = 2m\beta; \quad \delta + \varepsilon = 2n\gamma.$$

(6.) Denoting the *halves* of the *new sides* by a', b', c' (so that the *arc* EF $= 2a'$, &c.), the equations XIII. show also, by IV. and IX., that

$$\text{XIV.} \ldots \cos a = r\cos a', \quad \cos b = r\cos b', \quad \cos c = r\cos c';$$

the *cosines of the half-sides* of the *new* (or *bisected*) *triangle*, DEF, are therefore *proportional to the cosines of the sides* of the *old* (or *bisecting*) *triangle* ABC.

(7.) The equations IV. give, by 279, (1.),

$$\text{XV.} \ldots 2l = -(\beta\gamma + \gamma\beta), \quad 2m = -(\gamma\alpha + \alpha\gamma), \quad 2n = -(\alpha\beta + \beta\alpha);$$

we have therefore, by VII., the three following equations between quaternions,

$$\text{XVI.} \ldots \alpha\varepsilon = \zeta\alpha, \quad \beta\zeta = \delta\beta, \quad \gamma\delta = \varepsilon\gamma;$$

which may also be thus written,

$$\text{XVI}'. \ldots \varepsilon\alpha = \alpha\zeta, \quad \zeta\beta = \beta\delta, \quad \delta\gamma = \gamma\varepsilon,$$

and express in a new way the relations of *bisection* (5.).

(8.) We have therefore the equations between vectors,

$$\text{XVII.} \ldots \varepsilon = \alpha\zeta\alpha^{-1}, \quad \zeta = \beta\delta\beta^{-1}, \quad \delta = \gamma\varepsilon\gamma^{-1};$$

or

$$\text{XVII}'. \ldots \zeta = \alpha\varepsilon\alpha^{-1}, \quad \delta = \beta\zeta\beta^{-1}, \quad \varepsilon = \gamma\delta\gamma^{-1}.$$

(9.) Hence also, by V., or because α, β, γ are *unit-vectors*,

$$\text{XVIII.} \ldots \varepsilon = -\alpha\zeta\alpha, \quad \zeta = -\beta\delta\beta, \quad \delta = -\gamma\varepsilon\gamma;$$

or

$$\text{XVIII}'. \ldots \zeta = -\alpha\varepsilon\alpha, \quad \delta = -\beta\zeta\beta, \quad \varepsilon = -\gamma\delta\gamma.$$

(10.) In general, *whatever the length of the vector* α *may be*, the first equation XVII. expresses that *the line* ε *is* (comp. 138) *the reflexion of the line* ζ, *with respect to that vector* α; because it may be put (comp. 279) under the form,

$$\text{XIX.} \ldots \zeta\alpha^{-1} = \alpha^{-1}\varepsilon = \mathrm{K}\varepsilon\alpha^{-1}, \quad \text{or} \quad \text{XIX}'. \ldots \varepsilon\alpha^{-1} = \mathrm{K}\zeta\alpha^{-1}.$$

(11.) Another mode of arriving at the same *interpretation* of the equation

$\epsilon = \alpha\zeta\alpha^{-1}$, is to conceive ζ decomposed into two summand vectors, ζ' and ζ'', one parallel and the other perpendicular to α, in such a manner that

$$\text{XX.} \ldots \zeta = \zeta' + \zeta'', \quad \zeta' \parallel \alpha, \quad \zeta'' \perp \alpha;$$

for then we shall have, by 281, (10.), the transformations,

$$\text{XXI.} \ldots \epsilon = \alpha\zeta'\alpha^{-1} + \alpha\zeta''\alpha^{-1} = \zeta'\alpha\alpha^{-1} - \zeta''\alpha\alpha^{-1} = \zeta' - \zeta'';$$

the *parallel part* of ζ being thus *preserved*, but the *perpendicular part* being *reversed*, by the *operation* $\alpha(\ \ )\alpha^{-1}$.

(12.) Or we may *return* from $\epsilon = \alpha\zeta\alpha^{-1}$ to the form $\epsilon\alpha = \alpha\zeta$, that is, to the first equation XVI'.; and then this equation between quaternions will show, as suggested in (7.), that whatever may be the *length* of α, we must have,

$$\text{XXII.} \ldots \mathrm{T}\epsilon = \mathrm{T}\zeta, \quad \mathrm{Ax.}^{*}\, \epsilon\alpha = \mathrm{Ax.}\, \alpha\zeta, \quad \angle\, \epsilon\alpha = \angle\alpha\zeta;$$

so that the *two lines* ϵ, ζ are *equally long*, and the *rotation* from ϵ to α is *equal* to that from α to ζ; these two rotations being *similarly directed*, and in one *common plane*.

(13.) We may also write the equations XVII. XVII'. under the forms,

$$\text{XXIII.} \ldots \epsilon = \alpha^{-1}\zeta\alpha, \text{ \&c.}; \qquad \text{XXIII}'. \ldots \zeta = \alpha^{-1}\epsilon\alpha, \text{ \&c.}$$

(14.) Substituting this last expression for ζ in the second equation XVII'., we derive this new equation,

$$\text{XXIV.} \ldots \delta = \beta\alpha^{-1}\epsilon\alpha\beta^{-1}; \quad \text{or} \quad \text{XXIV}'. \ldots \epsilon = \alpha\beta^{-1}\delta\beta\alpha^{-1};$$

that is, more briefly,

$$\text{XXV.} \ldots \delta = q\epsilon q^{-1}, \quad \text{and} \quad \text{XXV}'. \ldots \epsilon = q^{-1}\delta q, \quad \text{if} \quad \text{XXVI.} \ldots q = \beta\alpha^{-1}.$$

(15.) An expression of this *form*, namely one with such a symbol as

$$\text{XXVII.} \ldots q(\ \ )q^{-1}$$

for an *operator*, occurred before, in 179, (1.), and in 191, (5.); and was seen to indicate a *conical rotation of the axis of the operand quaternion* (of which the *symbol* is to be conceived as being written *within the parentheses*), *round the axis of* q, *through an angle* $= 2\angle q$, without any change of the *angle*, or of the *tensor*, of that *operand;* so that a *vector* must *remain* a *vector*, after any *operation* of this sort, as being *still* a *right-angled quaternion* (290); or (comp. 223, (10.)) because

$$\text{XXVIII.} \ldots \mathrm{S}q\rho q^{-1} = \mathrm{S}q^{-1}q\rho = \mathrm{S}\rho = 0.$$

(16.) If then we conceive two *opposite points*, P' and P, to be determined on the unit-sphere, by the conditions of being respectively the *positive poles* of the two *opposite arcs*, AB and BA, so that

$$\text{XXIX.} \ldots \mathrm{OP}' = \mathrm{Ax.}\, \beta\alpha^{-1} = \mathrm{Ax.}\, q, \quad \text{and} \quad \mathrm{OP} = \mathrm{P'O} = \mathrm{Ax.}\, \alpha\beta^{-1} = \mathrm{Ax.}\, q^{-1},$$

we can infer from XXIV. that *the line* OD *may be derived from the line* OE, *by a conical rotation round the line* OP' *as an axis, through an angle equal to the double of the angle* AOB (if O be still the *centre* of the *sphere*).

(17.) And in like manner we can infer from XXIV'., that the line OE admits

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[*] It was remarked in 291, that this *characteristic* Ax. *can be dispensed with*, because it admits of being *replaced* by UV; but there may still be a convenience in employing it occasionally.

of being derived from OD, by an *equal* but *opposite conical rotation*, round the line OP as a *new positive axis*, through an angle equal to twice the angle BOA.

(18.) To illustrate these and other connected results, the annexed Figure 68 is drawn; in which P represents, as above, the positive pole of the arc BA, and arcs are drawn from it to D, E, F, meeting the great circle through A and B in the points R, S, T. (The other letters in the Figure are not, for the moment, required, but their significations will soon be explained.)

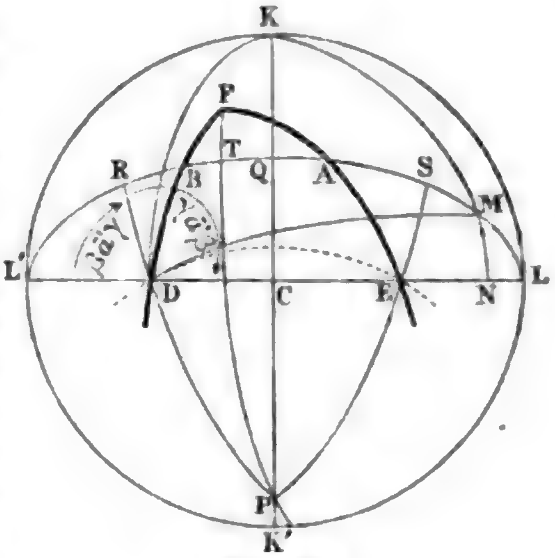

Fig. 68.

(19.) This being understood, we see, first, that because the arcs EF and FD are *bisected* (5.) at A and B, the *three arcual perpendiculars*, ES, FT, DR, let fall from E, F, D, on the great circle through A and B, are *equally long*: and that therefore the point P is the *interior pole of the small circle* DEF', if F' be the point diametrically *opposite to* F: so that a *conical rotation round this pole* P, or round the *axis* OP, would in fact bring the point D, or the line OD, *to the position* E, or OE, which is *one part* of the *theorem* (17.).

(20.) Again, the *quantity* of this *conical rotation*, is evidently measured by the *arc* RS of the *great circle* with P for *pole*; but the *bisections* above mentioned give (comp. 165) the two *arcual equations*,

XXX. . . $\frown$ RB = $\frown$ BT, $\frown$ TA = $\frown$ AS; whence XXXI. . . $\frown$ RS = 2 $\frown$ BA,

and the *other part* of the same theorem (17.) is proved.

(21.) The *point* F may be said to be the *reflexion, on the sphere, of the point* D, *with respect to the point* B, which *bisects the interval* between them; and thus we may say that *two successive reflexions of an arbitrary point upon a sphere* (as here from D to F, and then from F to E), *with respect to two given points* (B and A) *of a given great circle*, are jointly *equivalent to one conical rotation, round the pole* (P) *of that great circle*; or to the description of an *arc of a small circle*, round that *pole*, or *parallel* to that great circle: and that the *angular quantity* (DPE) of this *rotation* is *double* of that represented by the *arc* (BA) connecting the *two given points*; or is the *double of the angle* (BPA), which that *given arc subtends*, at the *same pole* (P).

(22.) There is, as we see, no difficulty in *geometrically proving* this *theorem of rotation*: but it is remarkable *how simply quaternions express it*: namely by the formula,

$$\text{XXXII.} \ldots \alpha . \beta^{-1} \rho \beta . \alpha^{-1} = \alpha \beta^{-1} . \rho . \beta \alpha^{-1},$$

in which α, β, ρ may denote *any three vectors*; and which, as we see by the *points*, involves essentially the *associative principle of multiplication*.

(23.) Instead of conceiving that the point D, or the line OD, has been *reflected* into the position F, or OF, *with respect to* the point B, or to the line OB, with a similar *successive reflexion* from F to E, we may conceive that a point has moved *along a small semicircle*, with B for *pole*, from D to F, as indicated in Fig. 69, and then along

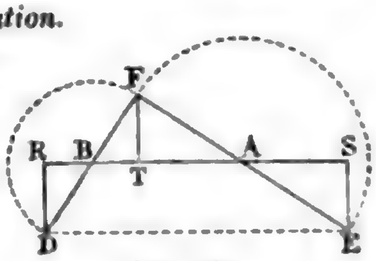

Fig. 69.

another small semicircle, with A for pole, from F to E; and we see that the *result*, or *effect*, of these *two successive and semicircular motions* is *equivalent* to a motion *along an arc* DE of a *third small circle*, which is *parallel* (as before) *to the great circle through* B and A, and has a *projection* RS *thereon*, which (still as before) is *double of the given arc* BA.

(24.) And instead of thus conceiving *two successive arcual motions of a point* D *upon a sphere*, or *two successive conical rotations of a radius* OD, considered as *compounding* themselves into *one resultant motion* of that *point*, or *rotation* of that *radius*, we may conceive an analogous *composition* of two successive *rotations of a solid body* (or *rigid system*), round *axes* passing *through a point* O, which is *fixed in space* (and *in the body*): and so obtain a *theorem* respecting *such* rotation, which easily suggests itself from what precedes, and on which we may perhaps return.

(25.) But to draw some additional consequences from the equations VII., &c., and from the recent Fig. 68, especially as regards the *Construction of the Fourth Proportional to three diplanar vectors*, let us first remark, generally, that when we have (as in 62) a *linear equation*, of the form

$$a\alpha + b\beta + c\gamma + d\delta = 0,$$

connecting *four co-initial vectors* $\alpha \, .. \, \delta$, whereof *no three are complanar*, then this *fifth vector*,

$$\varepsilon = a\alpha + b\beta = - c\gamma - d\delta,$$

is evidently *complanar* (22) with α, β, and *also* with γ, δ (comp. 294, (6.)); it is therefore part of the indefinite *line of intersection* of the *plane* AOB, COD, of these *two pairs* of vectors.

(26.) And if we *divide* this fifth vector ε by the two (generally unequal) *scalars*,

$$a + b, \quad \text{and} \quad - c - d,$$

the two (generally unequal) *vectors*,

$$(a\alpha + b\beta) : (a + b), \quad \text{and} \quad (c\gamma + d\delta) : (c + d),$$

which are obtained as the *quotients* of these two divisions, are (comp. 25, 64) the vectors of two (generally distinct) *points of intersection*, of *lines* with *planes*, namely the two following:

$$\text{AB} \cdot \text{OCD}, \quad \text{and} \quad \text{CD} \cdot \text{OAB}.$$

(27.) When the *two lines*, AB and CD, happen to *intersect each other*, the two last-mentioned *points coincide*; and thus we recover, in a new way, the *condition* (63), for the *complanarity* of the *four points* O, A, B, C, or for the *termino-complanarity* of the *four vectors* α, β, γ, δ; namely the equation

$$a + b + c + d = 0,$$

which may be compared with 294, XLV. and L.

(28.) Resuming now the recent equations VII., and introducing the new vector,

$$\text{XXXIII.} \ldots \lambda = l\alpha - m\beta = \tfrac{1}{2}(\varepsilon - \delta),$$

which gives,

$$\text{XXXIV.} \ldots \mathrm{S}\gamma\lambda = 0, \quad \text{and} \quad \text{XXXV.} \ldots \mathrm{T}\lambda = \sqrt{(r^2 - n^2)} = r \sin c',$$

we see that the two arcs BA, DE, prolonged, meet in a point L (comp. Fig. 68), for which OL $= \mathrm{U}\lambda$, and which is *distant by a quadrant from* C: a result which may be confirmed by elementary considerations, because (by a well-known theorem respecting *transversal arcs*) the *common bisector* BA of the two sides, DE and EF, must meet the *third side* in a point L, for which

$$\sin \mathrm{DL} = \sin \mathrm{EL}.$$

(29.) To prove *by quaternions* this last *equality of sines*, and to assign their common value, we have only to observe that by XXXIII.,

$$\text{XXXVI.} \ldots \ \mathrm{V}\delta\lambda = \mathrm{V}\varepsilon\lambda = \tfrac{1}{2}\mathrm{V}\delta\varepsilon\,;$$

in which,

$$\mathrm{T}\delta\lambda = \mathrm{T}\varepsilon\lambda = r^2 \sin c', \quad \text{and} \quad \mathrm{TV}\delta\varepsilon = r^2 \sin 2c'\,;$$

the *sines* in question are therefore (by 204, XIX.),

$$\text{XXXVI}'. \ldots \ \mathrm{TVU}\delta\lambda = \mathrm{TVU}\varepsilon\lambda = \tfrac{1}{2}r^2 \sin 2c' : r^2 \sin c' = \cos c'.$$

(30.) On similar principles, we may interpret the two *vector-equations*,

$$\text{XXXVII.} \ldots \ \mathrm{V}\beta\lambda = l\mathrm{V}\beta\alpha, \quad \mathrm{V}\alpha\lambda = m\mathrm{V}\beta\alpha,$$

in which

$$\text{XXXVIII.} \ldots \ \mathrm{T}\lambda : \mathrm{TV}\beta\alpha = r \sin c' : \sin c = \tan c' : \tan c,$$

an equivalent to the *trigonometric* equations,

$$\text{XXXIX.} \ldots \ \frac{\tan \mathrm{CD}}{\tan \mathrm{AB}} = \frac{\cos \mathrm{BC}}{\sin \mathrm{BL}} = \frac{\cos \mathrm{AC}}{\sin \mathrm{AL}}.$$

(31.) Accordingly, if we let fall the perpendicular CQ on AB (see again Fig. 68), so that Q bisects RS, and if we determine two new points M, N by the arcual equations,

$$\text{XL.} \ldots \ \frown \mathrm{LM} = \frown \mathrm{AB} = \frown \mathrm{QR}, \quad \frown \mathrm{LN} = \frown \mathrm{CD},$$

the arcs MR, ND will be *quadrants;* and because the angle at R is *right* by construction (18.), M is the *pole* of DR, and DM is a quadrant; whence D is the pole of MN and the angle LNM is right: conceiving then that the arcs CA and CB are drawn, we have three triangles, right-angled at Q and N, which show, by elementary principles, that the three trigonometric quotients in XXXIX. have in fact a common value, namely cos CQ, or cos L.

(32.) To prove this *last* result by *quaternions*, and *without* employing the auxiliary points M, N, Q, R, we have the transformations,

$$\text{XLI.} \ldots \ \cos \mathrm{L} = \mathrm{SU}\frac{\mathrm{V}\beta\alpha}{\mathrm{V}\delta\varepsilon} = \mathrm{SU}\,\frac{\mathrm{V}\beta\alpha}{\gamma\lambda} = \mathrm{T}\frac{\lambda}{\mathrm{V}\beta\alpha}\,.\,\mathrm{S}\frac{\beta\alpha}{\gamma\lambda} = \mathrm{T}\frac{\lambda}{\mathrm{V}\beta\alpha}\,;$$

because

$$\text{XLII.} \ldots \ \delta = n\gamma - \lambda, \quad \varepsilon = n\gamma + \lambda, \quad \mathrm{V}\delta\varepsilon = 2n\gamma\lambda, \quad \mathrm{UV}\delta\varepsilon = \mathrm{U}\gamma\lambda,$$

and

$$\text{XLIII.} \ldots \ \mathrm{S}\frac{\beta\alpha}{\gamma\lambda} = \frac{\mathrm{S}\beta\alpha\gamma\lambda}{(\gamma\lambda)^2} = -\,\mathrm{S}\beta\alpha^{-1}\gamma\lambda^{-1} = -\,\mathrm{S}\delta\lambda^{-1} = 1\,;$$

it being remembered that $\lambda \perp \gamma$, whence .

$$\mathrm{V}\gamma\lambda = \gamma\lambda = -\lambda\gamma, \quad (\gamma\lambda)^2 = -\gamma^2\lambda^2 = \lambda^2, \quad \mathrm{S}\gamma\lambda^{-1} = 0.$$

(33.) At the same time we see that if P be (as before) the positive pole of BA, and if K, K′ be the negative and positive poles of DE, while L′ is the negative (as L is the positive) pole of CQ, whereby all the letters in Fig. 68 have their significations determined, we may write,

$$\text{XLIV.} \ldots \ \mathrm{OP} = \mathrm{UV}\beta\alpha\,; \quad \mathrm{OK}' = \gamma\mathrm{U}\lambda\,; \quad \mathrm{OK} = -\,\gamma\mathrm{U}\lambda\,; \quad \mathrm{OL}' = -\,\mathrm{U}\lambda\,;$$

while

$$\mathrm{OL} = +\,\mathrm{U}\lambda, \text{ as before.}$$

(34.) Writing also,

$$\text{XLV.} \ldots \kappa = -\gamma\lambda, \quad \text{or} \quad \lambda = \gamma\kappa, \quad \text{and} \quad \mu = \beta\alpha^{-1}\lambda,$$

so that

$$\text{XLV'.} \ldots \mathrm{OK} = \mathrm{U}\kappa, \quad \text{and} \quad \mathrm{OM} = \mathrm{U}\mu,$$

we have

$$\text{XLVI.} \ldots \beta\alpha^{-1}.\gamma = \mu\lambda^{-1}.\lambda\kappa^{-1} = \mu\kappa^{-1};$$

this *fourth proportional, to the three equally long but diplanar vectors*, α, β, γ, is therefore a *versor*, of which the *representative arc* (162) is KM, and the *representative angle* (174) is KDM, or L'DR, or EDP; and we may write for this versor, or quaternion, the expression:

$$\text{XLVII.} \,.\, \beta\alpha^{-1}\gamma = \cos \mathrm{L'DR} + \mathrm{OD}.\sin \mathrm{L'DR}.$$

(35.) The *double* of this representative *angle* is the *sum* of the *two base-angles* of the *isosceles triangle* DPE; and because the two other triangles, EPF', F'PD, are *also* isosceles (19.), the *lune* FF' shows that this sum is what *remains*, when we *subtract the vertical angle* F, of the triangle DEF, from the *sum of the supplements* of the two base-angles D and E of that triangle; or when we subtract the *sum of the three angles* of the same triangle *from four right angles*. We have therefore this very simple expression for the *Angle of the Fourth Proportional:*

$$\text{XLVIII.} \ldots \angle\, \beta\alpha^{-1}\gamma = \mathrm{L'DR} = \pi - \tfrac{1}{2}(\mathrm{D} + \mathrm{E} + \mathrm{F}).$$

(36.) Or, if we introduce the *area*, or the *spherical excess*, say Σ, of the triangle DEF, writing thus

$$\text{XLIX.} \ldots \Sigma = \mathrm{D} + \mathrm{E} + \mathrm{F} - \pi,$$

we have these other expressions:

$$\text{L.} \ldots \angle\, \beta\alpha^{-1}\gamma = \tfrac{1}{2}\pi - \tfrac{1}{2}\Sigma; \qquad \text{LI.} \ldots \beta\alpha^{-1}\gamma = \sin \tfrac{1}{2}\Sigma + r^{-1}\delta \cos \tfrac{1}{2}\Sigma;$$

because

$$\mathrm{OD} = \mathrm{U}\delta = r^{-1}\delta, \text{ by XII.}$$

(37.) Having thus expressed $\beta\alpha^{-1}\gamma$, we require no new appeal to the Figure, in order to express this *other* fourth proportional, $\gamma\alpha^{-1}\beta$, which is the *negative* of its *conjugate*, or has an *opposite scalar*, but an *equal vector part* (comp. 204, (1.), and 295, (9.)): the geometrical difference being merely this, that because the rotation round α from β to γ has been supposed to be *negative*, the rotation round α from γ to β must be, on the contrary, *positive*.

(38.) We may thus write, at once,

$$\text{LII.} \ldots \gamma\alpha^{-1}\beta = -\mathrm{K}\beta\alpha^{-1}\gamma = -\sin \tfrac{1}{2}\Sigma + r^{-1}\delta \cos \tfrac{1}{2}\Sigma;$$

and we have, for the *angle* of this *new* fourth proportional, to the *same three vectors* α, β, γ, of which the *second* and *third* have merely *changed places* with each other, the formula:

$$\text{LIII.} \ldots \angle\, \gamma\alpha^{-1}\beta = \mathrm{RDL} = \tfrac{1}{2}(\mathrm{D} + \mathrm{E} + \mathrm{F}) = \tfrac{1}{2}\pi + \tfrac{1}{2}\Sigma.$$

(39.) But the *common vector part* of these *two* fourth proportionals is δ, by VII; we have therefore, by XI.,

$$\text{LIV.} \ldots r = \cos \tfrac{1}{2}\Sigma; \quad e = \pm \sin \tfrac{1}{2}\Sigma;$$

the upper sign being taken, when the rotation round α from β to γ is negative, as above supposed.

(40.) It follows by (6.) that when the *sides* $2a'$, $2b'$, $2c'$, of a spherical triangle

DEF, of which the *area* is Σ, are *bisected* by the *corners* A, B, C of *another* spherical triangle, of which the sides* are a, b, c, then

$$\text{LV.} \ldots \cos a : \cos a' = \cos b : \cos b' = \cos c : \cos c' = \cos \tfrac{1}{2}\Sigma.$$

(41.) It follows also, from what has been recently shown, that the *angle* RDK, or MDN, or the *arc* MN in Fig. 68, *represents the semi-area* of the *bisected triangle* DEF; whence, by the right-angled triangle LMN, we can infer that the *sine* of this *semi-area* is equal to the *sine of a side* of the *bisecting triangle* ABC, multiplied into the *sine of the perpendicular*, let fall upon that side from the opposite corner of the latter triangle; because we have

$$\text{LVI.} \ldots \sin \tfrac{1}{2}\Sigma = \sin \text{MN} = \sin \text{LM} \,.\, \sin \text{L} = \sin \text{AB} \,.\, \sin \text{CQ}.$$

(42.) The same conclusion can be drawn immediately, by quaternions, from the expression,

$$\text{LVII.} \ldots \sin \tfrac{1}{2}\Sigma = e = \mathrm{S}\alpha\beta\gamma = \mathrm{S}(\mathrm{V}\beta\alpha \,.\, \gamma^{-1}) = \mathrm{TV}\beta\alpha \,.\, \mathrm{SU}(\mathrm{V}\beta\alpha : \gamma);$$

in which one factor is the sine of AB, and the other factor is the cosine of CP, or the sine of CQ.

(43.) Under the same conditions, since

$$\text{LVIII.} \ldots \alpha = \mathrm{U}(\varepsilon + \zeta) = \tfrac{1}{2}l^{-1}(\varepsilon + \zeta), \ \&\text{c.},$$

we may write also,

$$\text{LIX.} \ldots \sin \tfrac{1}{2}\Sigma = \mathrm{SU}(\varepsilon + \zeta)\,(\zeta + \delta)\,(\delta + \varepsilon) = \mathrm{S}\delta\varepsilon\zeta : 4lmn;$$

in which, by IV. and XIII.,

$$\text{LX.} \ldots 4lmn = -\,\mathrm{S}(\delta + \varepsilon)\,(\varepsilon + \zeta) = r^2 - \mathrm{S}(\varepsilon\zeta + \zeta\delta + \delta\varepsilon).$$

(44.) Hence also, by LIV.,

$$\text{LXI.} \ldots \cos \tfrac{1}{2}\Sigma = r = (r^3 - r\mathrm{S}(\varepsilon\zeta + \zeta\delta + \delta\varepsilon)) : 4lmn;$$

$$\text{LXII.} \ldots \tan \tfrac{1}{2}\Sigma = \frac{e}{r} = \frac{\mathrm{S}\delta\varepsilon\zeta}{r^3 - r\mathrm{S}(\varepsilon\zeta + \zeta\delta + \delta\varepsilon)} = \frac{\mathrm{SU}\delta\varepsilon\zeta}{1 - \mathrm{SU}\varepsilon\zeta - \mathrm{SU}\zeta\delta - \mathrm{SU}\delta\varepsilon};$$

and under *this last form*, we have a *general expression for the tangent of half the spherical opening at* O, *of any triangular pyramid* ODEF, *whatever the lengths* $\mathrm{T}\delta$, $\mathrm{T}\varepsilon$, $\mathrm{T}\zeta$ *of the edges at* O *may be.*

(45.) As a verification, we have

$$\begin{aligned} \text{LXIII.} \ldots (4lmn)^2 &= -\tfrac{1}{4}(\varepsilon + \zeta)^2\,(\zeta + \delta)^2\,(\delta + \varepsilon)^2 \\ &= 2(r^2 - \mathrm{S}\varepsilon\zeta)\,(r^2 - \mathrm{S}\zeta\delta)\,(r^2 - \mathrm{S}\delta\varepsilon); \end{aligned}$$

but the elimination of $\tfrac{1}{2}\Sigma$ between LIX. LXI. gives,

$$\text{LXIV.} \ldots (4lmn)^2 = (\mathrm{S}\delta\varepsilon\zeta)^2 + (r^3 - r(\mathrm{S}\varepsilon\zeta + \mathrm{S}\zeta\delta + \mathrm{S}\delta\varepsilon))^2;$$

we ought then to find that

$$\text{LXV.} \ldots (\mathrm{S}\delta\varepsilon\zeta)^2 = r^6 - r^2\{(\mathrm{S}\varepsilon\zeta)^2 + (\mathrm{S}\zeta\delta)^2 + (\mathrm{S}\delta\varepsilon)^2\} - 2\mathrm{S}\varepsilon\zeta\mathrm{S}\zeta\delta\mathrm{S}\delta\varepsilon,$$

if $\delta^2 = \varepsilon^2 = \zeta^2 = -r^2$; and in fact this equality results immediately from the general formula 294, LIII.

(46.) Under the same condition, respecting the equal lengths of δ, ε, ζ, we have also the formula,

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* These *sides abc*, of the *bisecting triangle* ABC, have been hitherto supposed for simplicity (1.) to be *each less than a quadrant*, but it will be found that the *formula* LV. holds good, *without any such restriction*.

$$\text{LXVI.} \ldots -\mathrm{V}(\delta+\epsilon)(\epsilon+\zeta)(\zeta+\delta) = 2\delta(r^2 - \mathrm{S}\epsilon\zeta - \mathrm{S}\zeta\delta - \mathrm{S}\delta\epsilon) = 8lmn\delta;$$

whence other verifications may be derived.

(47.) If σ denote the *area** *of the bisecting triangle* ABC, the general principle LXII. enables us to infer that

$$\text{LXVII.} \ldots \tan\frac{\sigma}{2} = \frac{\mathrm{S}\alpha\beta\gamma}{1 - \mathrm{S}\beta\gamma - \mathrm{S}\gamma\alpha - \mathrm{S}\alpha\beta} = \frac{e}{1+l+m+n}$$
$$= \frac{\sin c \sin p}{1 + \cos a + \cos b + \cos c},$$

if p denote the perpendicular CQ from C on AB, so that

$$e = \sin c \sin p = \sin b \sin c \sin \mathrm{A} = \&c. \text{ (comp. 210, (21.)).}$$

(48.) But, by (IX.) and (XI.),

$$\text{LXVIII.} \ldots e^2 + (1+l+m+n)^2 = 2(1+l)(1+m)(1+n)$$
$$= \left(4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\right)^2;$$

hence the cosine and sine of the *new* semi-area are,

$$\text{LXIX.} \ldots \cos\frac{\sigma}{2} = \frac{1 + \cos a + \cos b + \cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}};$$

$$\text{LXX.} \ldots \sin\frac{\sigma}{2} = \frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin \mathrm{C}}{\cos\frac{c}{2}} = \&c.$$

(49.) Returning to the *bisected triangle*, DEF, the last formula gives,

$$\text{LXXI.} \ldots \sin\tfrac{1}{2}\Sigma = \frac{\sin a' \sin b' \sin \mathrm{F}}{\cos c'} = \sin p' \sin c \sec c',$$

if p' denote the perpendicular from F on the bisecting arc AB, or FT in Fig. 68; but $\cos\frac{1}{2}\Sigma = \cos c \sec c'$, by LV.; hence

$$\text{LXXII.} \ldots \tan\tfrac{1}{2}\Sigma = \sin p' \tan c = \sin \mathrm{FT} \,.\, \tan \mathrm{AB}.$$

Accordingly, in Fig. 68, we have, by spherical trigonometry,

$$\sin \mathrm{FT} = \sin \mathrm{ES} = \sin \mathrm{LE} \sin \mathrm{L} = \cos \mathrm{LN} \sin \mathrm{MN} \operatorname{cosec} \mathrm{LM} = \tan \mathrm{MN} \cot \mathrm{AB}.$$

(50.) The *arc* MN, which thus represents in *quantity* the semiarea of DEF, has its *pole* at the point D, and may be considered as the *representative arc* (162) of a certain *new quaternion*, Q, or of its *versor*, of which the *axis* is the *radius* OD, or $\mathrm{U}\delta$; and this new quaternion may be thus expressed:

$$\text{LXXIII.} \ldots Q = \delta\gamma\alpha\beta = -\delta^2 + \dot{\delta}\mathrm{S}\alpha\beta\gamma = r^2 + e\delta;$$

its tensor and versor being, respectively,

$$\text{LXXIV.} \ldots \mathrm{T}Q = r = \cos\tfrac{1}{2}\Sigma; \qquad \text{LXXV.} \ldots \mathrm{U}Q = \cos\tfrac{1}{2}\Sigma + \mathrm{OD} \,.\, \sin\tfrac{1}{2}\Sigma.$$

(51.) An important transformation of this last *versor* may be obtained as follows:

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* The reader will observe that the more usual symbol Σ, for this area of ABC, is *here* employed (36.) to denote the area of the *exscribed* triangle DEF.

$$\text{LXXVI.}\ldots\ \mathrm{U}Q = \mathrm{U}(\delta\gamma^{-1}.\,\alpha\zeta^{-1}.\,\zeta\beta^{-1}) = (\delta\varepsilon^{-1})^{\frac{1}{2}}\,(\varepsilon\zeta^{-1})^{\frac{1}{2}}\,(\zeta\delta^{-1})^{\frac{1}{2}};$$

so that

$$\text{LXXVII.}\ldots\ \tfrac{1}{2}\Sigma = \angle\,Q = \angle\,\delta\gamma\alpha\beta = \angle\,(\delta\varepsilon^{-1})^{\frac{1}{2}}\,(\varepsilon\zeta^{-1})^{\frac{1}{2}}\,(\zeta\delta^{-1})^{\frac{1}{2}};$$

these *powers* of quaternions, with *exponents* each $= \frac{1}{2}$, being interpreted as *square roots* (199, (1.)), or as equivalent to the symbols $\sqrt{}(\delta\varepsilon^{-1})$, &c.

(52.) The *conjugate* (or *reciprocal*) *versor*, $\mathrm{U}Q^{-1}$, which has NM for its *representative arc*, may be deduced from $\mathrm{U}Q$ by simply interchanging β and γ, or ε and ζ; the corresponding *quaternion* is,

$$\text{LXXVIII.}\ldots\ Q' = \mathrm{K}Q = \delta\beta\alpha\gamma = r^2 - e\delta;$$

and we have

$$\text{LXXIX.}\ldots\ \mathrm{U}Q' = \cos\tfrac{1}{2}\Sigma - \mathrm{OD}\,.\,\sin\tfrac{1}{2}\Sigma = (\delta\zeta^{-1})^{\frac{1}{2}}\,(\zeta\varepsilon^{-1})^{\frac{1}{2}}\,(\varepsilon\delta^{-1})^{\frac{1}{2}};$$

the rotation round D, from E to F, being still supposed to be negative.

(53.) Let H be any other point upon the sphere, and let $\mathrm{OH} = \eta$; also let Σ' be the area of the *new spherical triangle*, DFH; then the same reasoning shows that

$$\text{LXXX.}\ldots\ \cos\tfrac{1}{2}\Sigma' + \mathrm{OD}\,.\,\sin\tfrac{1}{2}\Sigma' = (\delta\zeta^{-1})^{\frac{1}{2}}\,(\zeta\eta^{-1})^{\frac{1}{2}}\,(\eta\delta^{-1})^{\frac{1}{2}},$$

if the rotation round D from F to H be negative; and therefore, by *multiplication* of the two *co-axal versors*, LXXVI. and LXXX., we have by LXXV. the analogous formula:

$$\text{LXXXI.}\ldots\ \cos\tfrac{1}{2}(\Sigma + \Sigma') + \mathrm{OD}\,.\,\sin\tfrac{1}{2}(\Sigma + \Sigma') = (\delta\varepsilon^{-1})^{\frac{1}{2}}\,(\varepsilon\zeta^{-1})^{\frac{1}{2}}\,(\zeta\eta^{-1})^{\frac{1}{2}}\,(\eta\delta^{-1})^{\frac{1}{2}};$$

where $\Sigma + \Sigma'$ denotes the *area of the spherical quadrilateral*, DEFH.

(54.) It is easy to *extend* this result to the *area* of *any spherical polygon*, or to the *spherical opening* (44.) of *any pyramid;* and we may even conceive an extension of it, as a *limit*, to the *area of any closed curve upon the sphere*, considered as decomposed into an *indefinite number of indefinitely small triangles*, with some *common vertex*, such as the point D, on the spheric surface, and with *indefinitely small arcs* EF, FH,.. *of the curve*, for their respective *bases:* or to the *spherical opening of any cone*, expressed thus as the *Angle of a Quaternion*, which is the *limit** *of the product of indefinitely many factors, each equal to the square-root of a quaternion, which differs indefinitely little from unity.*

(55.) To assist the recollection of this result, it may be stated as follows (comp. 180, (3.) for the definition of an *arcual sum*):—

"*The Arcual Sum of the Halves of the successive Sides, of any Spherical Polygon, is equal to an arc of a Great Circle, which has the Initial* (*or Final*) *Point of*

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* *This Limit* is closely *analogous* to a *definite integral*, of the ordinary kind; or rather, we may say that it *is* a *Definite Integral*, but one of a *new kind*, which could not easily have been introduced without Quaternions. In fact, if we did not employ the *non-commutative property* (168) of *quaternion multiplication*, the *Products* here considered would evidently become each *equal to unity:* so that they would furnish no expressions for spherical or other *areas*, and in short, it would be useless to speak of them. On the contrary, when that property or principle of multiplication is introduced, these *expressions of product-form* are found, as above, to have extremely useful significations in *spherical geometry;* and it will be seen that they suggest and embody a remarkable *theorem*, respecting the *resultant of rotations of a system*, round *any number of successive axes*, all passing *through one fixed point*, but in other respects succeeding each other with any gradual or sudden changes.

the Polygon for its Pole, and represents the Semi-area of the Figure;" it being understood that this *resultant arc* is *reversed* in *direction*, when the *half-sides* are (arcually) *added* in an *opposite order*.

(56.) As regards the *order* thus referred to, it may be observed that in the *arcual addition*, which corresponds to the *quaternion multiplication* in LXXVI., we conceive a point to *move, first*, from B to F, through *half* the *arc* DF; which half-side of the triangle DEF answers to the *right-hand factor*, or square-root, $(\zeta\delta^{-1})^{\frac{1}{2}}$. We then conceive the same point to move *next* from F to A, through half the arc FE, which answers to the factor placed immediately to the *left* of the former; having thus moved, on the whole, *so far*, through the *resultant arc* BA (as a *transvector*, 180, (3.)), or through any *equal arc* (163), such as ML in Fig. 68. And finally, we conceive a motion through half the arc ED, or through any arc *equal* to that half, such as the arc LN in the same Figure, to correspond to the extreme *left-hand factor* in the formula; the *final resultant* (or *total transvector arc*), which answers to the *product of the three square roots*, as *arranged* in the formula, being thus represented by the *final arc* MN, which has the point D for its *positive pole*, and the *half-area*, $\frac{1}{2}\Sigma$, for the *angle* (51.) of the *quaternion* (or *versor*) *product* which it *represents*.

(57.) Now the direction of *positive rotation* on the sphere has been supposed to be that round D, from F to E; and therefore *along the perimeter*, in the *order* DFE, as seen[*] from any point of the surface *within* the triangle: that is, in the order in which the *successive sides* DF, FE, ED have been taken, before *adding* (or *compounding*) their *halves*. And accordingly, in the *conjugate* (or *reciprocal*) formula LXXIX., we took the *opposite order*, DEF, in proceeding as usual from right-hand to left-hand factors, whereof the *former* are supposed to be multiplied *by*[†] the *latter*; while the result was, as we saw in (52.), a *new versor*, in the expression for which, the *area* Σ of the triangle was simply changed to its own *negative*.

(58.) To give an example of the reduction of the area to *zero*, we have only to conceive that the *three points* D, E, F are *co-arcual* (165), or situated *on one great circle*; or that the *three lines* δ, ϵ, ζ are *complanar*. For this case, by the laws[‡] of *complanar quaternions*, we have the formula,

$$\text{LXXXII.} \ldots (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\delta^{-1})^{\frac{1}{2}} = 1, \quad \text{if} \quad S\delta\epsilon\zeta = 0;$$

thus $\cos\frac{1}{2}\Sigma = 1$, and $\Sigma = 0$.

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[*] In this and other cases of the sort, the spectator is imagined to stand *on the point* of the sphere, *round which* the *rotation on the surface* is conceived to be performed; his body being *outside the sphere*. And similarly when we say, for example, that the *rotation round the line*, or *radius*, OA, *from the line* OB *to the line* OC, is *negative* (or left-handed), as in the recent Figures, we mean that such would appear to be the direction of that rotation, to a person standing thus with his *feet on* A, and with his *body* in the direction of OA *prolonged*: or else standing *on the centre* (or origin) O, with his *head at the point* A. Compare 174, II.; 177; and the Note to page 153.

[†] Compare the Notes to pages 146, 159.

[‡] Compare the Second Chapter of the Second Book.

(59.) Again, in (53.) let the point H be co-arcual with D and F, or let $S\delta\zeta\eta = 0$; then, because

$$\text{LXXXII}'. \ldots (\zeta\eta^{-1})^{\frac{1}{2}} (\eta\delta^{-1})^{\frac{1}{2}} = (\zeta\delta^{-1})^{\frac{1}{2}}, \quad \text{if} \quad S\delta\zeta\eta = 0,$$

the product of *four factors* LXXXI. reduces itself to the product of *three* factors LXXVI.; the *geometrical reason* being evidently that in this case the *added area* Σ' *vanishes*; so that the *quadrilateral* DEFH has only the *same* area as the *triangle* DEF.

(60.) But this *added* area (58.) may even have a *negative** *effect*, as for example when the new point H falls on the old side DE. Accordingly, if we write

$$\text{LXXXIII}. \ldots Q_1 = (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\eta^{-1})^{\frac{1}{2}} (\eta\epsilon^{-1})^{\frac{1}{2}},$$

and denote the product LXXXI. of four square-roots by Q_2, we shall have the transformation,

$$\text{LXXXIV}. \ldots Q_2 = (\delta\epsilon^{-1})^{\frac{1}{2}} \, Q_1 (\epsilon\delta^{-1})^{\frac{1}{2}}, \quad \text{if} \quad S\delta\epsilon\eta = 0;$$

which shows (comp. (15.)) that in this case the *angle* of the *quaternary product* Q_2 is that of the *ternary product* Q_1, or the half-area of the *triangle* EFH (= DEF − DHF), although the *axis* of Q_2 is *transferred* from the position of the axis of Q_1, by a *rotation* round the pole of the arc ED, which brings it from OE to OD.

(61.) From this example, it may be considered to be sufficiently evident, how the formula LXXXI. may be applied and extended, so as to represent (comp. (54.)) the *area of any closed figure on the sphere*, with *any assumed point* D *on the surface* as a sort of *spherical origin*; even when this *auxiliary point* is *not* situated *on the perimeter*, but is either *external* or *internal* thereto.

(62.) A *new quaternion* Q_0, with the *same axis* OD as the quaternion Q of (50.), but with a *double angle*, and with a *tensor* equal to *unity*, may be formed by simply *squaring the versor* U Q; and although this *squaring* cannot be effected by *removing the fractional exponents*,† in the formula LXXVI., yet it can easily be accomplished in other ways. For example we have, by LXXIII. LXXIV., and by VII. IX. X., the transformations:‡

$$\begin{aligned}\text{LXXXV}. \ldots Q_0 = \mathrm{U}\,Q^2 = r^{-2}(\delta\gamma\alpha\beta)^2 &= -\,\delta^{-2}\,.\,\gamma\alpha\beta\delta\,.\,\delta\gamma\alpha\beta \\ &= -\,(\gamma\alpha\beta)^2 = -\,(e-\delta)^2 = r^2 - e^2 + 2e\delta;\end{aligned}$$

and in fact, because $\delta = r\,.\,\text{OD}$, by XII., the trigonometric values LIV. for r and e enable us to write this last result under the form,

$$\text{LXXXVI}. \ldots Q_0 = -\,(\gamma\alpha\beta)^2 = \cos\Sigma + \text{OD}\,.\sin\Sigma.$$

(63.) To show its geometrical signification, let us conceive that ABC and LMN

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[*] In some investigations respecting *areas on a sphere*, it may be convenient to *distinguish* (comp. 28, 63) between the *two symbols* DEF and DFE, and to consider them as denoting two *opposite triangles*, of which the *sum* is *zero*. But for the present, we are content to express this *distinction*, by means of the two *conjugate quaternion products*, (51.) and (52.).

[†] Compare the Note to (54.).

[‡] The equation $\delta\gamma\alpha\beta = \gamma\alpha\beta\delta$ is *not valid generally*; but we have *here* $\delta = -\,\mathrm{V}\gamma\alpha\beta$; and in general, $q\rho = \rho q$, *if* $\rho \parallel \mathrm{V}q$.

have the same meanings in the new Fig. 70, as in Fig. 68; and that $A_1B_1M_1$ are three new points, determined by the three arcual equations (168),

$$\text{LXXXVII.}\quad \frown AC = \frown CA_1, \quad \frown BC = \frown CB_1, \quad \frown MN = \frown NM_1;$$

which easily conduct to this fourth equation of the same kind,

$$\text{LXXXVII}'.\ .\ .\ \frown LM_1 = \frown B_1A_1.$$

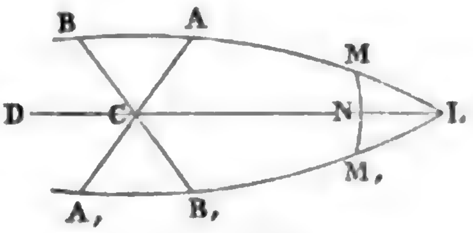

Fig. 70.

This *new arc* LM_1 *represents* thus (comp. 167, and Fig. 43) the product $\alpha_1\gamma^{-1}.\gamma\beta_1^{-1} = \gamma\alpha^{-1}.\beta\gamma^{-1}$; while the old arc ML, or its equal BA (31.), represents $\alpha\beta^{-1}$; whence the arc MM_1, which has its *pole* at D, and is numerically equal to the *whole area* Σ of DEF (because MN was seen to be equal (50.) to *half* that area), represents the product $\gamma\alpha^{-1}\beta\gamma^{-1}.\alpha\beta^{-1}$, or $-(\gamma\alpha\beta)^2$, or Q_0. The formula LXXXVI. has therefore been *interpreted*, and may be said to have been *proved anew*, by these simple geometrical considerations.

(64.) We see, at the same time, how to *interpret the symbol*,

$$\text{LXXXVIII.}\ .\ .\ Q_0 = \frac{\gamma}{\alpha}\frac{\beta}{\gamma}\frac{\alpha}{\beta};$$

namely as denoting a *versor*, of which the *axis* is directed *to*, or *from*, the *corner* D of a certain *auxiliary spherical triangle* DEF, whereof the *sides*, respectively *opposite* to D, E, F, are *bisected* (5.) by the *given points* A, B, C, according as the *rotation* round α from β to γ is *negative* or *positive*; and of which the *angle* represents, or is numerically equal to, the *area* Σ of that auxiliary triangle: at least if we *still* suppose, as we have hitherto for simplicity done (1.), that the *sides* of the *given* triangle ABC are *each less* than a *quadrant*.

298. The *case* when the *sides* of the *given* triangle are *all greater*, instead of being *all less*, than *quadrants*, may deserve next to be (although more briefly) considered; the case when they are *all equal* to *quadrants*, being reserved for a short subsequent Article: and *other* cases being easily referred to *these*, by *limits*, or by passing from a given line to its *opposite*.

(1.) Supposing now that

$$\text{I.}\ .\ .\ l < 0, \quad m < 0, \quad n < 0,$$

or that

$$\text{II.}\ .\ .\ a > \frac{\pi}{2}, \quad b > \frac{\pi}{2}, \quad c > \frac{\pi}{2},$$

we may still retain the recent equations IV. to XI.; XIII.; and XV. to XXVI., of 297; but we must *change the sign of the radical*, r, in the equations XII. and XIV., and also the *signs* of the *versors* $U\delta$, $U\epsilon$, $U\zeta$ in XII., if we desire that the *sides* of the *auxiliary triangle*, DEF, may still be *bisected* (as in Figures 67, 68) by the *corners* of the *given* triangle ABC, of which the sides a, b, c are now each *greater* than a *quadrant*. Thus, r being still the *common tensor* of δ, ϵ, ζ, and therefore being still supposed to be *itself* > 0, we must write now, under these *new conditions* I. or II., the new equations,

$$\text{III.}\ldots\ \mathrm{OD} = -\mathrm{U}\delta = -r^{-1}\delta; \quad \mathrm{OE} = -\mathrm{U}\epsilon = -r^{-1}\epsilon; \quad \mathrm{OF} = -\mathrm{U}\zeta = -r^{-1}\zeta;$$

$$\text{IV.}\ldots\ \cos a = -r\cos a', \quad \cos b = -r\cos b', \quad \cos c = -r\cos c'.$$

(2.) The equations IV. and VIII. of 297 still holding good, we may now write,

$$\text{V.}\ldots\ \pm 2r\cos a'\cos b'\cos c' = \cos a'^2 + \cos b'^2 + \cos c'^2 - 1,$$

according as we adopt positive values (297), or negative values (298), for the cosines l, m, n of the sides of the bisecting triangle; the value of r being still supposed to be positive.

(3.) It is not difficult to prove (comp. 297, LIV., LXIX.), that

$$\text{VI.}\ldots\ r = \pm\cos\tfrac{1}{2}\Sigma, \quad \text{according as} \quad l > 0, \text{ \&c.,} \quad \text{or} \quad l < 0, \text{ \&c.};$$

the recent formula V. may therefore be written *unambiguously* as follows:

$$\text{VII.}\ldots\ 2\cos a'\cos b'\cos c'\cos\tfrac{1}{2}\Sigma = \cos a'^2 + \cos b'^2 + \cos c'^2 - 1;$$

and the formula 297, LV. continues to hold good.

(4.) In like manner, we may write, without an ambiguous sign (comp. 297, LI.), the following *expression for the fourth proportional* $\beta\alpha^{-1}\gamma$ *to three unit-vectors* α, β, γ, the rotation round the first from the second to the third being negative:

$$\text{VIII.}\ldots\ \beta\alpha^{-1}\gamma = \sin\tfrac{1}{2}\Sigma + \mathrm{OD}.\cos\tfrac{1}{2}\Sigma;$$

where the scalar part changes sign, when the rotation is reversed.

(5.) It is, however, to be observed, that although this *formula* VIII. holds good, not only in the cases of the last article and of the present, but also in that which has been reserved for the next, namely when $l = 0$, &c.; yet because, in the *present case* (298) we have the area $\Sigma > \pi$, the *radius* OD is no longer the (positive) *axis* $\mathrm{U}\delta$ of the fourth proportional $\beta\alpha^{-1}\gamma$; nor is $\frac{1}{2}\pi - \frac{1}{2}\Sigma$ any longer, as in 297, L., the (positive) *angle* of that versor. On the contrary we have *now*, for this axis and angle, the expressions:

$$\text{IX.}\ldots\ \mathrm{Ax}.\,\beta\alpha^{-1}\gamma = \mathrm{DO} = -\mathrm{OD}; \qquad \text{X.}\ldots\ \angle\,\beta\alpha^{-1}\gamma = \tfrac{1}{2}(\Sigma - \pi).$$

(6.) To illustrate these results by a construction, we may remark that if, in Fig. 67, the bisecting arcs BC, CA, AB be supposed each greater than a quadrant, and if we proceed to form from it a new Figure, analogous to 68, the perpendicular CQ will also exceed a quadrant, and the poles P and K will fall *between* the points C and Q; also M and R will fall on the arcs LQ and QL′ *prolonged:* and although the arc KM, or the angle KDM, or L′DR, or EDP, may *still* be considered, as in 297, (34.), to *represent* the *versor* $\beta\alpha^{-1}\gamma$, yet the corresponding *rotation* round the point D is now of a *negative* character.

(7.) And as regards the *quantity* of this rotation, or the magnitude of the *angle* at D, it is again, as in Fig. 68, a base-angle of one of three isosceles triangles, with P for their common vertex; but we have now, as in Fig. 71, a *new arrangement*, in virtue of which this angle is to be found by halving what remains, when the sum of the supplements of the angles at D and E, in the triangle DEF, is subtracted *from* the angle at F, instead of our subtracting (as in 297, (35.)) the latter angle from the former sum; it is therefore *now*, in agreement with the recent expression X.,

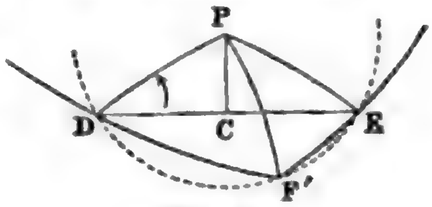

Fig. 71.

$$\text{XI.}\ldots\ \angle\,\beta\alpha^{-1}\gamma = \tfrac{1}{2}(\mathrm{D} + \mathrm{E} + \mathrm{F}) - \pi.$$

(8.) The negative of the conjugate of the formula VIII. gives,

$$\text{XII.} \ldots \gamma\alpha^{-1}\beta = -\sin\tfrac{1}{2}\Sigma + \mathrm{OD}.\cos\tfrac{1}{2}\Sigma;$$

and by taking the negative of the square of this equation, we are conducted to the following:

$$\text{XIII.} \ldots \frac{\gamma}{\alpha}\frac{\beta}{\gamma}\frac{\alpha}{\beta} = -(\gamma\alpha^{-1}\beta)^2 = \cos\Sigma + \mathrm{OD}.\sin\Sigma;$$

a result which had only been proved before (comp. 297, (62.), (64.)) for the case $\Sigma < \pi$; and in which it is still supposed that the rotation round α from β to γ is negative.

(9.) With the same direction of rotation, we have also the *conjugate* or *reciprocal* formula,

$$\text{XIV.} \ldots \frac{\beta}{\alpha}\frac{\gamma}{\beta}\frac{\alpha}{\gamma} = -(\beta\alpha^{-1}\gamma)^2 = \cos\Sigma - \mathrm{OD}.\sin\Sigma.$$

(10.) If it happened that only *one* side, as AB, of the *given* triangle ABC, was greater, while each of the two others was less than a quadrant, or that we had $l > 0$, $m > 0$, but $n < 0$; and if we wished to represent the fourth proportional to α, β, γ by means of the foregoing constructions: we should only have to introduce the point C′ *opposite* to C, or to change γ to $\gamma' = -\gamma$; for thus the *new* triangle ABC′ would have *each* side greater than a quadrant, and so would fall under the case of the present Article; after employing the construction for which, we should only have to change the resulting versor to its negative.

(11.) And in like manner, if we had l and m negative, but n positive, we might again substitute for C its opposite point C′, and so fall back on the construction of Art. 297: and similarly in other cases.

(12.) In general, if we *begin* with the equations 297, XII., attributing any *arbitrary* (but positive) value to the *common tensor*, r, of the three co-initial vectors δ, ϵ, ζ, of which the *versors*, or the *unit-vectors* $\mathrm{U}\delta$, &c., terminate at the corners of a *given* or *assumed triangle* DEF, with sides $= 2a'$, $2b'$, $2c'$, we may then suppose (comp. Fig. 67) that *another* triangle ABC, with sides denoted by a, b, c, and with their cosines denoted by l, m, n, is *derived* from this one, by the condition of *bisecting* its *sides;* and therefore by the equations (comp. 297, LVIII.),

$$\text{XV.} \ldots \mathrm{OA} = \alpha = \mathrm{U}(\epsilon + \zeta), \quad \mathrm{OB} = \beta = \mathrm{U}(\zeta + \delta), \quad \mathrm{OC} = \gamma = \mathrm{U}(\delta + \epsilon),$$

with the relations 297, IV. V. VI., as before; or by these other equations (comp. 297, XIII. XIV.),

$$\text{XVI.} \ldots \epsilon + \zeta = 2r\alpha\cos a', \quad \zeta + \delta = 2r\beta\cos b', \quad \delta + \epsilon = 2r\gamma\cos c'.$$

(13.) When *this* simple construction is adopted, we have at once (comp. 297, LX.), by merely taking *scalars of products of vectors*, and *without any reference to areas* (compare however 297, LXIX., and 298, VII.), the equations,

$$\begin{aligned}\text{XVII.} \ldots 4\cos a\cos b'\cos c' &= 4\cos b\cos c'\cos a' = 4\cos c\cos a'\cos b' \\ &= -r^{-2}\mathrm{S}(\zeta + \delta)(\delta + \epsilon) = \&\text{c.} = 1 + \cos 2a' + \cos 2b' + \cos 2c';\end{aligned}$$

or

$$\text{XVIII.} \ldots \frac{\cos a}{\cos a'}, = \frac{\cos b}{\cos b'} = \frac{\cos c}{\cos c'} = \frac{\cos a'^2 + \cos b'^2 + \cos c'^2 - 1}{2\cos a'\cos b'\cos c'};$$

which can indeed be otherwise deduced, by the known formulæ of spherical trigonometry.

(14.) We see, then, that *according as the sum of the squares of the cosines of the half-sides, of a given or assumed spherical triangle*, DEF, *is greater than unity, or equal to unity, or less than unity, the sides of the inscribed and bisecting triangle*, ABC, *are together less than quadrants, or together equal to quadrants, or together greater than quadrants.*

(15.) Conversely, *if* the sides of a *given* sphérical triangle ABC be thus *all less*, or *all greater* than *quadrants*, a triangle DEF, but only *one** such triangle, can be *exscribed* to it, so as to have its sides *bisected*, as above: the simplest process being to let fall a perpendicular, such as CQ in Fig. 68, from C on AB, &c.; and then to draw new arcs, through C, &c., perpendicular to these perpendiculars, and therefore coinciding in position with the sought sides DE, &c., of DEF.

(16.) The *trigonometrical results* of recent sub-articles, especially as regards the *area*† of a spherical triangle, are probably *all* well known, as certainly *some* of them are; but they are *here* brought forward only in connexion with *quaternion formulæ;* and as one of that class, which is not irrelevant to the present subject, and *includes* the formula 294, LIII., the following may be mentioned, wherein α, β, γ denote *any three vectors*, but the *order of the factors* is important:

$$\text{XIX} \;..\; (\alpha\beta\gamma)^2 = 2\alpha^2\beta^2\gamma^2 + \alpha^2(\beta\gamma)^2 + \beta^2(\alpha\gamma)^2 + \gamma^2(\alpha\beta)^2 - 4\alpha\gamma\,\mathrm{S}\alpha\beta\,\mathrm{S}\beta\gamma.$$

(17.) And if, as in 297, (1.), &c., we suppose that α, β, γ are three *unit-vectors*, OA, OB, OC, and denote, as in 297, (47.), by σ the area of the triangle ABC, the principle expressed by the recent formula XIII. may be stated under this apparently different, but essentially equivalent form:

$$\text{XX.} \;...\; \frac{\alpha+\beta}{\beta+\gamma} \cdot \frac{\gamma+\alpha}{\alpha+\beta} \cdot \frac{\beta+\gamma}{\gamma+\alpha} = \cos\sigma + \alpha\sin\sigma;$$

which admits of several verifications.

(18.) We may, for instance, transform it as follows (comp. 297, LXVII.):

$$\text{XXI.} \;...\; \frac{-(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}{\mathrm{K}(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)} = \frac{-2e + 2\alpha(1+l+m+n)}{+2e + 2\alpha(1+l+m+n)}$$

$$= \frac{1+l+m+n+e\alpha}{1+l+m+n-e\alpha} = \frac{1+\alpha\tan\frac{\sigma}{2}}{1-\alpha\tan\frac{\sigma}{2}} = \frac{\cos\frac{\sigma}{2}+\alpha\sin\frac{\sigma}{2}}{\cos\frac{\sigma}{2}-\alpha\sin\frac{\sigma}{2}}$$

$$= \left(\cos\frac{\sigma}{2} + \alpha\sin\frac{\sigma}{2}\right)^2 = \cos\sigma + \alpha\sin\sigma, \text{ as above.}$$

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* In the next Article, we shall consider a case of *indeterminateness*, or of the existence of indefinitely many exscribed triangles DEF: namely, when the sides of ABC are *all equal* to quadrants.

† This opportunity may be taken of referring to an interesting Note, to pages 96, 97 of *Luby's Trigonometry* (Dublin, 1852); in which an elegant construction, connected with the area of a spherical triangle, is acknowledged as having been mentioned to Dr. Luby, by a since deceased and lamented friend, the Rev. William Digby Sadleir, F. T. C. D. A construction nearly the same, described in the sub-articles to 297, was suggested to the present writer by quaternions, several years ago.

(19.) This seems to be a natural place for observing (comp. (16.)), that if α, β, γ, δ be *any four vectors*, the lately cited equation 294, LIII., and the square of the equation 294, XV., with δ written in it instead of ρ, conduct easily to the following very general and symmetric formula:

$$\begin{gathered}\text{XXII.}\ldots\ \alpha^2\beta^2\gamma^2\delta^2 + (\mathrm{S}\beta\gamma\,\mathrm{S}\alpha\delta)^2 + (\mathrm{S}\gamma\alpha\mathrm{S}\beta\delta)^2 + (\mathrm{S}\alpha\beta\mathrm{S}\gamma\delta)^2 \\ + 2\alpha^2\mathrm{S}\beta\gamma\mathrm{S}\beta\delta\mathrm{S}\gamma\delta + 2\beta^2\mathrm{S}\gamma\alpha\mathrm{S}\gamma\delta\mathrm{S}\alpha\delta + 2\gamma^2\mathrm{S}\alpha\beta\mathrm{S}\alpha\delta\mathrm{S}\beta\delta + 2\delta^2\mathrm{S}\alpha\beta\mathrm{S}\beta\gamma\mathrm{S}\gamma\alpha \\ = 2\mathrm{S}\gamma\alpha\mathrm{S}\alpha\beta\mathrm{S}\beta\delta\mathrm{S}\gamma\delta + 2\mathrm{S}\alpha\beta\mathrm{S}\beta\gamma\mathrm{S}\gamma\delta\mathrm{S}\alpha\delta + 2\mathrm{S}\beta\gamma\mathrm{S}\gamma\alpha\mathrm{S}\alpha\delta\mathrm{S}\beta\delta \\ + \beta^2\gamma^2(\mathrm{S}\alpha\delta)^2 + \gamma^2\alpha^2(\mathrm{S}\beta\delta)^2 + \alpha^2\beta^2(\mathrm{S}\gamma\delta)^2 \\ + \alpha^2\delta^2(\mathrm{S}\beta\gamma)^2 + \beta^2\delta^2(\mathrm{S}\gamma\alpha)^2 + \gamma^2\delta^2(\mathrm{S}\alpha\beta)^2.\end{gathered}$$

(20.) If then we take *any spherical quadrilateral* ABCD, and write

$$\text{XXIII.}\ldots\ l' = \cos \text{AD} = -\mathrm{SU}\alpha\delta, \quad m' = \cos \text{BD} = -\mathrm{SU}\beta\delta, \quad n' = \cos \text{CD} = \&\text{c.},$$

treating α, β, γ as the unit-vectors of the points A, B, C, and l, m, n as the cosines of the arcs BC, CA, AB, as in 297, (1.), we have the equation,

$$\begin{gathered}\text{XXIV.}\ldots\ 1 + l^2l'^2 + m^2m'^2 + n^2n'^2 + 2lm'n' + 2mn'l' + 2nl'm' + 2lmn \\ = 2mnm'n' + 2nln'l' + 2lml'm' \\ + l^2 + m^2 + n^2 + l'^2 + m'^2 + n'^2;\end{gathered}$$

which can be confirmed by elementary considerations,* but is here given merely as an *interpretation* of the quaternion formula XXII.

(21.) In squaring the lately cited equation 294, XV., we have used the two following formulæ of transformation (comp. 204, XXII., and 210, XVIII.), in which α, β, γ may be *any three vectors*, and which are often found to be useful:

$$\text{XXV.}\ldots\ (\mathrm{V}\alpha\beta)^2 = (\mathrm{S}\alpha\beta)^2 - \alpha^2\beta^2; \quad \text{XXVI.}\ldots\ \mathrm{S}(\mathrm{V}\beta\gamma\,.\,\mathrm{V}\gamma\alpha) = \gamma^2\mathrm{S}\alpha\beta - \mathrm{S}\beta\gamma\mathrm{S}\gamma\alpha.$$

299. The *two cases*, for which the *three sides* a, b, c, of the given triangle ABC, are *all less*, or *all greater*, than *quadrants*, having been considered in the two foregoing Articles, with a reduction, in 298, (10.) and (11.), of certain other cases to these, it only remains to consider that *third* principal *case*, for which the sides of that given triangle are *all equal to quadrants:* or to inquire what is, on our general principles, the *Fourth Proportional to Three Rectangular Vectors*. And we shall find, not only that *this* fourth proportional is *not itself a Vector*, but that it does not even contain any *vector part* (292) different from zero: although, as being found to be equal to a *Scalar*, it is still *included* (131, 276) in the general conception of a *Quaternion*.

(1.) In fact, if we suppose, in 297, (1.), that

$$\text{I.}\ldots\ l = 0,\ m = 0,\ n = 0, \quad \text{or that} \quad \text{II.}\ldots\ a = b = c = \frac{\pi}{2},$$

---

* A formula equivalent to this last *equation of seventeen terms*, connecting the *six cosines* of the arcs which join, two by two, the corners of a spherical quadrilateral ABCD, is given at page 407 of Carnot's *Géométrie de Position* (Paris, 1803).

or III. . . $S\beta\gamma = S\gamma\alpha = S\alpha\beta = 0$, while IV. . . $T\alpha = T\beta = T\gamma = 1$, the formulæ 297, VII. give,

$$\text{V.} \ldots \delta = 0, \quad \epsilon = 0, \quad \zeta = 0;$$

but these are the *vector parts* of the *three pairs* of *fourth proportionals* to the *three rectangular unit-lines*, α, β, γ, taken in all possible orders; and the same *evanescence of vector parts* must evidently take place, if the three given lines be only *at right angles* to each other, without being *equally long*.

(2.) Continuing, however, for simplicity, to suppose that they are unit lines, and that the rotation round α from β to γ is negative, as before, we see that we have now $r = 0$, and $e = 1$, in 297, (3.); and that thus *the six fourth proportionals reduce themselves to their scalar parts*, namely (here) to *positive or negative unity*. In this manner we find, under the supposed conditions, the values:

$$\text{VI.} \ldots \beta\alpha^{-1}\gamma = \gamma\beta^{-1}\alpha = \alpha\gamma^{-1}\beta = +1; \qquad \text{VI}'. \ldots \gamma\alpha^{-1}\beta = \alpha\beta^{-1}\gamma = \beta\gamma^{-1}\alpha = -1.$$

(3.) For example (comp. 295) we have, by the laws (182) of i, j, k, the values,

$$\text{VII.} \ldots ij^{-1}k = jk^{-1}i = ki^{-1}j = +1; \qquad \text{VII}'. \ldots kj^{-1}i = ik^{-1}j = ji^{-1}k = -1.$$

In fact, the *two fourth proportionals*, $ij^{-1}k$ and $kj^{-1}i$, are respectively equal to the *two ternary products*, $-ijk$ and $-kji$, and therefore to $+1$ and -1, by the laws included in the *Fundamental Formula* A (183).

(4.) To connect this important result with the *constructions* of the two last Articles, we may observe that when we seek, on the general plan of 298, (15.), to *exscribe a spherical triangle*, DEF, *to a given tri-quadrantal* (or *tri-rectangular*) *triangle*, ABC, as for instance to the triangle IJK (or JIK) of 181, in such a manner that the *sides* of the *new* triangle shall be *bisected* by the *corners* of the *old*, the problem is found to admit of *indefinitely many solutions*. *Any point* P may be assumed, in the *interior* of the given triangle ABC; and then, if its *reflexions* D, E, F be taken, with respect to the three *sides* a, b, c, so that (comp. Fig. 72) the *arcs* PD, PE, PF are *perpendicularly bisected* by those three sides, the three *other arcs* EF, FD, DE will be bisected by the *points* A, B, C, as required: because the arcs AE, AF have each the same length as AP, and the angles subtended at A by PE and PF are together equal to two right angles, &c.

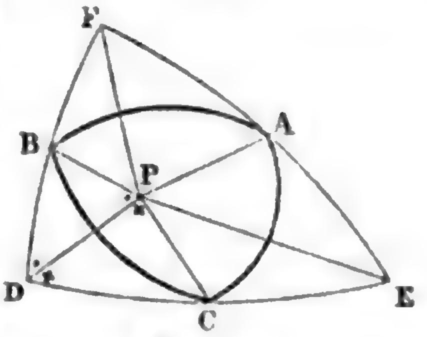

Fig. 72.

(5.) The *positions* of the *auxiliary points*, D, E, F, are therefore, in the present case, *indeterminate*, or *variable*; but the *sum* of the *angles* at those three points is *constant*, and equal to *four right angles*; because, by the *six isosceles triangles* on PD, PE, PF as bases, that sum of the three angles D, E, F is equal to the sum of the angles subtended by the sides of the given triangle ABC, at the assumed interior point P. The *spherical excess* of the triangle DEF is therefore equal to *two right angles*, and its area $\Sigma = \pi$; as may be otherwise seen from the same Figure 72, and might have been inferred from the formula 297, LV., or LVI.

(6.) The *radius* OD, in the formula 297, XLVII., for the fourth proportional $\beta\alpha^{-1}\gamma$, becomes therefore, in the present case, *indeterminate*; but because the *angle* L′DR, or $\frac{1}{2}(\pi - \Sigma)$, in the same equation, *vanishes*, the formula becomes simply

$\beta\alpha^{-1}\gamma = 1$, as in the recent equations VI.; and similarly in other examples, of the class here considered.

(7.) The conclusion, that *the Fourth Proportional to Three Rectangular Lines is a Scalar*, may in several other ways be deduced, from the principles of the present Book. For example, with the recent suppositions, we may write,

$$\text{VIII.} \ldots \beta\alpha^{-1} = -\gamma, \quad \gamma\beta^{-1} = -\alpha, \quad \alpha\gamma^{-1} = -\beta;$$
$$\text{VIII}'. \ldots \gamma\alpha^{-1} = +\beta, \quad \alpha\beta^{-1} = +\gamma, \quad \beta\gamma^{-1} = +\alpha;$$

the three fourth proportionals VI. are therefore equal, respectively, to $-\gamma^2$, $-\alpha^2$, $-\beta^2$, and consequently to $+1$; while the corresponding expressions VI'. are equal to $+\beta^2$, $+\gamma^2$, $+\alpha^2$, and therefore to -1.

(8.) Or (comp. (3.)) we may write *generally* the transformation (comp. 282, XXI.*),

$$\text{IX.} \ldots \beta\alpha^{-1}\gamma = \alpha^{-2}.\beta\alpha\gamma, \quad \text{if} \quad \alpha^{-2} = 1 : \alpha^2,$$

in which the factor α^{-2} is *always a scalar*, whatever vector α may be; while the *vector part* of the *ternary product* $\beta\alpha\gamma$ *vanishes*, by 294, III., when the recent *conditions of rectangularity* III. are satisfied.

(9.) Conversely, this *ternary product* $\beta\alpha\gamma$, and this *fourth proportional* $\beta\alpha^{-1}\gamma$, can *never reduce themselves to scalars, unless the three vectors* α, β, γ (supposed to be all *actual* (Art. 1)) *are perpendicular each to each.*

SECTION 8.—*On an equivalent Interpretation of the Fourth Proportional to Three Diplanar Vectors, deduced from the Principles of the Second Book.*

300. In the foregoing Section, we naturally employed the results of preceding Sections of the present Book, to assist ourselves in attaching a definite signification to the Fourth Proportional (297) to Three Diplanar Vectors; and thus, in order to *interpret the symbol* $\beta\alpha^{-1}\gamma$, we availed ourselves of the interpretations *previously* obtained, in this Third Book, of α^{-1} as a *line*, and of $\alpha\beta$, $\alpha\beta\gamma$ as *quaternions*. But it may be interesting, and not uninstructive, to inquire *how the equivalent symbol*,

$$\text{I.} \ldots (\beta : \alpha).\gamma, \quad \text{or} \quad \frac{\beta}{\alpha}\gamma, \quad \text{with } \gamma \textit{ not } \parallel\!| \; \alpha, \beta,$$

might have been interpreted, on the principles of the Second Book, without at first assuming as known, or even seeking to discover, *any interpretation* of the three lately mentioned symbols,

$$\text{II.} \ldots \alpha^{-1}, \quad \alpha\beta, \quad \alpha\beta\gamma.$$

It will be found that the inquiry conducts to an expression of the form,

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* The formula here referred to should have been printed as $\mathrm{R}\alpha = 1 : \alpha = \alpha^{-1}$.

$$\text{III.} \ldots (\beta : \alpha) . \gamma = \delta + eu;$$

where δ is the *same vector*, and e is the *same scalar*, as in the recent sub-articles to 297; while u is employed as a temporary symbol, to denote a certain *Fourth Proportional to Three Rectangular Unit Lines*, namely, to the three lines OQ, OL′, and OP in Fig. 68; so that, with reference to the construction represented by that Figure, we should be led, by the principles of the Second Book, to write the equation:

$$\text{IV.} \ldots (\mathrm{OB} : \mathrm{OA}) . \mathrm{OC} = \mathrm{OD} . \cos \tfrac{1}{2}\Sigma + (\mathrm{OL}' : \mathrm{OQ}) . \mathrm{OP} . \sin \tfrac{1}{2}\Sigma.$$

And when we proceed to consider *what signification* should be attached, on the principles of the same Second Book, to *that particular fourth proportional*, which is here the coefficient of $\sin \frac{1}{2}\Sigma$, and has been provisionally denoted by u, we find that although it may be regarded as being *in one sense* a *Line*, or at least *homogeneous with a line*, yet it *must not be equated to any Vector:* being rather *analogous, in Geometry, to the Scalar Unit of Algebra*, so that it may be naturally and conveniently *denoted* by the *usual symbol* 1, or + 1, or be *equated to Positive Unity*. But when we thus write $u = 1$, the last term of the formula III. or IV., of the present Article, becomes simply e, or $\sin \frac{1}{2}\Sigma$; and while this *term* (or *part*) of the result comes to be considered as a species of *Geometrical Scalar*, the *complete Expression for the General Fourth Proportional to Three Diplanar Vectors* takes the *Form of a Geometrical Quaternion:* and thus the *formula* 297, XLVII., or 298, VIII., is *reproduced*, at least if we substitute in it, for the present, $(\beta : \alpha) . \gamma$ for $\beta\alpha^{-1}\gamma$, to avoid the necessity of interpreting *here* the recent symbols II.

(1.) The construction of Fig. 68 being retained, but no principles peculiar to the Third Book being employed, we may write, with the same significations of c, p, &c., as before,

$$\text{V.} \ldots \mathrm{OB} : \mathrm{OA} = \mathrm{OR} : \mathrm{OQ} = \cos c + (\mathrm{OL}' : \mathrm{OQ}) \sin c;$$

$$\text{VI.} \ldots \mathrm{OC} = \mathrm{OQ} . \cos p + \mathrm{OP} . \sin p.$$

(2.) Admitting then, as is natural, for the purposes of the sought *interpretation*, that *distributive* property which has been *proved* (212) to hold good for the *multiplication of quaternions* (as it does for multiplication in algebra); and writing for abridgment,

$$\text{VII.} \ldots u = (\mathrm{OL}' : \mathrm{OQ}) . \mathrm{OP};$$

we have the *quadrinomial expression :*

$$\text{VIII.} \ldots (\mathrm{OB} : \mathrm{OA}) . \mathrm{OC} = \mathrm{OL}' . \sin c \cos p + \mathrm{OQ} . \cos c \cos p$$
$$+ \mathrm{OP} . \cos c \sin p + u . \sin c \sin p;$$

in which it may be observed that *the sum of the squares of the four coefficients of the* *three rectangular unit-vectors*, OQ, OL′, OP, *and of their fourth proportional*, *u*, *is equal to unity.*

(3.) But the coefficient of *this* fourth proportional, which may be regarded as a species of *fourth unit*, is

$$\text{IX.} \ldots \sin c \sin p = \sin \mathrm{MN} = \sin \tfrac{1}{2}\Sigma = e;$$

we must therefore expect to find that the *three other* coefficients in VIII., when divided by $\cos \frac{1}{2}\Sigma$, or by r, give quotients which are the cosines of the arcual distances of some point X upon the unit-sphere, from the three points L′, Q, P; or that a point X can be assigned, for which

$$\text{X.} \ldots \sin c \cos p = r \cos \mathrm{L'X}; \quad \cos c \cos p = r \cos \mathrm{QX}; \quad \cos c \sin p = r \cos \mathrm{PX}.$$

(4.) Accordingly it is found that these three last equations are satisfied, when we substitute D for X; and therefore that we have the transformation,

$$\text{XI.} \ldots \mathrm{OL'} . \sin c \cos p + \mathrm{OQ} . \cos c \cos p + \mathrm{OP} . \cos c \sin p = \mathrm{OD} . \cos \tfrac{1}{2}\Sigma = \delta,$$

whence follow the equations IV. and III.; and it only remains to study and interpret the *fourth unit*, *u*, which enters as a factor into the remaining part of the quadrinomial expression VIII., without employing any principles except those of the *Second Book:* and therefore *without using the Interpretations* 278, 284, of βa, &c.

301. In general, when two sets of three vectors, a, β, γ, and a', β', γ', are connected by the relation,

$$\text{I.} \ldots \frac{\beta}{a}\frac{\gamma}{\gamma'}\frac{a'}{\beta'} = 1, \quad \text{or} \quad \text{II.} \ldots \frac{\beta}{a}\frac{\gamma}{\gamma'} = \frac{\beta'}{a'},$$

it is natural to write this other equation,

$$\text{III.} \ldots \frac{\beta}{a}\gamma = \frac{\beta'}{a'}\gamma';$$

and to say that *these two fourth proportionals* (297), to a, β, γ, and to a', β', γ', are *equal to each other:* whatever the *full signification* of *each* of these two last *symbols* III., supposed for the moment to be *not yet fully known*, may be afterwards found to be. In short, we may propose to make it a *condition of the sought Interpretation*, on the principles of the Second Book, of the *phrase*,

"*Fourth Proportional to Three Vectors*,"

and of either of the two equivalent *Symbols* 300, I., that the recent *Equation* III. shall *follow* from I. or II.; just as, at the commencement of that Second Book, and *before* concluding (112) that the general *Geometric Quotient* $\beta : a$ of any *two lines in space* is a *Quaternion*, we made it a *condition* (103) of the *interpretation* of such a *quotient*, that the equation $(\beta : a) . a = \beta$ should be satisfied.

302. There are however *two tests* (comp. 287), to which the recent equation III. must be submitted, before its final adoption; in

order that we may be sure of its *consistency*, Ist, with the *previous interpretation* (226) of a Fourth Proportional to *Three Complanar Vectors*, as a *Line* in their common plane; and IInd, with the *general principle* of all *mathematical language* (105), that things *equal* to the *same* thing, are to be considered as equal to *each other*. And it is found, on trial, that *both* these tests *are borne:* so that they form no objection to our *adopting the equation* 301, III., *as true by definition*, whenever the preceding equation II., or I., is satisfied.

(1.) It may happen that the first member of that equation III. is equal to a *line* δ, as in 226; namely, when α, β, γ are *complanar*. In this case, we have by II. the equation,

$$\text{IV.} \ldots \frac{\delta}{\gamma'} = \frac{\delta}{\gamma}\frac{\gamma}{\gamma'} = \frac{\beta'}{\alpha'}, \quad \text{or} \quad \text{IV}'. \ldots \frac{\beta'}{\alpha'}\gamma' = \delta = \frac{\beta}{\alpha}\gamma;$$

so that α', β', γ' are *also* complanar (among themselves), and the line δ is *their* fourth proportional likewise: and the equation III. is satisfied, both members being *symbols for one common line*, δ, which is *in general* situated *in the intersection of the two planes*, $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$; although those planes may happen to *coincide*, without disturbing the truth of the equation.

(2.) Again, for the more general case of *diplanarity* of α, β, γ, we may conceive that the equation[*] II. co-exists with this other of the same form,

$$\text{V.} \ldots \frac{\beta}{\alpha}\frac{\gamma}{\gamma''} = \frac{\beta''}{\alpha''}; \quad \text{which gives} \quad \text{VI.} \ldots \frac{\beta}{\alpha}\gamma = \frac{\beta''}{\alpha''}\gamma'',$$

if the definition 301 be adopted. If then that definition be consistent with general principles of equality, we ought to find, by III. and VI., that this third equation between two fourth proportionals holds good:

$$\text{VII.} \ldots \frac{\beta'}{\alpha'}\gamma' = \frac{\beta''}{\alpha''}\gamma''; \quad \text{or that} \quad \text{VIII.} \ldots \frac{\beta'}{\alpha'}\frac{\gamma'}{\gamma''} = \frac{\beta''}{\alpha''},$$

when the equations II. and V. are satisfied. And accordingly, those two equations give, by the general principles of the Second Book, respecting quaternions considered as *quotients* of vectors, the transformation,

$$\frac{\beta'}{\alpha'}\frac{\gamma'}{\gamma''} = \frac{\beta}{\alpha}\frac{\gamma}{\gamma'}.\frac{\gamma'}{\gamma''} = \frac{\beta}{\alpha}\frac{\gamma}{\gamma''} = \frac{\beta''}{\alpha''}, \text{ as required.}$$

303. It is then permitted to *interpret the equation* 301, III., on the principles of the Second Book, as being simply a *transformation* (as it is in algebra) of the immediately preceding equation II., or I.; and therefore to write, generally,

$$\text{I.} \ldots q\gamma = q'\gamma', \quad \text{if} \quad \text{II.} \ldots q(\gamma : \gamma') = q';$$

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[*] In this and other cases of reference, the numeral cited is always supposed to be the one which (with the same number) has *last* occurred before, although perhaps it may have been in connexion with a shortly preceding Article. Compare 217, (1.).

where γ, γ' are *any two vectors*, and q, q' are *any two quaternions*, which satisfy this last *condition*. Now, if v and v' be any two *right quaternions*, we have (by 193, comp. 283) the equation,

$$\text{III.} \ldots \mathrm{I}v : \mathrm{I}v' = v : v' = vv'^{-1};$$

or

$$\text{IV.} \ldots v^{-1}(\mathrm{I}v : \mathrm{I}v') = v'^{-1}; \quad \text{whence} \quad \text{V.} \ldots v^{-1}.\mathrm{I}v = v'^{-1}.\mathrm{I}v',$$

by the principle which has just been enunciated. It follows, then, that "*if a right Line* ($\mathrm{I}v$) *be multiplied by the Reciprocal* (v^{-1}) *of the Right Quaternion* (v), *of which it is the Index, the Product* ($v^{-1}\mathrm{I}v$) *is independent of the Length, and of the Direction, of the Line thus operated on;*" or, in other words, that *this Product has one common Value, for all possible Lines* (α) *in Space:* which *common* or constant *value* may be regarded as a kind of *new Geometrical Unit*, and is equal to what we have lately denoted, in 300, III., and VII., by the temporary symbol u; because, in the last cited formula, the line OP is the index of the right quotient OQ : OL′. Retaining, then, for the moment, this *symbol*, u, we have, *for every line* α *in space, considered as the index of a right quaternion*, v, the *four equations:*

$$\text{VI.} \ldots v^{-1}\alpha = u; \qquad \text{VII.} \ldots \alpha = vu; \qquad \text{VIII.} \ldots v = \alpha : u;$$
$$\text{IX.} \ldots v^{-1} = u : \alpha;$$

in which it is understood that $\alpha = \mathrm{I}v$, and the three last are here regarded as being merely *transformations* of the first, which is deduced and interpreted as above. And hence it is easy to infer, that for *any given system of three rectangular lines* α, β, γ, we have the *general expression:*

$$\text{X.} \ldots (\beta : \alpha).\gamma = xu, \quad \text{if} \quad \alpha \perp \beta,\ \beta \perp \gamma,\ \gamma \perp \alpha;$$

where the *scalar co-efficient*, x, of the *new unit*, u, is determined by the equation,

$$\text{XI.} \ldots x = \pm(\mathrm{T}\beta : \mathrm{T}\alpha).\mathrm{T}\gamma, \quad \text{according as} \quad \text{XII.} \ldots \mathrm{U}\gamma = \pm\,\mathrm{Ax}.(\alpha : \beta).$$

This coefficient x is therefore *always equal*, in *magnitude* (or absolute *quantity*), to the *fourth proportional to the lengths* of the three given lines $\alpha\beta\gamma$; but it is *positively or negatively taken*, according as the *rotation round the third line* γ, *from the second line* β, *to the first line* α, is itself *positive* or *negative:* or in other words, according as the rotation *round the first* line, *from the second to the third*, is on the contrary *negative* or *positive* (compare 294, (3.)).

(1.) In illustration of the *constancy* of that fourth proportional which has been, for the present, denoted by u, while the *system* of the three rectangular unit-lines from which it is conceived to be derived is in any manner *turned about*, we may observe that the *three* equations, or proportions,

$$\text{XIII.} \ldots u : \gamma = \beta : \alpha ; \quad \gamma : \alpha = \alpha : -\gamma ; \quad \beta : -\gamma = \gamma : \beta,$$

conduct immediately to this *fourth* equation of the same kind,

$$\text{XIV.} \ldots u : \alpha = \gamma : \beta, \quad \text{or}^* \quad u = (\gamma : \beta) \,.\, \alpha ;$$

if we admit that this new quantity, or symbol, u, is to be operated on *at all*, or combined with *other* symbols, according to the general rules of vectors and quaternions.

(2.) It is, then, permitted to change the *three letters* α, β, γ, by a *cyclical permutation*, to the three other letters β, γ, α (considered again as representing *unit-lines*), without altering the *value* of the *fourth proportional*, u; or in other words, it is allowed to make the *system of the three rectangular lines revolve, through the third part of four right angles, round the interior and co-initial diagonal of the unit-cube*, of which they are three co-initial *edges*.

(3.) And it is still more evident, that no such change of value will take place, if we merely cause the system of the *two first lines* to revolve, *through any angle*, in its own plane, *round the third line* as an axis; since thus we shall merely substitute, for the factor $\beta : \alpha$, another factor *equal* thereto. But by *combining* these two last *modes* of rotation, we can represent *any rotation whatever*, round an origin supposed to be fixed.

(4.) And as regards the *scalar ratio* of any *one* fourth proportional, such as $\beta' : \alpha' \,.\, \gamma'$, to any *other*, of the kind here considered, such as $\beta : \alpha \,.\, \gamma$, or u, it is sufficient to suggest that, without any real change in the former, we are allowed to suppose it to be so *prepared*, that we shall have

$$\text{XV.} \ldots \alpha' = \alpha ; \quad \beta' = \beta ; \quad \gamma' = x\gamma ;$$

x being some scalar coefficient, and representing the ratio required.

304. In the more general case, when the three given lines are *not* rectangular, *nor* unit-lines, we may on similar principles determine their fourth proportional, without referring to Fig. 68, as follows. Without any real loss of generality, we may suppose that the planes of α, β and α, γ are perpendicular to each other; since this comes merely to substituting, if necessary, for the quotient $\beta : \alpha$, another quotient equal thereto. Having thus

$$\text{I.} \ldots \text{Ax.}\,(\beta : \alpha) \perp \text{Ax.}\,(\gamma : \alpha), \quad \text{let} \quad \text{II.} \ldots \beta = \beta' + \beta'', \quad \gamma = \gamma' + \gamma'',$$

where β' and γ' are parallel to α, but β'' and γ'' are perpendicular to it, and to each other; so that, by 203, I. and II., we shall have the expressions,

$$\text{III.} \ldots \beta' = \text{S}\frac{\beta}{\alpha} \,.\, \alpha, \quad \gamma' = \text{S}\frac{\gamma}{\alpha} \,.\, \alpha,$$

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* In equations of this form, the parentheses may be omitted, though for greater clearness they are here retained.

and $$\text{IV.}\ldots\ \beta'' = \mathrm{V}\frac{\beta}{\alpha}.\,\alpha, \quad \gamma'' = \mathrm{V}\frac{\gamma}{\alpha}.\,\alpha.$$

We may then deduce, by the distributive principle (300, (2.)), the transformations,

$$\text{V.}\ldots\ \frac{\beta}{\alpha}.\gamma = \left(\frac{\beta'}{\alpha} + \frac{\beta''}{\alpha}\right)(\gamma' + \gamma'')$$

$$= \frac{\beta'}{\alpha}\gamma' + \frac{\beta'}{\alpha}\gamma'' + \frac{\beta''}{\alpha}\gamma' + \frac{\beta''}{\alpha}\gamma'' = \delta + xu;$$

where

$$\text{VI.}\ldots\ \delta = \beta\mathrm{S}\frac{\gamma}{\alpha} + \gamma''\mathrm{S}\frac{\beta}{\alpha} = \gamma\mathrm{S}\frac{\beta}{\alpha} + \beta''\mathrm{S}\frac{\gamma}{\alpha}, \text{ and VII.}\ldots\ xu = \frac{\beta''}{\alpha}\gamma''.$$

The latter part, xu, is what we have called (300) the (geometrically) *scalar part*, of the sought fourth proportional; while the former part δ may (still) be called its *vector part:* and we see that *this part* is represented by a *line*, which is at once *in the two planes*, of β, γ'', and of γ, β''; or in two planes which may be generally *constructed* as follows, *without now assuming* that the planes $\alpha\beta$ and $\alpha\gamma$ are *rectangular*, as in I. Let γ' be the projection of the line γ on the plane of α, β, and operate on this projection by the quotient $\beta : \alpha$ as a multiplier; the *plane* which is drawn through the line $\beta : \alpha . \gamma'$ so obtained, at right angles to the plane $\alpha\beta$, is *one locus* for the sought *line* δ: and the plane through γ, which is perpendicular to the plane $\gamma\gamma'$, is *another locus* for that line. And as regards the *length* of this line, or vector part δ, and the *magnitude* (or quantity) of the scalar part xu, it is easy to prove that

$$\text{VIII.}\ldots\ \mathrm{T}\delta = t\cos s, \quad \text{and} \quad \text{IX.}\ldots\ x = \pm t \sin s,$$

where

$$\text{X.}\ldots\ t = \mathrm{T}\beta : \mathrm{T}\alpha . \mathrm{T}\gamma, \quad \text{and} \quad \text{XI.}\ldots\ \sin s = \sin c \sin p,$$

if c denote the angle between the two given lines α, β, and p the inclination of the third given line γ to their plane: the *sign* of the scalar *coefficient*, x, being positive or negative, according as the *rotation* round α from β to γ is negative or positive.

(1.) Comparing the recent construction with Fig. 68, we see that when the condition I. is satisfied, the four unit-lines $\mathrm{U}\gamma$, $\mathrm{U}\alpha$, $\mathrm{U}\beta$, $\mathrm{U}\delta$ take the directions of the four radii OC, OQ, OR, OD, which terminate at the four corners of what may be called a *tri-rectangular quadrilateral* CQRD on the sphere.

(2.) It may be remarked that the *area* of this *quadrilateral* is exactly equal to *half* the area Σ of the *triangle* DEF; which may be inferred, either from the circumstance that its *spherical excess* (over *four* right angles) is constructed by the angle MDN; or from the triangles DBR and KAS being together equal to the triangle ABF, so that the area of DKSR is Σ, and therefore that of CQRD is $\frac{1}{2}\Sigma$, as before.

(3.) The two *sides* CQ, QR of this quadrilateral, which are *remote* from the *obtuse* angle at D, being still called p and c, and the side CD which is *opposite* to c being still denoted by c', let the side DR which is opposite to p be now called p'; also let the *diagonals* CR, QD be denoted by d and d'; and let s denote the *spherical excess* ($\mathrm{CDR} - \frac{1}{2}\pi$), or the *area* of the quadrilateral. We shall then have the relations,

$$\text{XII.}\ldots \begin{cases} \cos d = \cos p \cos c; \quad \cos d' = \cos p \cos c'; \\ \tan c' = \cos p \tan c; \quad \tan p' = \cos c \tan p; \\ \cos s = \cos p \sec p' = \cos c \sec c' = \cos d \sec d'; \end{cases}$$

of which some have virtually occurred before, and all are easily proved by right-angled triangles, arcs being when necessary prolonged.

(4.) If we take now two points, A and B, on the side QR, which satisfy the arcual equation (comp. 297, XL., and Fig. 68),

$$\text{XIII.}\ldots \frown \mathrm{AB} = \frown \mathrm{QR};$$

and if we then join AC, and let fall on this new arc the perpendiculars BB′, DD′; it is easy to prove that the *projection* B′D′ of the side BD on the arc AC is *equal* to that arc, and that the *angle* DBB′ is right: so that we have the two new equations,

$$\text{XIV.}\ldots \frown \mathrm{B'D'} = \frown \mathrm{AC}; \qquad \text{XV.}\ldots \mathrm{DBB'} = \tfrac{1}{2}\pi;$$

and the *new quadrilateral* BB′D′D is also *tri-rectangular*.

(5.) Hence *the point* D *may be derived from the three points* ABC, *by any two of the four following conditions*: Ist, the *equality* XIII. of the *arcs* AB, QR; IInd, the corresponding equality XIV. of the arcs AC, B′D′; IIIrd, the *tri-rectangular character* of the *quadrilateral* CQRD; IVth, the corresponding character of BB′D′D.

(6.) In other words, this *derived point* D is the *common intersection of the four perpendiculars, to the four arcs* AB, AC, CQ, BB′, erected *at the four points* R, D′, C, B; CQ, BB′ being still the perpendiculars from C and B, on AB and AC; and R and D′ being deduced from Q and B′, by equal arcs, as above.

305. These consequences of the construction employed in 297, &c., are here mentioned merely in connexion with that theory of *fourth proportionals to vectors*, which they have thus served to illustrate; but they are perhaps numerous and interesting enough, to justify us in suggesting the *name*, "*Spherical Parallelogram*,"* for the quadrilateral CABD, or BACD, in Fig. 68 (or 67); and in proposing to say that D is the *Fourth Point*, which *completes* such a *parallelogram*, when the *three points* C, A, B, or B, A, C, are *given* upon the sphere, as *first*, *second*, and *third*. It must however be carefully observed, that the *analogy to the plane* is here thus far *imperfect*, that in the

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* By the same analogy, the quadrilateral CQRD, in Fig. 68, may be called a *Spherical Rectangle*.

general case, when the *three given points* are *not co-arcual*, but on the contrary are corners of a *spherical triangle* ABC, then if we take C, D, B, or B, D, C, for the *three first points* of a *new spherical parallelogram*, of the kind here considered, the *new fourth point*, say A_1, will *not coincide* with the *old second point* A; although it will *very nearly* do so, if the *sides* of the triangle ABC be *small:* the *deviation* AA_1 being in fact found to be small of the *third order*, if those sides of the given triangle be supposed to be small of the *first* order; and being always directed *towards the foot of the perpendicular*, let fall from A on BC.

(1.) To investigate the *law* of this *deviation*, let β, γ be still any two given unit-vectors, OB, OC, making with each other an angle equal to a, of which the cosine is l; and let ρ or OP be any third vector. Then, if we write,

$$\text{I.} \ldots \rho_1 = \phi(\rho) = \tfrac{1}{2}\mathrm{N}\rho \cdot \left(\frac{\beta}{\rho}\gamma + \frac{\gamma}{\rho}\beta\right), \quad \mathrm{OQ} = \mathrm{U}\rho, \quad \mathrm{OQ}_1 = \mathrm{U}\rho_1,$$

the new or *derived vector*, $\phi\rho$ or ρ_1, or OP_1, will be the *common vector part* of the *two fourth proportionals*, to ρ, β, γ, and to ρ, γ, β, *multiplied by the square of the length of* ρ; and BQOQ_1 will be what we have lately called a *spherical parallelogram*. We shall also have the transformation (compare 297, (2.)),

$$\text{II.} \ldots \rho_1 = \phi\rho = \beta\mathrm{S}\frac{\rho}{\gamma} + \gamma\mathrm{S}\frac{\rho}{\beta} - \rho\mathrm{S}\frac{\gamma}{\beta};$$

and the *distributive symbol of operation* ϕ will be such that

$$\text{III.} \ldots \phi\rho \parallel\!\mid \beta, \gamma, \quad \text{and} \quad \phi^2\rho = \rho, \quad \text{if} \quad \rho \parallel\!\mid \beta, \gamma;$$

but

$$\text{IV.} \ldots \phi\rho = -l\rho, \quad \text{if} \quad \rho \parallel \mathrm{Ax.}(\gamma : \beta).$$

(2.) This being understood, let

$$\text{V.} \ldots \rho = \rho' + \rho''; \quad \phi\rho' = \rho'_1; \quad \rho' \parallel\!\mid \beta, \gamma; \quad \rho'' \parallel \mathrm{Ax.}(\gamma : \beta);$$

so that ρ', or OP′, is the projection of ρ on the plane of $\beta\gamma$; and ρ'', or OP″, is the part (or component) of ρ, which is perpendicular to that plane. Then we shall have an indefinite *series of derived vectors*, $\rho_1, \rho_2, \rho_3, \ldots$ or rather *two* such series, succeeding each other *alternately*, as follows:

$$\text{VI.} \ldots \begin{cases} \rho_1 = \phi\rho = \rho'_1 - l\rho''; & \rho_2 = \phi^2\rho = \rho' + l^2\rho''; \\ \rho_3 = \phi^3\rho = \rho'_1 - l^3\rho''; & \rho_4 = \phi^4\rho = \rho' + l^4\rho''; \ \&\text{c.}; \end{cases}$$

the *two series of derived points*, $P_1, P_2, P_3, P_4, \ldots$ being thus *ranged, alternately, on the two perpendiculars*, PP′ and $P_1P'_1$, which are let fall *from the points* P and P_1, *on the given plane* BOC; and the intervals, $PP_2, P_1P_3, P_2P_4, \ldots$ forming a *geometrical progression*, in which each is equal to the one before it, multiplied by the *constant factor* $-l$, or by the *negative of the cosine* of the *given angle* BOC.

(3.) If then this angle be still supposed to be distinct from 0 and π, and also in general from the intermediate value $\frac{1}{2}\pi$, we shall have the *two limiting values*,

$$\text{VII.} \ldots \rho_{2n} = \rho', \quad \rho_{2n+1} = \rho'_1, \quad \text{if} \quad n = \infty;$$

or in words, *the derived points* $P_2, P_4, \ldots$ *of even orders, tend to the point* P′, and the *other derived points*, $P_1, P_3, \ldots$ *of odd orders*, tend to the *other point* P'_1, *as limiting* *positions:* these *two limit points* being the *feet of the two* (rectilinear) *perpendiculars*, let fall (as above) from P and P′ on the plane BOC.

(4.) But even the *first deviation* PP_2, is *small of the third order*, if the *length* $T\rho$ of the *line* OP be considered as *neither* large nor small, and if the *sides* of the spherical *triangle* BQC be *small* of the *first* order. For we have by VI. the following expressions for that deviation,

$$\text{VIII.} \ldots PP_2 = \rho_2 - \rho = (t^3 - 1)\rho'' = -\sin a^2 . \sin p_a . T\rho . U\rho'' ;$$

where p_a denotes the *inclination* of the *line* ρ to the *plane* $\beta\gamma$; or the *arcual perpendicular* from the *point* Q on the side BC, or a, of the triangle. The statements lately made (305) are therefore proved to have been correct.

(5.) And if we now resume and extend the *spherical construction*, and conceive that D_1 is deduced from BA_1C, as A_1 was from BDC, or D from BAC; while A_2 may be supposed to be deduced by the same rule from BD_1C, and D_2 from BA_2C, &c., through an indefinite *series of spherical parallelograms*, in which the *fourth point* of any *one* is treated as the *second point* of the *next*, while the *first* and *third* points remain *constant:* we see that the points A_1, A_2, . . are all situated *on the arcual perpendicular* let fall from A on BC; and that in like manner the points D_1, D_2, . . are all situated on that *other* arcual perpendicular, which is let fall from D on BC. We see also that *the ultimate positions*, A_∞ and D_∞, *coincide precisely with the feet of those two perpendiculars:* a remarkable theorem, which it would perhaps be difficult to prove, by any other method than that of the Quaternions, at least with calculations so simple as those which have been employed above.

(6.) It may be remarked that the construction of Fig. 68 might have been otherwise suggested (comp. 223, IV.), by the principles of the Second Book, if we had sought to assign the *fourth proportional* (297) *to three right quaternions;* for example, to *three right versors*, v, v', v'', whereof the *unit lines* α, β, γ should be supposed to be the *axes*. For the result would be in general a *quaternion* $v'v^{-1}v''$, with e for its *scalar part*, and with δ for the *index of its right part:* e and δ denoting the *same scalar*, and the *same vector*, as in the sub-articles to 297.

306. Quaternions may also be employed to furnish a *new construction*, which shall complete (comp. 305, (5.)) the *graphical* determination of the *two series* of *derived points*,

$$\text{I.} \ldots D,\ A_1,\ D_1,\ A_2,\ D_2,\ \&c.,$$

when the *three* points A, B, C are *given* upon the unit-sphere; and thus shall render *visible* (so to speak), with the help of a new Figure, the tendencies of those derived points to *approach*, alternately and indefinitely, to the *feet*, say D′ and A′, of the two *arcual perpendiculars* let fall from the two *opposite corners*, D and A, of the *first spherical parallelogram*, BACD, on its *given diagonal* BC; which *diagonal* (as we have seen) is common to *all* the successive parallelograms.

(1.) The given triangle ABC being supposed for simplicity to have its sides abc less than quadrants, as in 297, so that their cosines lmn are positive, let A′, B′, C′ be the feet of the perpendiculars let fall on these three sides from the points A, B, C; also let M and N be two auxiliary points, determined by the equations,

$$\text{II.}\ldots \frown BM = \frown MC, \qquad \frown AM = \frown MN;$$

so that the arcs AN and BC bisect each other in M. Let fall from N a perpendicular ND′ on BC, so that

$$\text{III.}\ldots \frown BD' = \frown A'C;$$

and let B″, C″ be two other auxiliary points, on the sides b and c, or on those sides prolonged, which satisfy these two other equations,

$$\text{IV.}\ldots \frown B'B'' = \frown AC, \qquad \frown C'C'' = \frown AB.$$

(2.) Then the *perpendiculars to these last sides*, CA and AB, erected *at these last points*, B″ and C″, will *intersect each other in the point* D, which *completes* (305) the *spherical parallelogram* BACD; and the *foot of the perpendicular from this point* D, on the third side BC of the given triangle, will *coincide* (comp. 305, (2.)) *with the foot* D′ *of the perpendicular on the same side from* N; so that this last perpendicular ND′ is *one locus of the point* D.

(3.) To obtain *another locus* for that point, adapted to our present purpose, let E denote now* that new point in which the two diagonals, AD and BC, intersect each other; then because (comp. 297, (2.)) we have the expression,

$$\text{V.}\ldots \mathrm{OD} = \mathrm{U}(m\beta + n\gamma - l\alpha),$$

we may write (comp. 297, (25.), and (30.)),

$$\text{VI.}\ldots \mathrm{OE} = \mathrm{U}(m\beta + n\gamma), \quad \text{whence} \quad \text{VII.}\ldots \sin BE : \sin EC = n : m = \cos BA' : \cos A'C;$$

the *diagonal* AD thus dividing the arc BC into *segments*, of which the *sines* are proportional to the *cosines* of the *adjacent sides* of the given triangle, or to the cosines of their *projections* BA′ and A′C on BC; so that the *greater segment* is adjacent to the *lesser side*, and the *middle point* M of BC (1.) lies *between the points* A′ and E.

(4.) The intersection E is therefore a known point, and the great circle through A and E is a *second known locus* for D; which point may therefore be found, as the *intersection of the arc* AE prolonged, *with the perpendicular* ND′ from N (1.). And because E lies (3.) *beyond the middle point* M of BC, with respect to the foot A′ of the perpendicular on BC from A, but (as it is easy to prove) *not so far beyond* M *as the point* D′, or in other words falls *between* M *and* D′ (when the arc BC is, as above supposed, *less than a quadrant*), the prolonged arc AE cuts ND′ *between* N *and* D′; or in other words, *the perpendicular distance of the sought fourth point* D, *from the given diagonal* BC *of the parallelogram, is less than the distance of the given second point* A, from the same given diagonal. (Compare the annexed Fig. 73.)

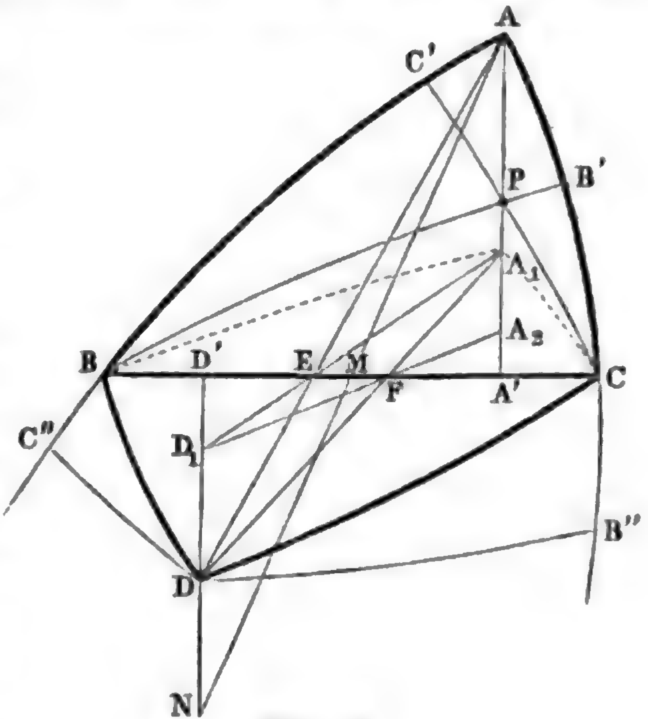

Fig. 73.

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* It will be observed that M, N, E have not here the same significations as in

(5.) Proceeding next (305) to *derive* a *new point* A_1 from B, D, C, as D has been derived from B, A, C, we see that we have only to determine a *new** *auxiliary point* F, by the equation,

$$\text{VIII.} \ldots \cap \mathrm{EM} = \cap \mathrm{MF};$$

and then to draw DF, and prolong it till it meets AA′ in the required point A_1, which will thus *complete the second parallelogram*, BDCA_1, with BC (as before) for a *given diagonal*.

(6.) In like manner, to complete (comp. 305, (5.)), the *third* parallelogram, $\mathrm{BA_1CD_1}$, with the *same* given diagonal BC, we have only to draw the arc $\mathrm{A_1E}$, and prolong it till it cuts ND′ in D_1; after which we should find the point A_2 of a *fourth* successive parallelogram $\mathrm{BD_1CA_2}$, by drawing $\mathrm{D_1F}$, and so on for ever.

(7.) The constant and indefinite *tendency*, of the *derived points* D, D_1, . . to the *limit-point* D′, and of the *other* (or *alternate*) derived points A_1, A_2, . . to the *other limit-point* A′, becomes therefore evident from this new construction; the *final* (or *limiting*) *results* of which, we may express by these two equations (comp. again 305, (5.)),

$$\text{IX.} \ldots \mathrm{D}_\infty = \mathrm{D}'; \quad \mathrm{A}_\infty = \mathrm{A}'.$$

(8.) But the *smallness* (305) of the *first deviation* $\mathrm{AA_1}$, when the *sides* of the given triangle ABC are *small*, becomes at the same time evident, by means of the same construction, with the help of the formula VII.; which shows that the *interval*† EM, or the *equal* interval MF (5.), is *small of the third order*, when the *sides* of the given triangle are supposed to be small of the *first* order: agreeing thus with the equation 305, VIII.

(9.) The theory of such *spherical parallelograms* admits of some interesting applications, especially in connexion with *spherical conics;* on which however we cannot enter here, beyond the mere *enunciation* of a *Theorem*,‡ of which (comp. 271) the proof by quaternions is easy :—

Fig. 68; and that the present letters C′ and C″ correspond to Q and R in that Figure.

* This new point, and the intersection of the perpendiculars of the given triangle, are evidently not the same in the new Figure 73, as the points denoted by the same letters, F and P, in the former Figure 68; although the four points A, B, C, D are conceived to bear to each other the same relations in the two Figures, and indeed in Fig. 67 also; BACD being, in *that* Figure also, what we have proposed to call a *spherical parallelogram*. Compare the Note to (3.).

† The formula VII. gives easily the relation,

$$\text{VII}'. \ldots \tan \mathrm{EM} = \tan \mathrm{MA}' \left(\tan \frac{a}{2} \right)^2;$$

hence the interval EM is small of the third order, in the case (8.) here supposed; and generally, if $a < \frac{\pi}{2}$, as in (1.), while b and c are unequal, the formula shows that this interval EM is less than MA′, or than D′M, so that E falls between M and D′, as in (4.).

‡ This Theorem was communicated to the Royal Irish Academy in June, 1845, as a consequence of the principles of Quaternions. See the *Proceedings* of that date (Vol. III., page 109).

"*If* KLMN *be any spherical quadrilateral, and* I *any point on the sphere; if also we complete the spherical parallelograms,*

$$\text{X.}\ldots\ \text{KILA},\quad \text{LIMB},\quad \text{MINC},\quad \text{NIKD},$$

and determine the poles E and F *of the diagonals* KM *and* LN *of the quadrilateral: then these two poles are the foci* of a spherical conic, inscribed in the derived quadrilateral* ABCD, *or touching its four sides.*"

(10.) Hence, in a notation† elsewhere proposed, we shall have, under these conditions of construction, the formula:

$$\text{XI.}\ldots\ \text{EF}\,(\,.\,.\,)\,\text{ABCD};\quad\text{or}\quad \text{XI}'.\ldots\ \text{EF}\,(\,.\,.\,)\,\text{BCDA};\ \&\text{c.}$$

(11.) Before closing this Article and Section, it seems not irrelevant to remark, that the projection γ' of the unit-vector γ, on the plane of α and β, is given by the formula,

$$\text{XII.}\ldots\ \gamma' = \frac{\alpha \sin a \cos \text{B} + \beta \sin b \cos \text{A}}{\sin c};$$

and that therefore the point P, in which (see again Fig. 73) the three arcual perpendiculars of the triangle ABC intersect, is on the vector,

$$\text{XIII.}\ldots\ \rho = \alpha \tan \text{A} + \beta \tan \text{B} + \gamma \tan \text{C}.$$

(12.) It may be added, as regards the construction in 305, (2.), that *the right lines,*

$$\text{XIV.}\ldots\ \text{PP}_1,\quad \text{P}_1\text{P}_2,\quad \text{P}_2\text{P}_3,\quad \text{P}_3\text{P}_4,\ \ldots$$

however far their series may be continued, *intersect the given plane* BOC, *alternately, in two points* S and T, of which the vectors are,

$$\text{XV.}\ldots\ \text{OS} = \frac{\rho'_1 + l\rho'}{1+l},\quad \text{OT} = \frac{\rho' + l\rho'_1}{1+l};$$

and which thus become *two fixed points in the plane*, when the position of the point P *in space* is *given*, or assumed.

SECTION 9.—*On a Third Method of interpreting a Product or Function of Vectors as a Quaternion; and on the Consistency of the Results of the Interpretation so obtained, with those which have been deduced from the two preceding Methods of the present Book.*

307. The *Conception* of the *Fourth Proportional to Three Rectangular Unit-Lines*, as being itself a species of *Fourth Unit in Geometry*, is eminently *characteristic of the present Calculus;* and offers a *Third Method of interpreting a Product of two Vectors* as a *Quaternion:* which is however found to be

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* In the language of modern geometry, the *conic* in question may be said to *touch eight given arcs; four real*, namely the *sides* AB, BC, CD, DA; and *four imaginary*, namely *two* from *each* of the *focal points*, E and F.

† Compare the Second Note to page 295.

consistent, in all its *results*, with the two former methods (278, 284) of the present Book; and admits of being easily extended to products of *three* or more *lines in space*, and generally to *Functions of Vectors* (289). In fact we have only to conceive* that *each proposed vector*, α, is *divided* by the *new* or *fourth unit*, u, above alluded to; and that the *quotient* so obtained, which is always (by 303, VIII.) the *right quaternion* $\mathrm{I}^{-1}\alpha$, whereof the vector α is the *index*, is *substituted* for that vector: the *resulting quaternion* being finally, if we think it convenient, *multiplied into the same fourth unit.* For in this way we shall merely *reproduce the process* of 284, or 289, although now as a consequence of a *different train of thought*, or of a *distinct* but *Consistent Interpretation:* which thus conducts, by a *new Method*, to the *same Rules of Calculation* as before.

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* It was in a somewhat analogous way that *Des Cartes* showed, in his *Geometria* (Schooten's Edition, Amsterdam, 1659), that all *products* and *powers* of *lines*, considered relatively to their *lengths* alone, and without any reference to their *directions*, could be *interpreted as lines*, by the suitable introduction of a line taken for *unity*, however high the *dimension* of the product or power might be. Thus (at page 3 of the cited work) the following remark occurs:—

"Ubi notandum est, quòd per a^2 vel b^3, similésve, communiter, non nisi lineas omnino simplices concipiam, licèt illas, ut nominibus in Algebra usitatis utar, Quadrata aut Cubos, &c. appellem."

But it was much more difficult to accomplish the corresponding *multiplication of directed lines in space*; on account of the *non-existence of any such line*, which is *symmetrically related to all other lines*, or *common to all possible planes* (comp. the Note to page 248). The *Unit of Vector-Multiplication* cannot properly be *itself* a *Vector*, if the *conception* of the *Symmetry of Space* is to be retained, and duly combined with the other elements of the question. This difficulty however disappears, at least in theory, when we come to consider that *new Unit*, of a *scalar* kind (300), which has been above denoted by the temporary symbol u, and has been obtained, in the foregoing Section, as a certain *Fourth Proportional to Three Rectangular Unit-Lines*, such as the *three co-initial edges*, AB, AC, AD of what we have called an *Unit-Cube*: for *this* fourth proportional, by the proposed *conception* of it, undergoes *no change*, when the cube ABCD is in any manner *moved*, or *turned*; and therefore may be considered to be *symmetrically related to all directions of lines in space*, or to all possible *vections* (or *translations*) of a *point*, or *body*. In fact, we *conceive* its *determination*, and the *distinction* of it (as $+u$) from the *opposite unit* of the *same kind* ($-u$), to depend *only* on the *usual assumption* of an *unit of length*, combined with the *selection of a hand* (as, for example, the *right* hand), *rotation towards which hand* shall be considered to be *positive*, and *contrasted* (as such) with rotation towards the *other hand*, round the *same arbitrary axis*. Now in whatever manner the supposed *cube* may be thrown about in space, the *conceived rotation round the edge* AB, *from* AC *to* AD, will have the *same character*, as *right-handed* or *left-handed*, at the *end* as at the *beginning* of the motion. If then the *fourth proportional* to these *three edges*, taken *in this order*, be *denoted* by $+u$, or simply by $+1$, *at one stage* of that arbitrary *motion*, it may (on the plan here considered) be denoted by the *same symbol*, at *every other stage*: while the *opposite character* of the (conceived) *rotation*, round the *same edge* AB, *from* AD *to* AC, leads us to regard the fourth proportional to AB, AD, AC as being on the contrary equal to $-u$, or to -1. It is true that this *conception* of a *new unit for space*, *symmetrically related* (as above) *to all linear directions* therein, may appear somewhat abstract and metaphysical; but readers who think it such can of course confine their attention to the *rules of calculation*, which have been above derived from it, and from other connected considerations: and which have (it is hoped) been stated and exemplified, in this and in a former Volume, with sufficient clearness and fullness.

(1.) The *equation of the unit-sphere*, $\rho^2 + 1 = 0$ (282, XIV.), may thus be conceived to be an *abridgment* of the following fuller equation:

$$\text{I.} \ldots \left(\frac{\rho}{u}\right)^2 = -1;$$

the *quotient* $\rho : u$ being considered as equal (by 303) to the *right quaternion*, $\mathrm{I}^{-1}\rho$, which must *here* be a *right versor* (154), because its *square* is negative unity.

(2.) The *equation of the ellipsoid*,

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2 \text{ (282, XIX.)},$$

may be supposed, in like manner, to be *abridged* from this other equation:

$$\text{II.} \ldots \mathrm{T}\left(\frac{\iota}{u}\frac{\rho}{u} + \frac{\rho}{u}\frac{\kappa}{u}\right) = \left(\frac{\kappa}{u}\right)^2 - \left(\frac{\iota}{u}\right)^2;$$

and similarly in other cases.

(3.) We might also write these equations, of the sphere and ellipsoid, under these other, but connected forms:

$$\text{III.} \ldots \frac{\rho}{u}\rho = -u; \qquad \text{IV.} \ldots \mathrm{T}\left(\frac{\iota}{u}\rho + \frac{\rho}{u}\kappa\right) = \frac{\kappa}{u}\kappa - \frac{\iota}{u}\iota;$$

with intepretations which easily offer themselves, on the principles of the foregoing Section.

(4.) It is, however, to be distinctly understood, that *we do not propose to adopt this Form of Notation*, in the *practice* of the present *Calculus:* and that we merely *suggest* it, in passing, as one which may serve to throw some additional light on the *Conception*, introduced in this Third Book, of a *Product of two Vectors* as a *Quaternion.*

(5.) In general, the *Notation of Products*, which has been employed throughout the greater part of the present Book and Chapter, appears to be *much more convenient*, for actual *use* in *calculation*, than any *Notation of Quotients:* either such as has been just now suggested for the sake of illustration, or such as was employed in the Second Book, in connexion with that *First Conception of a Quaternion* (112), to which that Book mainly related, as the *Quotient of two Vectors* (or of two directed lines in space). The *notations* of the two Books are, however, intimately *connected*, and the former was judged to be an useful preparation for the latter, even as regarded the *quotient-forms* of many of the expressions used: while the *Characteristics of Operation*, such as

$$\mathrm{S},\ \mathrm{V},\ \mathrm{T},\ \mathrm{U},\ \mathrm{K},\ \mathrm{N},$$

are employed according to exactly the *same laws* in both. In short, a reader of the Second Book has *nothing to unlearn* in the Third; although he may be supposed to have become prepared for the use of somewhat shorter and more convenient *processes*, than those before employed.

SECTION 10.—*On the Interpretation of a Power of a Vector as a Quaternion.*

308. The only symbols, of the kinds mentioned in 277, which we have not yet interpreted, are the *cube* α^3, and the *general power* α^t, of an *arbitrary vector base*, α, with an *arbitrary scalar exponent*, t; for we have already assigned interpretations (282, (1.), (14.), and 299, (8.)) for the *particular symbols* α^2, α^{-1}, α^{-2}, which are *included* in this last *form*. And we shall preserve those particular interpretations if we now *define*, in full consistency with the principles of the present and preceding Books, that this *Power* α^t is generally a *Quaternion*, which may be decomposed into *two factors*, of the *tensor* and *versor* kinds, as follows:

$$\mathrm{I.} \ldots \alpha^t = \mathrm{T}\alpha^t . \mathrm{U}\alpha^t;$$

$\mathrm{T}\alpha^t$ denoting the *arithmetical value* of the tth *power* of the *positive number* $\mathrm{T}\alpha$, which represents (as usual) the *length* of the *base-line* α; and $\mathrm{U}\alpha^t$ denoting a *versor*, which *causes any line* ρ, *perpendicular to that line* α, to *revolve round it as an axis, through t right angles*, or quadrants, and in a *positive* or *negative direction*, according as the *scalar exponent*, t, is *itself* a *positive* or *negative number* (comp. 234, (5.)).

(1.) As regards the omission of parentheses in the formula I., we may observe that the recent *definition*, or *interpretation*, of the symbol α^t, enables us to write (comp. 237, II. III.),

$$\mathrm{II.} \ldots \mathrm{T}(\alpha^t) = (\mathrm{T}\alpha)^t = \mathrm{T}\alpha^t; \qquad \mathrm{III.} \ldots \mathrm{U}(\alpha^t) = (\mathrm{U}\alpha)^t = \mathrm{U}\alpha^t.$$

(2.) The *axis* and *angle* of the *power* α^t, considered as a quaternion, are generally determined by the two following formulæ:

$$\mathrm{IV.} \ldots \mathrm{Ax}.\, \alpha^t = \pm\, \mathrm{U}\alpha; \qquad \mathrm{V.} \ldots \angle . \alpha^t = 2n\pi \pm \tfrac{1}{2}t\pi;$$

the *signs* accompanying each other, and the (positive or negative or null) integer, n, being so chosen as to bring the *angle* within the usual limits, 0 and π.

(3.) In general (comp. 235), we may speak of the (positive or negative) product $\frac{1}{2}t\pi$, as being the *amplitude* of the same *power*, with reference to the line a as an *axis of rotation;* and may write accordingly,

$$\text{VI.} \ldots \text{am.}\, a^t = \tfrac{1}{2}t\pi.$$

(4.) We may write also (comp. 234, VII. VIII.),

$$\text{VII.} \ldots \text{U}a^t = \cos\frac{t\pi}{2} + \text{U}a \,.\, \sin\frac{t\pi}{2}; \quad \text{or briefly,} \quad \text{VIII.} \ldots \text{U}a^t = \text{cas}\,\frac{t\pi}{2}.$$

(5.) In particular,

$$\text{IX.} \ldots \text{U}a^{2n} = \text{cas}\, n\pi = \pm 1; \qquad \text{IX}'. \ldots \text{U}a^{2n+1} = \pm\, \text{U}a;$$

upper or lower signs being taken, according as the number n (supposed to be whole) is even or odd. For example, we have thus the *cubes,*

$$\text{X.} \ldots \text{U}a^3 = -\,\text{U}a; \qquad \text{X}'. \ldots a^3 = -\,a\text{N}a.$$

(6.) The *conjugate* and *norm* of the power a^t may be thus expressed (it being remembered that to turn a line $\perp a$ through $-\frac{1}{2}t\pi$ round $+a$, is equivalent to turning that line through $+\frac{1}{2}t\pi$ round $-a$):

$$\text{XI.} \ldots \text{K}a^t = \text{T}a^t.\,\text{U}a^{-t} = (-\,a)^t; \qquad \text{XII.} \ldots \text{N}a^t = \text{T}a^{2t};$$

parentheses being unnecessary, because (by 295, VIII.) $\text{K}a = -\,a$.

(7.) The *scalar, vector,* and *reciprocal* of the same power are given by the formulæ:

$$\text{XIII.} \ldots \text{S}.a^t = \text{T}a^t.\cos\frac{t\pi}{2}; \qquad \text{XIV.} \ldots \text{V}.a^t = \text{T}a^t.\text{U}a\,.\sin\frac{t\pi}{2};$$

$$\text{XV.} \ldots 1 : a^t = \text{T}a^{-t}.\text{U}a^{-t} = a^{-t} = \text{K}a^t : \text{N}a^t \text{ (comp. 190, (3.))}.$$

(8.) If we decompose any vector ρ into parts ρ' and ρ'', which are respectively parallel and perpendicular to a, we have the general transformation:*

$$\text{XVI.} \ldots a^t\rho a^{-t} = a^t(\rho' + \rho'')\,a^{-t} = \rho' + \text{U}a^{2t}.\,\rho'',$$

= *the new vector obtained by causing* ρ *to revolve conically through an angular quantity expressed by* $t\pi$, *round the line* a *as an axis* (comp. 297, (15.)).

(9.) More generally (comp. 191, (5.)), if q be any *quaternion*, and if

$$\text{XVII.} \ldots a^tqa^{-t} = q',$$

the *new quaternion* q' is formed from q by such a *conical rotation of its own axis* Ax. q, through $t\pi$, round a, *without any change of its angle* $\angle\, q$, or of its tensor Tq.

(10.) Treating ijk as *three rectangular unit-lines* (295), the *symbol,* or *expression,*

$$\text{XVIII.} \ldots \rho = rk^tj^skj^{-s}k^{-t}, \quad \text{or} \quad \text{XIX.} \ldots \rho = rk^tj^{2s}k^{1-t},$$

in which

$$\text{XX.} \ldots r \geqq 0, \quad s \geqq 0, \quad s \leqq 1, \quad t \geqq 0, \quad t \leqq 2,$$

may represent any vector; the *length* or *tensor* of this line ρ being r; its *inclination*† to k being $s\pi$; and the *angle* through which the variable *plane* $k\rho$ may be conceived to have *revolved*, from the initial position ki, with an initial direction towards the position kj, being $t\pi$.

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* Compare the shortly following sub-article (11.).

† If we conceive (compare the first Note to page 322) that the two lines i and j are directed respectively towards the *south* and *west* points of the horizon, while the third line k is directed towards the *zenith*, then $s\pi$ is the *zenith-distance* of ρ; and $t\pi$ is the *azimuth* of the same line, measured *from south to west*, and thence (if necessary) through north and east, to south again.

(11.) In accomplishing the transformation XVI., and in passing from the expression XVIII. to the less symmetric but equivalent expression XIX., we employ the principle that

$$\text{XXI.} \ldots kj^{-s} = \mathrm{S}^{-1}0 = -\mathrm{K}(kj^{-s}) = j^s k;$$

which easily admits of extension, and may be confirmed by such transformations as VII. or VIII.

(12.) It is scarcely necessary to remark, that the definition or interpretation I., of the *power* a^t of *any vector* a, gives (as in algebra) the *exponential property*,

$$\text{XXII.} \ldots a^s a^t = a^{s+t},$$

whatever scalars may be denoted by s and t; and similarly when there are more than two factors of this form.

(13.) As verifications of the expression XVIII., considered as representing a *vector*, we may observe that it gives,

$$\text{XXIII.} \ldots \rho = -\mathrm{K}\rho; \quad \text{and} \quad \text{XXIV.} \ldots \rho^2 = -r^2.$$

(14.) More generally, it will be found that if u* be *any scalar*, we have the eminently simple transformation:

$$\text{XXV.} \ldots \rho^u = (rk^t j^s k j^{-s} k^{-t})^u = r^u k^t j^s k^u j^{-s} k^{-t}.$$

In fact, the two last expressions denote generally two *equal quaternions*, because they have, Ist, *equal tensors*, each $= r^u$; IInd, *equal angles*, each $= \angle(k^u)$; and IIIrd, *equal* (or coincident) *axes*, each formed from $\pm k$ by one *common system* of two *successive rotations*, one through $s\pi$ round j, and the other through $t\pi$ round k.

309. *Any quaternion*, q, which is not simply a *scalar*, may be brought to the *form* a^t, by a suitable choice of the *base*, a, and of the *exponent*, t; which latter may moreover be supposed to fall between the limits 0 and 2; since for this purpose we have only to write,

$$\text{I.} \ldots t = \frac{2\angle q}{\pi}; \qquad \text{II.} \ldots \mathrm{T}a = \mathrm{T}q^{\frac{1}{t}}; \qquad \text{III.} \ldots \mathrm{U}a = \mathrm{Ax}.\,q;$$

and thus *the general dependence of a Quaternion, on a Scalar and a Vector Element, presents itself in a new way* (comp. 17, 207, 292). When the proposed quaternion is a *versor*, $\mathrm{T}q = 1$, we have thus $\mathrm{T}a = 1$; or in other words, the *base* a, of the equivalent *power* a^t, is an *unit-line*. Conversely, *every versor* may be considered as a *power of an unit-line*, with a *scalar exponent*, t, which may be supposed to be in *general positive*, and *less than two*; so that we may write *generally*,

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* The employment of this letter u, to denote what we called, in the two preceding Sections, a *fourth unit*, &c., was stated to be a merely temporary one. In general, we shall henceforth simply *equate* that *scalar unit* to the *number one*; and *denote* it (when necessary to be denoted at all) by the *usual symbol*, 1, for that number.

$$\text{IV.} \ldots \mathrm{U}q = a^t, \quad \text{with} \quad \text{V.} \ldots a = \mathrm{Ax}.\,q = \mathrm{T}^{-1}1,$$

and

$$\text{VI.} \ldots t > 0, \quad t < 2;$$

although if this versor *degenerate* into 1 or -1, the *exponent* t becomes 0 or 2, and the *base* a has an indeterminate or *arbitrary direction*. And from such *transformations of versors* new methods may be deduced, for treating questions of *spherical trigonometry*, and generally of *spherical geometry*.

(1.) Conceive that P, Q, R, in Fig. 46, are replaced by A, B, C, with unit-vectors a, β, γ as usual; and let x, y, z be three scalars between 0 and 2, determined by the three equations,

$$\text{VII.} \ldots x\pi = 2\mathrm{A}, \quad y\pi = 2\mathrm{B}, \quad z\pi = 2\mathrm{C};$$

where A, B, C denote the angles of the spherical triangle. The three versors, indicated by the three arrows in the upper part of the Figure, come then to be thus denoted:

$$\text{VIII.} \ldots q = a^x; \quad q' = \beta^y; \quad q'q = \gamma^{2-z};$$

so that we have the equation,

$$\text{IX.} \ldots \beta^y a^x = \gamma^{2-z}; \quad \text{or} \quad \text{X.} \ldots \gamma^z \beta^y a^x = -1;$$

from which last, by easy divisions and multiplications, these two others immediately follow:

$$\text{X}'. \ldots a^x \gamma^z \beta^y = -1; \qquad \text{X}''. \ldots \beta^y a^x \gamma^z = -1;$$

the rotation round a from β to γ being again supposed to be negative.

(2.) In X. we may write (by 308, VIII.),

$$\text{XI.} \ldots a^x = \mathrm{c}a\mathrm{s}\mathrm{A}; \quad \beta^y = \mathrm{c}\beta\mathrm{s}\mathrm{B}; \quad \gamma^z = \mathrm{c}\gamma\mathrm{s}\mathrm{C};$$

and then the formula becomes, for *any spherical triangle*, in which the *order of rotation* is as above:

$$\text{XII.} \ldots \mathrm{c}\gamma\mathrm{s}\mathrm{C} \,.\, \mathrm{c}\beta\mathrm{s}\mathrm{B} \,.\, \mathrm{c}a\mathrm{s}\mathrm{A} = -1;$$

or (comp. IX.),

$$\text{XIII.} \ldots -\cos \mathrm{C} + \gamma \sin \mathrm{C} = (\cos \mathrm{B} + \beta \sin \mathrm{B})(\cos \mathrm{A} + a \sin \mathrm{A}).$$

(3.) Taking the scalars on both sides of this last equation, and remembering that $\mathrm{S}\beta a = -\cos c$, we thus immediately derive *one form* of the *fundamental equation of spherical trigonometry*; namely, the equation,

$$\text{XIV.} \ldots \cos \mathrm{C} + \cos \mathrm{A} \cos \mathrm{B} = \cos c \sin \mathrm{A} \sin \mathrm{B}.$$

(4.) Taking the vectors, we have this other formula:

$$\text{XV.} \ldots \gamma \sin \mathrm{C} = a \sin \mathrm{A} \cos \mathrm{B} + \beta \sin \mathrm{B} \cos \mathrm{A} + \mathrm{V}\beta a \sin \mathrm{A} \sin \mathrm{B};$$

which is easily seen to agree with 306, XII., and may also be usefully compared with the equation 210, XXXVII.

(5.) The result XV. may be eunnciated in the form of a *Theorem*, as follows:—

"*If there be any spherical triangle* ABC, *and three lines be drawn from the centre* O *of the sphere, one towards the point* A, *with a length* = sin A cos B; *another towards the point* B, *with a length* = sin B cos A; *and the third perpendicular to the plane* AOB, *and towards the same side of it as the point* C, *with a length* = sin *c* sin A sin B; *and if, with these three lines as edges, we construct a parallelepiped: the intermediate diagonal from* O *will be directed towards* C, *and will have a length* = sin C."

(6.) Dividing both members of the same equation XV. by ρ, and taking scalars, we find that if P be *any fourth point* on the sphere, and Q the *foot of the perpendicular* let fall from this point *on the arc* AB, this perpendicular PQ being considered as *positive* when C and P are situated *at one common side* of that arc (or in one *common hemisphere*, of the two into which the great circle through A and B divides the spheric surface), we have then,

XVI. . . $\sin \mathrm{C} \cos \mathrm{PC} = \sin \mathrm{A} \cos \mathrm{B} \cos \mathrm{PA} + \sin \mathrm{B} \cos \mathrm{A} \cos \mathrm{PB} + \sin \mathrm{A} \sin \mathrm{B} \sin c \sin \mathrm{PQ}$;

a formula which might have been derived from the equation 210, XXXVIII., by first cyclically changing *abc*ABC to *bca*BCA, and then passing from the former triangle to its *polar*, or *supplementary*: and from which many less general equations may be deduced, by assigning particular positions to P.

(7.) For example, if we conceive the point P to be the *centre of the circumscribed small circle* ABC, and denote by R the *arcual radius* of that circle, and by S the *semisum of the three angles*, so that $2\mathrm{S} = \mathrm{A} + \mathrm{B} + \mathrm{C} = \pi + \sigma$, if σ again denote, as in 297, (47.), the *area* of the triangle* ABC, whence

XVII. . . $\mathrm{PA} = \mathrm{PB} = \mathrm{PC} = R$, and $\sin \mathrm{PQ} = \sin R \sin (\mathrm{S} - \mathrm{C})$,

the formula XVI. gives easily,

XVIII. . . $2 \cot R \sin \frac{\sigma}{2} = \sin \mathrm{A} \sin \mathrm{B} \sin c$;

a relation between radius and area, which agrees with known results, and from which we may, by 297, LXX., &c., deduce the known equation:

XIX. . . $e \tan R = 4 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}$;

in which we have still, as in 297, (47.), &c.,

XX. . . $e = (\mathrm{S}\alpha\beta\gamma =) \sin a \sin b \sin \mathrm{C} = \&c.$

(8.) In like manner we might have supposed, in the corresponding general equation 210, XXXVIII., that P was placed at the centre of the *inscribed* small circle, and that the arcual radius of *that* circle was r, the semisum of the *sides* being s; and thus should have with ease deduced this other known relation, which is a sort of polar reciprocal of XVIII.,

XXI. . . $2 \tan r . \sin s = e$.

But these results are mentioned here, only to exemplify the fertility of the formulæ, to which the present calculus conducts, and from which the theorem in (5.) was early seen to be a consequence.

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* Compare the Note to the cited sub-article.

(9.) We might *develope the ternary product* in the equation XII., as we developed the *binary product* XIII.; compare scalar and vector parts; and operate on the latter, by the symbol $S.\rho^{-1}$. *New general theorems*, or at least new general *forms*, would thus arise, of which it may be sufficient in this place to have merely suggested the investigation.

(10.) As regards the *order of rotation* (1.) (2.), it is clear, from a mere *inspection* of the formula XV., that the rotation round γ from β to α, or that round C from B to A, *must be positive, when that equation* XV. *holds good;* at least if the *angles* A, B, C, of the *triangle* ABC, be (as usual) treated as *positive:* because the rotation round the *line* $V\beta\alpha$ from β to α is *always* positive (by 281, (3.)).

(11.) If, then, for any *given spherical triangle*, ABC, with *angles* still supposed to be *positive*, the *rotation* round C from B to A should happen to be (on the contrary) *negative*, we should be obliged to *modify the formula* XV.; which could be done, for example, so as to restore its correctness, by *interchanging α with β*, and at the same time A with B.

(12.) There is, however, a *sense* in which the *formula* might be considered as still *remaining true*, without any change in the mode of *writing* it; namely, if we were to *interpret the symbols* A, B, C as denoting *negative angles*, for the *case* last supposed (11.). Accordingly, if we take the *reciprocal* of the equation X., we get this other equation,

$$\text{XXII.} \ldots \alpha^{-x}\beta^{-y}\gamma^{-z} = -1;$$

where x, y, z are *positive*, as before, and therefore the *new exponents*, $-x$, $-y$, $-z$, are *negative*, if the rotation round α from β to γ be *itself* negative, as in (1.).

(13.) On the whole, then, if α, β, γ be *any given system of three co-initial and diplanar unit-lines*, OA, OB, OC, we can *always assign a system of three scalars*, x, y, z, which shall satisfy the *exponential equation* X., and shall have *relations of the form* VII. *to the spherical angles* A, B, C; but these three scalars, if determined so as to fall *between the limits* ± 2, will be *all positive*, or *all negative*, according as the *rotation* round α from β to γ is *negative*, as in (1.), or *positive*, as in (11.).

(14.) As regards the *limits* just mentioned, or the *inequalities*,

$$\text{XXIII.} \ldots x < 2, \quad y < 2, \quad z < 2; \qquad x > -2, \quad y > -2, \quad z > -2,$$

they are introduced with a view to render the problem of finding the *exponents* xyz in the formula X. *determinate;* for since we have, by 308,

$$\text{XXIV.} \ldots \alpha^4 = \beta^4 = \gamma^4 = +1, \quad \text{if} \quad T\alpha = T\beta = T\gamma = 1,$$

we might otherwise *add any multiple* (positive or negative) *of the number four*, to the value of the exponent of *any unit-line*, and the value of the resulting *power* would not be altered.

(15.) If we admitted exponents $= \pm 2$, we might render the problem of satisfying the equation X. *indeterminate* in another way; for it would then be sufficient to suppose that *any one* of the three exponents was thus equal to $+2$, or -2, and that the *two others* were each $= 0$; or else that *all three* were of the form ± 2.

(16.) When it was lately said (13.), that the *exponents*, x, y, z, in the formula X., if *limited* as above, would have one *common sign*, the *case* was tacitly excluded, for which those exponents, or *some* of them, when multiplied each by a quadrant, give angles *not* equal to those of the spherical *triangle* ABC, whether positively *or* negatively taken; *but* equal to the *supplements* of those angles, or to the *negatives* of those supplements.

(17.) In fact, it is evident (because $\alpha^2 = \beta^2 = \gamma^2 = -1$), that the equation X., or the reciprocal equation XXII., if it be satisfied by any *one system* of values of xyz, will *still* be satisfied, when we divide or multiply *any two* of the three exponential *factors*, by the *squares* of the two *unit-vectors*, of which those factors are supposed to be *powers:* or in other words, if we *subtract* or *add* the *number two*, in *each of two exponents*.

(18.) We may, for example, derive from XXII. this other equation:

$$\text{XXV.} \ldots \alpha^{2-x}\beta^{2-y}\gamma^{-z} = -1; \quad \text{or} \quad \text{XXVI.} \ldots \alpha^{2-x}\beta^{2-y} = \gamma^{z-2};$$

which, when the rotation is as supposed in (1.), so that xyz are *positive*, may be interpreted as follows.

(19.) Conceive a *lune* CC′, with points A and B on its two bounding semicircles, and with a negative rotation round A from B to C; or, what comes to the same thing, with a positive rotation round A from B to C′. Then, on the plan illustrated by Figures 45 and 46, the *supplements* $\pi - \text{A}$, $\pi - \text{B}$, of the angles A and B in the triangle ABC, or the angles at the *same points* A and B in the *co-lunar* triangle ABC′, will *represent two versors*, a *multiplier*, and a *multiplicand*, which are precisely those denoted, in XXVI., by the *two factors*, α^{2-x} and β^{2-y}; and the *product* of these two factors, taken in *this order*, is that *third versor*, which has its *axis* directed to C′, and is *represented*, on the same general plan (177), by the *external angle of the lune, at that point* C′; which, in *quantity*, is equal to the external angle of the same lune at C, or to the angle $\pi - \text{C}$. This *product* is therefore equal to that *power of the unit-line* OC′, or $-\gamma$, which has its *exponent* $= \frac{2}{\pi}(\pi - \text{C}) = 2 - z$; we have therefore, by this *construction*, the equation,

$$\text{XXVII.} \ldots \alpha^{2-x}\beta^{2-y} = (-\gamma)^{2-z};$$

which (by 308, (6.)) agrees with the recent formula XXVI.

310. The equation,

$$\text{I.} \ldots \gamma^{\frac{2\text{C}}{\pi}}\,\beta^{\frac{2\text{B}}{\pi}}\,\alpha^{\frac{2\text{A}}{\pi}} = -1,$$

which results from 309, (1.), and in which α, β, γ are the unit-vectors OA, OB, OC of any three points on the unit-sphere; while the three scalars A, B, C, in the exponents of the three factors, represent generally the angular quantities of rotation, round those three unit-lines, or radii, α, β, γ, from the plane AOC to the plane AOB, from BOA to BOC, and from COB to COA, and are positive or negative according as these rotations of planes are themselves positive or negative: must be regarded as an important formula, in the applications of the present Calculus. It *includes*, for example, the whole doctrine of *Spherical Triangles;* not merely because it conducts, as we have seen (309, (3.)), to *one form of the fundamental scalar equation of spherical trigonometry*, namely to the equation,

$$\text{II.} \ldots \cos \text{C} + \cos \text{A} \cos \text{B} = \cos c \sin \text{A} \sin \text{B};$$

but also because it gives a *vector equation* (309, (4.)), which serves to *connect* the *angles*, or the *rotations*, A, B, C, with the *directions** *of the radii*, α, β, γ, or OA, OB, OC, for any system of *three diverging right lines* from one origin. It may, therefore, be not improper to make here a few additional remarks, respecting the nature, evidence and extension of the recent formula I.

(1.) Multiplying both members of the equation I., by the inverse exponential $\gamma^{-\frac{2C}{\pi}}$, we have the transformation (comp. 309, (1.)):

$$\text{III.} \ldots \beta^{\frac{2B}{\pi}} \alpha^{\frac{2A}{\pi}} = -\gamma^{-\frac{2C}{\pi}} = \gamma^{\frac{2(\pi - C)}{\pi}}.$$

(2.) Again, multiplying both members of I. *into*† $\alpha^{-\frac{2A}{\pi}}$, we obtain this other formula:

$$\text{IV.} \ldots \gamma^{\frac{2C}{\pi}} \beta^{\frac{2B}{\pi}} = -\alpha^{-\frac{2A}{\pi}} = \alpha^{\frac{2(\pi - A)}{\pi}}.$$

(3.) Multiplying this last equation IV. *by* $\alpha^{\frac{2A}{\pi}}$, and the equation III. *into* $\gamma^{\frac{2C}{\pi}}$, we derive these other forms:

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* This may be considered to be another instance of that habitual reference to *direction*, as *distinguished* from mere *quantity* (or magnitude), although *combined* therewith, which pervades the present Calculus, and is eminently *characteristic* of it; whereas *Des Cartes*, on the contrary, had aimed to reduce all problems of geometry to the determination of the *lengths of right lines*: although (as all who use his *co-ordinates* are of course well aware) a certain reference to *direction* is even in *his* theory inevitable, in connexion with the interpretation of *negative roots* (by him called *inverse* or *false* roots) of equations. Thus in the first sentence of Schooten's recently cited translation (1659) of the *Geometry* of Des Cartes, we find it said:

> "Omian Geometriæ Problemata facilè ad hujusmodi terminos reduci possunt, ut deinde ad illorum constructionem, opus tantum sit rectarum quarundam longitudinem cognoscere."

The very different *view of geometry*, to which the present writer has been led, makes it the more proper to express here the profound admiration with which he regards the cited Treatise of Des Cartes: containing as it does the germs of so large a portion of all that has since been done in mathematical science, even as concerns *imaginary roots* of equations, considered as marks of *geometrical impossibility*.

† For the distinction between multiplying a quaternion *into* and *by* a factor, see the Notes to pages 146, 159.

$$\text{V.} \ldots \alpha^{\frac{2\text{A}}{\pi}} \gamma^{\frac{2\text{C}}{\pi}} \beta^{\frac{2\text{B}}{\pi}} = -1; \qquad \text{VI.} \ldots \beta^{\frac{2\text{B}}{\pi}} \alpha^{\frac{2\text{A}}{\pi}} \gamma^{\frac{2\text{C}}{\pi}} = -1;$$

so that *cyclical permutation of the letters*, α, β, γ, and A, B, C, *is allowed in the equation* I.; as indeed was to be expected, from the nature of the theorem which that equation expresses.

(4.) From either V. or VI. we can deduce the formula:

$$\text{VII.} \ldots \alpha^{\frac{2\text{A}}{\pi}} \gamma^{\frac{2\text{C}}{\pi}} = -\beta^{-\frac{2\text{B}}{\pi}} = \beta^{\frac{2(\pi-\text{B})}{\pi}};$$

by comparing which with III. and IV., we see that *cyclical permutation* of letters is permitted, in *these* equations also.

(5.) Taking the *reciprocal* (or *conjugate*) of the equation I., we obtain (compare 309, XXII.) this other equation:

$$\text{VIII.} \ldots \alpha^{-\frac{2\text{A}}{\pi}} \beta^{-\frac{2\text{B}}{\pi}} \gamma^{-\frac{2\text{C}}{\pi}} = -1;$$

or

$$\text{IX.} \ldots \alpha^{\frac{2(\pi-\text{A})}{\pi}} \beta^{\frac{2(\pi-\text{B})}{\pi}} \gamma^{\frac{2(\pi-\text{C})}{\pi}} = +1;$$

in which cyclical permutation of letters is again allowed, and from which (or from III.) we can at once derive the formula,

$$\text{X.} \ldots \alpha^{-\frac{2\text{A}}{\pi}} \beta^{-\frac{2\text{B}}{\pi}} = -\gamma^{\frac{2\text{C}}{\pi}}.$$

(6.) The equation X. may also be thus written (comp. 309, XXVII.):

$$\text{XI.} \ldots \alpha^{\frac{2(\pi-\text{A})}{\pi}} {}^{\frac{2(\pi-\text{B})}{\pi}} = \gamma^{-\frac{2(\pi-\text{C})}{\pi}} = (-\gamma)^{\frac{2(\pi-\text{C})}{\pi}}.$$

(7.) And all the foregoing equations may be *interpreted* (comp. 309, (19.)), and at the same time *proved*, by a reference to that general *construction* (177) for the *multiplication of versors*, which the Figures 45 and 46 were designed to illustrate; if we bear in mind that a *power* a^t, of an *unit-line* a, with a *scalar exponent*, t, is (by 308, 309) a *versor*, which has the *effect* of *turning a line* $\perp$ *a, through t right angles, round a* as an *axis of rotation*.

(8.) The principle expressed by the equation I., from which all the subsequent equations have been deduced, may be stated in the following manner, if we adopt the *definition* proposed in an earlier part of this work (180, (4.)), for the *spherical sum* of two angles on a spheric surface:

"*For any spherical triangle, the Spherical Sum of the three angles, if taken in a suitable Order, is equal to Two Right Angles.*"

(9.) In fact, when the rotation round A from B to C is negative, if we *spherically add* the angle B to the angle A, the *spherical sum* so obtained is (by the definition referred to) equal to the *external angle at* C; if then we *add* to *this sum*, or *supplement* of C, the angle C *itself*, we get a *final* or *total sum*, which is exactly equal to π; *addition of spherical angles* at *one vertex*, and therefore in *one plane*, being accomplished in the *usual* manner; but the *spherical summation* of angles with *different vertices* being performed according to those *new rules*, which were deduced in the Ninth Section of Book II., Chapter I.; and were connected (180, (5.)) with the conception of *angular transvection*, or of the *composition of angular motions, in different and successive planes*.

(10.) Without pretending to attach importance to the following *notation*, we may just propose it in passing, as one which may serve to recall and represent the *conception* here referred to. Using a *plus in parentheses*, as a *symbol* or *characteristic* of such *spherical addition of angles*, the formula I. may be *abridged* as follows:

$$\text{XII.} \ldots \text{C}(+)\text{B}(+)\text{A} = \pi\,;$$

the *symbol of an added angle* being written to the *left* of the symbol of the *angle to which it is added* (comp. 264, (4.)); because *such addition corresponds* (as above) to a *multiplication of versors*, and we have agreed to write the *symbol of the multiplier* to the *left** of the symbol of the *multiplicand*, in every *multiplication of quaternions.*

311. There is, however, *another view* of the important equation 310, I., according to which it is connected rather with *addition of arcs* (180, (3.)), than with *addition of angles* (180, (4.)); and may be *interpreted*, and *proved anew*, with the help of the *supplementary* or *polar triangle*, A′B′C′, as follows.

(1.) The rotation round A from B to C being still supposed to be negative, let A′, B′, C′ be (as in 175) the positive poles of the sides BC, CA, AB; and let α', β', γ' be their unit-vectors. Then, because the rotation round α from γ' to β' is positive (by 180, (2.)), and is in quantity the supplement of the spherical angle A, the *product* $\gamma'\beta'$ will be (by 281, (2.), (3.)) a *versor*, of which α is the *axis*, and A the *angle;* with similar results for the two other products, $\alpha'\gamma'$, $\beta'\alpha'$.

(2.) If then we write (comp. 291),

$$\text{I.} \ldots \alpha' = \text{UV}\beta\gamma, \quad \beta' = \text{UV}\gamma\alpha, \quad \gamma' = \text{UV}\alpha\beta,$$

supposing that

$$\text{II.} \ldots \text{T}\alpha = \text{T}\beta = \text{T}\gamma = 1, \quad \text{and} \quad \text{III.} \ldots \text{S}\alpha\beta\gamma > 0,$$

we shall have (comp. again 180, (2.)),

$$\text{IV.} \ldots \alpha = \text{UV}\gamma'\beta', \quad \beta = \text{UV}\alpha'\gamma', \quad \gamma = \text{UV}\beta'\alpha',$$

and

$$\text{V.} \ldots \text{A} = \angle\,\gamma'\beta', \quad \text{B} = \angle\,\alpha'\gamma', \quad \text{C} = \angle\,\beta'\alpha'\,;$$

whence (by 308 or 309) we have the following *exponential expressions* for these three last *products of unit-lines*,

$$\text{VI.} \ldots \gamma'\beta' = \alpha^{\frac{2\text{A}}{\pi}}; \quad \alpha'\gamma' = \beta^{\frac{2\text{B}}{\pi}}; \quad \beta'\alpha' = \gamma^{\frac{2\text{C}}{\pi}}.$$

(3.) Multiplying these three expressions, in an inverted order, we have, therefore, the new product:

$$\text{VII.} \ldots \gamma^{\frac{2\text{C}}{\pi}}\,\beta^{\frac{2\text{B}}{\pi}}\,\alpha^{\frac{2\text{A}}{\pi}} = \beta'\alpha' \,.\, \alpha'\gamma' \,.\, \gamma'\beta' = \gamma'^2\beta'^2\alpha'^2 = -1\,;$$

and the equation 310, I. is in this way *proved anew.*

(4.) And because, instead of VI., we might have written,

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* Compare the Note to page 146.

$$\text{VIII.} \ldots \alpha^{\frac{2\text{A}}{\pi}} = -\frac{\gamma'}{\beta'}; \quad \beta^{\frac{2\text{B}}{\pi}} = -\frac{\alpha'}{\gamma'}; \quad \gamma^{\frac{2\text{C}}{\pi}} = -\frac{\beta'}{\alpha'},$$

we see that the *equation* to be proved may be reduced to the form of the *identity*

$$\text{IX.} \ldots \frac{\beta'}{\alpha'}\frac{\alpha'}{\gamma'}\frac{\gamma'}{\beta'} = +1;$$

and may be *interpreted* as expressing, what is evident, that if a point be supposed to move first along the side B′C′, of the polar triangle A′B′C′, from B′ to C′; then along the successive side C′A′, from C′ to A′; and finally along the remaining side A′B′, from A′ to B′, it will thus have *returned* to the position from which it *set out*, or will *on the whole* have *not changed place* at all.

(5.) In *this view*, then, we perform what we have elsewhere called an *addition of arcs* (instead of *angles* as in 310); and in a *notation* already used (264, (4.)), we may express the result by the formula,

$$\text{X.} \ldots \frown \text{A'B'} + \frown \text{C'A'} + \frown \text{B'C'} = 0;$$

each of the the two *left-hand symbols* denoting an *arc*, which is conceived to be *added* (as a *successive vector-arc*, 180, (3.)), to the arc whose symbol immediately *follows* it, or is written *next* it, but towards the *right-hand*.

(6.) The expressions VI. or VIII., for the *exponential factors* in 310, I., show in a new way the necessity of attending to the *order* of those factors, in that formula: for if we should *invert that order, without altering* (as in 310, VIII.) *the exponents*, we may now see that we should obtain this *new product*:

$$\text{XI.} \ldots \alpha^{\frac{2\text{A}}{\pi}} \beta^{\frac{2\text{B}}{\pi}} \gamma^{\frac{2\text{C}}{\pi}} = -\frac{\gamma'}{\beta'}\frac{\alpha'}{\gamma'}\frac{\beta'}{\alpha'} = +(\gamma'\beta'\alpha')^2;$$

which, on account of the *diplanarity* of the lines α', β', γ', is *not equal to negative unity*, but to a certain *other versor*; the properties of which may be inferred from what was shown in 297, (64.), and in 298, (8.), but upon which we cannot here delay.

312. In general (comp. 221), *an equation*, such as

$$\text{I.} \ldots q' = q,$$

between two quaternions, includes a system of four scalar equations*, such as the following:

$$\text{II.} \ldots \text{S}q' = \text{S}q; \quad \text{S}\alpha q' = \text{S}\alpha q; \quad \text{S}\beta q' = \text{S}\beta q; \quad \text{S}\gamma q' = \text{S}\gamma q;$$

where α, β, γ may be *any three actual and diplanar vectors*: and conversely, if α, β, γ be any three *such* vectors, then the *four* scalar equations II. reproduce, and are sufficiently replaced by, the *one* quaternion equation I. But an *equation between two vectors* is equivalent only to a system of *three scalar equations*, such as the *three last* equations II.; for example, in 294, (12.), the *one vector equation* XXII. is equivalent to the *three scalar equations* XXI., under the immediately preceding *condition of diplanarity* XX. In like manner, an *equation between two versors of quaternions*,* such as the equation

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* The *propriety*, which such results as this establish, for the use of the *name*, QUATERNIONS, as applied to this whole Calculus, on account of its essential connexion with the *number* FOUR, does not require to be again insisted on.

$$\text{III.} \ldots \mathrm{U}q' = \mathrm{U}q,$$

includes generally a system of *three*, but of *not more than three*, *scalar* equations; because the *versor* Uq *depends* generally (comp. 157) on a *system of three scalars*, namely the *two* which determine its *axis* Ax. q, and the *one* which determines its *angle* $\angle\, q$; or because the *versor equation* III. requires to be *combined* with the *tensor equation*,

$$\text{IV.} \ldots \mathrm{T}q' = \mathrm{T}q, \qquad \text{compare 187 (13.)},$$

in order to reproduce the *quaternion equation* I. Now the recent equation, 310, I., is evidently of this *versor-form* III., if α, β, γ be still supposed to be *unit-lines*. If then we *met that equation*, or if one of its *form* had occurred to us, without any knowledge of its *geometrical signification*, we might propose to *resolve it*, with respect to the *three scalars* A, B, C, treated as *three unknown quantities*. The few following remarks, on the problem thus proposed, may be not out of place, nor uninstructive, here.

(1.) Writing for abridgment,

$$\text{V.} \ldots \cot \mathrm{A} = t, \quad \cot \mathrm{B} = u, \quad \cot \mathrm{C} = v,$$

and

$$\text{VI.} \ldots s = -\operatorname{cosec} \mathrm{A} \operatorname{cosec} \mathrm{B} \operatorname{cosec} \mathrm{C},$$

the equation to be resolved becomes (by 308, VII., or 309, XII.),

$$\text{VII.} \ldots (v + \gamma)(u + \beta)(t + \alpha) = s;$$

in which the *tensors* on both sides are already equal, because

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* An equation, $\mathrm{U}\rho' = \mathrm{U}\rho$, or $\mathrm{UV}q' = \mathrm{UV}q$, between two *versors of vectors* (156), or between the *axes* of two quaternions (291), is equivalent only to a system of *two scalar equations*; because the *direction of an axis*, or of a *vector*, depends on a system of *two angular elements* (111).

$$\text{VIII.} \ldots s^2 = (v^2+1)(u^2+1)(t^2+1).$$

(2.) Multiplying the equation VII. *by* $t + \alpha$, and *into* $t - \alpha$, and dividing the result by $t^2 + 1$, we have this new equation of the same *form*, but differing by *cyclical permutation* (comp. 310, (3.)):

$$\text{IX.} \ldots (t+\alpha)(v+\gamma)(u+\beta) = s;$$

and in like manner,

$$\text{X.} \ldots (u+\beta)(t+\alpha)(v+\gamma) = s.$$

(3.) Taking the half difference of the two last equations, and observing that (by 279, IV., and 294, II.)

$$\text{XI.} \ldots \begin{cases} \tfrac{1}{2}(\beta\alpha\gamma - \alpha\gamma\beta) = \mathrm{V}.\beta\mathrm{V}\alpha\gamma = \gamma\mathrm{S}\alpha\beta - \alpha\mathrm{S}\beta\gamma, \\ \tfrac{1}{2}(\beta\alpha - \alpha\beta) = \mathrm{V}\beta\alpha, \quad \tfrac{1}{2}(\beta\gamma - \gamma\beta) = \mathrm{V}\beta\gamma, \end{cases}$$

we arrive at this new equation, of *vector form* :

$$\text{XII.} \ldots 0 = v\mathrm{V}\beta\alpha + t\mathrm{V}\beta\gamma + \gamma\mathrm{S}\alpha\beta - \alpha\mathrm{S}\beta\gamma;$$

which is equivalent only to a system of *two scalar equations*, because it gives $0 = 0$, when operated on by $\mathrm{S}.\beta$ (comp. 294, (9.)).

(4.) It enables us, however, to *determine the two scalars*, t and v; for if we operate on it by $\mathrm{S}.\alpha$, we get (comp. 298, XXVI.),

$$\text{XIII.} \ldots t\mathrm{S}\alpha\beta\gamma = \alpha^2\mathrm{S}\beta\gamma - \mathrm{S}\beta\alpha\,\mathrm{S}\alpha\gamma = \mathrm{S}(\mathrm{V}\beta\alpha.\mathrm{V}\alpha\gamma);$$

and if we operate on the same equation XII. by $\mathrm{S}.\gamma$, we get in like manner,

$$\text{XIV.} \ldots v\mathrm{S}\alpha\beta\gamma = \gamma^2\mathrm{S}\alpha\beta - \mathrm{S}\alpha\gamma\,\mathrm{S}\gamma\beta = \mathrm{S}(\mathrm{V}\alpha\gamma.\mathrm{V}\gamma\beta).$$

(5.) Processes quite similar give the analogous result,

$$\text{XV.} \ldots u\mathrm{S}\alpha\beta\gamma = \beta^2\mathrm{S}\gamma\alpha - \mathrm{S}\gamma\beta\,\mathrm{S}\beta\alpha = \mathrm{S}(\mathrm{V}\gamma\beta.\mathrm{V}\beta\alpha):$$

and thus the *problem* is *resolved*, in the sense that *expressions have been found* for the *three sought scalars* t, u, v, or for the *cotangents* V. of the *three sought angles* A, B, C : whence the *fourth scalar*, s, in the quaternion equation VII., can easily be deduced, as follows.

(6.) Since (by 294, (6.), changing δ to α, and afterwards cyclically permuting) we have, for *any three vectors* α, β, γ, the general transformations,

$$\text{XVI.} \ldots \alpha\mathrm{S}\alpha\beta\gamma = \mathrm{V}(\mathrm{V}\beta\alpha.\mathrm{V}\alpha\gamma), \quad \beta\mathrm{S}\alpha\beta\gamma = \mathrm{V}(\mathrm{V}\gamma\beta.\mathrm{V}\beta\alpha),$$
$$\gamma\mathrm{S}\alpha\beta\gamma = \mathrm{V}(\alpha\gamma.\mathrm{V}\gamma\beta),$$

the expressions XIII. XV. XIV. give,

$$\text{XVII.} \ldots \begin{cases} (t+\alpha)\mathrm{S}\alpha\beta\gamma = \mathrm{V}\beta\alpha.\mathrm{V}\alpha\gamma; \\ (u+\beta)\mathrm{S}\alpha\beta\gamma = \mathrm{V}\gamma\beta.\mathrm{V}\beta\alpha; \\ (v+\gamma)\mathrm{S}\alpha\beta\gamma = \mathrm{V}\alpha\gamma.\mathrm{V}\gamma\beta; \end{cases}$$

whence, by VII ,

$$\text{XVIII.} \ldots s(\mathrm{S}\alpha\beta\gamma)^3 = (\mathrm{V}\gamma\beta)^2(\mathrm{V}\beta\alpha)^2(\mathrm{V}\alpha\gamma)^2;$$

and thus the remaining scalar, s, is also entirely determined.

(7.) And the equation VIII. may be verified, by observing that the expressions XVII. give,

$$\text{XIX.} \ldots \begin{cases} (t^2+1)(\mathrm{S}\alpha\beta\gamma)^2 = (\mathrm{V}\beta\alpha)^2(\mathrm{V}\alpha\gamma)^2; \\ (u^2+1)(\mathrm{S}\alpha\beta\gamma)^2 = (\mathrm{V}\gamma\beta)^2(\mathrm{V}\beta\alpha)^2; \\ (v^2+1)(\mathrm{S}\alpha\beta\gamma)^2 = (\mathrm{V}\alpha\gamma)^2(\mathrm{V}\gamma\beta)^2. \end{cases}$$

(8.) The equations XIII. XIV. XV. XVI. give, by elimination of $\mathrm{S}\alpha\beta\gamma$, these new expressions:

$$\text{XX.} \ldots \alpha t^{-1} = (\mathrm{V : S})\,(\mathrm{V}\beta\alpha\,.\,\mathrm{V}\alpha\gamma); \quad \beta u^{-1} = (\mathrm{V : S})\,(\mathrm{V}\gamma\beta\,.\,\mathrm{V}\beta\alpha);$$
$$\gamma v^{-1} = (\mathrm{V : S})\,(\mathrm{V}\alpha\gamma\,.\,\mathrm{V}\gamma\beta);$$

by comparing which with the formula 281, XXVIII., after suppressing (291) the characteristic I, we find that the *three scalars*, t, u, v, are either Ist, the *cotangents of the angles opposite to the sides* a, b, c, *of the spherical triangle in which the three given unit-lines* α, β, γ *terminate*, or IInd, the *negatives of those cotangents*, the *angles themselves* of that triangle being as usual supposed to be *positive* (309, (10.)), according as the *rotation* round α from β to γ is *negative* or *positive*: that is (294, (8.)), according as $\mathrm{S}\alpha\beta\gamma >$ or < 0; or finally, by XVIII., according as the *fourth scalar*, s, is negative or positive, because the second member of that equation XVIII. is *always negative*, as being the product of three *squares of vectors* (282, 292).

(9.) In the Ist case, which is that of 309, (1.), we see then *anew*, by V. and VI., that we are *permitted to interpret the scalars* A, B, C, *in the exponential formula* 310, I., *as equal to the angles of the spherical triangle* (8.), which are usually denoted by the *same letters*. But we see also, that we may *add any even multiples of* π *to those three angles*, *without disturbing the exponential equation*; or any *one even*, and *two odd* multiples of π, in any order, so as to preserve a *positive product of cosecants*, because s is, for *this* case, *negative* in VI., by (8.).

(10.) In the IInd case, which is that of 309, (11.), we may, for similar reasons, *interpret the scalars* A, B, C, in the formula 310, I., as equal to the *negatives of the angles of the triangle*; and as thus having, what VI. *now* requires, because s is now *positive* (8.), a *negative product of cosecants*, while their *cotangents* have the values required. But we may also *add*, as in (9.), *any multiples of* π, to the scalars thus found for the formula, provided that the *number* of the *odd* multiples, so added, is itself *even* (0 or 2).

(11.) The conclusions of 309, or 310, respecting the *interpretation of the exponential formula*, are therefere confirmed, and might have been anticipated, by the present *new analysis*: in conducting which it is evident that we have been dealing with *real scalars*, and with *real vectors*, only.

(12.) If this last *restriction* were *removed*, and *imaginary values admitted*, in the solution of the *quaternion equation* VII., we might have begun by operating, as in II., on that equation, by the *four characteristics*,

$$\text{XXI.} \ldots \mathrm{S}, \quad \mathrm{S}\,.\,\alpha, \quad \mathrm{S}\,.\,\beta, \quad \text{and} \quad \mathrm{S}\,.\,\gamma;$$

which would have given, with the significations 297, (1.), (3.), of l, m, n, and e, and therefore with the following *relation* between those *four scalar data*,

$$\text{XXII.} \ldots e^2 = 1 - l^2 - m^2 - n^2 + 2lmn,$$

a system of *four scalar equations*, involving the *four sought scalars*, s, t, u, v; from which it might have been required to deduce the (real or imaginary) values of those four scalars, by the ordinary processes of *algebra*.

(13.) The four scalar equations, so obtained, are the following:

$$\text{XXIII.} \ldots \begin{cases} 0 = e + lt + mu + nv - tuv + s; \\ 0 = et + mtu + ntv + uv - l; \\ 0 = -eu + ltu + tv + nuv + m - 2ln; \\ 0 = ev + tu + ltv + muv - n; \end{cases}$$

eliminating uv and u between the three last of which, we find, with the help of XXII., the determinant,

$$\text{XXIV.}\ldots 0 = \begin{vmatrix} 1, & mt, & ntv + et - l \\ m, & t, & ltv + ev - n \\ n, & lt - e, & tv + m - 2ln \end{vmatrix} = e(t^2 + 1)(ev - n + lm);$$

and analogous eliminations give,

$$\text{XXV.}\ldots 0 = e(t^2 + 1)(eu - m + nl),$$

and $$\text{XXVI.}\ldots 0 = (t^2 + 1)\{e^2uv - (m - nl)(n - lm) + (1 - l^2)(et - l + mn)\}.$$

(14.) Rejecting then the factor $t^2 + 1$ we find, as the *only real solution* of the problem (12.), the following system of values:

$$\text{XXVII.}\ldots et = l - mn; \quad eu = m - nl; \quad ev = n - lm;$$

and $$\text{XXVIII.}\ldots e^3s = -(1 - l^2)(1 - m^2)(1 - n^2);$$

which correspond precisely to those otherwise found before, in (4.) (5.) (6.), and might therefore serve to *reproduce the interpretation* of the *exponential formula* (310).

(15.) But on the purely *algebraic side*, it is found, by a similar analysis, that the *four equations* XXIII. are *satisfied also* by a system of *four imaginary solutions*, represented by the following formulæ:

$$\text{XXIX.}\ldots \begin{cases} t^2 + 1 = 0; \quad v^2 + 1 = 0; \\ s = tuv - lt - mu - nv - e = 0; \end{cases}$$

which it may be sufficient to have mentioned in passing, since they do not appear to have any such *geometrical* interest, as to deserve to be dwelt on here: though, as regards the *consistency* of the different processes employed, it may be remembered that in passing (2.) from the equation VII. to IX., after certain preliminary multiplications, we *divided by* $t^2 + 1$, as we were entitled to do, when seeking only for *real* solutions, because t was supposed to be a *scalar*.

(16.) This seems to be a natural occasion for remarking that the following *general transformation* exists, *whatever three vectors* may be denoted by α, β, γ:

$$\text{XXX.}\ldots \text{S}(\text{V}\beta\gamma.\text{V}\gamma\alpha.\text{V}\alpha\beta) = -(\text{S}\alpha\beta\gamma)^2;$$

which proves in a new way (comp. 180), that *the rotation round the line* $\text{V}\beta\gamma$, *from* $\text{V}\gamma\alpha$ *to* $\text{V}\alpha\beta$, *is always positive;* or is directed in the *same sense* (281, (3.)), as the rotation round $\text{V}\alpha\beta$ from α to β, &c.

(17.) In like manner we have generally,

$$\text{XXXI.}\ldots \text{S}(\text{V}\alpha\beta.\text{V}\gamma\alpha.\text{V}\beta\gamma) = +(\text{S}\alpha\beta\gamma)^2,$$

and $$\text{XXXII.}\ldots \text{S}(\text{V}\gamma\beta.\text{V}\alpha\gamma.\text{V}\beta\alpha) = +(\text{S}\alpha\beta\gamma)^2;$$

so that *the rotation round* $\text{V}\gamma\beta$ *from* $\text{V}\alpha\gamma$ *to* $\text{V}\beta\alpha$ *is negative*, whatever arrangement the three diplanar vectors α, β, γ may have among themselves.

(18.) If then A″, B″, C″ be the *negative poles* of the *three successive sides*, BC, CA, AB, of *any spherical triangle*, the *rotation round* A″ *from* B″ *to* C″ *is negative:* which is entirely consistent with the *opposite result* (180), respecting the system of the three *positive poles* A′, B′, C′.

(19.) A *quantitative interpretation* of the equation XXX. may also be easily assigned: for we may infer from it (by 281, (4.), and 294, (3.)) that *if* OABC *be any pyramid, and if normals* OA′, OB′, OC′ *to the three faces* BOC, COA, AOB *have their lengths numerically equal to the areas of those faces* (as bearing the same ratios to *units*, &c,), then (with a similar reference to units) *the volume of the new pyramid*, OA'B'C', *will be three quarters of the square of the volume of the old pyramid*, OABC.

313. But an allusion was made, in 310, to an *extension* of the *exponential formula* which has lately been under discussion; and in fact, that formula admits of being easily extended, from *triangles* to *polygons* upon the sphere: for we may write, generally,

$$\text{I.} \ldots a_n^{\frac{2A_n}{\pi}} a_{n-1}^{\frac{2A_{n-1}}{\pi}} \ldots a_2^{\frac{2A_2}{\pi}} a_1^{\frac{2A_1}{\pi}} = (-1)^n,$$

if $A_1A_2 \ldots A_{n-1} A_n$ be *any spherical polygon*, and if the scalars $A_1, A_2, \ldots$ in the exponents denote the positive or negative *angles* of that polygon, considered as the *rotations* $A_nA_1A_2$, $A_1A_2A_3, \ldots$ namely those from A_1A_n to A_1A_2, &c.; while n is any positive whole number* > 2.

(1.) One mode of proving this extended formula is the following. Let $OC = \gamma$ be the unit-vector of an arbitrary point C on the spheric surface; and conceive that arcs of great circles are drawn from this point C to the n successive corners of the polygon. We shall thus have a system of n spherical triangles, and each angle of the polygon will (generally) be decomposed into two (positive or negative) *partial angles*, which may be thus denoted:

$$\text{II.} \ldots CA_1A_2 = A_1', \quad CA_2A_3 = A_2', \ldots;$$
$$\text{III.} \ldots A_nA_1C = A_1'', \quad A_1A_2C = A_2'', \ldots;$$

so that, with attention to signs of angles in the additions,

$$\text{IV.} \ldots A_1 = A_1' + A_1'', \quad A_2 = A_2' + A_2'', \text{ \&c.}$$

Also let

$$\text{V.} \ldots A_2CA_1 = C_1, \quad A_3CA_2 = C_2, \text{ \&c.};$$

and therefore

$$\text{VI.} \ldots C_1 + C_2 + \ldots + C_n = \text{an even multiple of } \pi,$$

which reduces itself to 2π in the simple case of a polygon with no re-entrant angles, and with the point C in its interior.

(2.) Then, for the triangle CA_1A_2, of which the angles are C_1, A_1', A_2'', we have, by 310, III., the equation,

$$\text{VII.} \ldots a_2^{\frac{2A_2''}{\pi}} a_1^{\frac{2A_1'}{\pi}} = -\gamma^{-\frac{2C_1}{\pi}};$$

and in like manner, for the triangle CA_2A_3, we have

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* The formula admits of interpretation, even for the case $n = 2$.

$$\text{VIII.}\ldots \alpha_3^{\frac{2\mathrm{A}_3}{\pi}}\,\alpha_2^{\frac{2\mathrm{A}_2'}{\pi}} = -\gamma^{-\frac{2\mathrm{C}_2}{\pi}}, \text{ \&c.}$$

But, when we multiply VII. by VIII., we obtain, by IV., the product,

$$\text{IX.}\ldots \alpha_3^{\frac{2\mathrm{A}_3''}{\pi}}\,\alpha_2^{\frac{2\mathrm{A}_2}{\pi}}\,\alpha_1^{\frac{2\mathrm{A}_1'}{\pi}} = +\gamma^{-\frac{2(\mathrm{C}_1+\mathrm{C}_2)}{\pi}};$$

and so proceeding, we have at last, by VI., a product of the form,

$$\text{X.}\ldots \alpha_1^{\frac{2\mathrm{A}_1''}{\pi}}\,\alpha_n^{\frac{2\mathrm{A}_n}{\pi}}\ldots\alpha_2^{\frac{2\mathrm{A}_2}{\pi}}\,\alpha_1^{\frac{2\mathrm{A}_1'}{\pi}} = (-1)^n;$$

which reduces itself to I., when it is multiplied *by* $\alpha^{-\frac{2\mathrm{A}_1''}{\pi}}$, and *into* $\alpha^{\frac{2\mathrm{A}_1''}{\pi}}$ (comp. 310, (3.)). The theorem is therefore proved.

(3.) In words (comp. 310, (8.)), "*the spherical sum of the successive angles of any spherical polygon, if taken in a suitable order, is equal to a multiple of two right angles, which is odd or even, according as the number of the sides* (or corners) of the polygon *is itself odd or even*": the *definition* formerly given (180, (4.)), of a *Spherical Sum of Angles*, being of course retained. And the *reasoning* may be briefly stated thus. When an arbitrary point C is taken on the spherical surface, as in (1.), the *spherical sum* of the *two partial angles*, at the ends of *any one side*, is the *supplement* of the angle which *that side subtends, at the point* C; but the *sum* of all *such* subtended angles is either *four right angles*, or some whole *multiple* thereof: therefore the *sum* of their *supplements* can differ only by some such multiple from $n\pi$, if n be the number of the sides.

(4.) Whatever that *number* may be, if we denote by p_n the *exponential product* in the formula I., we have for *every vector* ρ, and for *every quaternion* q, the equations:

$$\text{XI.}\ldots p_n\rho p_n^{-1} = \rho; \qquad \text{XII.}\ldots p_n q p_n^{-1} = q;$$

whereof the former may (by 308, (8.), be thus interpreted:—

"*If any line* OP, *drawn from the centre* O *of a sphere, be made to revolve conically round any n radii,* $\mathrm{OA}_1, \ldots \mathrm{OA}_n$, *as n successive axes of rotation, through angles equal respectively to the doubles of the angles of the spherical polygon* $\mathrm{A}_1 \ldots \mathrm{A}_n$, *the line will be brought back to its initial position, by the composition of these n rotations.*"

(5.) Another way of proving the extended formula I., for any spherical *polygon*, is analogous to that which was employed in 311 for the case of a *triangle* on a sphere, and may be stated as follows. Let A_1', A_2', $\ldots \mathrm{A}_n'$ be the positive poles of the arcs $\mathrm{A}_1\mathrm{A}_2$, $\mathrm{A}_2\mathrm{A}_3, \ldots \mathrm{A}_n\mathrm{A}_1$; and let α_1', α_2', $\ldots \alpha_n'$ be the unit-vectors of those n poles. Then the point A_1 is the positive pole of the new arc $\mathrm{A}_1'\mathrm{A}_n'$, and the angle A_1 of the polygon at that point is measured by the supplement of that arc; with similar results for other corners of the polygon. Thus we have the system of expressions (comp. 311, VI.):

$$\text{XIII.}\ldots \alpha_1^{\frac{2\mathrm{A}_1}{\pi}} = \alpha_1'\alpha_n'; \ldots \alpha_n^{\frac{2\mathrm{A}_n}{\pi}} = \alpha_n'\alpha_{n-1}';$$

so that the *product of powers* in I. is equal to the following *product of n squares of unit-lines*, and therefore to the nth *power of negative unity*,

$$\text{XIV.}\ldots\ \alpha'_n\alpha'_{n-1}\,.\,\alpha'_{n-1}\alpha'_{n-2}\ldots\alpha'_2\alpha'_1\,.\,\alpha'_1\alpha'_n=(-1)^n;$$

and thus the extended theorem is proved anew.

(6.) This latter process may be translated into another *theorem of rotation*, on which it is possible that we may briefly return,* in the Second and last Chapter of this Third Book, but upon which we cannot here delay.

(7.) It may be remarked however here (comp. 309, XII.), that the *extended exponential formula* I. may be thus written:

$$\text{XV.}\ldots\ \mathrm{c}\alpha_n\mathrm{s}\,\mathrm{A}_n\,.\,\mathrm{c}\alpha_{n-1}\mathrm{s}\,\mathrm{A}_{n-1}\ldots\mathrm{c}\alpha_2\mathrm{s}\,\mathrm{A}_2\,.\,\mathrm{c}\alpha_1\mathrm{s}\,\mathrm{A}_1=(-1)^n.$$

(8.) For example, if ABCD be any *spherical quadrilateral*, of which the *angles* (suitably measured) are denoted by A, . . D, so that A represents the positive or negative *rotation* from AD to AB, &c., while α, β, γ, δ are the unit vectors of its corners, then

$$\text{XVI.}\ldots\ \mathrm{c}\delta\mathrm{s}\,\mathrm{D}\,.\,\mathrm{c}\gamma\mathrm{s}\,\mathrm{C}\,.\,\mathrm{c}\beta\mathrm{s}\,\mathrm{B}\,.\,\mathrm{c}\alpha\mathrm{s}\,\mathrm{A}=+1.$$

(9.) Hence (comp. 309, XIII.), we may write also,

$$\text{XVII.}\ldots\ (\cos\mathrm{C}-\gamma\sin\mathrm{C})(\cos\mathrm{D}-\delta\sin\mathrm{D})=(\cos\mathrm{B}+\beta\sin\mathrm{B})(\cos\mathrm{A}+\alpha\sin\mathrm{A});$$

and therefore, by taking scalars on both sides, and changing signs,

$$\text{XVIII.}\ldots\ -\cos\mathrm{C}\cos\mathrm{D}+\sin\mathrm{C}\sin\mathrm{D}\cos\mathrm{CD}=-\cos\mathrm{B}\cos\mathrm{A}+\sin\mathrm{B}\sin\mathrm{A}\cos\mathrm{BA};$$

in fact, each member of this last formula is equal (by 309, XIV.) to the cosine of the angle AEB, or CED, if the opposite sides AD, BC of the quadrilateral intersect in E.

(10.) Let $\rho=\mathrm{OP}$ be the unit vector of *any fifth point*, P, upon the spheric surface; then operating by $\mathrm{S}\,.\,\rho$ on XVII., we obtain this other general formula,

$$\text{XIX.}\ldots\ \begin{cases}0=\sin\mathrm{A}\cos\mathrm{B}\cos\mathrm{AP}+\sin\mathrm{B}\cos\mathrm{A}\cos\mathrm{BP}+\sin\mathrm{A}\sin\mathrm{B}\sin\mathrm{AB}\sin\mathrm{PQ}\\ \quad+\sin\mathrm{C}\cos\mathrm{D}\cos\mathrm{CP}+\sin\mathrm{D}\cos\mathrm{C}\cos\mathrm{DP}+\sin\mathrm{C}\sin\mathrm{D}\sin\mathrm{CD}\sin\mathrm{PR};\end{cases}$$

in which the *sines of the sides* AB, CD are treated as *always positive;* but the *sines of the perpendiculars* PQ and PR, on those two sides, are regarded as *positive* or *negative*, according as the *rotations* round P, from A to B and from C to D, are *negative* or *positive:* and hence, by assigning particular positions to P, several other but less general equations of *spherical tetragonometry* can be derived.

(11.) For example, if we place P at the *intersection*, say F, of the *opposite sides* AB, CD, the two last perpendiculars will vanish, and *two* of the *six terms* will *disappear*, from the general formula XIX.; and a similar *reduction to four terms* will occur, if we make the arbitrary point P the *pole* of a *side*, or of a *diagonal*.

314. The definition of the *power* α^t, which was assigned in 308, enables us to form some useful expressions, by quaternions, for *circular*, *elliptic*, and *spiral loci*, in a given plane, or in space, a few of which may be mentioned here.

(1.) Let α be any given unit-vector OA, and β any other given line OB, perpendicular to it; then, by the definition (308), if we write,

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* Compare 297, (24.).

$$\text{I.}\ldots\ \mathrm{OP} = \rho = \alpha^t\beta, \quad \mathrm{T}\alpha = 1, \quad \mathrm{S}\alpha\beta = 0,$$

the *locus of the point* P will be the *circumference of a circle*, with O for *centre*, and OB for *radius*, and in a *plane perpendicular to* OA.

(2.) If we *retain* the condition $\mathrm{T}\alpha = 1$, but *not* the condition $\mathrm{S}\alpha\beta = 0$, then the *product* $\alpha^t\beta$ will be in general a *quaternion*, and not merely a *vector*; but if we take its *vector-part* (292), we can form this *new* vector-expression,

$$\text{II.}\ldots\ \mathrm{OP} = \rho = \mathrm{V}.\,\alpha^t\beta = \beta\cos x + \gamma\sin x,$$

where $\text{III.}\ldots\ 2x = t\pi$, and $\text{IV.}\ldots\ \gamma = \mathrm{OC} = \mathrm{V}\alpha\beta$;

and *now* the *locus* of P is a *plane ellipse*, with its *centre* at O, and with OB and OC for its *major* and *minor semiaxes*: while the angular quantity, x, is what is often called the *excentric anomaly*.

(3.) If we write, under the same conditions (2.),

$$\text{V.}\ldots\ \mathrm{OB}' = \beta' = \mathrm{V}\beta\alpha : \alpha = \alpha^{-1}\gamma, \quad \text{and} \quad \text{VI.}\ldots\ \mathrm{OP}' = \rho' = \mathrm{V}\rho\alpha : \alpha = \alpha\mathrm{V}\rho\alpha,$$

so that B′ and P′ are the *projections* (203) of B and P *on a plane drawn through* O, *at right angles to the unit-line* OA, we have then, by II., the equation,

$$\text{VII.}\ldots\ \rho' = \beta'\cos x + \gamma\sin x = \alpha^t\beta';$$

so that the *locus of this projected point* P′ is a *circle*, with OB′ and OC for two rectangular radii.

(4.) Under the same conditions, the *elliptic locus* (2.), of the point P *itself*, is the *section* of the *right cylinder* (compare 203, (5.)),

$$\text{VIII.}\ldots\ \mathrm{TV}\alpha\rho = \mathrm{TV}\alpha\beta = \mathrm{T}\gamma,$$

made by the *plane*,

$$\text{IX.}\ldots\ 0 = \mathrm{S}\gamma\beta\rho, \quad \text{or} \quad \text{IX}'.\ldots\ \beta^2\mathrm{S}\alpha\rho = \mathrm{S}\alpha\beta\mathrm{S}\beta\rho \ (\text{comp. 298, XXVI.});$$

as a confirmation of which last form we have, by II. and IV.,

$$\text{X.}\ldots\ \mathrm{S}\alpha\rho = \mathrm{S}\alpha\beta\cos x, \quad \mathrm{S}\beta\rho = \beta^2\cos x.$$

(5.) If we *retain* the condition $\mathrm{S}\alpha\beta = 0$ (1.), but *not now* the condition $\mathrm{T}\alpha = 1$, we may again write the equation I. for ρ; but the *locus of* P will now be a *logarithmic spiral*, with O for its *pole*, in the plane perpendicular to OA; because *equal angular motions*, of the *turning line* OP, correspond now to *equal multiplications of the length* of that line ρ.

(6.) For example, when the scalar exponent t is increased by 4, so that the *revolving unit line*,

$$\text{XI.}\ldots\ \mathrm{U}\rho = \mathrm{U}\alpha^t.\,\mathrm{U}\beta$$

returns (comp. 309, XXIV.) to the *direction* which it had before the increase of t was made, the *length* $\mathrm{T}\rho$ of the *turning line* ρ *itself*, or of the *radius vector of the locus*, is *multiplied by* $\mathrm{T}\alpha^4$; which constant and positive scalar is *not now* equal to *unity*.

(7.) If we *reject both* the conditions (1.),

$$\mathrm{T}\alpha = 1, \quad \text{and} \quad \mathrm{S}\alpha\beta = 0,$$

so that the *line* α, or the *base* of the *power* α^t, is now *neither an unit-line*, *nor perpendicular to* β, namely to the line on which that power *operates*, as a *factor*, we must *again* take *vector parts*, but we have now this *new* expression:

$$\text{XII.}\ldots\ \mathrm{OP} = \rho = \mathrm{V}.\,\alpha^t\beta = a^t(\beta\cos x + \gamma\sin x)$$

in which we have written, for abridgment,

$$\text{XIII.} \ldots a = \mathrm{T}a, \quad \gamma = \mathrm{V}(\mathrm{U}a . \beta).$$

(8.) In this more complex case, the *locus* of P is *still* a *plane curve*, and may be said to be now an *elliptic* logarithmic spiral;* for if we *suppress* the *scalar factor*, a^t, we fall back on the *form* II., and have again an *ellipse* as the locus: but when we *take account* of that factor, we find (comp. (2.)) that *equal increments of excentric anomaly* (x), in the *auxiliary ellipse* so determined, correspond to *equal multiplications of the length* ($\mathrm{T}\rho$), of the vector of the *new spiral.*

(9.) We may also project B and P, as in (3.), into points B′ and P′, on the plane through O perpendicular to OA, which plane still contains the extremity C of the auxiliary vector γ; and then, since it is easily proved that $\gamma = \mathrm{U}a . \beta'$, the equation of the *projected spiral* becomes (with $\mathrm{T}a >$ or < 1),

$$\text{XIV.} \ldots \rho' = a^t(\beta' \cos x + \gamma \sin x) = a^t \beta' ;$$

so that we are brought back to the case (5.), and the *projected* curve is seen to be a logarithmic spiral, of the known and *ordinary* kind.

(10.) Several *spirals of double curvature* are easily represented, on the same general plan, by merely introducing a *vector-term proportional to t*, combined or not with a *constant* vector-term, in each of the expressions above given, for the *variable* vector ρ. For example, the equation,

$$\text{XV.} \ldots \rho = cta + a^t\beta, \quad \text{with} \quad \mathrm{T}a = 1, \quad \text{and} \quad \mathrm{S}a\beta = 0,$$

while c is *any constant scalar* different from zero, represents a *helix*, on the right circular cylinder VIII.

(11.) And if we introduce a new and variable scalar, u, as a *factor* in the right-hand term, and so write,

$$\text{XVI.} \ldots \rho = cta + ua^t\beta,$$

we shall have an expression for a *variable vector* ρ, considered as depending on *two variable scalars* (t and u), which thus becomes (99) the expression for a *vector of a surface*: namely of that important *Screw Surface*, which is the *locus of the perpendiculars*, let fall from the various points of a *given helix, on the axis* of the cylinder of revolution, on which that helix, or spiral curve, is traced.

315. Without at present pursuing farther the study of these *loci* by quaternions, it may be remarked that the definition (308) of the power a^t, especially for the case when $\mathrm{T}a = 1$, combined with the laws (182) of i, j, k, and with the identification (295) of those three important right versors with their own indices, enables us to establish the following among other *transformations*, which will be found useful on several occasions.

(1.) Let a be *any unit-vector*, and let t be *any scalar;* then,

$$\text{I.} \ldots \mathrm{S} . a^{-t} = \mathrm{S} . a^t; \qquad \text{II.} \ldots \mathrm{S} . a^{-t-1} = \mathrm{S} . a^{t+1} = -\mathrm{S} . a^{t-1};$$

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* The *usual* logarithmic spiral might perhaps be called, by contrast to *this* one, a *circular logarithmic spiral.* Compare the following sub-article (9.), respecting the *projection* of what is here called an *elliptic* logarithmic spiral.

$$\text{III.} \ldots \alpha^t = \text{S}\,.\,\alpha^t + \alpha\text{S}\,.\,\alpha^{t-1}; \qquad \text{IV.} \ldots \alpha^{-t} = \text{S}\,.\,\alpha^t - \alpha\text{S}\,.\,\alpha^{t-1};$$

$$\text{V.} \ldots (\text{S}\,.\,\alpha^t)^2 + (\text{S}\,.\,\alpha^{t-1})^2 = \alpha^t\alpha^{-t} = 1.$$

(2.) Let α and ι *be any two unit-vectors*, and let t be still *any scalar;* then

$$\text{VI.} \ldots \text{S}\,.\,\alpha^t = \text{S}\,.\,\iota^t; \qquad \text{VII.} \ldots \text{V}\,.\,\alpha^t = \alpha\text{S}\,.\,\alpha^{t-1};$$

$$\text{VIII.} \ldots \alpha\text{V}\,.\,\alpha^t = \alpha^2\text{S}\,.\,\alpha^{t-1} = \text{S}\,.\,\alpha^{t+1}.$$

(3.) Hence, by the laws of i, j, k,

$$\text{IX.} \ldots i\text{V}.i^t = j\text{V}.j^t = k\text{V}.k^t = \text{S}\,.\,\alpha^{t+1}.$$

(4.) We have also, by the same principles and laws,

$$\text{X.} \ldots i\text{V}.j^t = \text{V}.k^t; \quad j\text{V}.k^t = \text{V}.i^t; \quad k\text{V}.i^t = \text{V}.j^t;$$

$$\text{XI.} \ldots j\text{V}.i^t = -\text{V}.k^t; \quad k\text{V}.j^t = -\text{V}.i^t; \quad i\text{V}.k^t = -\text{V}.j^t.$$

(5.) The expression 308, (10.), for an *arbitrary vector* ρ, may be put under the following form:

$$\text{XII.} \ldots \rho = r\text{V}.k^{2s+1} + rk^{2t}\text{V}.i^{2s}.$$

(6.) And it may be expanded as follows:

$$\text{XIII.} \ldots \rho = r\{(i\cos t\pi + j\sin t\pi)\sin s\pi + k\cos s\pi\}.$$

(7.) We shall return, briefly, in the Second Chapter of this Book, on some of these last expressions, in connexion with *differentials* and *derivatives* of *powers of vectors;* but, for the purposes of the present Section, they may suffice.

SECTION 11.—*On Powers and Logarithms of Diplanar Quaternions; with some Additional Formulæ.*

316. We shall conclude the present Chapter with a short Supplementary Section, in which the recent definition (308) of a *power of a vector*, with a *scalar exponent*, shall be extended so as to include the *general case*, of a *Power of a Quaternion*, with a *Quaternion Exponent*, even when the two quaternions so combined are *diplanar:* and a connected *definition* shall be given (consistent with the less general one of the same kind, which was assigned in the Second Chapter of the Second Book), for the *Logarithm of a Quaternion in an arbitrary Plane:*[*] together with a few additional Formulæ, which could not be so conveniently introduced before.

(1.) We propose, then, to write, *generally*,

$$\text{I.} \ldots \epsilon^q = 1 + \frac{q}{1} + \frac{q^2}{1.2} + \frac{q^3}{1.2.3} + \&\text{c.};$$

q being *any quaternion*, and ϵ being the real and known base of the natural (or Napierian) system of logarithms, of real and positive scalars: so that (as usual),

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[*] The quaternions considered, in the Chapter referred to, were all supposed to be in the plane of the right versor i. But see the Second Note to page 265.

$$\text{II.} \ldots \varepsilon = \varepsilon^1 = 1 + \frac{1}{1} + \frac{1^2}{1\,.\,2} + \&\text{c.} = 2{\cdot}71828\ldots$$

(Compare 240, (1.) and (2.).)

(2.) We shall also write, for any quaternion q, the following expression for what we shall call its *principal logarithm*, or simply its *Logarithm*:

$$\text{III.} \ldots \mathrm{l}q = \mathrm{lT}q + \angle\, q\,.\,\mathrm{UV}q\,;$$

and thus shall have (comp. 243) the equation,

$$\text{IV.} \ldots \varepsilon^{\mathrm{l}q} = q.$$

(3.) When q is any *actual* quaternion (144), which does not *degenerate* (131) into a *negative scalar*, the formula III. assigns a *definite value* for the *logarithm*, $\mathrm{l}q$; which is such (comp. again 243) that

$$\text{V.} \ldots \mathrm{Sl}q = \mathrm{lT}q\,; \qquad \text{VI.} \ldots \mathrm{Vl}q = \angle\, q\,.\,\mathrm{UV}q\,;$$
$$\text{VII.} \ldots \mathrm{UVl}q = \mathrm{UV}q\,; \qquad \text{VIII.} \ldots \mathrm{TVl}q = \angle\, q\,;$$

the *scalar part of the logarithm* being thus the (natural) *logarithm of the tensor*; and the *vector part* of the same logarithm $\mathrm{l}q$ being constructed by a *line* in the direction of the *axis* Ax. q, of which the *length* bears, to the assumed *unit of length*, the same *ratio* as that which the *angle* $\angle\, q$ bears, to the usual *unit of angle* (comp. 241, (2.), (4.)).

(4.) If it were merely required to *satisfy the equation*,

$$\text{IX.} \ldots \varepsilon^{q'} = q,$$

in which q is supposed to be a *given* and *actual* quaternion, which is not equal to any negative scalar (3.), we might do this by writing (compare again 243),

$$\text{X.} \ldots q' = (\log q)_n = \mathrm{l}q + 2n\pi\mathrm{UV}q,$$

where n is *any whole number*, positive or negative or null; and in this view, what we have called *the logarithm*, $\mathrm{l}q$, of the quaternion q, is only what may be considered as the *simplest solution* of the *exponential equation* IX., and may, as such, be thus denoted:

$$\text{XI.} \ldots \mathrm{l}q = (\log q)_0.$$

(5.) The *excepted case* (3.), where q is a *negative scalar*, becomes on this plan a case of *indetermination*, but *not* of *impossibility*: since we have, for example, by the definition III., the following expression for the *logarithm of negative unity*,

$$\text{XII.} \ldots \mathrm{l}(-1) = \pi\sqrt{-1}\,;$$

which in its *form* agrees with old and well-known results, but is here *interpreted* as signifying *any unit-vector*, of which the *length* bears to the *unit* of length the ratio of π to 1 (comp. 243, VII.).

(6.) We propose also to write, generally, for *any two quaternions*, q and q', *even if diplanar*, the following expression (comp. 243, (4.)) for what may be called the *principal value* of the *power*, or simply *the Power*, in which the former quaternion q is the *base*, while the latter quaternion q' is the *exponent*:

$$\text{XIII.} \ldots q^{q'} = \varepsilon^{q'\mathrm{l}q}\,;$$

and thus this *quaternion power* receives, *in general*, with the help of the definitions I. and III., a perfectly *definite signification*.

(7.) When the *base*, q, becomes a *vector*, ρ, its *angle* becomes a *right angle*; the definition III. gives therefore, for this case,

$$\text{XIV.} \ldots \ \mathrm{l}\rho = \mathrm{lT}\rho + \frac{\pi}{2}\,\mathrm{U}\rho\,;$$

and this is the quaternion which is to be multiplied by q', in the expression,

$$\text{XV.} \ldots \ \rho^{q'} = \varepsilon^{q'\mathrm{l}\rho}.$$

(8.) When, for the same *vector-base*, the *exponent* q' becomes a *scalar*, t, the last formula becomes:

$$\text{XVI.} \ldots \ \rho^{t} = \varepsilon^{t\mathrm{l}\rho} = \mathrm{T}\rho^{t}\,.\,\varepsilon^{x\mathrm{U}\rho}, \quad \text{if} \quad 2x = t\pi\,;$$

and because, by I., the relation $(\mathrm{U}\rho)^2 = -1$ gives,

$$\text{XVII.} \ldots \ \varepsilon^{x\mathrm{U}\rho} = \cos x + \mathrm{U}\rho \sin x, \quad \text{or briefly,} \quad \text{XVII}'. \ldots \ \varepsilon^{x\mathrm{U}\rho} = \mathrm{c}\rho\mathrm{s}\,x,$$

we see that the former definition, 308, I., of the power a^t, is in this way *reproduced*, as one which is *included* in the *more general definition* XIII., of the power $q^{q'}$; for we may write, by the last mentioned definition,

$$\text{XVIII.} \ldots \ (\mathrm{U}\rho)^{t} = \varepsilon^{x\mathrm{U}\rho} = \mathrm{c}\rho\mathrm{s}\,\frac{t\pi}{2} \ \text{(comp. 234, VIII.)},$$

with the recent values XVI. and XVII., of x and $\varepsilon^{x\mathrm{U}\rho}$.

(9.) In the present theory of *diplanar quaternions*, we cannot expect to find that the *sum of the logarithms* of any two proposed *factors*, shall be *generally equal* to the *logarithm of the product*; but for the simpler and earlier case of *complanar* quaternions, that *algebraic property* may be considered to exist, with due modifications for *multiplicity of value*.*

(10.) The definition III. enables us, however, to establish *generally* the very simple formula (comp. 243, II. III.):

$$\text{XIX.} \ldots \ \mathrm{l}q = \mathrm{l}(\mathrm{T}q\,.\,\mathrm{U}q) = \mathrm{lT}q + \mathrm{lU}q\,;$$

in which (comp. (3.)),

$$\text{XX.} \ldots \ \mathrm{lU}q = \angle\, q\,.\,\mathrm{UV}q = \mathrm{Vl}q\,; \quad \text{XXI.} \ldots \ \mathrm{TlU}q = \angle\, q\,; \quad \text{XXII.} \ldots \ \mathrm{UlU}q = \mathrm{UV}q.$$

(11.) We have also generally, by XIII., for *any scalar exponent*, t, and *any quaternion base*, q, the *power*,

$$\text{XXIII.} \ldots \ q^{t} = \varepsilon^{t\mathrm{l}q} = (\mathrm{T}q)^{t}\,.\,(\cos t\angle\, q + \mathrm{UV}q\,.\,\sin t\angle\, q)\,;$$

or briefly,

$$\text{XXIII}'. \ldots \ q^{t} = \mathrm{T}q^{t}\,.\,\mathrm{c}v\mathrm{s}\,t\angle\, q, \quad \text{if} \quad v = \mathrm{UV}q\,;$$

in which the parentheses about $\mathrm{T}q$ may be omitted, because

$$\text{XXIV.} \ldots \ \mathrm{T}(q^{t}) = (\mathrm{T}q)^{t} = \mathrm{T}q^{t} \ \text{(comp. 237, II.)}.$$

(12.) When the base and exponent of a power are *two rectangular vectors*, ρ and ρ', then, whatever their *lengths* may be, the product $\rho'\mathrm{l}\rho$ is, by XIV., a *vector*; but ε^{α} is always a *versor*,

$$\text{XXV.} \ldots \ \varepsilon^{\alpha} = \cos \mathrm{T}\alpha + \mathrm{U}\alpha \sin \mathrm{T}\alpha, \ \text{if } \alpha \text{ be any vector;}$$

we have therefore,

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* In 243, (3.), it might have been observed, that *every value of each member* of the formula IX., there given, is *one of the values* of the *other member*; and a similar remark applies to the formulæ I. and II. of 236.

$$\text{XXVI.} \ldots \mathrm{T}.\rho^{\rho'} = 1, \quad \text{if} \quad \mathrm{S}.\rho\rho' = 0\,;$$

or in words, the *power* $\rho^{\rho'}$ *is a versor*, under this condition of *rectangularity*.

(13.) For example (comp. 242, (7.),* and the shortly following formula XXVIII.),

$$\text{XXVII.} \ldots i^j = \epsilon^{j\,l\,i} = -k\,; \quad j^i = \epsilon^{i\,l\,j} = +k\,;$$

and generally, if the *base* be an *unit-line*, and the *exponent* a line of *any length*, but *perpendicular to the base*, the *axis of the power* is a line *perpendicular to both*; unless the *direction* of that *axis* becomes *indeterminate*, by the *power* reducing itself to a *scalar*, which in certain cases may happen.

(14.) Thus, whatever scalar c may be, we may write,

$$\text{XXVIII.} \ldots i^{cj} = \epsilon^{cj\,l\,i} = \epsilon^{-\frac{1}{2}ck\pi} = \cos\frac{c\pi}{2} - k\sin\frac{c}{2}\,;$$

this *power*, then, is a *versor* (12.), and its *axis* is *generally* the line $\mp k$; but in the *case* when c is any *whole* and *even number*, this versor *degenerates* into positive or negative *unity* (153), and the axis becomes *indeterminate* (131).

(15.) If, for any *real* quaternion q, we write again,

$$\text{XXIX.} \ldots \mathrm{UV}q = \upsilon, \quad \text{and therefore} \quad \text{XXX.} \ldots \upsilon q = q\upsilon, \quad \text{and} \quad \text{XXXI.} \ldots \upsilon^2 = -1,$$

the process of 239 will hold good, when we change i to υ; the *series*, denoted in I. by ϵ^q, is therefore *always at last convergent*,† *however great* (but finite) *the tensor* Tq *may be*; and in like manner the two following other series, derived from it, which represent (comp. 242, (3.)) what we shall *call*, *generally*, by analogy to known expressions, the *cosine* and *sine* of the *quaternion* q, are always *ultimately convergent*:

$$\text{XXXII.} \ldots \cos q = \tfrac{1}{2}(\epsilon^{\upsilon q} + \epsilon^{-\upsilon q}) = 1 - \frac{q^2}{1.2} + \frac{q^4}{1.2.3.4} - \&\text{c.}\,;$$

$$\text{XXXIII.} \ldots \sin q = \frac{1}{2\upsilon}(\epsilon^{\upsilon q} - \epsilon^{-\upsilon q}) = \frac{q}{1} - \frac{q^3}{1.2.3} + \frac{q^5}{1.2.3.4.5} - \&\text{c.}$$

(16.) We shall also *define* that the *secant*, *cosecant*, *tangent*, and *cotangent* of a *quaternion*, supposed still to be *real*, are the functions:

$$\text{XXXIV.} \ldots \sec q = \frac{2}{\epsilon^{\upsilon q} + \epsilon^{-\upsilon q}}\,; \quad \operatorname{cosec} q = \frac{2\upsilon}{\epsilon^{\upsilon q} - \epsilon^{-\upsilon q}}\,;$$

$$\text{XXXV.} \ldots \tan q = \frac{\upsilon^{-1}(\epsilon^{\upsilon q} - \epsilon^{-\upsilon q})}{\epsilon^{\upsilon q} + \epsilon^{-\upsilon q}}\,; \quad \cot q = \frac{\upsilon(\epsilon^{\upsilon q} + \epsilon^{-\upsilon q})}{\epsilon^{\upsilon q} - \epsilon^{-\upsilon q}}\,;$$

and thus shall have the usual relations, $\sec q = 1 : \cos q$, &c.

(17.) We shall also have,

$$\text{XXXVI.} \ldots \epsilon^{\upsilon q} = \cos q + \upsilon \sin q, \quad \epsilon^{-\upsilon q} = \cos q - \upsilon \sin q\,;$$

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* In the theory of *complanar quaternions*, it was found convenient to admit a certain *multiplicity of value* for a *power*, when the *exponent* was not a *whole number*; and therefore a *notation* for the *principal value of a power* was employed, with which the conventions of the present Section enable us now to dispense.

† In fact, it can be proved that this final convergence exists, even when the quaternion is *imaginary*, or when it is replaced by a *biquaternion* (214, (8.)); but we have no occasion here to consider any but *real* quaternions.

and therefore, as in trigonometry (comp. 315, (1.)),

$$\text{XXXVII.} \ldots (\cos q)^2 + (\sin q)^2 = \varepsilon^{\nu q} . \varepsilon^{-\nu q} = \varepsilon^{\circ} = 1,$$

whatever quaternion q may be.

(18.) And *all the formulæ of trigonometry*, for *cosines* and *sines* of *sums* of *two or more arcs*, &c., will thus hold good *for quaternions also*, *provided* that the quaternions to be combined are *in any common plane;* for example,

$$\text{XXXVIII.} \ldots \cos(q' + q) = \cos q' \cos q - \sin q' \sin q, \quad \text{if} \quad q' \mathbin{|||} q.$$

(19.) This *condition of complanarity* is here a *necessary* one; because (comp. (9.)) it is necessary for the establishment of the *exponential relation* between *sums* and *powers*.

(20.) Thus, we may indeed write,

$$\text{XXXIX.} \ldots \varepsilon^{q'+q} = \varepsilon^{q'} . \varepsilon^{q}, \quad \textit{if} \quad q \mathbin{|||} q;$$

but, *in general*, the developments of these two expressions give the difference,

$$\text{XL.} \ldots \varepsilon^{q'+q} - \varepsilon^{q'} \varepsilon^{q} = \frac{qq' - q'q}{2} + \text{terms of third and higher dimensions};$$

and

$$\text{XLI.} \ldots \tfrac{1}{2}(qq' - q'q) = \text{V}(\text{V}q . \text{V}q'),$$

an expression which does not vanish, when the quaternions q and q' are *diplanar*.

(21.) A few supplementary formulæ, connected with the present Chapter, may be appended here, as was mentioned at the commencement of this Article (316). And first it may be remarked, as connected with the theory of *powers of vectors*, that if α, β, γ be *any three unit-lines*, OA, OB, OC, and if σ denote the *area* of the spherical triangle ABC, then the formula 298, XX. may be thus written:

$$\text{XLII.} \ldots \frac{\alpha + \beta}{\beta + \gamma} . \frac{\gamma + \alpha}{\alpha + \beta} . \frac{\beta + \gamma}{\gamma + \alpha} = \alpha^{\frac{2\sigma}{\pi}};$$

the exponent being here a scalar.

(22.) The immediately preceding formula, 298, XIX., gives for *any three vectors*, the relation:

$$\text{XLIII.} \ldots (\text{U}\alpha\beta\gamma)^2 + (\text{U}\beta\gamma)^2 + (\text{U}\alpha\gamma)^2 + (\text{U}\alpha\beta)^2 + 4\text{U}\alpha\gamma . \text{SU}\alpha\beta . \text{SU}\beta\gamma = -2;$$

for example, if α, β, γ be made equal to i, j, k, the first member of this equation becomes, $1 - 1 - 1 - 1 + 0 = -2$.

(23.) The following is a much more complex identity, involving as it does not only *three arbitrary vectors* α, β, γ, but also *four arbitrary scalars*, a, b, c, and r; but it has some geometrical applications, and a student would find it a good exercise in *transformations*, to investigate a proof of it for himself. To abridge notation, the three vectors α, β, γ, and the three scalars a, b, c, are considered as each composing a *cycle*, with respect to which are formed *sums* Σ, and *products* Π, on a plan which may be thus exemplified:

$$\text{XLIV.} \ldots \Sigma a\text{V}\beta\gamma = a\text{V}\beta\gamma + b\text{V}\gamma\alpha + c\text{V}\alpha\beta; \quad \Pi a^2 = a^2b^2c^2.$$

This being understood, the formula to be proved is the following:

$$\begin{aligned} \text{XLV.} \ldots (\text{S}\alpha\beta\gamma)^2 + (\Sigma a\text{V}\beta\gamma)^2 + r^2(\Sigma\text{V}\beta\gamma)^2 - r^2(\Sigma a(\beta - \gamma))^2 \\ + 2\Pi(r^2 + \text{S}\beta\gamma + bc) = 2\Pi(r^2 + a^2) + 2\Pi a^2 \\ + \Sigma(r^2 + a^2 + \alpha^2)\{(\text{V}\beta\gamma)^2 + 2bc(r^2 + \text{S}\beta\gamma) - r^2(\beta - \gamma)^2\}; \end{aligned}$$

the sign of summation in the last line governing all that follows it.

(24.) For example, by making the *four scalars* a, b, c, r each $= 0$, this formula gives, for *any three vectors* α, β, γ, the relation,

$$\text{XLVI.} \ldots (\mathrm{S}\alpha\beta\gamma)^2 + 2\Pi\mathrm{S}\beta\gamma = 2\Pi\alpha^2 + \Sigma\,.\,\alpha^2(\mathrm{V}\beta\gamma)^2\,;$$

which agrees with the very useful equation 294, LIII., because

$$\text{XLVII.} \ldots \alpha^2(\mathrm{V}\beta\gamma)^2 = \alpha^2\{(\mathrm{S}\beta\gamma)^2 - \beta^2\gamma^2\} = (\alpha\mathrm{S}\beta\gamma)^2 - \Pi\alpha^2.$$

(25.) Let α, β, γ be the *vectors of three points* A, B, C, which are *exterior to a given sphere*, of which the *radius* is r, and the *equation* is,

$$\text{XLVIII.} \ldots \rho^2 + r^2 = 0 \text{ (comp. 282, XIII.)};$$

and let a, b, c denote the *lengths of the tangents* to that sphere, which are drawn from those three points respectively. We shall then have the relations:

$$\text{XLIX.} \ldots \alpha^2 + a^2 = \beta^2 + b^2 = \gamma^2 + c^2 = -r^2\,;$$

thus $r^2 + a^2 = -\alpha^2$, &c., and the second member of the formula XLV. vanishes; the first member of that formula is therefore *also* equal to zero, for these significations of the letters: and thus a *theorem* is obtained, which is found to be extremely useful, in the investigation by quaternions of the system of the *eight* (real or imaginary) *small circles, which touch a given set of three small circles on a sphere.*

(26.) We cannot enter upon *that* investigation here; but may remark that because the vector ρ of the foot P, of the perpendicular OP let fall the origin O on the right line AB, is given by the expression,

$$\text{L.} \ldots \rho = \alpha\mathrm{S}\frac{\beta}{\beta - \alpha} + \beta\mathrm{S}\frac{\alpha}{\alpha - \beta} = \frac{\mathrm{V}\beta\alpha}{\alpha - \beta},$$

as may be proved in various ways, the *condition of contact* of that *right line* AB with the *sphere* XLVIII. is expressed by the equation,

$$\text{LI.} \ldots \mathrm{TV}\beta\alpha = r\mathrm{T}(\alpha - \beta); \quad \text{or} \quad \text{LII.} \ldots (\mathrm{V}\beta\alpha)^2 = r^2(\alpha - \beta)^2\,;$$

or by another easy transformation, with the help of XLIX.,

$$\text{LIII.} \ldots (r^2 + \mathrm{S}\alpha\beta)^2 = (r^2 + \alpha^2)\,(r^2 + \beta^2) = a^2b^2.$$

(27.) This last equation evidently admits of decomposition into *two factors*, representing *two alternative conditions*, namely,

$$\text{LIV.} \ldots r^2 + \mathrm{S}\alpha\beta - ab = 0\,; \qquad \text{LV.} \ldots r^2 + \mathrm{S}\alpha\beta + ab = 0\,;$$

and if we still consider the *tangents* a and b (25.) as *positive*, it is easy to prove, in several different ways, that the *first* or the *second* factor is to be selected, according as the *point* P, at which the *line* AB *touches the sphere*, *does* or *does not* fall *between the points* A and B; or in other words, according as the *length* of that line is equal to the *sum*, or to the *difference*, of those two tangents.

(28.) In fact we have, for the first case,

$$\text{LVI.} \ldots \mathrm{T}(\beta - \alpha) = b + a, \quad \text{or} \quad 0 = (\beta - \alpha)^2 + (b + a)^2 = -2\,(r^2 + \mathrm{S}\alpha\beta - ab),$$

in virtue of the relations XLIX.; but, for the second case,

$$\text{LVII.} \ldots \mathrm{T}(\beta - \alpha) = \pm(b - a), \quad \text{or} \quad 0 = (\beta - \alpha)^2 + (b - a)^2 = -2\,(r^2 + \mathrm{S}\alpha\beta + ab)\,;$$

and it may be remarked, that we might in this way have been led to find the system of the *two conditions* (27.), and thence the equation LIII., or its transformations, LII. and LI.

(29.) We may conceive a *cone of tangents* from A, *circumscribing* the *sphere* XLVIII., and touching it *along a small circle*, of which the *plane*, or the *polar plane of the point* A, is easily found to have for its equation,

$$\text{LVIII.} \ldots S\alpha\rho + r^2 = 0 \text{ (comp. 294, (28.), and 215, (10.));}$$

and in like manner the equation,

$$\text{LIX.} \ldots S\beta\rho + r^2 = 0,$$

represents the polar plane of the point B, which plane cuts the sphere in a *second small circle:* and *these two circles touch each other*, when *either* of the two conditions (27.) is satisfied; such *contact* being *external* for the case LIV., but *internal* for the case LV.

(30.) The *condition of contact* (26.), of the *line and sphere*, might have been otherwise found, as the condition of *equality of roots* in the *quadratic equation* (comp. 216, (2.)),

$$\text{LX.} \ldots 0 = (x\alpha + y\beta)^2 + (x + y)^2 r^2,$$

or
$$\text{LXI.} \ldots 0 = x^2(r^2 + \alpha^2) + 2xy(r^2 + S\alpha\beta) + y^2(r^2 + \beta^2);$$

the *contact* being thus considered here as a case of *coincidence of intersections*.

(31.) The *equation of conjugation* (comp. 215, (13.)), which expresses that each of the two points A and B is in the polar plane of the other, is (with the present notations),

$$\text{LXII.} \ldots r^2 + S\alpha\beta = 0;$$

the *equal* but *opposite roots* of LXI., which then exist if the line cuts the sphere, answering here to the well-known *harmonic division* of the *secant line* AB (comp. 215, (16.)), which thus connects *two conjugate points.*

(32.) In like manner, from the quadratic equation* 216, III., we get this analogous equation,

$$\text{LXIII.} \ldots S\frac{\lambda}{\alpha} S\frac{\mu}{\alpha} - S\left(V\frac{\lambda}{\beta} . V\frac{\mu}{\beta}\right) = 1,$$

connecting the vectors λ, μ of any two points L, M, which are *conjugate relatively to the ellipsoid* 216, II.; and if we place the point L *on the surface*, the equation LXIII. will represent the *tangent plane at that point* L, considered as the *locus of the conjugate point* M; whence it is easy to deduce the *normal*, at any point of the ellipsoid. But all researches respecting *normals to surfaces* can be better conducted, in connexion with the *Differential Calculus of Quaternions*, to which we shall next proceed.

(33.) It may however be added here, as regards *Powers of Quaternions* with *scalar exponents* (11.), that the symbol $q^t r q^{-t}$ represents a quaternion formed from r, by a conical rotation of its axis round that of q, through an angle $= 2t \angle q$; and that both members of the equation,

$$\text{LXIV.} \ldots (qrq^{-1})^t = qr^tq^{-1},$$

are symbols of one common quaternion.

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* Corrected as in the first Note to page 298.

CHAPTER II.

ON DIFFERENTIALS AND DEVELOPMENTS OF FUNCTIONS OF QUATERNIONS; AND ON SOME APPLICATIONS OF QUATERNIONS, TO GEOMETRICAL AND PHYSICAL QUESTIONS.

SECTION 1.—*On the Definition of Simultaneous Differentials.*

317. IN the foregoing Chapter of the present Book, and in several parts of the Book preceding it, we have taken occasion to exhibit, as we went along, a considerable variety of *Examples*, of the *Geometrical Application of Quaternions*: but these have been given, chiefly as assisting to impress on the reader the *meanings of new notations*, or of *new combinations of symbols*, when such presented themselves in turn to our notice. In this concluding Chapter, we desire to offer a few *additional examples*, of the same *geometrical* kind, but dealing, more freely than before, with *tangents* and *normals* to *curves* and *surfaces*; and to give at least some *specimens*, of the application of quaternions to *Physical Inquiries*. But it seems necessary that we should first establish here some *Principles*, and some *Notations*, respecting *Differentials of Quaternions*, and of their *Functions*, generally.

318. The *usual definitions*, of *differential coefficients*, and of *derived functions*, are found to be inapplicable generally to the present Calculus, on account of the (generally) *non-commutative* character of quaternion-multiplication (168, 191). It becomes, therefore, necessary to have recourse to a *new Definition of Differentials*, which yet ought to be so framed, as to be *consistent with*, and to *include*, the *usual Rules of Differentiation*: because *scalars* (131), as well as *vectors* (292), have been seen to be *included*, under the general *Conception of Quaternions*.

319. In seeking for such a new definition, it is natural to

go back to the first principles of the whole subject of Differentials: and to consider how the great Inventor of *Fluxions* might be supposed to have dealt with the question, if he had been *deprived* of that powerful resource of *common calculation*, which is supplied by the *commutative property* of *algebraic multiplication;* or by the familiar equation,

$$xy = yx,$$

considered as a *general* one, or as subsisting for *every pair of factors*, x and y; while *limits* should still be *allowed*, but *infinitesimals* be still *excluded:* and indeed the *fluxions themselves* should be regarded as *generally finite*,[*] according to what seems to have been the ultimate *view* of NEWTON.

320. The answer to this question, which a study of the Principia appears to suggest, is contained in the following *Definition*, which we believe to be a perfectly general one, as regards the *older* Calculus, and which we propose to *adopt* for Quaternions:—

"*Simultaneous Differentials* (or *Corresponding Fluxions*) are *Limits of Equimultiples*[†] *of Simultaneous and Decreasing Differences.*"

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[*] Compare the remarks annexed to the Second Lemma of the Second Book of the Principia (Third Edition, London, 1726); and especially the following passage (page 244):

"Neque enim spectatur in hoc Lemmate magnitudo momentorum, sed prima nascentium proportio. Eodem recidit si loco momentorum usurpentur vel velocitates incrementorum ac decrementorum (quas etiam motus, mutationes et fluxiones quantitatum nominare licet) vel finitæ quævis quantitates velocitatibus hisce proportionales."

[†] As regards the notion of *multiplying* such *differences*, or generally any quantities which all *diminish together*, in order to render their *ultimate relations* more evident, it may be suggested by various parts of the *Principia* of Sir Isaac Newton; but especially by the First Section of the First Book. See for example the Seventh Lemma (p. 31), under which such expressions as the following occur: "intelligantur semper AB et AD ad puncta longinqua b et d produci," . . . . "ideoque rectæ semper finitæ Ab, Ad, . . ." The direction, "ad puncta longinqua produci," is repeated in connexion with the Eighth and Ninth Lemmas of the same Book and Section; while under the former of those two Lemmas we meet the expression, "triangula semper finita," applied to the *magnified representations* of *three triangles*, which all *diminish indefinitely together:* and under the latter Lemma the words occur, "manente longitudine Ae," where Ae is a *finite* and *constant line*, obtained by a *constantly increasing multiplication* of a *constantly diminishing line* AE (page 33 of the edition cited).

And conversely, whenever any *simultaneous differences*, of any system of variables, all *tend* to *vanish together*, according to any *law*, or system of laws; then, if any *equimultiples* of those decreasing differences all *tend* together to any system of finite *limits*, those *Limits* are *said* to be *Simultaneous Differentials* of the related *Variables* of the *System;* and are *denoted*, as such, by prefixing the *letter* d, as a *characteristic of differentiation*, to the *Symbol* of each such *variable*.

321. More fully and symbolically, let

$$\text{I.} \ldots q,\ r,\ s, \ldots$$

denote *any system of connected variables* (quaternions or others); and let

$$\text{II.} \ldots \Delta q,\ \Delta r,\ \Delta s, \ldots$$

denote, as usual, a system of their *connected* (or *simultaneous*) *differences;* in such a manner that the sums,

$$\text{III.} \ldots q+\Delta q, \quad r+\Delta r, \quad s+\Delta s, \ldots$$

shall be a *new system of variables*, satisfying the *same laws of connexion*, whatever they may be, as those which are satisfied by the *old system* I. Then, in *returning gradually* from the new system to the old one, or in proceeding gradually from the old to the new, the simultaneous *differences* II. can all be made (in general) to *approach together to zero*, since it is evident that they may *all vanish together*. But *if*, while the *differences themselves* are thus supposed to *decrease** *indefinitely together*, we *multiply them all* by some one *common* but *increasing number*, n, the *system of their equimultiples*,

$$\text{IV.} \ldots n\Delta q, \quad n\Delta r, \quad n\Delta s, \ldots$$

may tend to become *equal* to some determined *system of finite limits*. And *when this happens*, as in all ordinary cases it may be *made* to do, by a suitable *adjustment* of the *increase* of n to the *decrease* of Δq, &c., the *limits* thus obtained are *said* to be *simultaneous differentials* of the related variables, q, r, s; and are *denoted*, as such, by the symbols,

$$\text{V.} \ldots \mathrm{d}q, \quad \mathrm{d}r, \quad \mathrm{d}s, \ldots$$

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* A *quaternion* may be said to *decrease*, when its *tensor* decreases; and to decrease *indefinitely*, when that tensor *tends to zero*.

SECTION 2.—*Elementary Illustrations of the Definition, from Algebra and Geometry.*

322. To leave no possible doubt, or obscurity, on the *import* of the foregoing *Definition*, we shall here apply it to determine the *differential of a square*, in *algebra*, and that of a *rectangle*, in *geometry;* in doing which we shall show, that while for such cases the *old rules* are *reproduced*, the *differentials* treated of *need not be small;* and that it would be a *vitiation*, and *not* a *correction*, of the results, if any additional *terms* were introduced into their expressions, for the purpose of rendering *all the differentials equal* to the corresponding *differences:* though *some* of them may be *assumed* to be so, namely, in the first Example, *one*, and in the second Example, *two*.

(1.) In Algebra, then, let us consider the equation,

$$\text{I.} \ldots y = x^2,$$

which gives,

$$\text{II.} \ldots y + \Delta y = (x + \Delta x)^2,$$

and therefore, as usual,*

$$\text{III.} \ldots \Delta y = 2x\Delta x + \Delta x^2;$$

or what comes to the same thing,

$$\text{IV.} \ldots n\Delta y = 2xn\Delta x + n^{-1}(n\Delta x)^2,$$

where n is an *arbitrary multiplier*, which may be supposed, for simplicity, to be a positive whole number.

(2.) Conceive now that while the *differences* Δx and Δy, remaining always connected with each other and with x by the equation III., *decrease*, and *tend together to zero*, the *number n increases*, in the transformed equation IV., and *tends to infinity*, in such a manner that the *product*, or *multiple*, $n\Delta x$, tends to some *finite limit* a; which may happen, for example, by our obliging Δx to satisfy always the condition,

$$\text{V.} \ldots \Delta x = n^{-1}a, \quad \text{or} \quad n\Delta x = a,$$

after a previous *selection* of some *given* and *finite value* for a.

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* We write here, as is common, Δx^2 to denote $(\Delta x)^2$; while $\Delta . x^2$ would be written, on the same known plan, for $\Delta (x^2)$, or Δy. In like manner we shall write $\mathrm{d}x^2$, as usual, for $(\mathrm{d}x)^2$; and shall denote $\mathrm{d}(x^2)$ by $\mathrm{d}.x^2$. Compare the notations $\mathrm{S}q^2$, $\mathrm{S}.q^2$, and $\mathrm{V}q^2$, $\mathrm{V}.q^2$, in 199 and 204.

(3.) We shall then have, with this last condition V., the following expression by IV., for the *equimultiple* $n\Delta y$, of the *other difference*, Δy:

$$\text{VI.} \ldots n\Delta y = 2xa + n^{-1}a^2 = b + n^{-1}a^2, \quad \text{if} \quad b = 2xa.$$

But because a, and therefore a^2, is *given* and *finite*, (2.), while the number n increases indefinitely, the *term* $n^{-1}a^2$, in this expression VI. for $n\Delta y$, indefinitely *tends* to *zero*, and its *limit* is *rigorously null*. Hence the *two finite quantities*, a and b (since x is supposed to be finite), are *two simultaneous limits*, to which, under the supposed conditions, the *two equimultiples*, $n\Delta x$ and $n\Delta y$, *tend*;[*] they are, therefore, by the *definition* (320), *simultaneous differentials* of x and y: and we may *write* accordingly (321),

$$\text{VII.} \ldots \mathrm{d}x = a, \quad \mathrm{d}y = b = 2xa;$$

or, as usual, after elimination of a,

$$\text{VIII.} \ldots \mathrm{d}y = \mathrm{d}.x^2 = 2x\mathrm{d}x.$$

(4.) And it would *not improve*, *but vitiate*, according to the adopted *definition* (320), this usual expression for the *differential of the square* of a variable x in algebra, if we were to *add* to it the *term* $\mathrm{d}x^2$, in imitation of the formula III. for the *difference* $\Delta.x^2$. For this would come to supposing that, for a *given* and *finite value*, a, of $\mathrm{d}x$, or of $n\Delta x$, the *term* $n^{-1}a^2$, or $n^{-1}\mathrm{d}x^2$, in the expression VI. for $n\Delta y$, *could fail to tend to zero*, while the *number*, n, by which the *square* of $\mathrm{d}x$ is *divided*, *increases without limit*, or *tends* (as above) *to infinity*.

(5.) As an *arithmetical example*, let there be the *given values*,

$$\text{IX.} \ldots x = 2, \quad y = x^2 = 4, \quad \mathrm{d}x = 1000;$$

and let it be required to compute, as a consequence of the *definition* (320), the *arithrithmetical value* of the *simultaneous differential*, $\mathrm{d}y$. We have now the following *equimultiples* of *simultaneous differences*,

$$\text{X.} \ldots n\Delta x = \mathrm{d}x = 1000; \quad n\Delta y = 4000 + 1000000\,n^{-1};$$

but the *limit* of the nth *part* of a *million* (or of *any greater*, but *given* and *finite number*) is *exactly zero*, if n increase *without limit*; the required *value* of $\mathrm{d}y$ is, therefore, *rigorously*, in this example,

$$\text{XI.} \ldots \mathrm{d}y = 4000.$$

(6.) And we see that these two *simultaneous differentials*,

$$\text{XII.} \ldots \mathrm{d}x = 1000, \quad \mathrm{d}y = 4000,$$

are *not*, in *this* example, even *approximately equal* to the two *simultaneous differences*,

$$\text{XIII.} \ldots \Delta x = \mathrm{d}x = 1000, \quad \Delta y = 1002^2 - 2^2 = 1004000,$$

which answer to the value $n = 1$; although, no doubt, from the very *conception* of simultaneous *differentials*, as embodied in the *definition* (320), they must admit of having such *equisubmultiples* of themselves taken,

$$\text{XIV.} \ldots n^{-1}\mathrm{d}x \quad \text{and} \quad n^{-1}\mathrm{d}y,$$

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[*] In this case, indeed, the multiple $n\Delta x$ has by V. a *constant value*, namely a; but it is found convenient to extend the use of the word, *limit*, so as to *include* the case of constants: or to say, generally, that a *constant* is *its own limit*.

as to be ***nearly equal***, for ***large values of the number n***, to ***some system*** of simultaneous and ***decreasing differences***,

$$\text{XV.} \ldots \Delta x \quad \text{and} \quad \Delta y;$$

and ***more and more nearly equal*** to such a ***system***, even in the way of ***ratio***, as they ***all*** become ***smaller and smaller together***, and ***tend*** together to ***vanish***.

(7.) For example, while the ***differentials themselves*** retain the ***constant values*** XII., their ***millionth parts*** are, respectively,

$$\text{XVI.} \ldots n^{-1}\mathrm{d}x = 0{\cdot}001, \quad \text{and} \quad n^{-1}\mathrm{d}y = 0{\cdot}004, \quad \text{if} \quad n = 1000000;$$

and the same value of the number n gives, by X., the equally rigorous values of two ***simultaneous differences***, as follows,

$$\text{XVII.} \ldots \Delta x = 0{\cdot}001, \quad \text{and} \quad \Delta y = 0{\cdot}004001;$$

so that ***these values*** of the ***decreasing differences*** XV. may already be considered to be ***nearly equal*** to the two ***equisubmultiples***, XIV. or XVI., of the two ***simultaneous differentials***, XII. And it is evident that ***this approximation*** would be ***improved***, by taking ***higher values*** of the ***number***, n, without the ***rigorous*** and ***constant values*** XII., of dx and dy, being ***at all affected*** thereby.

(8.) It is, however, evident also, that after assuming $y = x^2$, and $x = 2$, as in IX., we ***might*** have ***assumed any other finite value for the differential*** dx, ***instead*** of the value 1000; and should then have ***deduced a different*** (but still ***finite***) ***value*** for the ***other differential***, dy, and ***not*** the ***formerly*** deduced value, 4000: but there would always exist, ***in this example***, or for ***this form*** of the ***function***, y, and for ***this value*** of the ***variable***, x, the ***rigorous relation*** between the ***two simultaneous differentials***, dx and dy,

$$\text{XVIII.} \ldots \mathrm{d}y = 4\mathrm{d}x,$$

which is obviously a ***case*** of the equation VIII., and can be proved by similar reasonings.

323. Proceeding to the promised *Example from Geometry* (322), we shall *again* see that differences and differentials are not in general to be *confounded* with each other, and that the latter (like the former) *need not be small*. But we shall *also* see that the *differentials* (*like the differences*), which enter into a statement of relation, or into the enunciation of a proposition, respecting quantities which *vary together*, according to any *law* or *laws*, *need not even be homogeneous among themselves:* it being sufficient that *each separately* should be *homogeneous with the variable* to which it *corresponds*, and of which it *is* the *differential*, as *line* of *line*, or *area* of *area*. It will also be seen that the *definition* (320) enables us to *construct the differential of a rectangle*, as the *sum of two other* (*finite*) *rectangles*, *without any reference to units of length*, or of *area*, and without even the *thought* of employing *any numerical calculation* whatever.

(1.) Let, then, as in the annexed Figure 74, ABCD be any given rectangle, and let BE and DG be any arbitrary but given and finite increments of its sides, AB and AD. Complete the increased rectangle GAEF, or briefly AF, which will thus exceed the given rectangle AC, or CA, by the sum of the three partial rectangles, CE, CF, CG; or by what we may call the *gnomon*,* CBEFGDC. On the diagonal CF take a point I, so that the line CI may be any arbitrarily selected submultiple of that diagonal; and draw through I, as in the Figure, lines HM, KL, parallel to the sides AD, AB; and therefore intercepting, on the sides AB, AD prolonged, equisubmultiples BH, DK of the two given increments, BE, DG, of those two given sides.

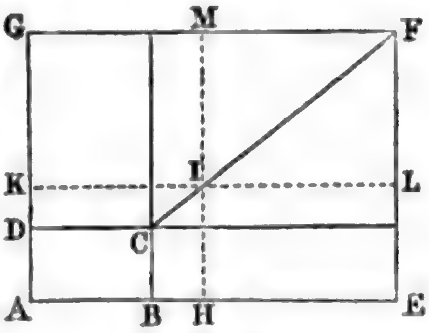

Fig. 74.

(2.) Conceive now that, in this construction, the *point* I *approaches to* C, or that we take a series of new points I, on the given diagonal CF, nearer and nearer to the given point C, by taking the *line* CI successively a *smaller and smaller part* of that diagonal. Then the *two new linear intervals*, BH, DK, and the *new gnomon*, CBHIKDC, or the *sum* of the *three new partial rectangles*, CH, CI, CK, will *all* indefinitely *decrease*, and will *tend* to *vanish together:* remaining, however, always a *system* of *three simultaneous differences* (or *increments*), of the two given *sides*, AB, AD, and of the given *area*, or *rectangle*, AC.

(3.) But the *given increments*, BE and DG, of the two given *sides*, are always (by the *construction*) *equimultiples* of the *two first* of the *three new* and *decreasing differences;* they *may*, therefore, by the *definition* (320), be *arbitrarily taken* as *two simultaneous differentials* of the *two sides*, AB and AD, *provided that* we *then treat*, as the corresponding or *simultaneous differential of the rectangle* AC, the *limit of the equimultiple of the new gnomon* (2.), or of the decreasing *difference* between the *two rectangles*, AC and AI, whereof the *first* is *given*.

(4.) We are then, *first*, to *increase* this *new gnomon*, or the *difference* of AC, AI, or the *sum* (2.) of the three partial rectangles, CH, CI, CK, in the *ratio* of BE to BH, or of DG to DK; and *secondly*, to seek the *limit of the area so increased*. For *this last limit* will, by the *definition* (320), be exactly and rigorously *equal* to the sought *differential of the rectangle* AC; *if* the *given* and finite *increments*, BE and DG, be *assumed* (as by (3.) they may) to be the *differentials of the sides*, AB, AD.

(5.) Now when we thus increase the *two new* partial rectangles, CH and CK, we get precisely the *two old* partial rectangles, CE and CG; which, as being *given* and *constant*, must be considered to be *their own limits*.† But when we *increase*, in the *same ratio*, the *other new partial rectangle* CI, we do *not recover* the *old partial rectangle* CF, corresponding to it; but obtain the *new* rectangle CL, or the equal rectangle CM, which is *not constant*, but *diminishes indefinitely* as the point I approaches to C; in such a manner that the *limit of the area*, of this new rectangle CL or CM, is *rigorously null*.

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* The word, *gnomon*, is here used with a slightly more extended signification, than in the Second Book of Euclid.

† Compare the Note to page 395.

(6.) *If*, then, the ***given increments***, BE, DG, be ***still assumed*** to be the ***differentials of the given sides*** AB, AD (an ***assumption*** which has been seen to be ***permitted***), the ***differential of the given area***, or ***rectangle***, AC, is ***proved*** (***not assumed***) to be, as a necessary ***consequence of the definition*** (320), exactly and ***rigorously equal to the sum of the two partial rectangles*** CE and CG; because ***such*** is the ***limit*** (5.) ***of the multiple of the new gnomon*** (2.), in the construction.

(7.) And if any one were to suppose that he could ***improve*** this known ***value*** for the ***differential of a rectangle***, by ***adding to it*** the ***rectangle*** CF, as a new ***term***, or ***part***, so as to make it ***equal*** to the ***old*** or ***given gnomon*** (1.), he would (the ***definition*** being granted) ***commit a geometrical error***, equivalent to that of supposing that the ***two similar rectangles*** CI and CF, bear to each other the ***simple ratio***, instead of bearing (as they do) the ***duplicate ratio***, of their ***homologous sides***.

SECTION 3.—*On some general Consequences of the Definition.*

324. Let there be any proposed equation of the form,

$$\text{I.} \ldots Q = F(q, r, \ldots);$$

and let dq, dr, ... be any *assumed* (but generally *finite*) and *simultaneous differentials* of the *variables*, q, r, ... whether scalars, or vectors, or quaternions, on which Q is supposed to *depend*, by the equation I. Then the *corresponding* (or simultaneous) *differential* of their *function*, Q, is equal (by the definition 320, compare 321) to the following *limit*:

$$\text{II.} \ldots \mathrm{d}Q = \lim_{n=\infty} n\{F(q + n^{-1}\mathrm{d}q,\ r + n^{-1}\mathrm{d}r, \ldots) - F(q, r, \ldots)\};$$

where n is any whole number (or other positive* scalar) which, as the formula expresses, is conceived to become indefinitely greater and greater, and so to tend to infinity. And if, in particular, we consider the function Q as involving only *one* variable q, so that

$$\text{III.} \ldots Q = f(q) = fq,$$

then

$$\text{IV.} \ldots \mathrm{d}Q = \mathrm{d}fq = \lim_{n=\infty} n\{f(q + n^{-1}\mathrm{d}q) - fq\};$$

a *formula for the differential of a single explicit function of a single variable*, which agrees perfectly with those given, near the end of the First Book, for the differentials of a *vector*, and of a *scalar*, considered each as a function (100) of a *single sca-*

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* Except in some rare cases of ***discontinuity***, not at present under our consideration, this ***scalar*** n may as well be conceived to tend to ***negative infinity***.

lar variable, t: but which is now *extended*, as a consequence of the *general definition* (320), to the case when the connected *variables*, q, Q, and their *differentials*, $\mathrm{d}q$, $\mathrm{d}Q$, are *quaternions*: with an analogous application, of the still *more general Formula of Differentiation* II., to *Functions of several Quaternions.*

(1.) As an example of the use of the formula IV., let the function of q be its *square*, so that

$$\text{V.} \ldots \; Q = fq = q^2.$$

Then, by the formula,

$$\text{VI.} \ldots \; \mathrm{d}Q = \mathrm{d}fq = \lim_{n=\infty} . \; n\{(q + n^{-1}\mathrm{d}q)^2 - q^2\}$$
$$= \lim_{n=\infty} . \; (q \,.\, \mathrm{d}q + \mathrm{d}q \,.\, q + n^{-1}\mathrm{d}q^2),$$

where $\mathrm{d}q^2$ signifies[*] the square of $\mathrm{d}q$; that is,

$$\text{VII.} \ldots \; \mathrm{d} \,.\, q^2 = q \,.\, \mathrm{d}q + \mathrm{d}q \,.\, q\,;$$

or without the points[†] between q and $\mathrm{d}q$,

$$\text{VII}'. \ldots \; \mathrm{d} \,.\, q^2 = q\mathrm{d}q + \mathrm{d}q \;\; q\,;$$

an expression for the *differential of the square of a quaternion*, which does not *in general* admit of any further *reduction*: because q and $\mathrm{d}q$ are not generally *commutative*, as *factors* in multiplication. When, however, it *happens*, as in algebra, that $q \,.\, \mathrm{d}q = \mathrm{d}q \,.\, q$, by the two quaternions q and $\mathrm{d}q$ being *complanar*, the expression VII. then evidently reproduces the *usual form*, 322, VIII., or becomes,

$$\text{VIII.} \ldots \; \mathrm{d} \,.\, q^2 = 2q\mathrm{d}q, \quad \textit{if} \quad \mathrm{d}q \parallel\!| \; q \; (123).$$

(2.) As another example, let the function be the *reciprocal*,

$$\text{IX.} \ldots \; Q = fq = q^{-1}.$$

Then, because

$$\text{X.} \ldots \; f(q + n^{-1}\mathrm{d}q) - fq = (q + n^{-1}\mathrm{d}q)^{-1} - q^{-1}$$
$$= (q + n^{-1}\mathrm{d}q)^{-1}\{q - (q + n^{-1}\mathrm{d}q)\}q^{-1}$$
$$= -n^{-1}(q + n^{-1}\mathrm{d}q)^{-1} \,.\, \mathrm{d}q \,.\, q^{-1},$$

of which, when multiplied by n, the limit is $-q^{-1}\mathrm{d}q \,.\, q^{-1}$, we have the following expression for the *differential of the reciprocal of a quaternion*,

$$\text{XI.} \ldots \; \mathrm{d} \,.\, q^{-1} = -q^{-1} \,.\, \mathrm{d}q \,.\, q^{-1}\,;$$

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[*] Compare the Note to page 394.

[†] The *point* between d and q^2, in the first member of VII., is *indispensable*, to distinguish the *differential of the square* from the *square of the differential*. But just as this latter *square* is denoted briefly by $\mathrm{d}q^2$, so the *products*, $q \,.\, \mathrm{d}q$ and $\mathrm{d}q \,.\, q$, may be written as $q\mathrm{d}q$ and $\mathrm{d}q \; q$; the *symbol*, $\mathrm{d}q$, being thus treated as a *whole* one, or as if it were a *single letter*. Yet, for greater clearness of expression, we shall *retain the point* between q and $\mathrm{d}q$, in several (though not in all) of the subsequent formulæ, leaving it to the student to *omit* it, at his pleasure.

or without the *points*[*] in the second member, dq being treated (as in VII′.) as a *whole symbol,*

$$\text{XI}'\ldots\ \mathrm{d}.q^{-1} = -q^{-1}\mathrm{d}q\ q^{-1};$$

an expression which does not *generally* admit of being any farther *reduced,* but becomes, as in the ordinary calculus,

$$\text{XII}\ldots\ \mathrm{d}.q^{-1} = -q^{-2}\mathrm{d}q, \quad \textit{if} \quad \mathrm{d}q \parallel\parallel q,$$

that is, for the *case of complanarity,* of the quaternion and its differential.

325. Other *Examples of Quaternion Differentiation* will be given in the following Section; but the two foregoing may serve sufficiently to exhibit the nature of the operation, and to show the *analogy* of its results to those of the older Calculus, while exemplifying also the *distinction* which generally exists between them. And we shall here proceed to explain a *notation,* which (at least in the *statement* of the present theory of differentials) appears to possess some advantages; and will enable us to offer a still more brief *symbolical definition,* of the *differential of a function* fq, than before.

(1.) We have *defined* (320, 324), that if dq be called the *differential* of a (quaternion or other) *variable,* q, then the *limit of the multiple,*

$$\text{I}\ldots\ n\{f(q+n^{-1}\mathrm{d}q) - fq\},$$

of an *indefinitely decreasing difference* of the *function* fq, of that (single) variable q, when taken relatively to an *indefinite increase* of the *multiplying number,* n, is the corresponding or simultaneous *differential of that function,* and is *denoted,* as such, by the *symbol* dfq.

(2.) But *before* we thus pass to the *limit,* relatively to n, and while that *multiplier,* n, is still considered and treated as *finite,* the *multiple* I. is evidently a *function of that number,* n, *as well* as of the *two independent variables,* q and dq. And we propose to *denote* (at least for the present) *this new function* of the *three variables,*

$$\text{II}\ldots\ n,\ q, \text{ and } \mathrm{d}q,$$

of which the *form depends,* according to the *law* expressed by the formula I., *on the form of the given function,* f, by the *new symbol,*

$$\text{III}\ldots\ f_n(q, \mathrm{d}q);$$

in such a manner as to write, for *any two variables,* q and q', and *any number,* n, the *equation,*

$$\text{IV}\ldots\ f_n(q, q') = n\{f(q+n^{-1}q') - fq\};$$

which may obviously be also written thus,

$$\text{V}\ldots\ f(q+n^{-1}q') = fq + n^{-1}f_n(q, q'),$$

and is here regarded as *rigorously exact,* in virtue of the *definitions,* and without anything whatever being *neglected,* as *small.*

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[*] Compare the Note immediately preceding.

(3.) For example, it appears from the little calculation in 324, (1.), that,

$$\text{VI.} \ldots f_n(q, q') = qq' + q'q + n^{-1}q'^2, \quad \text{if } fq = q^2;$$

and from 324, (2.), that,

$$\text{VII.} \ldots f_n(q, q') = -(q + n^{-1}q')^{-1}q'q^{-1}, \quad \text{if } fq = q^{-1}.$$

(4.) And the definition of d*fq* may now be briefly thus expressed:

$$\text{VIII.} \ldots \mathrm{d}fq = f_\infty(q, \mathrm{d}q);$$

or, if the *sub-index* $_\infty$ be *understood*, we may write, still more simply,

$$\text{IX.} \ldots \mathrm{d}fq = f(q, \mathrm{d}q);$$

this last expression, $f(q, \mathrm{d}q)$, or $f(q, q')$, denoting thus a *function of two independent variables*, q and q', of which the *form* is *derived*[*] or *deduced* (comp. (2.)), from the *given* or *proposed form* of the function fq of a *single variable*, q, according to a *law* which it is one of the main objects of the *Differential Calculus* (at least as regards Quaternions) to study.

326. One of the most important *general properties*, of the *functions of this class* $f(q, q')$, is that they are all *distributive* with respect to the *second independent variable*, q', which is introduced in the foregoing process of what we have called *derivation*,[†] from some *given function* fq, of a *single variable*, q: a theorem which may be proved as follows, whether the two independent variables be, or be not, quaternions.

(1.) Let q'' be any *third* independent variable, and let n be *any number*; then the formula 325, V. gives the three following equations, resulting from the *law of derivation* of $f_n(q, q')$ from fq:

$$\text{I.} \ldots f(q + n^{-1}q'') = fq + n^{-1}f_n(q, q'');$$

$$\text{II.} \ldots f(q + n^{-1}q'' + n^{-1}q') = f(q + n^{-1}q'') + n^{-1}f_n(q + n^{-1}q'', q');$$

$$\text{III.} \ldots f(q + n^{-1}q' + n^{-1}q'') = fq + n^{-1}f_n(q, q' + q'');$$

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[*] It was remarked, or hinted, in 318, that the *usual definition* of a *derived function*, namely, that given by Lagrange in the *Calcul des Fonctions*, cannot be taken as a *foundation* for a differential calculus of *quaternions*: although *such derived functions* of *scalars* present themselves occasionally in the applications of that calculus, as in 100, (3.) and (4.), and in some analogous but more general cases, which will be noticed soon. The *present Law of Derivation* is of an entirely different kind, since it conducts, as we see, from a *given* function of *one* variable, to a *derived* function of *two* variables, which are in general *independent* of each other. The function $f_n(q, q')$, of the *three* variables, n, q, q', may *also* be called a *derived function*, since it is *deduced*, by the *fixed law* IV., from the *same given function* fq, although it has in general a *less simple form* than *its own limit*, $f_\infty(q, q')$, or $f(q, q')$.

[†] Compare the Note immediately preceding.

by comparing which we see at once that

$$\text{IV.}\ldots f_n(q,\, q'+q'') = f_n(q+n^{-1}q'',\, q') + f_n(q,\, q''),$$

the *form* of the *original function*, fq, and the *values* of the *four variables*, q, q', q'', and n, remaining altogether *arbitrary:* except that n is supposed to be a *number*, or at least a *scalar*, while q, q', q'' *may* (or may *not*) *be quaternions.*

(2.) For example, if we take the particular function $fq = q^2$, which gives the form 325, VI. of the derived function $f_n(q,\, q')$, we have

$$\text{V.}\ldots f_n(q,\, q'') = qq'' + q''q + n^{-1}q''^2;$$

$$\text{VI.}\ldots f_n(q,\, q'+q'') = q(q'+q'') + (q'+q'').q + n^{-1}(q'+q'')^2;$$

and therefore

$$\begin{aligned}\text{VII.}\ldots f_n(q,\, q'+q'') - f_n(q,\, q'') &= qq' + q'q + n^{-1}(q'^2 + q'q'' + q''q') \\ &= (q+n^{-1}q'')\,q' + q'\,(q+n^{-1}q'') + n^{-1}q'^2 \\ &= f_n(q+n^{-1}q'',\, q'),\end{aligned}$$

as required by the formula IV.

(3.) Admitting then that formula as proved, for *all* values of the number n, we have only to conceive that *number* (or *scalar*) to *tend to infinity*, in order to deduce this *limiting form* of the equation:

$$\text{VIII.}\ldots f_\infty(q,\, q'+q'') = f_\infty(q,\, q') + f_\infty(q,\, q'');$$

or simply, with the *abridged notation* of 325, (4.),

$$\text{IX.}\ldots f(q,\, q'+q'') = f(q,\, q') + f(q,\, q'');$$

which contains the expression of the *functional property*, above asserted to exist.

(4.) For example, by what has been already shown (comp. 325, (3.) and (4.)),

$$\text{X.}\ldots \text{if } fq = q^2, \quad \text{then} \quad f(q,\, q') = qq' + q'q;$$

$$\text{and XI.}\ldots \text{if } fq = q^{-1}, \quad \text{then} \quad f(q,\, q') = -q^{-1}q'q^{-1};$$

in *each* of which instances we see that the *derived function* $f(q,\, q')$ is *distributive relatively to* q', although it is only in the *first* of them that it happens to be distributive with respect to q *also.*

(5.) It follows at once from the formula IX. that we have generally*

$$\text{XII.}\ldots f(q,\, 0) = 0;$$

This $f = 0$)

and it is not difficult to prove, as a result including this, that

$$\text{XIII.}\ldots f(q,\, xq') = xf(q,\, q'), \text{ if } x \text{ be any } \textit{scalar}.$$

(6.) As a confirmation of this last result, we may observe that the *definition* of $f(q,\, q')$ may be expressed by the following formula (comp. 324, IV., and 325, IX.):

$$\text{XIV.}\ldots f(q,\, q') = \lim_{n=\infty}.\, n\{f(q+n^{-1}q') - fq\};$$

we have therefore, if x be any finite scalar, and $m = x^{-1}n$,

$$\text{XV.}\ldots f(q,\, xq') = x.\lim_{m=\infty}.\, m\{f(q+m^{-1}q') - fq\};$$

a transformation which gives the recent property XIII., since it is evident that the letter m may be written instead of n, in the formula of definition XIV.

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* We abstract here from some exceptional cases of *discontinuity*, &c.

327. Resuming then the general expression 325, IX., or writing anew,

$$\text{I.} \ldots \mathrm{d}fq = f(q, \mathrm{d}q),$$

we see (by 326, IX.) that *this derived function*, d*fq*, of *q* and d*q*, is always (as in the examples 324, VII. and XI.) *distributive with respect to that differential* d*q*, considered as an *independent variable*, whatever the *form* of the *given function fq* may be. We see also (by 326, XIII.), that *if the differential* d*q of the variable*, *q*, *be multiplied by any scalar*, *x*, *the differential* d*fq*, *of the function fq*, comes to be *multiplied*, at the same time, by the *same scalar*, or that

$$\text{II.} \ldots f(q, x\mathrm{d}q) = xf(q, \mathrm{d}q), \text{ if } x \text{ be any scalar.}$$

And in fact it is evident, from the very *conception* and *definition* (320) of *simultaneous differentials*, that every *system* of *such* differentials must admit of being *all changed together* to any system of *equimultiples*, or *equisubmultiples*, of themselves, *without ceasing* to be *simultaneous* differentials: or more generally, that it is *permitted to multiply all the differentials of a system*, by *any common scalar*.

(1.) It follows that the *quotient*,

$$\text{III.} \ldots \mathrm{d}fq : \mathrm{d}q = f(q, \mathrm{d}q) : \mathrm{d}q,$$

of the *two simultaneous differentials*, d*fq* and d*q*, *does not change* when the differential d*q* is thus *multiplied by any scalar*; and consequently that this quotient III. is *independent of the tensor* Td*q*, although it is *not generally independent of the versor* Ud*q*, if *q* and d*q* be *quaternions*: except that it remains in general *unchanged*, when we merely change that *versor* to its own *opposite* (or negative), or to —Ud*q*, because this comes to multiplying d*q* by − 1, which is a scalar.

(2.) For example, the quotient,

$$\text{IV.} \ldots \mathrm{d}.q^2 : \mathrm{d}q = q + \mathrm{d}q.q.\mathrm{d}q^{-1} = q + \mathrm{Ud}q.q.\mathrm{Ud}q^{-1},$$

in which $\mathrm{d}q^{-1}$ and $\mathrm{Ud}q^{-1}$ denote the reciprocals of d*q* and Ud*q*, is very far from being independent of d*q*, or at least of Ud*q*; since it represents, as we see, the *sum* of the *given* quaternion *q*, and of a certain *other* quaternion, which latter, in its *geometrical interpretation* (comp. 191, (5.)), may be considered as being *derived* from *q*, by a *conical rotation* of A*x*.*q* round A*x*.d*q*, through an *angle* $= 2\angle \mathrm{d}q$: so that both the *axis* and the *quantity* of this rotation *depend* on the versor Ud*q*, and *vary with that versor*.

(3.) In general we may, if we please, say that the *quotient* III. is a *Differential Quotient*; but we ought not to call it a *Differential Coefficient* (comp. 318), because d*fq* does *not* generally admit of *decomposition into two factors*, whereof *one* shall be the *differential* d*q*, and the *other* a *function of q alone*.

(4.) And for the same reason, we ought not to call that *Quotient* a *Derived Function* (comp. again 318), unless in so speaking we understand a *Function of Two** *independent Variables*, namely of q and U dq, as before.

(5.) When, however, a *quaternion*, q, is considered as a *function of a scalar variable*, t, so that we have an equation of the form,

$$\text{V.} \ldots q = ft, \text{ where } t \text{ denotes a scalar,}$$

it is *then* permitted (comp. 100, (8.) and (4.)) to write,

$$\begin{aligned}\text{VI.} \ldots \mathrm{d}q : \mathrm{d}t = \mathrm{d}ft : \mathrm{d}t &= \lim_{n=\infty} . \frac{n}{\mathrm{d}t}\left\{f\left(t + \frac{\mathrm{d}t}{n}\right) - ft\right\} \\ &= \lim_{h=0} . h^{-1}\{f(t+h) - ft\} \\ &= f't = \mathrm{D}_t ft = \mathrm{D}_t q\,;\end{aligned}$$

and to *call this limit*, as usual, a *derived function of* t, because *it is* (in fact) a *function of that scalar variable*, t, *alone*, and is *independent of the scalar differential*, dt.

(6.) We may also *write*, under these circumstances, the *differential equation*,

$$\text{VII.} \ldots \mathrm{d}q = \mathrm{D}_t q . \mathrm{d}t, \quad \text{or} \quad \text{VIII.} \ldots \mathrm{d}fq = f't . \mathrm{d}t,$$

and may *call the derived quaternion*, $\mathrm{D}_t q$, or $f't$, as usual, a *differential coefficient* in *this* formula, because the *scalar differential*, dt, is (in fact) *multiplied by it*, in the expression thus found for the *quaternion differential*, dq or dft.

(7.) But as regards the *logic* of the question (comp. again 100, (3.)), it is important to remember that *we* regard this *derived function*, or *differential coefficient*,

$$\text{IX.} \ldots f't, \quad \text{or} \quad \mathrm{D}_t ft, \quad \text{or} \quad \mathrm{D}_t q,$$

as being an *actual quotient* VI., obtained by *dividing an actual quaternion*,

$$\text{X.} \ldots \mathrm{d}ft, \quad \text{or} \quad \mathrm{d}q,$$

by an actual scalar, dt, of which the *value* is altogether *arbitrary*, and may (if we choose) be supposed to be *large* (comp. 322); while the *dividend quaternion* X. *depends, for its value, on the values of the two independent scalars*, t and dt, *and on the form of the function* ft, according to the *law* which is expressed by the *general formula* 324, IV., for the *differentiation of explicit functions of any single variable.*

328. It is easy to conceive that similar remarks apply to *quaternion functions of more variables than one;* and that when the *differential* of *such* a *function* is expressed (comp. 324, II.) under the form,

$$\text{I.} \ldots \mathrm{d}Q = \mathrm{d}F(q, r, s, \ldots) = F(q, r, s, \ldots \mathrm{d}q, \mathrm{d}r, \mathrm{d}s, \ldots),$$

the *new function* F is always *distributive*, with respect to *each* separately of the *differentials*, dq, dr, ds, . .; being also *homogeneous of the first dimension* (comp. 327), with respect to *all* those differentials, considered as a *system;* in such a manner that, whatever may be the *form* of the *given quaternion function*, Q, or F, the *derived* function* F, or the *third member* of the *formula* I., must possess this general *functional property* (comp. 326, XIII., and 327, II.),

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* Compare the Note to 325, (4.).

$$\text{II.} \ldots F(q, r, s, \ldots x\text{d}q, x\text{d}r, x\text{d}s \ldots) = xF(q, r, s, \ldots \text{d}q, \text{d}r, \text{d}s, \ldots),$$

where x may be *any scalar:* so that *products*, as well as *squares*, of the *differentials* $\text{d}q$, $\text{d}r$, &c., of q, r, &c. considered as so many *variables* on which Q *depends*, are *excluded* from the *expanded expression* of the *differential* $\text{d}Q$ *of the function* Q.

(1.) For example, if the function to be differentiated be a *product* of *two* quaternions,

$$\text{III.} \ldots Q = F(q, r) = qr,$$

then it is easily found from the general formula 324, II., that (because the *limit* of $n^{-1}.\text{d}q.\text{d}r$ is *null*, when the *number* n *increases without limit*) the differential of the function is,

$$\text{IV.} \ldots \text{d}Q = \text{d}.qr = \text{d}F(q, r) = F(q, r, \text{d}q, \text{d}r) = q.\text{d}r + \text{d}q.r;$$

with analogous results, for differentials of products of more than two quaternions.

(2.) Again, if we take this other function,

$$\text{V.} \ldots Q = F(q, r) = q^{-1}r,$$

then, applying the same general formula 324, II., and observing that we have, for *all values* of the *number* (or other *scalar*), n, and of the *four quaternions*, q, r, q', r', the identical *transformation* (comp. 324, (2.)),

$$\text{VI.} \ldots n\{(q + n^{-1}q')^{-1}(r + n^{-1}r') - q^{-1}r\}$$
$$= q^{-1}r' - (q + n^{-1}q')^{-1}q'q^{-1}(r + n^{-1}r'),$$

we find, as the required *limit*, when n tends to *infinity*, the following *differential of the function :*

$$\text{VII.} \ldots \text{d}Q = \text{d}.q^{-1}r = \text{d}F(q, r) = F(q, r, \text{d}q, \text{d}r) = q^{-1}.\text{d}r - q^{-1}.\text{d}q.q^{-1}r;$$

which is again, like the expression IV., *distributive* with respect to *each* of the *differentials* $\text{d}q$, $\text{d}r$, of the *variables* q, r, and does *not* involve the *product* of those two differentials: although these two differential expressions, IV. and VII., are both entirely *rigorous*, and are not *in any way* dependent on any supposition that the *tensors* of $\text{d}q$ and $\text{d}r$ are *small* (comp. again 322).

329. In thus differentiating a function of more variables than one, we are led to consider what may be called *Partial Differentials of Functions of two or more Quaternions;* which may be thus *denoted*,

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* Compare the Note last referred to.

$$\text{I.} \ldots \mathrm{d}_q Q,\ \mathrm{d}_r Q,\ \mathrm{d}_s Q, \ldots$$

if Q be a function, as above, of $q, r, s, \ldots$ which is here supposed to be differentiated with respect to *each* variable *separately, as if* the *others* were *constant.* And then, if $\mathrm{d}Q$ denote, as before, what may be called, by contrast, the *Total Differential* of the function Q, we shall have the *General Formula,*

$$\text{II.} \ldots \mathrm{d}Q = \mathrm{d}_q Q + \mathrm{d}_r Q + \mathrm{d}_s Q + \ldots;$$

or, briefly and symbolically,

$$\text{III.} \ldots \mathrm{d} = \mathrm{d}_q + \mathrm{d}_r + \mathrm{d}_s + \ldots,$$

if $q, r, s, \ldots$ denote the quaternion variables on which the *quaternion function* depends, of which the total differential is to be taken; whether those *variables* be all *independent*, or be *connected* with each other, by any *relation* or relations.

(1.) For example (comp. 328, (1.)),

$$\text{IV.} \ldots \text{if } Q = qr, \text{ then } \mathrm{d}_q Q = \mathrm{d}q \,.\, r, \text{ and } \mathrm{d}_r Q = q \,.\, \mathrm{d}r;$$

and the *sum* of these *two partial differentials* of Q makes up its *total differential* $\mathrm{d}Q$, as otherwise found above.

(2.) Again (comp. 328, (2.)),

$$\text{V.} \ldots \text{if } Q = q^{-1}r, \text{ then } \mathrm{d}_q Q = -q^{-1}\mathrm{d}q \,.\, q^{-1}r; \quad \mathrm{d}_r Q = q^{-1}\mathrm{d}r;$$

and $\mathrm{d}_q Q + \mathrm{d}_r Q =$ the same $\mathrm{d}Q$ as that which was otherwise found before, for this form of the function Q.

(3.) To exemplify the possibility of a *relation* existing between the *variables* q and r, let those variables be now supposed *equal* to each other in V.; we shall then have $Q = 1$, $\mathrm{d}Q = 0$; and accordingly we have here $\mathrm{d}_q Q = -q^{-1}\mathrm{d}q = -\mathrm{d}_r Q$.

(4.) Again, in IV., let $qr = c =$ any *constant* quaternion; we shall then again have $0 = \mathrm{d}Q = \mathrm{d}_q Q + \mathrm{d}_r Q$; and may infer that

$$\text{VI.} \ldots \mathrm{d}r = -q^{-1} \,.\, \mathrm{d}q \,.\, r, \quad \text{if } qr = c = \text{const.};$$

a result which evidently agrees with, and includes, the expression 324, XI., for the *differential of a reciprocal.*

(5.) A *quaternion*, q, may happen to be expressed as a *function of two or more scalar variables*, $t, u, \ldots$; and then it will have, as such, by the present Article, its *partial differentials*, $\mathrm{d}_t q$, $\mathrm{d}_u q$, &c. But because, by 327, VII., we may in *this* case write,

$$\text{VII.} \ldots \mathrm{d}_t q = \mathrm{D}_t q \,.\, \mathrm{d}t, \quad \mathrm{d}_u q = \mathrm{D}_u q \,.\, \mathrm{d}u, \ldots$$

where the *coefficients* are *independent* of the *differentials* (as in the ordinary calculus), we shall have (by II.) an expression for the *total differential* $\mathrm{d}q$, of the form,

$$\text{VIII.} \ldots \mathrm{d}q = \mathrm{d}_t q + \mathrm{d}_u q + \ldots = \mathrm{D}_t q \,.\, \mathrm{d}t + \mathrm{D}_u q \,.\, \mathrm{d}u + \ldots;$$

and may at pleasure say, *under the conditions here supposed*, that the *derived quaternions*,

$$\text{IX.}\ldots \mathrm{D}_t q,\quad \mathrm{D}_u q,\ \ldots$$

are either the *Partial Derivatives*, or the *Partial Differential Coefficients*, of the *Quaternion Function*,

$$\text{X.}\ldots q = F(t,\ u,\ \ldots);$$

with analogous remarks for the *case*, when the *quaternion*, q, *degenerates* (comp. 289) into a *vector*, ρ.

330. In general, it may be considered as evident, from the definition in 320, that the *differential of a constant* is *zero;* so that if Q be changed to *any constant quaternion*, c, in the equation 324, I., then $\mathrm{d}Q$ is to be *replaced by* 0, in the *differentiated equation*, 324, II. And if there be given any *system of equations*, connecting the *quaternion variables*, q, r, s, . . . we may treat the corresponding *system of differentiated equations*, as holding good, for the *system of simultaneous differentials*, $\mathrm{d}q$, $\mathrm{d}r$, $\mathrm{d}s$, . . . ; and may therefore, legitimately in theory, whenever in practice it shall be found to be possible, *eliminate* any one or more of those *differentials*, between the equations of this system.

(1.) As an example, let there be the two equations,

$$\text{I.}\ldots qr = c,\quad \text{and II.}\ldots s = r^2,$$

where c denotes a constant quaternion. Then (comp. 328, (1.), and 324, (1.)) we have the two differentiated equations corresponding,

$$\text{III.}\ldots q\,.\,\mathrm{d}r + \mathrm{d}q\,.\,r = 0;\qquad \text{IV.}\ldots \mathrm{d}s = r\,.\,\mathrm{d}r + \mathrm{d}r\,.\,r;$$

in which the *points** might be omitted. The former gives,

$$\text{V.}\ldots \mathrm{d}r = -q^{-1}\mathrm{d}q\,.\,r,\ \text{as in 329, VI.};$$

and when we substitute this value in the latter, we thereby *eliminate the differential* $\mathrm{d}r$, and obtain this *new differential equation*,

$$\text{VI.}\ldots \mathrm{d}s = -rq^{-1}\,.\,\mathrm{d}q\,.\,r - q^{-1}\,.\,\mathrm{d}q\,.\,r^2.$$

(2.) The equation I. gives also the expression,

$$\text{VII.}\ldots r = q^{-1}c;$$

the equation II. gives therefore this other expression,

$$\text{VIII.}\ldots s = (q^{-1}c)^2 = q^{-1}cq^{-1}c,$$

by *elimination before differentiation*. And if, in the formula VI., we substitute the expressions VII. and VIII. for r and s, we get this *other* differential equation,

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* Compare the second Note to 324, (1.).

$$\text{IX.} \ldots \mathrm{d}.(q^{-1}c)^2 = -q^{-1}cq^{-1}.\mathrm{d}q.q^{-1}c - q^{-1}.\mathrm{d}q.q^{-1}cq^{-1}c;$$

which might have been otherwise obtained (comp. again 324, (1.) and (2.)), under the form,

$$\text{X.} \ldots \mathrm{d}.(q^{-1}c)^2 = q^{-1}c.\mathrm{d}(q^{-1}c) + \mathrm{d}(q^{-1}c).q^{-1}c.$$

331. *No special rules* are required, for the *differentiation of functions of functions* of quaternions; but it may be instructive to show, briefly, how the consideration of *such* differentiation conducts (comp. 326) to a *general property of functions of the class* $f(q, q')$; and how that property can be *otherwise* established.

(1.) Let f, ϕ, and ψ denote any functional operators, such that

$$\text{I.} \ldots \psi q = \phi(fq);$$

then writing

$$\text{II.} \ldots r = fq, \quad \text{and III.} \ldots s = \phi r, \quad \text{we have IV.} \ldots s = \psi q;$$

$$\text{whence V.} \ldots \mathrm{d}s = \mathrm{d}\psi q = \mathrm{d}\phi r.$$

That is, we may (as usual) *differentiate the compound function*, $\phi(fq)$, *as if* fq *were an independent variable*, r; and *then*, in the expression so found, *replace the differential* dfq *by its value*, obtained by *differentiating the simple function*, fq. For this comes virtually to the *elimination of the differential* dr, or of the *symbol* dfq, in a way which we have seen to be permitted (330).

(2.) But, by the *definitions* of dfq and $f_n(q, q')$, we saw (325, VIII. IX.) that the differential dfq might generally be denoted by $f_\infty(q, \mathrm{d}q)$, or briefly by $f(q, \mathrm{d}q)$; whence dϕr and dψq may also, by an extension of the same notation, be represented by the analogous symbols, $\phi_\infty(r, \mathrm{d}r)$ and $\psi_\infty(q, \mathrm{d}q)$, or simply by $\phi(r, \mathrm{d}r)$ and $\psi(q, \mathrm{d}q)$.

(3.) We ought, therefore, to find that

$$\text{VI.} \ldots \psi_\infty(q, \mathrm{d}q) = \phi_\infty(fq, f_\infty(q, \mathrm{d}q)), \quad \text{if} \quad \psi q = \phi(fq);$$

or briefly that

$$\text{VII.} \ldots \psi(q, q') = \phi(fq, f(q, q')), \quad \text{if} \quad \psi q = \phi fq,$$

for *any two quaternions*, q, q', and *any two functions*, f, ϕ; provided that the functions $f_n(q, q')$, $\phi_n(q, q')$, $\psi_n(q, q')$ are *deduced* (or *derived*) from the functions fq, ϕq, ψq, according to the *law* expressed by the formula 325, IV.; and that then the *limits* to which these *derived functions* $f_n(q, q')$, &c. *tend*, when the *number* n tends to *infinity*, are denoted by these *other functional symbols*, $f(q, q')$, &c.

(4.) To prove this *otherwise*, or to establish this *general property* VII., *of functions of this class* $f(q, q')$, *without any use of differentials*, we may observe that the general and rigorous *transformation* 325, V., of the formula 325, IV. by which the functions $f_n(q, q')$ are *defined*, gives for *all* values of n the equation:

$$\begin{aligned}\text{VIII.} \ldots \phi f(q + n^{-1}q') &= \phi(fq + n^{-1}f_n(q, q')) \\ &= \phi fq + n^{-1}\phi_n(fq, f_n(q, q'));\end{aligned}$$

but also, by the same general transformation,

$$\text{IX.} \ldots \psi(q + n^{-1}q') = \psi q + n^{-1}\psi_n(q, q');$$

hence *generally*, for *all values of the number n*, as well as for *all values of the two independent quaternions*, q, q', and for *all forms of the two functions*, f, ϕ, we may write,

$$\text{X.} \ldots \psi_n(q, q') = \phi_n(fq, f_n(q, q')), \quad \text{if} \quad \psi q = \phi f q;$$

an equation of which the *limiting form*, for $n = \infty$, is (with the notations used) the equation VII. which was to be proved.

(5.) It is scarcely worth while to verify the general formula X., by any particular example: yet, merely as an exercise, it may be remarked that if we take the forms,

$$\text{XI.} \ldots fq = q^2, \quad \phi q = q^2, \quad \psi q = q^4,$$

of which the two first give, by 325, VI., the common derived form,

$$\text{XII.} \ldots f_n(q, q') = \phi_n(q, q') = qq' + q'q + n^{-1}q'^2,$$

the formula X. becomes,

$$\text{XIII.} \ldots \psi_n(q, q') = \phi_n(q^2, qq' + q'q + n^{-1}q'^2)$$
$$= q^2(qq' + q'q + n^{-1}q'^2) + (qq' + q'q + n^{-1}q'^2)q^2 + n^{-1}(qq' + q'q + n^{-1}q'^2)^2;$$

which agrees with the value deduced immediately from the function ψq or q^4, by the definition 325, IV., namely,

$$\text{XIV.} \ldots \psi_n(q, q') = n\{(q + n^{-1}q')^4 - q^4\}$$
$$= n\{(q^2 + n^{-1}(qq' + q'q + n^{-1}q'^2))^2 - (q^2)^2\}.$$

(6.) In general, the *theorem*, or *rule*, for *differentiating* as in (1.) a *function of a function*, of a quaternion or other variable, may be briefly and symbolically expressed by the formula,

$$\text{XV.} \ldots \mathrm{d}(\phi f)q = \mathrm{d}\phi(fq);$$

and if we did not otherwise know it, a *proof* of its correctness would be supplied, by the recent proof of the correctness of the equivalent formula VII.

SECTION 4.—*Examples of Quaternion Differentiation.*

332. It will now be easy and useful to give a short collection of *Examples of Differentiation of Quaternion Functions and Equations*, additional to and inclusive of those which have incidentally occurred already, in treating of the *principles* of the subject.

(1.) If c be any *constant* quaternion (as in 330), then

$$\text{I.} \ldots \mathrm{d}c = 0; \quad \text{II.} \ldots \mathrm{d}(fq + c) = \mathrm{d}fq;$$
$$\text{III.} \ldots \mathrm{d}.cfq = c\mathrm{d}fq; \quad \text{IV.} \ldots \mathrm{d}(fq.c) = \mathrm{d}fq.c.$$

(2.) In general,

$$\text{V.} \ldots \mathrm{d}(fq + \phi q + \ldots) = \mathrm{d}fq + \mathrm{d}\phi q + \ldots; \quad \text{or briefly, VI.} \ldots \mathrm{d}\Sigma = \Sigma\mathrm{d},$$

if Σ be used as a mark of summation.

(3.) Also, $\text{VII.} \ldots \mathrm{d}(fq \cdot \phi q) = \mathrm{d}fq \cdot \phi q + fq \cdot \mathrm{d}\phi q;$

and similarly for a product of more functions than two: the *rule* being simply, to *differentiate each factor separately, in its own place*, or without disturbing the *order* of the *factors* (comp. 318, 319); and then to *add together the partial results* (comp. 329).

(4.) In particular, if m be any positive whole number,

$$\text{VIII.} \ldots \mathrm{d}.q^m = q^{m-1}dq + q^{m-2}\mathrm{d}q.q \ldots + q\mathrm{d}q.q^{m-2} + \mathrm{d}q.q^{m-1};$$

and because we have seen (324, (2.)) that

$$\text{IX.} \ldots \mathrm{d}.q^{-1} = -q^{-1}.\mathrm{d}q.q^{-1},$$

we have this analogous expression for the differential of a *power* of a *quaternion*, with a *negative* but *whole exponent*,

$$\text{X.} \ldots \mathrm{d}.q^{-m} = -q^{-m}(\mathrm{d}.q^m).q^{-m}$$
$$= -q^{-1}\mathrm{d}q.q^{-m} - q^{-2}\mathrm{d}q.q^{1-m} - \ldots - q^{1-m}\mathrm{d}q.q^{-2} - q^{-m}\mathrm{d}q.q^{-1}.$$

(5.) To *differentiate a square root*, we are to *resolve the linear equation*,*

$$\text{XI.} \ldots q^{\frac{1}{2}}.\mathrm{d}.q^{\frac{1}{2}} + \mathrm{d}.q^{\frac{1}{2}}.q^{\frac{1}{2}} = \mathrm{d}q; \quad \text{or} \quad \text{XI'.} \ldots rr' + r'r = q',$$

if we write, for abridgment,

$$\text{XII.} \ldots r = q^{\frac{1}{2}}, \quad q' = \mathrm{d}q, \quad r' = \mathrm{d}.q^{\frac{1}{2}} = \mathrm{d}r.$$

(6.) Writing also, for this purpose,

$$\text{XIII.} \ldots s = \mathrm{K}r = \mathrm{K}.q^{\frac{1}{2}},$$

whence (by 190, 196) it will follow that

$$\text{XIV.} \ldots rs = \mathrm{N}r = \mathrm{T}r^2 = \mathrm{T}q, \quad \text{and XV.} \ldots r + s = 2\mathrm{S}r = 2\mathrm{S}.q^{\frac{1}{2}},$$

the *product* and *sum* of these two *conjugate quaternions*, r and s, being thus *scalars* (140, 145), we have, by XI'.,

$= r^{-1}rr's + r^{-1}r'rs$

$$\text{XVI.} \ldots r^{-1}q's = r's + sr';$$

$= r's + r^{-1}(rs)r'$
$= r's + sr'$

whence, by addition,

$$\text{XVII.} \ldots q' + r^{-1}q's = (r+s)r' + r'(r+s) = 2r'(r+s);$$

and finally,

$$\text{XVIII.} \ldots r' = \frac{q' + r^{-1}q's}{2(r+s)}, \quad \text{or} \quad \text{XIX.} \ldots \mathrm{d}.q^{\frac{1}{2}} = \frac{\mathrm{d}q + q^{-\frac{1}{2}}\mathrm{d}q.\mathrm{K}.q^{\frac{1}{2}}}{4\mathrm{S}.q^{\frac{1}{2}}}; \quad +$$

an expression for the differential of the square-root of a quaternion, which will be found to admit of many transformations, not needful to be considered here.

(7.) In the three last sub-articles, as in the three preceding them, it has been supposed, for the sake of generality, that q and $\mathrm{d}q$ are two *diplanar quaternions;* but if in any application they *happen*, on the contrary, to be *complanar*, the expressions are then *simplified*, and take *usual*, or *algebraic forms*, as follows:

$$\text{XX.} \ldots \mathrm{d}.q^m = mq^{m-1}\mathrm{d}q; \qquad \text{XXI.} \ldots \mathrm{d}.q^{-m} = -mq^{-m-1}\mathrm{d}q;$$

and $\quad \text{XXII.} \ldots \mathrm{d}.q^{\frac{1}{2}} = \frac{1}{2}q^{-\frac{1}{2}}\mathrm{d}q, \quad$ *if* $\quad \text{XXIII.} \ldots \mathrm{d}q \,|||\, q$ (123);

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* Although such *solution of a linear equation*, or equation of the *first degree*, in quaternions, is easily enough accomplished in the present instance, yet in general the problem presents difficulties, without the consideration of which the theory of *differentiation of implicit functions of quaternions* would be entirely incomplete. But a *general method*, for the solution of *all such equations*, will be sketched in a subsequent Section.

when $q \,|||\, dq$ $\qquad d.q^{\frac{1}{2}} = \qquad q^{-\frac{1}{2}} \dfrac{q^{\frac{1}{2}}dq + Kq^{\frac{1}{2}}dq}{2(q^{\frac{1}{2}} + Kq^{\frac{1}{2}})} = \frac{1}{2}q^{-\frac{1}{2}}dq.$

because, when q' is complanar with q, and therefore with $q^{\frac{1}{2}}$, or with r, in the expression XVIII., the numerator of that expression may be written as $r^{-1}q'(r+s)$.

(8.) More generally, if x be *any scalar exponent*, we may write, as in the ordinary calculus, but still under the *condition of complanarity* XXIII.,

$$\text{XXIV.} \ldots \mathrm{d}\,.\,q^x = xq^{x-1}\mathrm{d}q; \quad \text{or} \quad \text{XXV.} \ldots q\mathrm{d}\,.\,q^x = xq^x\,\mathrm{d}x.$$

333. The *functions* of quaternions, which have been lately differentiated, may be said to be of *algebraic form;* the following are a few examples of differentials of what may be called, by contrast, *transcendental functions of quaternions:* the condition of *complanarity* ($\mathrm{d}q \parallel\!| q$) being however *here* supposed to be satisfied, in order that the expressions may not become too complex. In fact, *with this simplification*, they will be found to assume, for the most part, the *known* and *usual forms*, of the *ordinary differential calculus.*

(1.) Admitting the definitions in 316, and supposing throughout that $\mathrm{d}q \parallel\!| q$. we have the usual expressions for the differentials of ε^q and $\mathrm{l}q$, namely,

$$\text{I.} \ldots \mathrm{d}.\,\varepsilon^q = \varepsilon^q\mathrm{d}q\,; \qquad \text{II.} \ldots \mathrm{dl}q = q^{-1}\mathrm{d}q.$$

(2.) We have also, by the same system of definitions (316),

$$\text{III.} \ldots \mathrm{d}\sin q = \cos q\mathrm{d}q\,; \qquad \text{IV.} \ldots \mathrm{d}\cos q = -\sin q\mathrm{d}q\,;\ \&\text{c.}$$

(3.) Also, if r and $\mathrm{d}r$ be complanar with q and $\mathrm{d}q$, then, by 316,

$$\text{IV}'. \ldots \mathrm{d}\,.\,q^r = \mathrm{d}.\,\varepsilon^{r\mathrm{l}q} = q^r\mathrm{d}.r\mathrm{l}q = q^r(\mathrm{l}q\mathrm{d}r + q^{-1}r\mathrm{d}q)\,;$$

or in the notation of partial differentials (329),

$$\text{V.} \ldots \mathrm{d}_q.q^r = rq^{r-1}\mathrm{d}q, \quad \text{and} \quad \text{VI.} \ldots \mathrm{d}_r.q^r = q^r\mathrm{l}q\mathrm{d}r.$$

(4.) In particular, if the *base* q be a *given* or *constant vector*, a, and if the *exponent* r be a *variable scalar*, t, then (by the value 316, XIV. of $\mathrm{l}\rho$) the recent formula IV. becomes,

$$\text{VII.} \ldots \mathrm{d}\,.\,a^t = \left(\mathrm{lT}a + \frac{\pi}{2}\,\mathrm{U}a\right)a^t\mathrm{d}t.$$

(5.) If then the base a be a *given unit line*, so that $\mathrm{lT}a = 0$, and $\mathrm{U}a = a$, we may write simply,

$$\text{VIII.} \ldots \mathrm{d}\,.\,a^t = \frac{\pi}{2}\,a^{t+1}\,\mathrm{d}t, \quad \text{if} \quad \mathrm{d}a = 0, \quad \text{and} \quad \mathrm{T}a = 1.$$

(6.) This useful formula, for the differential of a power of a constant unit line, with a variable scalar exponent, may be obtained more rapidly from the equation 308, VII., which gives,

$$\text{IX.} \ldots a^t = \cos\frac{t\pi}{2} + a\sin\frac{t\pi}{2}, \quad \text{if} \quad \mathrm{T}a = 1\,;$$

since it is evident that the differential of this expression is equal to the expression itself multiplied by $\frac{1}{2}\pi a\mathrm{d}t$, because $a^2 = -1$.

(7.) The formula VIII. admits also of a simple *geometrical interpretation*, connected with the *rotation* through *t right angles*, in a *plane perpendicular to a*, of which rotation, or *version*, the *power* a^t, or the *versor* $\mathrm{U}a^t$, is considered (308) to be the *instrument*,* or *agent*, or *operator* (comp. 293).

334. Besides *algebraical* and *transcendental forms*, there are *other results of operation* on a quaternion, q, or on a function thereof, which may be regarded as forming a *new class* (or kind) of *functions*, arising out of the *principles* and *rules* of the *Quaternion Calculus itself*: namely those which we have denoted in former Chapters by the *symbols*,

$$\text{I.} \ldots \mathrm{K}q,\ \mathrm{S}q,\ \mathrm{V}q,\ \mathrm{N}q,\ \mathrm{T}q,\ \mathrm{U}q,$$

or by symbols formed through *combinations* of the same *signs of operation*, such as

$$\text{II.} \ldots \mathrm{SU}q,\ \mathrm{VU}q,\ \mathrm{UV}q,\ \&\text{c}.$$

And it is essential that we should know how to *differentiate* expressions *of these forms*, which can be done in the following manner, with the help of the principles of the present and former Chapters, and *without* now assuming the *complanarity*, $\mathrm{d}q \parallel\!| q$.

(1.) In general, let f represent, for a moment, *any distributive symbol*, so that for any two quaternions, q and q', we shall have the equation,

$$\text{III.} \ldots f(q+q') = fq + fq';$$

and therefore also† (comp. 326, (5.)),

$$\text{IV.} \ldots f(xq) = xfq, \text{ if } x \text{ be any scalar.}$$

(2.) Then, with the notation 325, IV., we shall have

$$\text{V.} \ldots f_n(q, q') = n\{f(q + n^{-1}q') - fq\} = fq';$$

and therefore, by 325, VIII., for any *such* function fq, we shall have the differential expression,

$$\text{VI.} \ldots \mathrm{d}fq = f\mathrm{d}q.$$

(3.) But S, V, K have been seen to be *distributive symbols* (197, 207); we can therefore infer at once that

$$\text{VII.} \ldots \mathrm{dK}q = \mathrm{Kd}q; \qquad \text{VIII.} \ldots \mathrm{dS}q = \mathrm{Sd}q; \qquad \text{IX.} \ldots \mathrm{dV}q = \mathrm{Vd}q;$$

or in words, that *the differentials of the conjugate, the scalar, and the vector of a quaternion are*, respectively, *the conjugate, the scalar, and the vector of the differential of that quaternion.*

(4.) To find the *differential of the norm*, $\mathrm{N}q$, or to deduce an *expression for* $\mathrm{dN}q$, we have (by VII. and 145) the equation,

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* Compare the second Note to page 133.

† In *quaternions* the equation III. is not a necessary consequence of IV., although the latter is so of the former; for example, the equation IV., but *not* the equation III., will be satisfied, if we assume $fq = qcq^{-1}c'q$, where c and c' are any two constant quaternions, which do not degenerate into scalars.

$$\text{X.}\ \ldots\ \mathrm{dN}q = \mathrm{d}\,.\,q\mathrm{K}q = \mathrm{d}q\,.\,\mathrm{K}q + q\,.\,\mathrm{Kd}q;$$

but $q\mathrm{K}q' = \mathrm{K}\,.\,q'\mathrm{K}q$, by 145, and 192, II.;

and $(1+\mathrm{K})\,.\,q'\mathrm{K}q = 2\mathrm{S}\,.\,q'\mathrm{K}q = 2\mathrm{S}(\mathrm{K}q\,.\,q')$, by 196, II., and 198, I.;

therefore

$$\text{XI.}\ \ldots\ \mathrm{dN}q = 2\mathrm{S}(\mathrm{K}q\,.\,\mathrm{d}q).$$

(5.) Or we might have deduced this expression XI. for $\mathrm{dN}q$, more immediately, by the *general formula* 324, IV., from the earlier expression 200, VII., or 210, XX., for the *norm of a sum*, under the form,

$$\text{XI}'.\ \ldots\ \mathrm{dN}q = \lim_{n=\infty}.\,n\{\mathrm{N}(q+n^{-1}\mathrm{d}q) - \mathrm{N}q\}$$
$$= \lim_{n=\infty}.\,\{2\mathrm{S}(\mathrm{K}q\,.\,\mathrm{d}q) + n^{-1}\mathrm{Nd}q\}$$
$$= 2\mathrm{S}(\mathrm{K}q\,.\,\mathrm{d}q),$$

as before.

(6.) The *tensor*, $\mathrm{T}q$, is the *square-root* (190) *of the norm*, $\mathrm{N}q$; and because $\mathrm{T}q$ and $\mathrm{N}q$ are scalars, the formula 332, XXII. may be applied; which gives, for the *differential of the tensor* of a quaternion, the expression (comp. 158),

$$\text{XII.}\ \ldots\ \mathrm{dT}q = \frac{\mathrm{dN}q}{2\mathrm{T}q} = \mathrm{S}(\mathrm{KU}q.\mathrm{d}q) = \mathrm{S}\frac{\mathrm{d}q}{\mathrm{U}q},$$

a result which is more easily *remembered*, under the form,

$$\text{XIII.}\ \ldots\ \frac{\mathrm{dT}q}{\mathrm{T}q} = \mathrm{S}\frac{\mathrm{d}q}{q}.$$

(7.) The *versor* $\mathrm{U}q$ is equal (by 188) to the *quotient*, $q : \mathrm{T}q$, of the quaternion q divided by its tensor $\mathrm{T}q$; hence the *differential of the versor* is,

$$\text{XIV.}\ \ldots\ \mathrm{dU}q = \mathrm{d}\frac{q}{\mathrm{T}q} = \left(\frac{\mathrm{d}q}{q} - \mathrm{S}\frac{\mathrm{d}q}{q}\right)\frac{q}{\mathrm{T}q} = \mathrm{V}\frac{\mathrm{d}q}{q}\,.\,\mathrm{U}q;$$

whence follows at once this formula, analogous to XIII., and like it easily remembered,

$$\text{XV.}\ \ldots\ \frac{\mathrm{dU}q}{\mathrm{U}q} = \mathrm{V}\frac{\mathrm{d}q}{q}.$$

(8.) We might also have observed that because (by 188), we have generally $q = \mathrm{T}q\,.\,\mathrm{U}q$, therefore (by 332, (3.)) we have also,

$$\text{XVI.}\ \ldots\ \mathrm{d}q = \mathrm{dT}q\,.\,\mathrm{U}q + \mathrm{T}q\,.\,\mathrm{dU}q,$$

and

$$\text{XVII.}\ \ldots\ \frac{\mathrm{d}q}{q} = \frac{\mathrm{dT}q}{\mathrm{T}q} + \frac{\mathrm{dU}q}{\mathrm{U}q};$$

if then we have in any manner established the equation XIII., we can immediately deduce XV.; and conversely, the former equation would follow at once from the latter.

(9.) It may be considered as remarkable, that we should thus have *generally*, or *for any two quaternions*, q and $\mathrm{d}q$, the formula:*

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* When the connexion of the theory of *normals to surfaces*, with the *differential calculus of quaternions*, shall have been (even briefly) explained in a subsequent Section, the student will perhaps be able to perceive, in this formula XVIII., a recognition, though not a very direct one, of the geometrical principle, that the *radii* of a *sphere* are its *normals*.

$$d\frac{q}{Tq} = \frac{dq}{Tq} - $$ [illegible]

$$\text{XVIII.}\ldots\ \mathrm{S}\,(\mathrm{dU}q : \mathrm{U}q) = 0\,; \quad \text{or} \quad \text{XVIII}'.\ldots\ \mathrm{dU}q : \mathrm{U}q = \mathrm{S}^{-1}0\,;$$

but this *vector character* of the *quotient* $\mathrm{dU}q : \mathrm{U}q$ can easily be confirmed, as follows. Taking the *conjugate* of that quotient, we have, by VII. (comp. 192, II.; 158; and 324, XI.),

$$\text{XIX.}\ldots\ \mathrm{K}\,(\mathrm{dU}q\,.\,\mathrm{U}q^{-1}) = \mathrm{KU}q^{-1}\,.\,\mathrm{dKU}q = \mathrm{U}q\,.\,\mathrm{d}(\mathrm{U}q^{-1}) = -\,\mathrm{dU}q\,.\,\mathrm{U}q^{-1}\,;$$

whence

$$\text{XX.}\ldots\ (1+\mathrm{K})\,(\mathrm{dU}q\,.\,\mathrm{U}q^{-1}) = 0\,;$$

which agrees (by 196, II.) with XVIII.

(10.) The *scalar character* of the *tensor*, $\mathrm{T}q$, enables us always to write, as in the ordinary calculus,

$$\text{XXI.}\ldots\ \mathrm{dlT}q = \mathrm{dT}q : \mathrm{T}q\,;$$

but $\mathrm{lT}q = \mathrm{Sl}q$, by 316, V.; the recent formula XIII. may therefore by VIII. be thus written,

$$\text{XXII.}\ldots\ \mathrm{Sdl}q = \mathrm{dSl}q = \mathrm{dT}q : \mathrm{d}q = \mathrm{S}(\mathrm{d}q : q)\,; \quad \text{or} \quad \text{XXII}'.\ldots\ \mathrm{dl}q - q^{-1}\mathrm{d}q = \mathrm{S}^{-1}0.$$

(11.) When $\mathrm{d}q \,|||\, q$, this last difference vanishes, by 333, II.; and the equation XV. takes the form,

$$\text{XXIII.}\ldots\ \mathrm{dlU}q = \mathrm{Vdl}q = \mathrm{dVl}q.$$

And in fact we have *generally*, $\mathrm{lU}q = \mathrm{Vl}q$, by 316, XX., although the *differentials* of these two equal expressions do not *separately* coincide with the members of the recent formula XV., when q and $\mathrm{d}q$ are *diplanar*. We may however write generally (comp. XXII.),

$$\text{XXIV.}\ldots\ \mathrm{dlU}q - \mathrm{dU}q : \mathrm{U}q = \mathrm{V}(\mathrm{dl}q - \mathrm{d}q : q) = \mathrm{dl}q - \mathrm{d}q : q.$$

335. We have now differentiated the *six simple functions* 334, I., which are formed by the operation of the *six characteristics*,

$$\mathrm{K},\ \mathrm{S},\ \mathrm{V},\ \mathrm{N},\ \mathrm{T},\ \mathrm{U}\,;$$

and as regards the differentiation of the *compound functions* 334, II., which are formed by *combinations* of those former operations, it is easy on the same principles to determine them, as may be seen in the few following examples.

(1.) The *axis* $\mathrm{Ax}.\,q$ of a quaternion has been seen (291) to admit of being represented by the *combination* $\mathrm{UV}q$; the *differential* of this axis may therefore, by 334, IX. and XIV., be thus expressed:

$$\text{I.}\ldots\ \mathrm{d}\,(\mathrm{Ax}.\,q) = \mathrm{dUV}q = \mathrm{V}\,(\mathrm{Vd}q : \mathrm{V}q)\,.\,\mathrm{UV}q\,;$$

whence

$$\text{II.}\ldots\ \frac{\mathrm{d}\,(\mathrm{Ax}.\,q)}{\mathrm{Ax}.\,q} = \frac{\mathrm{dUV}q}{\mathrm{UV}q} = \mathrm{V}\,\frac{\mathrm{Vd}q}{\mathrm{V}q}.$$

The *differential of the axis* is therefore, *generally*, a *line perpendicular to that axis*, or situated *in the plane of the quaternion*; but it *vanishes*, when the *plane* (and therefore the *axis*) of that quaternion is *constant*; or when the quaternion and its differential are *complanar*.

(2.) Hence,

$$\text{III.}\ldots\ \mathrm{dUV}q = 0, \quad \text{if} \quad \text{IV.}\ldots\ \mathrm{d}q \,|||\, q\,;$$

and conversely this *complanarity* IV. may be expressed by the *equation* III.

(3.) It is easy to prove, on similar principles, that

$$\text{V.} \ldots \mathrm{dVU}q = \mathrm{VdU}q = \mathrm{V}\left(\mathrm{V}\frac{\mathrm{d}q}{q}\,.\,\mathrm{U}q\right);$$

and

$$\text{VI.} \ldots \mathrm{dSU}q = \mathrm{SdU}q = \mathrm{S}\left(\mathrm{V}\frac{\mathrm{d}q}{q}\,.\,\mathrm{U}q\right).$$

(4.) But in general, for any two quaternions, q and q', we have (comp. 223, (5.)) the transformations,

$$\text{VII.} \ldots \mathrm{S}(\mathrm{V}q'\,.\,q) = \mathrm{S}(\mathrm{V}q'\,.\,\mathrm{V}q) = \mathrm{S}\,.\,q'\mathrm{V}q;$$

and when we thus suppress the characteristic V before $\mathrm{d}q : q$, and insert it before $\mathrm{U}q$, under the sign S in the last expression VI., we may replace the new factor $\mathrm{VU}q$ by $\mathrm{TVU}q\,.\,\mathrm{UVU}q$ (188), or by $\mathrm{TVU}q\,.\,\mathrm{UV}q$ (274, XIII.), or by $-\mathrm{TVU}q : \mathrm{UV}q$ (204, V.), where the scalar factor $\mathrm{TVU}q$ may be taken outside (by 196, VIII.); also for $q^{-1} : \mathrm{UV}q$ we may substitute $1 : (\mathrm{UV}q\,.\,q)$, or $1 : q\mathrm{UV}q$, because $\mathrm{UV}q \;|||\; q$; the formula VI. may therefore be thus written,

$$\text{VIII.} \ldots \mathrm{dSU}q = -\mathrm{S}\frac{\mathrm{d}q}{q\mathrm{UV}q}\,.\,\mathrm{TVU}q.$$

(5.) Now it may be remembered, that among the earliest *connexions* of quaternions with *trigonometry*, the following formulæ occurred (196, XVI., and 204, XIX.),

$$\text{IX.} \ldots \mathrm{SU}q = \cos\angle q, \quad \mathrm{TVU}q = \sin\angle q;$$

we had also, in 316, these expressions for the *angle* of a quaternion,

$$\text{X.} \ldots \angle q = \mathrm{TVl}q = \mathrm{TlU}q;$$

we may therefore establish the following expression for the *differential of the angle* of a quaternion,

$$\text{XI.} \ldots \mathrm{d}\angle q = \mathrm{dTVl}q = \mathrm{dTlU}q = \mathrm{S}\frac{\mathrm{d}q}{q\mathrm{UV}q}.$$

(6.) The following is another way of arriving at the same result, through the differentiation of the *sine* instead of the *cosine* of the angle, or through the calculation of $\mathrm{dTVU}q$, instead of $\mathrm{dSU}q$. For this purpose, it is only necessary to remark that we have, by 334, XII. XIV., and by some easy transformations of the kind lately employed in (4.), the formula,

$$\text{XII.} \ldots \mathrm{dTVU}q = \mathrm{S}\frac{\mathrm{VdU}q}{\mathrm{UVU}q} = \mathrm{S}\frac{\mathrm{dU}q}{\mathrm{UV}q} = \mathrm{S}\left(\mathrm{V}\frac{\mathrm{d}q}{q}\,.\,\frac{\mathrm{U}q}{\mathrm{UV}q}\right) = \mathrm{S}\frac{\mathrm{d}q}{q\mathrm{UV}q}\,.\,\mathrm{SU}q;$$

dividing which by $\mathrm{SU}q$, and attending to IX. and X., we arrive again at the expression XI., for the differential of the angle of a quaternion.

(7.) Eliminating $\mathrm{S}(\mathrm{d}q : q\mathrm{UV}q)$ between VIII. and XII., we obtain the *differential equation*,

$$\text{XIII.} \ldots \mathrm{SU}q\,.\,\mathrm{dSU}q + \mathrm{TVU}q\,.\,\mathrm{dTVU}q = 0;$$

of which, on account of the *scalar* character of the differentiated variables, the *integral* is evidently of the *form*,

$$\text{XIV.} \ldots (\mathrm{SU}q)^2 + (\mathrm{TVU}q)^2 = \text{const.};$$

and accordingly we saw, in 204, XX., that the sum in the first member of this equation is constantly equal to positive unity.

(8.) The formula XI. may also be thus written,

$$\text{XV.} \ldots \mathrm{d}\angle q = \mathrm{S}(\mathrm{V}(\mathrm{d}q : q) : \mathrm{UV}q);$$

with the verification, that when we suppose $\mathrm{d}q \mathbin{|||} q$, as in IV., and therefore $\mathrm{dUV}q = 0$ by III., the expression under the sign S becomes the differential of the quotient, $\mathrm{Vl}q : \mathrm{UV}q$, and therefore, by 316, VI., of the angle $\angle q$ itself.

336. An important application of the foregoing principles and rules consists in the *differentiation of scalar functions of vectors*, when those functions are defined and expressed according to the laws and notations of quaternions. It will be found, in fact, that *such* differentiations play a very extensive part, in the applications of quaternions to *geometry*; but, for the moment, we shall treat them *here*, as merely exercises of calculation. The following are a few examples.

(1.) Let ρ denote, in these sub-articles, a *variable vector*; and let the following equation be proposed,

$$\text{I.} \ldots r^2 + \rho^2 = 0, \quad \text{in which} \quad \mathrm{V}r = 0,$$

so that r is a (generally variable) *scalar*. Differentiating, and observing that, by 279, III., $\rho\rho' + \rho'\rho = 2\mathrm{S}\rho\rho'$, if ρ' be any *second vector*, such as we suppose $\mathrm{d}\rho$ to be, we have, by 322, VIII., and 324, VII., the equation,

$$\text{II.} \ldots r\mathrm{d}r + \mathrm{S}\rho\mathrm{d}\rho = 0; \quad \text{or} \quad \text{III.} \ldots \mathrm{d}r = -r^{-1}\mathrm{S}\rho\mathrm{d}\rho = r\mathrm{S}\rho^{-1}\mathrm{d}\rho.$$

In fact, if r be supposed *positive*, it is here, by 282, II., the *tensor* of ρ; so that this last expression III. for $\mathrm{d}r$ is included in the general formula, 334, XIII.

(2.) If this tensor, r, be *constant*, the differential equation II. becomes simply,

$$\text{IV.} \ldots \mathrm{S}\rho\mathrm{d}\rho = 0, \quad \text{if} \quad -\rho^2 = \text{const.}, \quad \text{or if} \quad \mathrm{dT}\rho = 0.$$

(3.) Again, let the proposed equation be (comp. 282, XIX.),

$$\text{V.} \ldots r^2 = \mathrm{T}(\iota\rho + \rho\kappa), \quad \text{with} \quad \mathrm{d}\iota = 0, \quad \mathrm{d}\kappa = 0,$$

so that ι and κ are here *two constant vectors*. Then, squaring and differentiating, we have (by 334, XI., because $\mathrm{K}\iota\rho = \rho\iota$, &c.),

$$\text{VI.} \ldots 2r^3\mathrm{d}r = \tfrac{1}{2}\mathrm{dN}(\iota\rho + \rho\kappa) = \mathrm{S}(\rho\iota + \kappa\rho)(\iota\mathrm{d}\rho + \mathrm{d}\rho\kappa)$$
$$= (\iota^2 + \kappa^2)\,\mathrm{S}\rho\mathrm{d}\rho + 2\mathrm{S}\kappa\rho\iota\mathrm{d}\rho;$$

$S\alpha\beta\gamma\delta = S\beta\gamma\delta\alpha$

or more briefly,

$$\text{VII.} \ldots 2r^{-1}\mathrm{d}r = \mathrm{S}\nu\mathrm{d}\rho,$$

if ν be an *auxiliary vector*, determined by the equation,

$$\text{VIII.} \ldots r^4\nu = (\iota^2 + \kappa^2)\,\rho + 2\mathrm{V}\kappa\rho\iota;$$

which admits of several transformations.

(4.) For example we may write, by 295, VII.,

$$\text{IX.} \ldots r^4\nu = (\iota^2 + \kappa^2)\,\rho + \kappa\rho\iota + \iota\rho\kappa$$
$$= \iota(\iota\rho + \rho\kappa) + \kappa(\rho\iota + \kappa\rho);$$

or, by 294, III., and 282, XII.,

$$\text{X.} \ldots r^4\nu = (\iota^2 + \kappa^2)\,\rho + 2(\kappa\mathrm{S}\iota\rho - \rho\mathrm{S}\iota\kappa + \iota\mathrm{S}\kappa\rho)$$
$$= (\iota - \kappa)^2\rho + 2(\iota\mathrm{S}\kappa\rho + \kappa\mathrm{S}\iota\rho); \ \&\text{c.}$$

(5.) The equation V. gives (comp. 190, V.), when squared without differentiation,

$$\text{XI.} \ldots r^4 = \mathrm{N}(\iota\rho + \rho\kappa) = (\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho)$$
$$= (\iota^2 + \kappa^2)\rho^2 + \iota\widehat{\rho\kappa\rho} + \widehat{\rho\kappa\rho}\iota$$
$$= (\iota^2 + \kappa^2)\rho^2 + 2\mathrm{S}\iota\rho\kappa\rho$$
$$= (\iota - \kappa)^2\rho^2 + 4\mathrm{S}\iota\rho\,\mathrm{S}\kappa\rho = \&\text{c.},$$

by transformations of the same kind as before; we have therefore, by the recent expressions for $r^4\nu$, the following remarkably simple *relation* between the *two variable vectors*, ρ and ν,

$$\text{XII.} \ldots \mathrm{S}\nu\rho = 1; \quad \text{or} \quad \text{XII}'. \ldots \mathrm{S}\rho\nu = 1.$$

(6.) When the *scalar*, r, is *constant*, we have, by VII., the *differential equation*,

$$\text{XIII.} \ldots \mathrm{S}\nu d\rho = 0; \quad \text{whence also} \quad \text{XIV.} \ldots \mathrm{S}\rho d\nu = 0, \text{ by XII.};$$

a *relation of reciprocity* thus existing, between the *two vectors* ρ and ν, of which the *geometrical signification* will soon be seen.

(7.) Meanwhile, supposing r again to *vary*, we see that the last expression VI. for $2r^3dr$ may be otherwise obtained, by taking half the differential of either of the two last expanded expressions XI. for r^4; it being remembered, in all these little calculations, that *cyclical permutation of factors, under the sign* S, *is permitted* (223, (10.)), even if those factors be *quaternions*, and whatever their *number* may be: and that if they be *vectors*, and if their number be *odd*, it is then permitted, *under the sign* V, to *invert their order* (295, (9.)), and so to write, for instance, $\mathrm{V}\iota\rho\kappa$ instead of $\mathrm{V}\kappa\rho\iota$, in the formula VIII.

(8.) As another example of a *scalar function of a vector*, let p denote the *proximity* (or *nearness*) of a *variable point* P to the *origin* O; so that

$$\text{XV.} \ldots p = (-\rho^2)^{-\frac{1}{2}} = \mathrm{T}\rho^{-1}, \quad \text{or} \quad \text{XV}'. \ldots p^{-2} + \rho^2 = 0.$$

Then,

$$\text{XVI.} \ldots dp = \mathrm{S}\nu d\rho, \quad \text{if} \quad \text{XVII.} \ldots \nu = p^3\rho = p^2\mathrm{U}\rho;$$

ν being here a *new auxiliary vector*, distinct from the one lately considered (VIII.), and having (as we see) the *same versor* (or the *same direction*) as the vector ρ *itself*, but having its *tensor* equal to the *square of the proximity of* P *to* O; or equal to the *inverse square of the distance*, of one of those two points from the other.

337. On the other hand, we have often occasion, in the applications, to consider *vectors as functions of scalars*, as in 99, but now with *forms* arising out of operations on *quaternions*, and therefore such as had not been considered in the First Book. And whenever we have thus an expression such as either of the two following,

$$\text{I.} \ldots \rho = \phi(t), \quad \text{or} \quad \text{II.} \ldots \rho = \phi(s, t),$$

for the *variable vector* of a *curve*, or of a *surface* (comp. again 99), s and t being *two variable scalars*, and $\phi(t)$ and $\phi(s, t)$ denoting *any functions of vector form*, whereof the *latter* is here supposed to be entirely *independent** of the *former*, we may then employ (comp. 100,

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* We are therefore not employing *here* the temporary notation of some recent Articles, according to which we should have had, $d\phi q = \phi(q, dq)$.

(4.) and (9.), and the more recent sub-articles, 327, (5.), (6.), and 329, (5.)) the *notation of derivatives*, total or partial; and so may write, as the *differentiated equations*, resulting from the forms I. and II. respectively, the following:

$$\text{III.} \ldots \mathrm{d}\rho = \phi' t . \mathrm{d}t = \rho' \mathrm{d}t = \mathrm{D}_t \rho . \mathrm{d}t;$$

$$\text{IV.} \ldots \mathrm{d}\rho = \mathrm{d}_s \rho + \mathrm{d}_t \rho = \mathrm{D}_s \rho . \mathrm{d}s + \mathrm{D}_t \rho . \mathrm{d}t;$$

of which the geometrical significations have been already partially seen, in the sub-articles to 100, and will soon be more fully developed.

(1.) Thus, for the *circular locus*, 314, (1.), for which

$$\text{V.} \ldots \rho = \alpha^t \beta, \quad \mathrm{T}\alpha = 1, \quad \mathrm{S}\alpha\beta = 0,$$

we have, by 333, VIII., the following *derived vector*,

$$\text{VI.} \ldots \rho' = \mathrm{D}_t \rho = \frac{\pi}{2} \alpha^{t+1} \beta = \frac{\pi}{2} \alpha \rho.$$

(2.) And for the *elliptic locus*, 314, (2.), for which

$$\text{VII.} \ldots \rho = \mathrm{V} . \alpha^t \beta, \quad \mathrm{T}\alpha = 1, \quad \text{but } \mathit{not} \ \mathrm{S}\alpha\beta = 0,$$

we have, in like manner, this other derived vector,

$$\text{VIII.} \ldots \rho' = \mathrm{D}_t \rho = \frac{\pi}{2} \mathrm{V} . \alpha^{t+1} \beta.$$

(3.) As an example of a *vector-function* of *more scalars than one*, let us resume the expression (308, XVIII.),

$$\text{IX.} \ldots \rho = r k^t j^s k j^{-s} k^{-t};$$

in which we shall now suppose that the tensor r is *given*, so that ρ is the *variable vector* of a point upon a *given spheric surface*, of which the *radius* is r, and the *centre* is at the origin; while s and t are *two independent scalar variables*, with respect to which the *two partial derivatives* of the vector ρ are to be determined.

(4.) The derivation relatively to t is easy; for, since ijk are *vector-units* (295), and since we have generally, by 333, VIII.,

$$\text{X.} \ldots \mathrm{d} . \alpha^x = \frac{\pi}{2} \alpha^{x+1} \mathrm{d}x, \quad \text{and therefore} \quad \text{XI.} \ldots \mathrm{D}_t . \alpha^x = \frac{\pi}{2} \alpha^{x+1} \mathrm{D}_t x,$$

if $\mathrm{T}\alpha = 1$, and if x be any scalar function of t, we may write, at once, by 279, IV.,

$$\text{XII.} \ldots \mathrm{D}_t \rho = \frac{\pi}{2} (k\rho - \rho k) = \pi \mathrm{V} k \rho;$$

and we see that

$$\text{XIII.} \ldots \mathrm{S}\rho \mathrm{D}_t \rho = 0,$$

a result which was to be expected, on account of the equation,

$$\text{XIV.} \ldots \rho^2 + r^2 = 0,$$

which follows, by 308, XXIV., from the recent expression IX. for ρ.

(5.) To form an expression of about the same degree of simplicity, for the *other* partial derivative of ρ, we may observe that $j^{s+1} k j^{-s}$ is equal to its own vector part (its scalar vanishing); hence

$$\text{XV.} \ldots \mathrm{D}_s\rho = \pi k^t jk^{-t}\rho; \quad \text{or} \quad \text{XVI.} \ldots \mathrm{D}_s\rho = \pi k^{2t} j\rho = \pi jk^{-2t}\rho,$$

by the transformation 308, (11.). And because the scalar of $k^t jk^{-t}$ is zero, we have thus the equation,

$$\text{XVII.} \ldots \mathrm{S}\rho\mathrm{D}_s\rho = 0,$$

which is analogous to XIII., and might have been otherwise obtained, by taking the derivative of XIV. with respect to the variable scalar s.

(6.) The partial derivative $\mathrm{D}_s\rho$ must be a *vector*; hence, by XV. or XVI., ρ must be *perpendicular* to the vector $k^t jk^{-t}$, or $k^{2t}j$, or jk^{-2t}; a result which, under the last form, is easily confirmed by the expression 315, XII. for ρ. In fact that expression gives, by 315, (3.) and (4.), and by the recent values XII. XVI., these *other forms* for the two partial derivatives of ρ, which have been above considered:

$$\text{XVIII.} \ldots \mathrm{D}_t\rho = \pi r k^{2t}\mathrm{V}.j^{2s}; \qquad \text{XIX.} \ldots \mathrm{D}_s\rho = \pi r(k^{2t}\mathrm{V}.i^{2s+1} - \mathrm{V}.k^{2s});$$

which might have been immediately obtained, by partial derivations, from the expression 315, XII. itself, and of which *both* are *vector-forms*.

(7.) And hence, or immediately by *derivating* the expanded expression 315, XIII., we obtain these new forms for the partial derivatives of ρ:

$$\text{XX.} \ldots \mathrm{D}_t\rho = \pi r(j\cos t\pi - i\sin t\pi)\sin s\pi;$$

$$\text{XXI.} \ldots \mathrm{D}_s\rho = \pi r\{(i\cos t\pi + j\sin t\pi)\cos s\pi - k\sin s\pi\}.$$

(8.) We may add that not only is the variable vector ρ *perpendicular to each* of the *two derived vectors*, $\mathrm{D}_s\rho$ and $\mathrm{D}_t\rho$, but also *they* are perpendicular to *each other*; for we may write, by XII. and XVI.,

$$\text{XXII.} \ldots \mathrm{S}(\mathrm{D}_s\rho\,.\,\mathrm{D}_t\rho) = -\pi^2\mathrm{S}.k^{2t}j\rho^2k = \pi^2r^2\mathrm{S}.k^{2t}i = 0;$$

and the same conclusion may be drawn from the expressions XX. and XXI.

(9.) A *vector* may be considered as a function of *three independent scalar variables*, such as r, s, t; or rather it *must* be so considered, if it is to admit of being the vector of an *arbitrary point of space*: and then it will have a *total differential* (329) of the *trinomial form*,

$$\text{XXIII.} \ldots \mathrm{d}\rho = \mathrm{d}_r\rho + \mathrm{d}_s\rho + \mathrm{d}_t\rho = \mathrm{D}_r\rho\,.\,\mathrm{d}r + \mathrm{D}_s\rho\,.\,\mathrm{d}s + \mathrm{D}_t\rho\,.\,\mathrm{d}t;$$

and will thus have *three* partial derivatives*.

(10.) For example, when ρ has the expression IX., we have this *third* partial derivative,

$$\text{XXIV.} \ldots \mathrm{D}_r\rho = r^{-1}\rho = \mathrm{U}\rho,$$

which may also be thus more fully written (comp. again 315, XIII.),

$$\text{XXV.} \ldots \mathrm{D}_r\rho = k^t j^s kj^{-s}k^{-t} = (i\cos t\pi + j\sin t\pi)\sin s\pi + k\cos s\pi;$$

and we see that the *three derived vectors*,

$$\text{XXVI.} \ldots \mathrm{D}_r\rho,\ \mathrm{D}_s\rho,\ \mathrm{D}_t\rho,$$

compose here a *rectangular system*.

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* That is to say, *three of the first order*; for we shall soon have occasion to consider *successive differentials*, of functions of one or more variables, and so shall be conducted to the consideration of *orders* of differentials and derivatives, *higher than the first*.

SECTION 5.—*On Successive Differentials, and Developments, of Functions of Quaternions.*

338. There will now be no difficulty in the *successive differentiation*, total or partial, of functions of one or more quaternions; and *such* differentiation will be found to be *useful*, as in the ordinary calculus, in connexion with *developments of functions:* besides that it is *necessary* for many of those *geometrical and physical applications* of differentials of quaternions, on which we have not entered yet. A few *examples* of successive differentiation may serve to show, more easily than any general precepts, the nature and effects of the operation; and we shall begin, for simplicity, with *explicit functions of one quaternion variable.*

(1.) Take then the *square*, q^2, of a quaternion, as a function fq, which is to be *twice* differentiated. We saw, in 324, VII., that a *first* differentiation gave the equation,

$$\text{I.} \ldots \mathrm{d}fq = \mathrm{d}\,.\,q^2 = q\,.\,\mathrm{d}q + \mathrm{d}q\,.\,q;$$

but we are now to differentiate *again*, in order to form the *second differential* d^2fq of the *function* q^2, treating the differential of the variable q as *still* equal to $\mathrm{d}q$, and *in general* writing $\mathrm{dd}q = \mathrm{d}^2q$, where d^2q is a *new arbitrary quaternion*, of which the *tensor*, Td^2q, *need not be small* (comp. 322). And thus we get, *in general*, this *twice differentiated expression*, or *differential of the second order*,

$$\text{II.} \ldots \mathrm{d}^2fq = \mathrm{d}^2\,.\,q^2 = q\,.\,\mathrm{d}^2q + 2\mathrm{d}q^2 + \mathrm{d}^2q\,.\,q.$$

(2.) The *second differential of the reciprocal* of a quaternion is *generally* (comp. 324, XI.),

$$\text{III.} \ldots \mathrm{d}^2\,.\,q^{-1} = 2(q^{-1}\mathrm{d}q)^2\,q^{-1} - q^{-1}\mathrm{d}^2q\,.\,q^{-1}.$$

(3.) If ρ be a *variable vector*, then (comp. 336, (1.)) we have, for the first and second differentials of its square, the expressions:

$$\text{IV.} \ldots \mathrm{d}\,.\,\rho^2 = 2\mathrm{S}\rho\mathrm{d}\rho; \qquad \text{V.} \ldots \mathrm{d}^2\,.\,\rho^2 = 2\mathrm{S}\rho\mathrm{d}^2\rho + 2\mathrm{d}\rho^2.$$

(4.) If $f\rho$ be any *other scalar function* of a variable vector ρ, and if (comp. again the sub-articles to 336) its *first* differential be put under the *form*,

$$\text{VI.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho, \text{ when } \nu \text{ is another variable vector,}$$

then the *second* differential of the same function may be expressed as follows:

$$\text{VII.} \ldots \mathrm{d}^2f\rho = 2\mathrm{S}\nu\mathrm{d}^2\rho + 2\mathrm{Sd}\nu\mathrm{d}\rho;$$

in which we have written, briefly, $\mathrm{Sd}\nu\mathrm{d}\rho$, instead of $\mathrm{S}(\mathrm{d}\nu\,.\,\mathrm{d}\rho)$.

(5.) The following very simple equation will be found useful, in the theory of *motions*, performed under the influence of *central forces:*

$$\text{VIII.} \ldots \mathrm{dV}\rho\mathrm{d}\rho = \mathrm{V}\rho\mathrm{d}^2\rho; \quad \text{because } \mathrm{V}\,.\,\mathrm{d}\rho^2 = 0.$$

(6.) As an example of the second differential of a *quaternion*, considered as a

[illegible] $= +2q^{-1}dq]^2 - q^{-1}q$

function of a scalar variable (comp. 333, VIII., and 337, (1.)), the following may be assigned, in which a denotes a given unit line, so that $a^2 = -1$, $\mathrm{d}a = 0$, but x is a variable scalar:

$$\text{IX.} \ldots \mathrm{d}^2 . a^x = \mathrm{d}\left(\frac{\pi}{2} a^{x+1} \mathrm{d}x\right) = \frac{\pi}{2} a^{x+1} \mathrm{d}^2 x - \left(\frac{\pi}{2}\right)^2 a^x \mathrm{d}x^2.$$

(7.) The second differential of the *product* of *any two functions* of a *quaternion* q may be expressed as follows (comp. II.):

$$\text{X.} \ldots \mathrm{d}^2(fq . \phi q) = \mathrm{d}^2 fq . \phi q + 2\mathrm{d}fq . \mathrm{d}\phi q + fq . \mathrm{d}^2 \phi q.$$

339. The second differential, $\mathrm{d}^2 q$, of the variable quaternion q, enters *generally* (as has been seen) into the expression of the second differential $\mathrm{d}^2 fq$, of the function fq, as a *new* and *arbitrary* quaternion: but, for that very reason, it is *permitted*, and it is frequently found to be *convenient*, to *assume* that this *second* differential $\mathrm{d}^2 q$ is equal to *zero*: or, what comes to the same thing, that the *first* differential $\mathrm{d}q$ is *constant*. And when we make this *new supposition*,

$$\text{I.} \ldots \mathrm{d}q = \textit{constant}, \quad \text{or} \quad \text{I}'. \ldots \mathrm{d}^2 q = 0,$$

the expressions for $\mathrm{d}^2 fq$ become of course more simple, as in the following examples.

(1.) With this last supposition, I. or I'., we have the following *second* differentials, of the *square* and the *reciprocal* of a quaternion:

$$\text{II.} \ldots \mathrm{d}^2 . q^2 = 2\mathrm{d}q^2; \qquad \text{III.} \ldots \mathrm{d}^2 . q^{-1} = 2(q^{-1}\mathrm{d}q)^2 q^{-1} = 2q^{-1}(\mathrm{d}q . q^{-1})^2.$$ 2

(2.) Again, if we suppose that c_0, c_1, c_2 are any three *constant* quaternions, and take the function,

$$\text{IV.} \ldots fq = c_0 q c_1 q c_2,$$

we find, under the same condition I. or I'., that its first and second differentials are,

$$\text{V.} \ldots \mathrm{d}fq = c_0 \mathrm{d}q . c_1 q c_2 + c_0 q c_1 \mathrm{d}q . c_2; \qquad \text{VI.} \ldots \mathrm{d}^2 fq = 2c_0 \mathrm{d}q . c_1 \mathrm{d}q . c_2;$$

in writing which, the *points** may be omitted.

(3.) The *first* differential, $\mathrm{d}q$, remaining still entirely *arbitrary* (comp. 322, (8.), and 325, (2.)), so that no supposition is made that its *tensor* $\mathrm{Td}q$ is *small*, although we *now* suppose this differential $\mathrm{d}q$ to be *constant* (I.) we have *rigorously*,

$$\text{VII.} \ldots (q + \mathrm{d}q)^2 = q^2 + \mathrm{d} . q^2 + \tfrac{1}{2}\mathrm{d}^2 . q^2;$$

an equation which may be also written thus,

$$\text{VIII.} \ldots (q + \mathrm{d}q)^2 = (1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2) . q^2.$$

(4.) And in like manner we shall have, more generally, under the same condition of *constancy* of $\mathrm{d}q$, the equation,

$$\text{IX.} \ldots f(q + \mathrm{d}q) = (1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2) fq,$$

if the function fq be the *sum* of *any number of monomes*, *each* separately of the *form*

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* Compare the second Note to page 399.

IV., and therefore each *rational*, *integral*, and *homogeneous* of the *second dimension*, with respect to the variable quaternion, q; or of *such* monomes, combined with others of the *first* dimension, and with *constant terms:* that is, if a_0, b_0, b_1, b'_0, b'_1, . . and c_0, c_1, c_2, c'_0, c'_1, c'_2, . . be *any constant quaternions*, and

$$\text{X.} \ldots fq = a_0 + \Sigma b_0 q b_1 + \Sigma c_0 q c_1 q c_2.$$

340. It is easy to carry on the operation of differentiating, to the *third* and *higher orders;* remembering only that if, *in any former stage*, we have *denoted* the *first* differentials of q, $\mathrm{d}q$, . . by $\mathrm{d}q, \mathrm{d}^2q$, . . we then *continue* so to denote them, in every *subsequent* stage of the *successive differentiation:* and that *if* we find it convenient to treat any *one* differential as *constant*, we must then treat *all* its *successive* differentials as *vanishing*. A few examples may be given, chiefly with a view to the *extension* of the recent formula 339, IX., for the *function* $f(q + \mathrm{d}q)$ *of a sum*, of *any two quaternions*, q and $\mathrm{d}q$, to *polynomial forms*, of *dimensions* higher than the *second*.

(1.) The *third differential of a square* is *generally* (comp. 338, II.),

$$\text{I.} \ldots \mathrm{d}^3 . q^2 = q . \mathrm{d}^3 q + \mathrm{d}^3 q . q + 3(\mathrm{d}q . \mathrm{d}^2 q + \mathrm{d}^2 q . \mathrm{d}q).$$

(2.) More generally, the *third* differential of a *product of two quaternion functions* (comp. 338, X.) may be thus expressed:

$$\text{II.} \ldots \mathrm{d}^3 (fq . \phi q) = \mathrm{d}^3 fq . \phi q + 3\mathrm{d}^2 fq . \mathrm{d}\phi q + 3\mathrm{d}fq . \mathrm{d}^2 \phi q + fq . \mathrm{d}^3 \phi q.$$

(3.) More generally still, the nth differential of a *product* is, as in the ordinary calculus,

$$\text{III.} \ldots \mathrm{d}^n (fq . \phi q) = \mathrm{d}^n fq . \phi q + n\mathrm{d}^{n-1} fq . \mathrm{d}\phi q + n_2 \mathrm{d}^{n-2} fq . \mathrm{d}^2 \phi q + \ldots + fq . \mathrm{d}^n \phi q,$$

$$\text{if} \qquad n_2 = \frac{n(n-1)}{2}, \quad n_3 = \frac{n(n-1)(n-2)}{2 . 3}, \quad \&\text{c.};$$

the only thing *peculiar* to quaternions being, that we are obliged to *retain* (generally) the *order of the factors*, in each term of this expansion III.

(4.) Hence, in particular, denoting briefly the function fq by r, and changing ϕq to q,

$$\text{IV.} \ldots \mathrm{d}^n . rq = \mathrm{d}^n r . q + n\mathrm{d}^{n-1} r . \mathrm{d}q, \quad \text{if} \quad \mathrm{d}^2 q = 0.$$

(5.) Hence also, under this *condition* that $\mathrm{d}q$ is constant, if c be any *other* constant quaternion, we have the transformation,

$$\text{V.} \ldots \left(1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2 + \frac{1}{2.3}\mathrm{d}^3 + \ldots + \frac{1}{2.3\ldots n}\mathrm{d}^n\right) . rqc =$$

$$\left(1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2 + \frac{1}{2.3}\mathrm{d}^3 + \ldots + \frac{1}{2.3\ldots(n-1)}\mathrm{d}^{n-1}\right) r . (q + \mathrm{d}q)c, \quad \text{if} \quad \mathrm{d}^n r = 0.$$

(6.) Hence, by 339, (4.), it is easy to infer that if we *interpret* the *symbol* ε^{d} by the *equation* (comp. 316, I.),

$$\text{VI.} \ldots \varepsilon^{\mathrm{d}} = 1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2 + \frac{1}{2.3}\mathrm{d}^3 + \&\text{c.},$$

that is, if we interpret this *other* symbol $\varepsilon^{\mathrm{d}} fq$, as concisely denoting the *series* which is formed from fq, by *operating* on it with this symbolic development; and if the *function* fq, thus operated on, be *any finite polynome*, involving (like the expression 339, X.) *no fractional nor negative exponents;* we may then write, as an *extension* of a recent equation (339, IX.), the formula:

$$\text{VII.} \ldots \epsilon^{\mathrm{d}} fq = f(q + \mathrm{d}q), \quad \text{if} \quad \mathrm{d}^2 q = 0;$$

which is here a perfectly *rigorous* one, *all* the *terms* of this *expansion* for a *function of a sum* of two quaternions, q and $\mathrm{d}q$, becoming separately equal to *zero*, as soon as the symbolic *exponent* of d becomes greater than the *dimension* of the polynome.

(7.) We shall soon see that there is a *sense*, in which this *exponential transformation* VII. may be *extended*, to *other functional forms* which are *not* composed as above: and that thus an *analogue of Taylor's Theorem* can be established *for Quaternions*. Meanwhile it may be observed that by changing $\mathrm{d}q$ to Δq, in the *finite expansion* obtained as above, we may write the formula as follows:

$$\text{VIII.} \ldots \epsilon^{\mathrm{d}} fq = f(q + \Delta q) = (1 + \Delta) fq, \quad \text{or briefly,} \quad \text{IX.} \ldots \epsilon^{\mathrm{d}} = 1 + \Delta;$$

which last *symbolical equation* may be *operated on*, or *transformed*, as in the *usual* calculus of *differences* and *differentials*. For instance, it being understood that we treat $\Delta^2 q$ as well as $\mathrm{d}^2 q$ as vanishing, we have thus (for any positive and whole exponent m), the two following transformations of IX.,

$$\text{X.} \ldots \Delta^m = (\epsilon^{\mathrm{d}} - 1)^m, \quad \text{and} \quad \text{XI.} \ldots \mathrm{d}^m = (\log(1 + \Delta))^m;$$

the *results of operating*, with the *symbols* thus *equated*, on any *polynomial function* fq, of the kind above described, being always *finite expansions*, which are *rigorously equal* to each other.

341. Let Fx and ϕx be *any two functions* of a *scalar variable*, of which both *vanish with that variable;* so that they satisfy the two conditions,

$$\text{I.} \ldots F0 = 0, \quad \phi 0 = 0.$$

Then the *three simultaneous values*,

$$\text{II.} \ldots x, \quad Fx, \quad \phi x,$$

of the variable and the two functions, are at the same time (comp. 320, 321) *three simultaneous differences*, as compared with this *other system* of three simultaneous values,

$$\text{III.} \ldots 0, \quad F0, \quad \phi 0.$$

If, then, any *equimultiples*,

$$\text{IV.} \ldots nx, \quad nFx, \quad n\phi x,$$

of the three values II., can be made, by any suitable *increase* of the *number*, n, combined with a *decrease* of the *variable*, x, to *tend together* to any *system of limits*, those *limits* must (by the *definition* in 320, compare again 321) admit of being considered as a *system of simultaneous differentials*,

$$\text{V.} \ldots \mathrm{d}x, \quad \mathrm{d}Fx, \quad \mathrm{d}\phi x,$$

answering to the *system of initial values* III.; and must be *proportional* to the *ultimate values* of the connected *system of derivatives*,

$$\text{VI.} \ldots 1, \quad F'x, \quad \phi' x, \text{ when } x \textit{ tends} \text{ to } \textit{zero}.$$

We may therefore write, as *expressions* for those *ultimate values* of the two last *derived functions*,

$$\text{VII.} \ldots F'0 = \lim_{n=\infty} . nF\frac{1}{n}, \quad \phi'0 = \lim_{n=\infty} . n\phi\frac{1}{n}, \quad \text{if} \quad F0 = \phi 0 = 0.$$

And *even if these* last values *vanish*, or if the *two new conditions*

$$\text{VIII.} \ldots F'0 = 0, \quad \phi'0 = 0,$$

are satisfied, so that x, $F'x$, and $\phi' x$ are now (comp. II.) a *new system* of *simultaneous differences*, we may *still* establish the following *equation of limits of quotients*, which is *independent* of these *last* conditions VIII.,

$$\text{IX.} \ldots \lim_{x=0} (Fx : \phi x) = \lim_{x=0} (F'x : \phi' x), \quad \text{if} \quad F0 = \phi 0 = 0\,;$$

it being understood that, in certain *cases*, these *two quotients* may *both vanish with* x; or may *tend together to infinity*, when x *tends*, as before, *to zero*.

(1.) This theorem is so important, that it will not be useless to confirm it by a *geometrical illustration*, which may at the same time serve for a *geometrical proof*; at least for the extensive case where *both* the *functions* fx and ϕx are of *scalar forms*, and consequently may be *represented*, or *constructed*, by the *corresponding ordinates*, XY and XZ (or ordinates answering to one *common abscissa* OX), of *two curves* OyY and OzZ, which are *in one plane*, and set out from (or pass through) one *common origin* O, as in the annexed Figure 75. We shall afterwards see that the result, so obtained, can be *extended to quaternion functions*.

(2.) Suppose then, first, that the ordinates of these two curves are *proportional*, or that they bear to each other one *fixed* and *constant ratio*; so that the equation,

$$\text{X.} \ldots \mathrm{XY} : \mathrm{XZ} = xy : xz,$$

is satisfied for *every pair* of *abscissæ*, OX and Ox, however *great* or *small* the corresponding *ordinates* may be. Prolonging then (if necessary) the *chord* Yy of the *first curve*, to meet the *axis* of abscissæ in some point t, and so to determine a *subsecant* tX, we see at once (by similar triangles) that the *corresponding chord* Zz of thes *econd curve* will meet the same axis in the *same point*, t; and therefore that it will determine (*rigorously*) the *same subsecant*, tX.

(3.) Hence, if the point x be conceived to approach to X, so that the *secant* Yyt of the *first* curve tends to coincide with the *tangent* YT to *that* curve at the point Y, the *secant* Zzt of the *second* curve must tend to coincide with the line ZT, which line therefore must be the *tangent* to that second curve: or in other words, *corresponding subtangents coincide*, and of course are *equal*, under the supposed *condition* X., of a constant *proportionality of ordinates.*

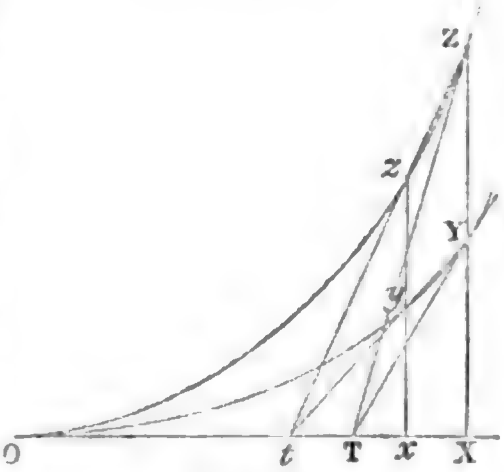

Fig. 75.

(4.) Suppose next that corresponding ordinates only *tend* to bear a *given* or *constant ratio* to each other; or that their (now) *variable* ratio tends to a given or fixed *limit*, when the common abscissa is indefinitely diminished, or when the *point* X *tends* to O; and let T be still the variable point in which the tangent to the *first* curve at Y meets the axis, so that the line TX is still the *first subtangent*. Then the corresponding tangent to the *second* curve at Z will *not* in general pass through the point T, but will meet the axis in some *different* point U. But the *ratio* of the two *corresponding subtangents*, TX and UX, which *had* been a ratio of *equality*, when the condition of *proportionality* X. was satisfied *rigorously*, will now at least *tend* to *such* a ratio; so that we shall have, under this *new condition*, of *tendency* to proportionality of ordinates, the *limiting equation*,

$$\text{XI.} \ldots \lim (\mathrm{TX} : \mathrm{UX}) = 1;$$

whence the equation IX. results, under the *geometrical form*,

$$\text{XII.} \ldots \lim (\tan \mathrm{XTY} : \tan \mathrm{XUZ}) = \lim (\mathrm{XY} : \mathrm{XZ}).$$

(5.) We might also have observed that, when the *proportion* X. is *rigorous*, corresponding *areas*[*] (such as $x$XY$y$ and $x$XZ$z$) of the two curves are then *exactly* in the *given* ratio of the *ordinates;* so that this *other* equation, or proportion,

$$\text{XIII.} \ldots \mathrm{OXY}y\mathrm{O} : \mathrm{OXZ}z\mathrm{O} = \mathrm{XY} : \mathrm{XZ},$$

is then *also* rigorous. Hence if we only suppose, as in (4.), that the ordinates *tend* to some fixed *limiting ratio*, the *areas* must tend to the *same;* so that *if* the *second member* of the equation IX. have *any definite value*, as a *limit*, the *first* member must have the *same:* whereas the recent proof, by *subtangents*, served rather to show that *if* the *first* (or left hand) *limit* in IX. *existed*, then the *second* limit in that equation existed *also*, and was *equal* to the first.

(6.) If the *function* Fx be a *quaternion*, we may (by 221) express it as follows,

$$\text{XIV.} \ldots Fx = W + iX + jY + kZ,$$

where W, X, Y, Z are *four scalar functions* of x, of which *each* separately can be

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[*] Compare the Fourth Lemma of the First Book of the Principia; and see especially its Corollary, in which the reasoning of the present sub-article is virtually anticipated.

constructed, as the ordinate of a plane curve; and the recent *geometrical** reasoning will thus apply to each of them, and therefore to their *linear combination* Fx: which *quaternion function* reduces itself to a *vector function* of x, when $W = 0$.

(7.) And if ψx were *another* quaternion or vector function, we might first *substitute* it for Fx, and then *eliminate* the *scalar* function ϕx; so that a limiting equation of the *form* IX. may thus be proved to hold good, when *both* the functions compared are *vectors*, or *quaternions*, supposed still to *vanish* with x.

(8.) The general considerations, however, on which the equation IX. was lately established, appear to be more simple and direct; and it is evident that they give, in like manner, this other but analogous equation, in which $F''x$ and $\phi''x$ are *second derivatives*, and the conditions VIII. are now supposed to be satisfied:

$$\text{XV.} \ldots \lim_{x=0} (F'x : \phi'x) = \lim_{x=0} (F''x : \phi''x), \quad \text{if} \quad F'0 = 0,\ \phi'0 = 0.$$

And so we might proceed, as long as *successive derivatives, of higher orders*, continue to vanish together.

(9.) Hence, in particular, if we take this *scalar form*,

$$\text{XVI.} \ldots \phi x = \frac{x^m}{2 . 3 \ldots m},$$

which evidently gives the values,

$$\text{XVII.} \ldots \phi 0 = 0, \quad \phi' 0 = 0, \quad \phi'' 0 = 0, \ldots \phi^{(m-1)} 0 = 0, \quad \phi^{(m)} 0 = 1,$$

and if we suppose that the function Fx is such that

$$\text{XVIII.} \ldots F0 = 0, \quad F'0 = 0, \quad F''0 = 0, \ldots F^{(m-1)} 0 = 0,$$

while $F^{(m)}0$ has any finite value, we may then establish this limiting equation:

$$\text{XIX.} \ldots \lim_{x=0} (Fx : \phi x) = F^{(m)} 0;$$

in which the *function* Fx, and the *value* $F^{(m)}0$, are *here* supposed to be *generally quaternions*; although they may *happen*, in particular cases, to *reduce* themselves (292) to *vectors*, or to *scalars*.

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* Instead of the equation IX., it has become usual, in modern works on the Differential Calculus, to give one of the following form (deduced from principles of Lagrange):

$$\frac{F(x)}{\phi(x)} = \frac{F'(\theta x)}{\phi'(\theta x)}, \quad \text{if} \quad F(0) = \phi(0) = 0;$$

θ denoting *some proper fraction*, or quantity between 0 and 1. And a *geometrical illustration*, which is also a *geometrical proof*, when the *functions* Fx and ϕx can be *constructed* (or conceived to be constructed) as the *ordinates of two plane curves*, is sometimes derived from the *axiom* (or *geometrical intuition*), that the *chord* of any finite and *plane arc* must be *parallel to the tangent*, drawn *at some point* of that *finite arc*. But this *parallelism* no longer *exists*, in general, when the *curve* is one of *double curvature*; and accordingly the *equation* in this Note is *not generally true*, when the *functions* are *quaternions*; or even when *one* of them is a quaternion, or a *vector*.

342. It will now be easy to *extend* the *Exponential Transformation* 340, VII.; and to show that there is a *sense* in which that very important *Formula*,

$$\text{I.} \ldots \varepsilon^{\mathrm{d}} fq = f(q + \mathrm{d}q), \quad \text{if} \quad \mathrm{d}^2 q = 0,$$

which is, in fact, a known* mode of expressing the *Series* or *Theorem* of *Taylor*, holds good for *Quaternion Functions generally*, and not merely for those functions of *finite* and *polynomial form*, with *positive* and *whole exponents*, for which it was lately deduced, in 340, (6.). For let fq and $f(q + \mathrm{d}q)$ denote *any two states*, or *values*, of which *neither* is *infinite*, of *any function of a quaternion;* and of the m first differentials,

$$\text{II.} \ldots \mathrm{d}fq, \quad \mathrm{d}^2 fq, \ldots \quad \mathrm{d}^m fq, \quad \text{in which} \quad \mathrm{d}q = \text{const.},$$

let it be supposed that *no one* is *infinite*, and that the *last* of them is different from *zero;* while all that precede it, and the functions fq and $f(q + \mathrm{d}q)$ themselves, may or may not happen to vanish. Let the first m *terms*, of the *exponential development* of the *symbol* $(\varepsilon^{\mathrm{d}} - 1)fq$, be denoted briefly by q_1, q_2, .. q_m; and let r_m denote what may be called the *remainder of the series*, or the *correction* which must be conceived to be added to the *sum* of these m terms, in order to produce the *exact value* of the *difference*,

$$\text{III.} \ldots \Delta fq = f(q + \Delta q) - fq = f(q + \mathrm{d}q) - fq;$$

in such a manner that we shall have *rigorously*, by the *notations* employed, the equation,

$$\text{IV.} \ldots f(q + \mathrm{d}q) = fq + q_1 + q_2 + \ldots + q_m + r_m, \text{ where } q_m = \frac{\mathrm{d}^m fq}{2.3\ldots m};$$

this term q_m being different from *zero*, but *no one* of the terms being *infinite*, by what has been above supposed. Then we shall prove, *as a Theorem*, that

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* Lacroix, for instance, in page 168 of the First Volume of his larger Treatise on the Differential and Integral Calculus (Paris, 1810), presents the Theorem of Taylor under the form,

$$u' = u + \frac{\mathrm{d}u}{1} + \frac{\mathrm{d}^2 u}{1.2} + \frac{\mathrm{d}^3 u}{1.2.3} + \frac{\mathrm{d}^4 u}{1.2.3.4} + \&\text{c.};$$

where u' denotes the value which the function u receives, when the variable x receives the *arbitrary increment* dx (l'accroissement quelconque dx).

$$\text{V.} \ldots \lim (\mathrm{T}r_m : \mathrm{T}q_m) = 0, \quad \text{if} \quad \lim. \mathrm{T}\mathrm{d}q = 0;$$

or in words, that *the tensor of the remainder may be made to bear as small a ratio as we please, to the tensor of the last term retained, by diminishing the tensor, without changing the versor, of the differential* (or *difference*) dq. And *this* very general result, which will soon be seen to extend to functions of *several* quaternions, is in the present Calculus that *analogue* of Taylor's theorem to which we lately alluded (in 340, (7.)); and it may be called, for the sake of reference, "*Taylor's Theorem adapted to Quaternions.*"

(1.) Writing

$$\text{VI.} \ldots Fx = f(q + x\mathrm{d}q) - fq - x\mathrm{d}fq - \frac{x^2}{2}\mathrm{d}^2fq - \ldots - \frac{x^{m-1}}{2.3..(m-1)}\mathrm{d}^{m-1}fq,$$

we shall have the following successive derivatives with respect to x,

$$\text{VII.} \ldots \begin{cases} F'x = \mathrm{d}f(q + x\mathrm{d}q) - \mathrm{d}fq - x\mathrm{d}^2fq - \ldots - \dfrac{x^{m-2}}{2.3\ldots(m-2)}\mathrm{d}^{m-1}fq; \\ F''x = \mathrm{d}^2f(q + x\mathrm{d}q) - \mathrm{d}^2fq - \ldots - \dfrac{x^{m-3}}{2.3..(m-3)}\mathrm{d}^{m-1}fq; \ldots \\ F^{(m-1)}x = \mathrm{d}^{m-1}f(q + x\mathrm{d}q) - \mathrm{d}^{m-1}fq; \text{ and finally,} \\ F^{(m)}x = \mathrm{d}^mf(q + x\mathrm{d}q); \end{cases}$$

because, by 327, VI., and 324, IV.,

$$\text{VIII.} \ldots \mathrm{D}f(q + x\mathrm{d}q) = \lim_{n=\infty} . n\{f(q + x\mathrm{d}q + n^{-1}\mathrm{d}q) - f(q + x\mathrm{d}q)\} = \mathrm{d}f(q + x\mathrm{d}q),$$

and in like manner,

$$\text{IX.} \ldots \mathrm{D}^2f(q + x\mathrm{d}q) = \mathrm{d}^2f(q + x\mathrm{d}q), \text{ \&c.};$$

the mark of derivation D referring to the scalar variable x, while d operates on q alone, and not here on x, nor on dq.

(2.) We have therefore, by VI. and VII., the values,

$$\text{X.} \ldots F0 = 0, \quad F'0 = 0, \quad F''0 = 0, \ldots F^{(m-1)}0 = 0, \quad F^{(m)}0 = \mathrm{d}^mfq;$$

whence, by 341, XIX., we have this limiting equation,

$$\text{XI.} \ldots \lim_{x=0} . \left(Fx : \frac{x^m}{2.3\ldots m} \right) = \mathrm{d}^mfq;$$

or

$$\text{XII.} \ldots \lim_{x=0} (Fx : \psi x) = 1, \quad \text{if} \quad \psi x = \left(\frac{x^m\mathrm{d}^mfq}{2.3\ldots m} \right).$$

(3.) But these two functions, Fx and ψx, are formed by IV. from $q_m + r_m$ and q_m, by changing dq to xdq; and instead of thus *multiplying* dq by a *decreasing scalar*, x, we may *diminish* its *tensor* Tdq, without changing its *versor* Udq. We may therefore say that, when this is done, the *quotient* $(q_m + r_m) : q_m$ *tends* to *unity*, or this other quotient $r_m : q_m$ to *zero*, as its *limit*; or in other words, the *limiting equation* V. holds good.

(4.) As an *example*, let the *function* fq be the *reciprocal*, q^{-1}; then (comp. 339, III.) its mth differential is (for $\mathrm{d}q = \text{const.}$),

$$\text{XIII.} \ldots \mathrm{d}^m fq = \mathrm{d}^m . q^{-1} = 2.3 \ldots m . q^{-1}(-r)^m, \quad \text{if} \quad r = \mathrm{d}q . q^{-1};$$

and it is easy to prove, *without differentials*, that

$$\text{XIV.} \ldots (q + rq)^{-1} = q^{-1}(1+r)^{-1} = q^{-1}\{1 - r + r^2 - \ldots + (-r)^m + (-r)^{m+1}(1+r)^{-1}\};$$

we have therefore here

$$\text{XV.} \ldots q_m = q^{-1}(-r)^m, \quad r_m = -q_m r(1+r)^{-1}, \quad \mathrm{T}(r_m : q_m) = \mathrm{T}r . \mathrm{T}(1+r)^{-1};$$

and this last tensor indefinitely diminishes with $\mathrm{Td}q$, the quaternion q being supposed to have some given value different from zero.

(5.) In general, if we establish the following equation,

$$\text{XVI.} \ldots f(q + n^{-1}\mathrm{d}q) = fq + n^{-1}\mathrm{d}fq + \frac{n^{-2}}{2}\mathrm{d}^2 fq + \ldots + \frac{n^{1-m}}{2.3\ldots(m-1)}\mathrm{d}^{m-1}fq$$
$$+ \frac{n^{-m}}{2.3\ldots m} f_n^{(m)}(q, \mathrm{d}q),$$

as a *definitional extension* of the equation 325, V.; and if we suppose that neither the function fq itself, nor any one of its differentials as far as $\mathrm{d}^{m-1}fq$ is infinite; the result contained in the limiting equation XI. may then be expressed by the formula,

$$\text{XVII.} \ldots f_\infty^{(m)}(q, \mathrm{d}q) = \mathrm{d}^m fq;$$

which for the particular value $m = 1$, if we suppress the upper index, coincides with the form 325, VIII. of the definition $\mathrm{d}fx$, but for *higher* values of m contains a *theorem:* namely (when $\mathrm{d}^m fq$ is supposed *neither* to *vanish*, *nor* to become *infinite*), what we have called *Taylor's Theorem adapted to Quaternions.*

343. That very important theorem may be applied to cases, in which a *quaternion* (as in 327, (5.)), or a *vector* (as in 337), is expressed as a *function of a scalar;* also to *transcendental forms* (333), whenever the differentiations can be effected; and to those *new forms* (334), which result from the *peculiar operations* of the present Calculus itself. A few such applications may here be given.

(1.) Taking first this transcendental and quaternion function of a variable scalar,

$$\text{I.} \ldots q = ft = a^t, \quad \text{with} \quad \mathrm{T}a = 1, \quad \mathrm{d}a = 0, \quad \mathrm{d}t = \text{const.},$$

we have, by 333, VIII., the general term,

$$\text{II.} \ldots q_m = \frac{\mathrm{d}^m . a^t}{2.3\ldots m} = \frac{a^t}{2.3\ldots m}\left(\frac{\pi a \mathrm{d}t}{2}\right)^m = \frac{a^t(xa)^m}{2.3\ldots m}, \quad \text{if} \quad 2x = \pi \mathrm{d}t;$$

dividing then $\epsilon^{\mathrm{d}} . a^t$ by a^t, we obtain an *infinite series*, which is found to be *correct*, and *convergent;* namely (comp. 308, (4.)),

$$\text{III.} \ldots a^{\mathrm{d}t} = 1 + xa + \frac{(xa)^2}{2} + \ldots + \frac{(xa)^m}{2.3\ldots m} + \ldots = \epsilon^{xa} = \cos\frac{\pi \mathrm{d}t}{2} + a\sin\frac{\pi \mathrm{d}t}{2}.$$

(2.) Correct and *finite expansions*, for $\mathrm{S}(q + \mathrm{d}q)$, $\mathrm{V}(q + \mathrm{d}q)$, $\mathrm{K}(q + \mathrm{d}q)$, and $\mathrm{N}(q + \mathrm{d}q)$, are obtained when we operate with ϵ^{d} on $\mathrm{S}q$, $\mathrm{V}q$, $\mathrm{K}q$, and $\mathrm{N}q$; for example ($\mathrm{d}q$ being still constant), the third and higher differentials of $\mathrm{N}q$ vanish by 334, XI., and we have

$$\text{IV.} \ldots \epsilon^{\mathrm{d}}\mathrm{N}q = (1 + \mathrm{d} + \tfrac{1}{2}\mathrm{d}^2)\mathrm{N}q = \mathrm{N}q + 2\mathrm{S}(\mathrm{K}q\,.\,\mathrm{d}q) + \mathrm{N}\mathrm{d}q = \mathrm{N}(q + \mathrm{d}q);$$

an expression for the *norm of a sum*, which agrees with 210, XX., and with 200, VII.

(3.) To develope, on like principles, the *tensor and versor of a sum*, let us again write r for $\mathrm{d}q : q$, and denote the scalar and vector parts of this quotient by s and v; so that, by 334, XIII. and XV.,

$$\text{V.} \ldots s = \mathrm{S}r = \mathrm{S}\frac{\mathrm{d}q}{q} = \frac{\mathrm{dT}q}{\mathrm{T}q}; \qquad \text{VI.} \ldots v = \mathrm{V}r = \mathrm{V}\frac{\mathrm{d}q}{q} = \frac{\mathrm{dU}q}{\mathrm{U}q}.$$

(4.) Then writing also, for abridgment, as in a known notation of *factorials*,

$$\text{VII.} \ldots [-\overset{m}{1}] = (-1)\,.\,(-2)\,.\,(-3)\,.\,\ldots\,.\,(-m),$$

we shall have, by 342, XIII., $\mathrm{d}q$ being still treated as constant, the equation,

$$\text{VIII.} \ldots \mathrm{d}^m(s + v) = \mathrm{d}^m r = [-\overset{m}{1}]\,r^{m+1} = [-\overset{m}{1}]\,(s + v)^{m+1},$$

of which it is easy to separate the scalar and vector parts; for example,

$$\text{IX.} \ldots \mathrm{d}s = -\mathrm{S}\,.\,(s + v)^2 = -(s^2 + v^2); \qquad \mathrm{d}v = -\mathrm{V}\,.\,(s + v)^2 = -2sv.$$

(5.) We have also, by V. and VI.,

$$\text{X.} \ldots \frac{\mathrm{d}^m\mathrm{T}q}{\mathrm{T}q} = (s + \mathrm{d})\frac{\mathrm{d}^{m-1}\mathrm{T}q}{\mathrm{T}q} = \ldots = (s + \mathrm{d})^m 1; \quad \therefore \mathrm{d}^m(\mathrm{T}q) = (s+\mathrm{d})^m . 1$$

$$\text{XI.} \ldots \frac{\mathrm{d}^m\mathrm{U}q}{\mathrm{U}q} = (v + \mathrm{d})\frac{\mathrm{d}^{m-1}\mathrm{U}q}{\mathrm{U}q} = \ldots = (v + \mathrm{d})^m 1; \therefore \mathrm{d}^m(\mathrm{U}q) = (v+\mathrm{d})^m . 1$$

the notation being such that we have, for instance, by IX.,

$$\text{XII.} \ldots (s + \mathrm{d})1 = s; \quad (s + \mathrm{d})^2 1 = (s + \mathrm{d})s = s^2 + \mathrm{d}s = -v^2;$$

$$\text{XIII.} \ldots (v + \mathrm{d})1 = v; \quad (v + \mathrm{d})^2 1 = (v + \mathrm{d})v = v^2 + \mathrm{d}v = v^2 - 2sv.$$

(6.) The *exponential formula* 342, I., gives, therefore,

$$\text{XIV.} \ldots \mathrm{T}(q + \mathrm{d}q) = \epsilon^{\mathrm{d}}\mathrm{T}q = \epsilon^{s+\mathrm{d}}1\,.\,\mathrm{T}q;$$

$$\text{XV.} \ldots \mathrm{U}(q + \mathrm{d}q) = \epsilon^{\mathrm{d}}\mathrm{U}q = \epsilon^{v+\mathrm{d}}1\,.\,\mathrm{U}q;$$

or, dividing and substituting,

$$\text{XVI.} \ldots \mathrm{T}(1 + s + v) = \epsilon^{s+\mathrm{d}}1; \qquad \text{XVII.} \ldots \mathrm{U}(1 + s + v) = \epsilon^{v+\mathrm{d}}1; \qquad \frac{\mathrm{d}q}{q} = s + v$$

s and v being here a *scalar* and a *vector*, which are entirely *independent* of each other; but of which, in the applications, the *tensors* must not be taken *too large*, in order that the *series* may *converge*.

(7.) The *symbolical expressions*, XVI. and XVII., for those two *series*, may be *developed* by (4.) and (5.); thus, if we only write down the terms which do not exceed the *second dimension*, with respect to s and v, we have by XII. and XIII. the development,

$$\text{XVIII.} \ldots \mathrm{T}(1 + s + v) = 1 + s - \tfrac{1}{2}v^2 + \ldots,$$

$$\text{XIX.} \ldots \mathrm{U}(1 + s + v) = 1 + v + (\tfrac{1}{2}v^2 - sv) + \ldots;$$

of which accordingly the *product* is $1 + s + v$, to the same order of approximation.

(8.) A *function of a sum* of two quaternions can sometimes be developed, *without differentials*, by processes of a more *algebraical character*; and when this happens, we may *compare* the result with the *form* given by *Taylor's Series*, as adapted to quaternions in 342, and so *deduce* the *values* of the *successive differentials* of the function; for example, we can *infer* the expression 342, XIII. for $\mathrm{d}^m\,.\,q^{-1}$, from the *series* 342, XIV., for the *reciprocal of a sum*.

(9.) And not only may we *verify* the recent developments, XVIII. and XIX., by comparing them with the more *algebraical forms*,

$$\text{XX.}\ldots\ \mathrm{T}(1+s+v)=(1+s+v)^{\frac{1}{2}}\,(1+s-v)^{\frac{1}{2}},$$
$$\text{XXI.}\ldots\ \mathrm{U}(1+s+v)=(1+s+v)^{\frac{1}{2}}\,(1+s-v)^{-\frac{1}{2}},$$

but also, if the first of these for example (when expanded by *ordinary processes*, which are in *this* case applicable) have given us, *without* differentials,

$$\text{XXII.}\ldots\ \mathrm{T}(q+q')=(1+s-\tfrac{1}{2}v^2\ldots)\mathrm{T}q,\quad\text{where}\quad s=\mathrm{S}q'q^{-1},\quad\text{and}\quad v=\mathrm{V}q'q^{-1},$$

we can then *infer* the values of the *first* and *second differentials* of the *tensor* of a quaternion, as follows:

$$\text{XXIII.}\ldots\ \mathrm{dT}q=\mathrm{S}\frac{\mathrm{d}q}{q}.\mathrm{T}q;\quad \mathrm{d}^2\mathrm{T}q=-\left(\mathrm{V}\frac{\mathrm{d}q}{q}\right)^2\mathrm{T}q;$$

whereof the first agrees with 334, XII. or XIII., and the second can be deduced from it, under the form,

$$\text{XXIV.}\ldots\ \mathrm{d}^2\mathrm{T}q=\mathrm{d}\left(\mathrm{S}\frac{\mathrm{d}q}{q}.\mathrm{T}q\right)=\left(\left(\mathrm{S}\frac{\mathrm{d}q}{q}\right)^2-\mathrm{S}.\left(\frac{\mathrm{d}q}{q}\right)^2\right)\mathrm{T}q.$$

(10.) In general, if we can only develope a function $f(q+q')$ as far as the term or terms which are of the *first dimension* relatively to q', we shall still obtain thus an expression for the *first differential* $\mathrm{d}fq$, by merely writing $\mathrm{d}q$ in the place of q'. But we have not chosen (comp. 100, (14.)) to regard *this property of the differential of a function* as the *fundamental one*, or to adopt it as the *definition* of $\mathrm{d}fq$; because we have not chosen to *postulate* the *general possibility* of such *developments of functions of quaternion sums*, of which in fact it is in many cases *difficult to discover the laws*, or even to *prove the existence*, except in some such way as that above explained.

(11.) This opportunity may be taken to observe, that (with recent notations) we have, by VIII., the symbolical expression,

$$\text{XXV.}\ldots\ \varepsilon^{s+v+\mathrm{d}}1=1+s+v;\quad\text{or}\quad\text{XXVI.}\ldots\ \varepsilon^{r+\mathrm{d}}1=1+r.$$

344. *Successive differentials* are also connected with *successive differences*, by laws which it is easy to investigate, and on which only a few words need here be said.

(1.) We can easily prove, from the definition 324, IV. of $\mathrm{d}fq$, that if $\mathrm{d}q$ be constant,

$$\text{I.}\ldots\ \mathrm{d}^2fq=\lim_{n=\infty}.\,n^2\{f(q+2n^{-1}\mathrm{d}q)-2f(q+n^{-1}\mathrm{d}q)+fq\};$$

with analogous expressions for differentials of higher orders.

(2.) Hence we may say (comp. 340, X.) that the *successive differentials*,

$$\text{II.}\ldots\ \mathrm{d}fq,\quad \mathrm{d}^2fq,\quad \mathrm{d}^3fq,\ldots\quad\text{for}\quad \mathrm{d}^2q=0,$$

are *limits* to which the following *multiples of successive differences*,

$$\text{III.}\ldots\ n\Delta fq,\quad n^2\Delta^2fq,\quad n^3\Delta^3fq,\ldots\quad\text{for}\quad \Delta^2q=0,$$

all simultaneously *tend*, when the multiple $n\Delta q$ is either constantly *equal* to $\mathrm{d}q$, or at least *tends* to become equal thereto, while the number n increases indefinitely.

(3.) And hence we might prove, in a new way, that *if the function* $f(q+\mathrm{d}q)$

can be developed, in a *series* proceeding according to *ascending* and *whole dimensions* with respect to dq, the *parts* of this series, which are of those *successive* dimensions, must follow the *law* expressed by *Taylor's Theorem** *adapted to Quaternions* (342).

345. It is easy to conceive that the foregoing results may be extended (comp. 338), to the successive differentiations of functions of *several quaternions;* and that thus there arises, in each such case, a *system of successive differentials, total and partial:* as also a system of *partial derivatives*, of orders higher than the first, when a *quaternion*, or a *vector*, is regarded (comp. 337) as a function of *several scalars*.

(1.) The *general* expression for the *second total differential*,

$$\text{I.} \ldots \mathrm{d}^2 Q = \mathrm{d}^2 F(q, r, \ldots),$$

involves $\mathrm{d}^2 q$, $\mathrm{d}^2 r$, . . ; but it is often convenient to suppose that all *these* second differentials *vanish*, or that the *first* differentials dq, dr, . . are *constant ;* and *then* $\mathrm{d}^m Q$, or $\mathrm{d}^m F(q, r, \ldots)$, becomes a rational, integral, and homogeneous function of the mth dimension, of those first differentials dq, dr, . . , which may (comp. 329, III.) be thus denoted,

$$\text{II.} \ldots \mathrm{d}^m Q = (\mathrm{d}_q + \mathrm{d}_r + \ldots)^m Q; \quad \text{or briefly,} \quad \text{III.} \ldots \mathrm{d}^m = (\mathrm{d}_q + \mathrm{d}_r + \ldots)^m,$$

in developing which *symbolical power*, the *multinomial theorem* of *algebra* may be employed : because we have generally, for quaternions as in the ordinary calculus,

$$\text{IV.} \ldots \mathrm{d}_r \mathrm{d}_q = \mathrm{d}_q \mathrm{d}_r.$$

(2.) For example, if we denote dq and dr by q' and r', and suppose

$$\text{V.} \ldots Q = rqr, \quad \text{then} \quad \text{VI.} \ldots \mathrm{d}_q Q = rq'r; \quad \text{VII.} \ldots \mathrm{d}_r Q = r'qr + rqr';$$

and

$$\text{VIII.} \ldots \mathrm{d}_r \mathrm{d}_q Q = \mathrm{d}_q \mathrm{d}_r Q = r'q'r + rq'r'.$$

And in general, *each* of the two equated symbols IV. gives, by its operation on $F(q, r)$, the *limit* of this other function, or product (comp. 344, I.),

$$\text{IX.} \ldots nn' \{ F(q + n^{-1}\mathrm{d}q, r + n'^{-1}\mathrm{d}r) - F(q, r + n'^{-1}\mathrm{d}r) - F(q + n^{-1}\mathrm{d}q, r) + F(q, r) \};$$

in which the numbers n and n' are supposed to tend to infinity.

(3.) We may also write, for functions of *several* quaternions,

$$\text{X.} \ldots Q + \Delta Q = F(q + \mathrm{d}q, r + \mathrm{d}r, \ldots) = \epsilon^{\mathrm{d}_q + \mathrm{d}_r + \ldots} F(q, r);$$

or briefly,

$$\text{XI.} \ldots 1 + \Delta = \epsilon^{\mathrm{d}_q + \mathrm{d}_r + \ldots} = \epsilon^{\mathrm{d}};$$

with *interpretations* and *transformations* analogous to those which have occurred already, for functions of a *single* quaternion.

(4.) Finally, as an example of *successive* and *partial derivation*, if we resume the vector expression 308, XVIII. (comp. 315, XII. and XIII.), namely,

$$\text{XII.} \ldots \rho = rk^t j^s k j^{-s} k^{-t},$$

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* Some remarks on the adaptation and proof of this important theorem will be found in the *Lectures*, pages 589, &c.

which has been seen to be capable of representing the vector of *any point of space*, we may observe that it gives, *without trigonometry*, by the principle mentioned in 308, (11.), and by the sub-articles to 315, not only the form,

$$\text{XIII.} \ldots \rho = rk^t j^{2s} k^{1-t}, \text{ as in 308, XIX.},$$

but also, if a be *any vector unit*,

$$\text{XIV.} \ldots \rho = rk^{t+1} j^{-2s} k^{-t} = rk^t (k\text{S}\,.\,a^{2s} + i\text{S}\,.\,a^{2s-1})\,.\,k^{-t};$$

whence $\text{XV.} \ldots \rho = r\text{V}.\,k^{2s+1} + rk^{2t}\text{V}.\,i^{2s}$, as in 315, XII.

(5.) We have therefore the following new expressions (compare the sub-articles to 337), for the two *partial derivatives* of the *first order*, of this variable vector ρ, taken with respect to s and t:

$$\text{XVI.} \ldots \text{D}_s\rho = \pi rk^t j^s i j^{-s} k^{-t} = -\pi\rho k^t j k^{-t},$$

with the verification, that

$$\text{XVII.} \ldots \rho\text{D}_s\rho = \pi r^2\,.\,k^t j^s k j^{-s} k^{-t}\,.\,k^t j^s i j^{-s} k^{-t} = \pi r^2 k^t j k^{-t};$$

and $\text{XVIII.} \ldots \text{D}_t\rho = \pi rk^{2t}\text{V}.\,j^{2s} = \pi rk^{2t} j\text{S}\,.\,a^{2s-1} = r^{-1}\rho\text{D}_s\rho\,.\,\text{S}\,.\,a^{2s-1}$,

whence $\text{XIX.} \ldots \rho\text{D}_t\rho = -r\text{D}_s\rho\,.\,\text{S}\,.\,a^{2s-1}$, and $\text{XX.} \ldots \text{D}_s\rho\,.\,\text{D}_t\rho = \pi^2 r\rho\text{S}\,.\,a^{2s-1}$;

while $\text{XXI.} \ldots \text{D}_r\rho = r^{-1}\rho = k^t j^s k j^{-s} k^{-t}$, as in 337, XXV.;

so that we have the following *ternary product* of these *derived vectors* of the first order,

$$\text{XXII.} \ldots \text{D}_r\rho\,.\,\text{D}_s\rho\,.\,\text{D}_t\rho = \pi^2\rho^2\text{S}\,.\,a^{2s-1} = \pi r^3\text{D}_s\text{S}\,.\,a^{2s};$$

the *scalar character* of which product depends (comp. 299, (9.)) on the circumstance, that the vectors thus multiplied compose (337, (10.)) a *rectangular system*.

(6.) It is easy then to infer, for the *six partial derivatives* of ρ, of the *second order*, taken with respect to the same three scalar variables, r, s, t, the expressions:

$$\text{XXIII.} \ldots \text{D}_r^2\rho = 0; \quad \text{D}_r\text{D}_s\rho = \text{D}_s\text{D}_r\rho = r^{-1}\text{D}_s\rho; \quad \text{D}_r\text{D}_t\rho = \text{D}_t\text{D}_r\rho = r^{-1}\text{D}_t\rho;$$

$$\text{XXIV.} \ldots \text{D}_s^2\rho = -\pi^2\rho; \quad \text{D}_s\text{D}_t\rho = \text{D}_t\text{D}_s\rho = \pi^2 rk^{2t}\text{V}.\,j^{2s+1}; \quad \text{D}_t^2\rho = -\pi^2 rk^{2t}\text{V}.\,i^{2s}.$$

(7.) The three *partial differentials* of the *first order*, of the same variable vector ρ, are the following:

$$\text{XXV.} \ldots \text{d}_r\rho = r^{-1}\rho\text{d}r; \quad \text{d}_s\rho = \text{D}_s\rho\,.\,\text{d}s; \quad \text{d}_t\rho = \text{D}_t\rho\,.\,\text{d}t;$$

with the products,

$$\text{XXVI.} \ldots \text{d}_s\rho\,.\,\text{d}_t\rho = -\pi r\rho\text{dS}\,.\,a^{2s}\,.\,\text{d}t;$$

$$\text{XXVII.} \ldots \text{d}_r\rho\,.\,d_s\rho\,.\,\text{d}_t\rho = \pi r^2\text{d}r\,.\,\text{dS}\,.\,a^{2s}\,.\,\text{d}t.$$

(8.) These *differential vectors*, $\text{d}_r\rho$, $\text{d}_s\rho$, $\text{d}_t\rho$, are (in the present theory) *generally finite;* $\text{d}_r\rho$, like $\text{D}_r\rho$, being a line in the direction of ρ, or of the *radius* of this *sphere* round the origin, at least if $\text{d}r$, like r, be positive; while $\text{d}_s\rho$, like $\text{D}_s\rho$, is (comp. 100, (9.)) a *tangent to the meridian* of that spheric surface, for which r and t are *constant;* but $\text{d}_t\rho$, like $\text{D}_t\rho$, is on the contrary a *tangent to the small circle* (or *parallel*), on the same sphere, for which r and s are constant.

(9.) Treating only the *radius* r as constant, and writing $\rho = \text{OP}$, if we pass from the point P, or (s, t), to *another point* Q, or $(s + \Delta s, t)$, on the *same meridian*, the *chord* PQ is represented by the *finite difference*, $\Delta_s\rho$; and in like manner, if we pass from P to a point R, or $(s, t + \Delta t)$, on the *same parallel*, the new chord PR is represented by the *other* partial and finite difference, $\Delta_t\rho$; while the point $(s + \Delta s,\ t + \Delta t)$ may be denoted by S.

(10.) If now the *two points* Q and R be conceived to *approach to* P, and to come to be *very near it*, the *chords* PQ and PR will *very nearly coincide* with the two corresponding *arcs* of meridian and parallel; or with the *tangents* to the same two circles at P, so drawn as to have the *lengths* of those two arcs: or finally with the *differential* and *tangential vectors*, $d_s\rho$ and $d_t\rho$, if we suppose (as we may, comp. 322) that the two *arbitrary* and *scalar differentials*, ds and dt, are so *assumed* as to be constantly *equal* to the two *differences*, Δs and Δt, and consequently to *diminish with them*.

(11.) Whether the differentials ds and dt be *large* or *small*, the *product* $d_s\rho \,.\, d_t\rho$, like the product $D_s\rho \,.\, D_t\rho$, represents *rigorously* a *normal vector* (as in XXVI. and XX.); of which the *length* bears to *the unit of length* (comp. 281) the *same ratio*, as that which the *rectangle* under the *two perpendicular tangents*, $d_s\rho$ and $d_t\rho$, to the *sphere*, bears to the *unit of area*. Hence, with the recent suppositions (10.), we may regard this *product* $d_s\rho \,.\, d_t\rho$ as representing, with a continually and indefinitely increasing accuracy, even in the way of ratio, what we may call the *directed element of spheric surface*, PQRS, considered as thus *represented* (or constructed) by a *normal* at P; and the *tensor* of the same product, namely (by XXVI.),

$$\text{XXVIII.} \ldots \mathrm{T}(d_s\rho \,.\, d_t\rho) = -\pi r^2 d\mathrm{S} \,.\, a^{2s} \,.\, dt,$$

in which the *negative sign* is retained, because $\mathrm{S} \,.\, a^{2s}$ decreases from $+1$ to -1, while s increases from 0 to 1, is an expression on the same plan for what we may call by contrast the *undirected element of spheric area*, or that element considered with reference merely to *quantity*, and *not* with reference to *direction*.

(12.) *Integrating*, then, this last *differential expression* XXVIII., from $t = 0$ to $t = 2$, and from $s = s_0$ to $s = s_1$, that is, taking the *limit of the sum* of all the *elements* PQRS between these bounding values, we find the following equation:

$$\text{XXIX.} \ldots \textit{Area of Spheric Zone} = 2\pi r^2 \mathrm{S}\,(a^{2s_0} - a^{2s_1});$$

whence

$$\text{XXX.} \ldots \textit{Area of Spheric Cap } (s) = 2\pi r^2 (1 - \mathrm{S} \,.\, a^{2s}) = 4\pi r^2 (\mathrm{TV} \,.\, a^s)^2;$$

and finally,

$$\text{XXXI.} \ldots \textit{Area of Sphere} = 4\pi r^2, \text{ as usual.}$$

(13.) In like manner the expression XXVII., with its sign changed (on account of the decrease of $\mathrm{S} \,.\, a^{2s}$, as in (11.)), represents the *element of volume*; and thus, by integrating from $r = r_0$ to $r = r_1$, from $s = 0$ to $s = 1$, and from $t = 0$ to $t = 2$, we obtain anew the known values:

$$\text{XXXII.} \ldots \textit{Volume of Spheric Shell} = \frac{4\pi}{3}(r_1^3 - r_0^3);$$

and

$$\text{XXXIII.} \ldots \textit{Volume of Sphere } (r) = \frac{4\pi r^3}{3}, \text{ as usual.}$$

(14.) *These* are however only specimens of what may be called *Scalar Integration*, although connected with *quaternion forms*; and it will be more *characteristic* of the present Calculus, if we apply it briefly to take the *Vector Integral*, or the *limit of the vector-sum* of the *directed elements* (11.), of a portion of a spheric surface: a problem which corresponds, in *hydrostatics*, to calculating the *resultant of the pressures* on that surface, each pressure having a *normal direction*, and a *quantity* proportional to the *element of area*.

(15.) For this purpose, we may employ the expression XXVI. with its sign changed, in order to denote an *inward normal*, or a *pressure* acting *from without*; and if we then substitute for ρ its value XV., and observe that

$$\text{XXXIV.} \ldots \int_0^2 k^{2t}\mathrm{d}t = 0, \text{ because } k^2 = -1,$$

and remember that $\mathrm{V}.\,k^{2t+1} = k\mathrm{S}\,.\,a^{2t}$, we easily deduce the expressions:

$$\text{XXXV.} \ldots \textit{Sum of Directed Elements of Elementary Zone} = \pi r^2 k\mathrm{d}\,.\,(\mathrm{S}\,.\,a^{2t})^2;$$

$$\text{XXXVI.} \ldots \textit{Sum of Directed Elements of Spheric Cap } (s) = -\pi r^2 k\,(1 - (\mathrm{S}.a^{2t})^2)$$
$$= \pi r^2 k\,(\mathrm{V}.\,a^{2t})^2 = \pi^{-1} k\,(\mathrm{D}_t\rho)^2 = \pi k\,(\mathrm{V}k\rho)^2.$$

(16.) But the *radius* of the *plane* and *circular base*, of the *spheric segment* corresponding, is $\mathrm{TV}k\rho$, so that its *area* is in *quantity* $= -\pi\,(\mathrm{V}k\rho)^2$; and the *common direction* of all its *inward normals* is that of $+k$; hence if we still represent the directed *elements* by normals thus drawn *inwards*, we have this new expression:

$$\text{XXXVII.} \ldots \textit{Sum of Directed Elements of Circular Base} = -\pi k\,(\mathrm{V}k\rho)^2;$$

comparing which with XXXVI., we arrive at the formula,

$$\text{XXXVIII.} \ldots \textit{Sum of Directed Elements of Spheric Segment} = \textit{Zero};$$

a result which may be greatly extended, and which evidently answers to a known case of *equilibrium* in *hydrostatics*.

(17.) These few *examples* may serve to show already, that *Differentials of Quaternions* (or of *Vectors*) may be *applied* to various *geometrical* and *physical questions*; and that, when so applied, it is *permitted* to treat them as *small*, if any *convenience* be gained thereby, as in cases of *integration* there always is. But we must now pass to an important investigation of another kind, with which *differentials* will be found to have only a sort of *indirect* or *suggestive* connexion.

SECTION 6.—*On the Differentiation of Implicit Functions of Quaternions; and on the General Inversion of a Linear Function, of a Vector or a Quaternion: with some connected Investigations.*

346. We saw, when differentiating the square-root of a quaternion (332, (5.) and (6.)), that it was necessary for that purpose to *resolve* a *linear equation*,* or an equation of the *first degree*; namely the equation,

$$\text{I} \;\ldots\; rr' + r'r = q',$$

in which r and q' represented two given quaternions, $q^{\frac{1}{2}}$ and $\mathrm{d}q$, while r' represented a sought quaternion, namely $\mathrm{d}r$ or $\mathrm{d}\,.\,q^{\frac{1}{2}}$. And generally, from the *linear* or *distributive form* (327), of the *quaternion differential*

$$\text{II.} \ldots \mathrm{d}Q = \mathrm{d}fq = f(q, \mathrm{d}q),$$

of any given and *explicit function* fq, when considered as depending on the differential $\mathrm{d}q$ of the quaternion variable q, we see that the *return* from the former differential to the latter, that is from dQ to dq, or the *differentiation* of the *inverse* or *implicit* function $f^{-1}Q$, requires for its accomplishment the *Solution of an Equation of the First Degree:* or what may be called the *Inversion of a Linear Function of a Quaternion.* We are therefore led to consider here that general Problem; to which accordingly, and to investigations connected with which, we shall devote the present Section, dismissing however now the *special* consideration of the *Differentials* above mentioned, or treating them only as *Quaternions*, sought or given, of which the *relations* to each other are to be studied.

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* Compare the Note to page 410.

347. Whatever the *particular form* of the given *linear* or *distributive function*, fq, may be, we can always *decompose* it as follows:

$$\text{I.} \ldots fq = f(\mathrm{S}q + \mathrm{V}q) = f\mathrm{S}q + f\mathrm{V}q = \mathrm{S}q.f1 + f\mathrm{V}q;$$

taking then separately scalars and vectors, or operating with S and V on the proposed *linear equation*,

$$\text{II.} \ldots fq = r,$$

where r is a *given* quaternion, and q a *sought* one, we can in general *eliminate* Sq, and so reduce the problem to the solution of a *linear* and *vector equation*, of the form,

$$\text{III.} \ldots \phi\rho = \sigma;$$

where σ is a *given vector*, but $\rho(= \mathrm{V}q)$ is a *sought* one, and ϕ is used as the characteristic of a *given linear and vector function* of a vector, which function we shall throughout suppose to be a *real* one, or to involve *no imaginary constants* in its composition. But, to every *such* function $\phi\rho$, there always *corresponds* what may be called a *conjugate* linear and vector function $\phi'\rho$, connected with it by the following *Equation of Conjugation*,

$$\text{IV.} \ldots \mathrm{S}\lambda\phi\rho = \mathrm{S}\rho\phi'\lambda;$$

where λ and ρ are *any two vectors.* Assuming then, as we may, that μ and ν are *two auxiliary* vectors, so chosen as to satisfy the equation,

$$\text{V.} \ldots \mathrm{V}\mu\nu = \sigma,$$

and therefore also,

$$\text{VI.} \ldots \mathrm{S}\lambda\sigma = \mathrm{S}\lambda\mu\nu, \quad \mathrm{S}\mu\sigma = 0, \quad \mathrm{S}\nu\sigma = 0,$$

where λ is a *third* auxiliary and arbitrary vector, we may (comp. 312) replace the *one* vector equation III. by the *three* scalar equations,

$$\text{VII.}\ldots\ \mathrm{S}\rho\phi'\lambda = \mathrm{S}\lambda\mu\nu,\quad \mathrm{S}\rho\phi'\mu = 0,\quad \mathrm{S}\rho\phi'\nu = 0.$$

And these give, by principles with which the reader is supposed to be already familiar,* the expression,

$$\text{VIII.}\ldots\ m\rho = \psi\sigma,\quad \text{or}\quad \text{IX.}\ldots\ \rho = \phi^{-1}\sigma = m^{-1}\psi\sigma,$$

if m be a *vector-constant*, and ψ an *auxiliary linear and vector function*, of which the *value* and the *form* are determined by the two following equations:

$$\text{X.}\ldots\ m\mathrm{S}\lambda\mu\nu = \mathrm{S}(\phi'\lambda.\phi'\mu.\phi'\nu);$$

$$\text{XI.}\ldots\ \psi(\mathrm{V}\mu\nu) = \mathrm{V}(\phi'\mu.\phi'\nu);$$

[illegible]

$m\rho = \mathrm{V}(\phi'\mu\phi'\nu)$

[illegible]

or briefly,

$$\text{X}'.\ldots\ m\mathrm{S}\lambda\mu\nu = \mathrm{S}.\phi'\lambda\phi'\mu\phi'\nu,$$

and

$$\text{XI}'.\ldots\ \psi\mathrm{V}\mu\nu = \mathrm{V}.\phi'\mu\phi'\nu.$$

And thus the proposed *Problem of Inversion*, of the linear and vector function ϕ, may be considered to be, in all its generality, *resolved;* because it is always possible so to *prepare* the second members of the equations X. and XI., that they shall take the *forms* indicated in the first members of those equations.

(1.) For example, if we *assume* any three diplanar vectors α, α', α'', and *deduce* from them three other vectors β_0, β'_0, β''_0, by the equations,

$$\text{XII.}\ldots\ \beta_0\mathrm{S}\alpha\alpha'\alpha'' = \mathrm{V}\alpha'\alpha'',\quad \beta'_0\mathrm{S}\alpha\alpha'\alpha'' = \mathrm{V}\alpha''\alpha,\quad \beta''_0\mathrm{S}\alpha\alpha'\alpha'' = \mathrm{V}\alpha\alpha',$$

then *any* vector ρ may, by 294, XV., be expressed as follows,

$$\text{XIII.}\ldots\ \rho = \beta_0\mathrm{S}\alpha\rho + \beta'_0\mathrm{S}\alpha'\rho + \beta''_0\mathrm{S}\alpha''\rho;$$

if then we write,

$$\text{XIV.}\ldots\ \beta = \phi\beta_0,\quad \beta' = \phi\beta_0',\quad \beta'' = \phi\beta''_0,$$

we shall have the following *General Expression*, or *Standard Trinomial Form*, for a *Linear and Vector Function of a Vector*,

$$\text{XV.}\ldots\ \phi\rho = \beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho + \beta''\mathrm{S}\alpha''\rho;$$

containing, as we see, *three vector constants*, β, β', β'', or *nine scalar constants*, such as

$$\text{XVI.}\ldots\ \mathrm{S}\alpha\beta,\ \mathrm{S}\alpha'\beta,\ \mathrm{S}\alpha''\beta;\quad \mathrm{S}\alpha\beta',\ \mathrm{S}\alpha'\beta',\ \mathrm{S}\alpha''\beta';\quad \mathrm{S}\alpha\beta'',\ \mathrm{S}\alpha'\beta'',\ \mathrm{S}\alpha''\beta'';$$

which may (and generally will) all *vary*, in passing from *one* linear and *vector function* $\phi\rho$ to *another* such function; but which are *all* supposed to be *real*, and *given*, for each *particular form* of that function.

(2.) Passing to what we have called the *conjugate* linear function $\phi'\rho$, the form XV. gives, by IV., the expression,

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* A student might find it useful, at this stage, to read again the Sixth Section of the preceding Chapter; or at least the early sub-articles to Art. 294, a *familiar* acquaintance with which is presumed in the present Section.

$$\text{XVII.} \ldots \phi'\rho = \alpha S\beta\rho + \alpha' S\beta'\rho + \alpha'' S\beta''\rho\,;$$

but

$$\text{V.}(\alpha S\beta\mu + \alpha' S\beta'\mu)\,(\alpha S\beta\nu + \alpha' S\beta'\nu) = \text{V}\alpha\alpha' \text{S}.\beta'(\nu S\beta\mu - \mu S\beta\nu)$$
$$= \text{V}\alpha\alpha' \text{S}.\beta' \text{V}.\beta \text{V}\mu\nu = \text{V}\alpha\alpha' \text{S}.\beta'\beta \text{V}\mu\nu\,;$$

therefore the transformation XI. succeeds, and gives,

$$\text{XVIII.} \ldots \psi\rho = \text{V}\alpha'\alpha'' S\beta''\beta'\rho + \text{V}\alpha''\alpha S\beta\beta''\rho + \text{V}\alpha\alpha' S\beta'\beta\rho,$$

as an expression for the *auxiliary* function ψ; of which the *conjugate* may be thus written,

$$\text{XIX.} \ldots \psi'\rho = \text{V}\beta'\beta'' S\alpha''\alpha'\rho + \text{V}\beta''\beta S\alpha\alpha''\rho + \text{V}\beta\beta' S\alpha'\alpha\rho\,;$$

so that ψ is changed to ψ', when ϕ is changed to ϕ', by interchanging each of the three *alphas* with the corresponding *beta*.

(3.) If we write, as in this whole investigation we propose to do,

$$\text{XX.} \ldots \lambda' = \text{V}\mu\nu, \quad \mu' = \text{V}\nu\lambda, \quad \nu' = \text{V}\lambda\mu,$$

the formulæ XI. and X. become,

$$\text{XXI.} \ldots \psi\lambda' = \text{V}.\phi'\mu\phi'\nu, \quad \text{and} \quad \text{XXII.} \ldots m S\lambda\lambda' = \text{S}.\phi'\lambda\psi\lambda',$$

with the same sort of abridgment of notation as in XI'.; and because the coefficient of $S\alpha\alpha'\alpha''$ in this last expression XXII. is by XVII. XVIII.,

$$S\beta\lambda S\beta''\beta'\lambda' + S\beta'\lambda S\beta\beta''\lambda' + S\beta''\lambda S\beta'\beta\lambda' = S\beta''\beta'\beta S\lambda\lambda',$$

the *division* by $S\lambda\lambda'$, or by $S\lambda\mu\nu$, succeeds, and we find the expression,

$$\text{XXIII.} \ldots m = S\alpha\alpha'\alpha'' S\beta''\beta'\beta\,;$$

which may also be thus written,

$$\text{XXIII}'. \ldots m = S\beta\beta'\beta'' S\alpha''\alpha'\alpha,$$

so that m does not change when we pass from ϕ to ϕ', on which account we may write also,

$$\text{XXIV.} \ldots m S\lambda\lambda' = \text{S}.\phi\lambda\psi'\lambda', \quad \text{or} \quad \text{XXIV}'. \ldots m S\lambda\mu\nu = \text{S}.\phi\lambda\phi\mu\phi\nu,$$

because, by (2.), we can deduce from XI. the conjugate expression,

$$\text{XXV.} \ldots \psi'\lambda' = \text{V}.\phi\mu\phi\nu.$$

(4.) We ought then to find that the *linear equation*,

$$\text{XXVI.} \ldots \sigma = \phi\rho = \beta S\alpha\rho + \beta' S\alpha'\rho + \beta'' S\alpha''\rho,$$

has its *solution* expressed (comp. VIII.) by the formula,

$$\text{XXVII.} \ldots \rho S\alpha\alpha'\alpha'' S\beta''\beta'\beta = \text{V}\alpha'\alpha'' S\beta''\beta'\sigma + \text{V}\alpha''\alpha S\beta\beta''\sigma + \text{V}\alpha\alpha' S\beta'\beta\sigma\,;$$

and accordingly, if we operate on the expression XXVI. for σ with the three symbols,

$$\text{XXVIII.} \ldots \text{S}.\beta''\beta', \quad \text{S}.\beta\beta'', \quad \text{S}.\beta'\beta,$$

we obtain the three scalar equations,

$$\text{XXIX.} \ldots S\beta''\beta'\sigma = S\beta''\beta'\beta S\alpha\rho, \ \&\text{c.},$$

from which the equation XXVII. follows immediately, without any introduction of the auxiliary vectors λ, μ, ν, although these are useful in the theory generally.

(5.) Conversely, if the equation XXVII. were *given*, and the value of σ *sought*, we might operate with the three symbols,

$$\text{XXX.} \ldots \text{S}.\alpha, \quad \text{S}.\beta, \quad \text{S}.\gamma,$$

and so obtain the three scalar equations XXIX., from which the expression XXVI. for σ would follow.

(6.) It will be found an useful check on formulæ of this sort, to consider each *beta*, in what we have called the *Standard Form* (1.) of $\phi\rho$, as being of the *first dimension;* for then we may say that ϕ and ϕ' are *also* of the *first* dimension, but ψ and ψ' of the *second*, and m of the *third;* and every formula, into which these symbols enter, will thus be *homogeneous:* α, α', α'', and λ, μ, ν, ρ, being *not counted*, in this mode of estimating *dimensions*, but σ being treated as of the *first* dimension, when it is taken as representing $\phi\rho$.

(7.) And although the *trinomial form* XV. has been seen to be *sufficiently general*, yet if we choose to take the more expanded form,

$$\text{XXXI.}\ldots\ \phi\rho = \Sigma\beta\mathrm{S}\alpha\rho, \quad \text{which gives} \quad \text{XXXII.}\ldots\ \phi'\rho = \Sigma\alpha\mathrm{S}\beta\rho,$$

any number of *terms* of $\phi\rho$, such as $\beta\mathrm{S}\alpha\rho$, $\beta'\mathrm{S}\alpha'\rho$, &c, being now included in the *sum* Σ, there is no difficulty in proving that the equations VIII. and IX. are satisfied, when we write,

$$\text{XXXIII.}\ldots\ \psi\rho = \Sigma\mathrm{V}\alpha\alpha'\mathrm{S}\beta'\beta\rho, \quad \text{with} \quad \text{XXXIV.}\ldots\ \psi'\rho = \Sigma\mathrm{V}\beta\beta'\mathrm{S}\alpha'\alpha\rho,$$

and

$$\text{XXXV.}\ldots\ m = \Sigma\mathrm{S}\alpha\alpha'\alpha''\mathrm{S}\beta''\beta'\beta = \Sigma\mathrm{S}\beta\beta'\beta''\mathrm{S}\alpha''\alpha'\alpha.$$

(8.) The important property (2.), that the auxiliary function ψ is changed to its own *conjugate* ψ, when ϕ is changed to ϕ', may be proved without any reference to the form $\Sigma\beta\mathrm{S}\alpha\rho$ of $\phi\rho$, by means of the definitions IV. and XI., of ϕ' and ψ, as follows. Whatever four vectors μ, ν, μ_1, and ν_1 may be, if we write

$$\text{XXXVI.}\ldots\ \lambda'_1 = \mathrm{V}\mu_1\nu_1, \quad \text{and} \quad \text{XXXVII.}\ldots\ \psi'\mathrm{V}\mu\nu = \mathrm{V}.\phi\mu\phi\nu,$$

adopting here this last equation as a *definition* of the *function* ψ', we may proceed to *prove* that it is *conjugate* to ψ, by observing that we have the transformations,

$$\begin{aligned}\text{XXXVIII.}\ldots\ \mathrm{S}\lambda'_1\psi'\lambda' &= \mathrm{S}(\mathrm{V}\mu_1\nu_1.\mathrm{V}.\phi\mu\phi\nu) = \mathrm{S}.\mu_1(\mathrm{V}.\nu_1\mathrm{V}.\phi\mu\phi\nu)\\ &= \mathrm{S}\mu_1\phi\nu.\mathrm{S}\nu_1\phi\mu - \mathrm{S}\mu_1\phi\mu.\mathrm{S}\nu_1\phi\nu\\ &= \mathrm{S}\mu\phi'\nu_1.\mathrm{S}\nu\phi'\mu_1 - \mathrm{S}\mu\phi'\mu_1.\mathrm{S}\nu\phi'\nu_1\\ &= \mathrm{S}.\mu(\mathrm{V}.\nu\mathrm{V}.\phi'\mu_1\phi'\nu_1) = \mathrm{S}(\mathrm{V}\mu\nu.\mathrm{V}.\phi'\mu_1\phi'\nu_1) = \mathrm{S}\lambda'\psi\lambda'_1;\end{aligned}$$

which establish the relation in question, between ψ and ψ'.

(9.) And the not less important property (3.), that m remains *unchanged* when we pass from ϕ to ϕ', may in like manner be proved, without reference to the *form* XV. or XXXI. of $\phi\rho$, by observing that we have by XXXVII., &c. the transformations,

$$\text{XXXIX.}\ldots\ \mathrm{S}.\phi\lambda\phi\mu\phi\nu = \mathrm{S}.\phi\lambda\psi'\lambda' = \mathrm{S}\lambda'\psi\phi\lambda = m\mathrm{S}\lambda'\lambda = m\mathrm{S}\lambda\mu\nu,$$

because the equations III. and VIII. give,

$$\text{XL.}\ldots\ \psi\phi\rho = m\rho, \quad \text{whatever vector } \rho \text{ may be;}$$

so that the value of this scalar constant m may now be derived from the *original* linear function ϕ, exactly as it was in X. or X′. from the *conjugate* function ϕ'.

348. It is found, then, that the *linear and vector equation*,

$$\text{I.}\ldots\ \phi\rho = \sigma, \quad \text{gives} \quad \text{II.}\ldots\ m\rho = \psi\sigma,$$

as its *formula of solution;* with the *general method*, above explained, of deducing m and ψ from ϕ. We have therefore the two *identities*,

$$\text{III.} \ldots m\sigma = \phi\psi\sigma, \quad m\rho = \psi\phi\rho\,;$$

or briefly and symbolically,

$$\text{III}'. \ldots m = \phi\psi = \psi\phi\,;$$

with which, by what has been shown, we may connect these *conjugate equations*,

$$\text{III}''. \ldots m = \phi'\psi' = \psi'\phi'.$$

Changing then successively μ and ν to $\psi'\mu$ and $\psi'\nu$, in the equation of definition of the auxiliary function ψ, or in the formula,

$$\psi \mathrm{V}\mu\nu = \mathrm{V}.\phi'\mu\phi'\nu, \qquad 347,\ \text{XI}'.,$$

we get these two other equations,

$$\text{IV.} \ldots -\psi\mathrm{V}.\nu\psi'\mu = m\mathrm{V}.\mu\phi'\nu\,; \qquad \text{V.} \ldots \psi\mathrm{V}.\psi'\mu\psi'\nu = m^2\mathrm{V}\mu\nu\,;$$

in the former of which the *points* may be omitted, while in each of them *accented* may be exchanged with unaccented symbols of operation: and we see that the *law of homogeneity* (347, (6.)) is preserved. And many other transformations of the same sort may be made, of which the following are a few examples.

(1.) Operating on V. by ψ^{-1}, or by $m^{-1}\phi$, we get this new formula,

$$\text{VI.} \ldots \mathrm{V}.\psi'\mu\psi'\nu = m\phi\mathrm{V}\mu\nu\,;$$

comparing which with the lately cited *definition* of ψ, we see that we *may change ϕ to ψ, if we at the same time change ψ to $m\phi$*, and therefore also m to m^2; ϕ' being then changed to ψ', and ψ' to $m\phi'$.

(2.) For example, we may thus pass from IV. and V. to the formulæ,

$$\text{VII.} \ldots -\phi\mathrm{V}\nu\phi'\mu = \mathrm{V}\mu\psi'\nu, \quad \text{and} \quad \text{VIII.} \ldots \phi\mathrm{V}.\phi'\mu\phi'\nu = m\mathrm{V}\mu\nu\,;$$

in which we see that the lately cited *law of homogeneity* is still observed.

(3.) The equation VII. might have been otherwise obtained, by interchanging μ and ν in IV., and operating with $-\,m^{-1}\phi$, or with $-\psi^{-1}$; and the formula VIII. may be at once deduced from the equation of definition of ψ, by operating on it with ϕ. In fact, our *rule of inversion*, of the *linear function* ϕ, may be said to be contained in the formula,

$$\text{IX.} \ldots \phi^{-1}\mathrm{V}\mu\nu = m^{-1}\mathrm{V}.\phi'\mu\phi'\nu\,;$$

where m is a scalar constant, as above.

(4.) By similar operations and substitutions,

$$\text{X.} \ldots \phi^2\mathrm{V}.\phi'\mu\phi'\nu = m\phi\mathrm{V}\mu\nu = \mathrm{V}.\psi'\mu\psi'\nu\,;$$
$$\text{XI.} \ldots m\phi\mathrm{V}.\phi'\mu\phi'\nu = m^2\mathrm{V}\mu\nu = \psi\mathrm{V}.\psi'\mu\psi'\nu\,;$$
$$\text{XII.} \ldots m^2\mathrm{V}.\phi'\mu\phi'\nu = m^2\psi\mathrm{V}\mu\nu = \psi^2\mathrm{V}.\psi'\mu\psi'\nu\,;$$
$$\text{XIII.} \ldots \mathrm{V}.\phi'^2\mu\phi'^2\nu = \psi\mathrm{V}.\phi'\mu\phi'\nu = \psi^2\mathrm{V}\mu\nu\,; \ \text{\&c.}$$

(5.) But we have also,

$$\text{XIV.} \ldots \ \mathrm{S}.\lambda\phi^2\rho = \mathrm{S}.\phi\rho\phi'\lambda = \mathrm{S}.\rho\phi'^2\lambda,$$

so that the *second functions* ϕ^2 and ϕ'^2 are *conjugate* (compare 347, IV.); hence, by XIII., ψ^2 is formed from ϕ^2, as ψ from ϕ; and generally it will be found, that if n be *any whole number*, and if we change ϕ to ϕ^n, we change at the same time ϕ' to ϕ'^n, ψ to ψ^n, ψ' to ψ'^n, and m to m^n.

(6.) It may also be remarked that the changes (1.) conduct to the equation,

$$\text{XV.} \ldots \ (\mathrm{S}.\phi\lambda\phi\mu\phi\nu)^2 = \mathrm{S}\lambda\mu\nu\mathrm{S}.\psi\lambda\psi\mu\psi\nu;$$

and to many other analogous formulæ.

349. The expressions,

$$\lambda'\phi\lambda + \mu'\phi\mu + \nu'\phi\nu, \quad \lambda'\psi\lambda + \mu'\psi\mu + \nu'\psi\nu$$

with the significations 347, XX. of λ', μ', ν', and others of the same type, are easily proved to *vanish* when λ, μ, ν are *complanar*, and therefore to be *divisible* by $\mathrm{S}\lambda\mu\nu$, since each such expression involves each of the three auxiliary vectors λ, μ, ν in the *first degree* only; the *quotients* of such divisions being therefore certain *constant quaternions*, independent of λ, μ, ν, and depending only on the *particular form* of ϕ, or on the (scalar or vector, but real) *constants*, which enter into the composition of that given function. Writing, then,

$$\text{I.} \ldots \ q_1 = (\lambda'\phi\lambda + \mu'\phi\mu + \nu'\phi\nu) : \mathrm{S}\lambda\mu\nu,$$

and

$$\text{II.} \ldots \ q_2 = (\lambda'\psi\lambda + \mu'\psi\mu + \nu'\psi\nu) : \mathrm{S}\lambda\mu\nu,$$

we shall find it useful to consider separately the scalar and vector *parts* of these two *quaternion constants*, q_1 and q_2; which constants are, respectively, of the *first* and *second dimensions*, in a sense lately explained.

(1.) Since $\mathrm{V}\lambda'\phi\lambda = \mu\mathrm{S}\nu\phi\lambda - \nu\mathrm{S}\lambda\phi'\mu$, &c., it follows that the vector parts of q_1 and q_2 change signs, when ϕ is changed to ϕ', and therefore ψ to ψ'. On the other hand, we may change the arbitrary vectors λ, μ, ν to λ', μ', ν', if we at the same time change λ' to $\mathrm{V}\mu'\nu'$, or to $-\lambda\mathrm{S}\lambda\mu\nu$, &c., and $\mathrm{S}\lambda\mu\nu$, or $\mathrm{S}\lambda\lambda'$, to $-(\mathrm{S}\lambda\mu\nu)^2$; dividing then by $-\mathrm{S}\lambda\mu\nu$, we find these new expressions,

$$\text{III.} \ldots \ q_1 = (\lambda\phi\lambda' + \mu\phi\mu' + \nu\phi\nu') : \mathrm{S}\lambda\mu\nu,$$
$$\text{IV.} \ldots \ q_2 = (\lambda\psi\lambda' + \mu\psi\mu' + \nu\psi\nu') : \mathrm{S}\lambda\mu\nu;$$

operating on which by S, we return to the scalars of the expressions I. and II., with ϕ and ψ changed to ϕ' and ψ'.

(2.) Hence the *conjugate quaternion constants*, $\mathrm{K}q_1$ and $\mathrm{K}q_2$, are obtained by passing to the *conjugate linear functions*; and thus we may write,

$$\text{V.} \ldots \mathrm{K}q_1 = (\lambda'\phi'\lambda + \mu'\phi'\mu + \nu'\phi'\nu) : \mathrm{S}\lambda\mu\nu\,;$$
$$\text{VI.} \ldots \mathrm{K}q_2 = (\lambda'\psi'\lambda + \mu'\psi'\mu + \nu'\psi'\nu) : \mathrm{S}\lambda\mu\nu\,;$$

or, interchanging λ with λ', &c., in the dividends,

$$\text{VII.} \ldots \mathrm{K}q_1 = (\lambda\phi'\lambda' + \mu\phi'\mu' + \nu\phi'\nu') : \mathrm{S}\lambda\mu\nu\,;$$
$$\text{VIII.} \ldots \mathrm{K}q_2 = (\lambda\psi'\mu' + \mu\psi'\mu' + \nu\psi'\nu') : \mathrm{S}\lambda\mu\nu\,;$$

where $\lambda' = \mathrm{V}\mu\nu$, &c., as before.

(3.) Operating with $\mathrm{V}.\rho$ on $\mathrm{V}q_1$, and observing that

$$\mathrm{V}.\rho\mathrm{V}\lambda'\phi\lambda = \phi(\lambda\mathrm{S}\lambda'\rho) - \lambda'\mathrm{S}\lambda\phi'\rho, \text{ \&c.},$$

while
$$\phi(\lambda\mathrm{S}\lambda'\rho + \mu\mathrm{S}\mu'\rho + \nu\mathrm{S}\nu'\rho) = \phi\rho\mathrm{S}\lambda\mu\nu,$$
and
$$\lambda'\mathrm{S}\lambda\phi'\rho + \mu'\mathrm{S}\mu\phi'\rho + \nu'\mathrm{S}\nu\phi'\rho = \phi'\rho\mathrm{S}\lambda\mu\nu,$$

with similar transformations for $\mathrm{V}.\rho\mathrm{V}q_2$, we find that

$$\text{IX.} \ldots \mathrm{V}.\rho\mathrm{V}q_1 = \phi\rho - \phi'\rho\,;$$
and
$$\text{X.} \ldots \mathrm{V}.\rho\mathrm{V}q_2 = \psi\rho - \psi'\rho.$$

(4.) Accordingly, since

$$\mathrm{S}\rho(\phi\rho - \phi'\rho) = -\mathrm{S}\rho(\phi\rho - \phi'\rho) = 0,$$

the vector $\phi\rho - \phi'\rho$, if it do not vanish, must be a line perpendicular to ρ, and therefore of the *form*,

$$\text{XI.} \ldots \phi\rho - \phi'\rho = 2\mathrm{V}\gamma\rho,$$

in which γ is some constant vector; so that we may write,

$$\text{XII.} \ldots \phi\rho = \phi_0\rho + \mathrm{V}\gamma\rho, \quad \phi'\rho = \phi_0\rho - \mathrm{V}\gamma\rho,$$

where the function $\phi_0\rho$ is *its own conjugate*, or is the *common self-conjugate part* of $\phi\rho$ and $\phi'\rho$; namely the part,

$$\text{XIII.} \ldots \phi_0\rho = \tfrac{1}{2}(\phi\rho + \phi'\rho).$$

And we see that, with this signification of γ,

$$\text{XIV.} \ldots \mathrm{V}(\lambda'\phi\lambda + \mu'\phi\mu + \nu'\phi\nu) = -2\gamma\mathrm{S}\lambda\mu\nu, \quad \text{or} \quad \text{XIV}'. \ldots \mathrm{V}q_1 = -2\gamma\,;$$

while we have, in like manner,

$$\text{XV.} \ldots \mathrm{V}(\lambda'\psi\lambda + \mu'\psi\mu + \nu'\psi\nu) = -2\delta\mathrm{S}\lambda\mu\nu, \quad \text{or} \quad \text{XV}'. \ldots \mathrm{V}q_2 = -2\delta,$$
if
$$\text{XVI.} \ldots \psi\rho - \psi'\rho = 2\mathrm{V}\delta\rho.$$

As a confirmation, the part ϕ_0 of ϕ has by (1.) no effect on $\mathrm{V}q_1$; and if we change $\phi\lambda$ to $\mathrm{V}\gamma\lambda$, &c., in the first member of XIV., we have thus,

$$(\lambda\mathrm{S}\gamma\lambda' + \mu\mathrm{S}\gamma\mu' + \nu\mathrm{S}\gamma\nu') - \gamma\mathrm{S}(\lambda\lambda' + \mu\mu' + \nu\nu') = \gamma\mathrm{S}\lambda\mu\nu - 3\gamma\mathrm{S}\lambda\mu\nu.$$

(5.) Since $\mathrm{V}\lambda'\psi'\lambda = -\phi\mathrm{V}\lambda\phi'\lambda'$, &c., by 348, VII., while we may write, by (1.), (2.), and (4.),

$$\text{XVII.} \ldots \mathrm{V}(\lambda\phi\lambda' + \mu\phi\mu' + \nu\phi\nu') = -2\gamma\mathrm{S}\lambda\mu\nu,$$
$$\text{XVIII.} \ldots \mathrm{V}(\lambda\psi\lambda' + \mu\psi\mu' + \nu\psi\nu') = -2\delta\mathrm{S}\lambda\mu\nu,$$
or
$$\text{XIX.} \ldots \mathrm{V}(\lambda\phi'\lambda' + \mu\phi'\mu' + \nu\phi'\nu') = +2\gamma\mathrm{S}\lambda\mu\nu,$$
and
$$\text{XX.} \ldots \mathrm{V}(\lambda'\psi'\lambda + \mu'\psi'\mu + \nu'\psi'\nu) = +2\delta\mathrm{S}\lambda\mu\nu,$$

we have this *relation* between the two new vector constants,

$$\text{XXI.} \ldots \delta = -\phi\gamma = -\phi'\gamma = -\phi_0\gamma\,;$$

for ϕ, ϕ', and ϕ_0 have all the *same effect*, on *this* particular vector, γ.

(6.) We may add that the vector constant γ is of the *first dimension*, and that δ is of the *second* dimension, with respect to the *betas* of the *standard form*; in fact, with that *form*, 347, XV., of $\phi\rho$, we have the expressions,

$$\text{XXII.}\ldots\ \gamma = \tfrac{1}{2}\mathrm{V}(\beta\alpha + \beta'\alpha' + \beta''\alpha''),$$

and $$\text{XXIII.}\ldots\ \delta = \tfrac{1}{2}\mathrm{V}(\mathrm{V}\beta'\beta''.\mathrm{V}\alpha'\alpha'' + \mathrm{V}\beta''\beta.\mathrm{V}\alpha''\alpha + \mathrm{V}\beta\beta'.\mathrm{V}\alpha\alpha').$$

(7.) If we denote by ψ_0 and m_0, what ψ and m become when ϕ is changed to ϕ_0, we easily find that

$$\text{XXIV.}\ldots\ \psi\rho = \psi_0\rho - \gamma\mathrm{S}\gamma\rho + \mathrm{V}\delta\rho; \qquad \text{XXV.}\ldots\ \psi'\rho = \psi_0\rho - \gamma\mathrm{S}\gamma\rho - \mathrm{V}\delta\rho;$$

so that the *self-conjugate part* of $\psi\rho$ contains a *term*, $-\gamma\mathrm{S}\gamma\rho$, which involves the vector γ, but only in the *second degree;* and in like manner,

$$\text{XXVI.}\ldots\ m = m_0 + \mathrm{S}\gamma\delta = m_0 - \mathrm{S}\gamma\phi\gamma;$$

γ again entering only in an *even degree*, because m remains unchanged, when we pass from ϕ to ϕ', or from γ to $-\gamma$.

(8.) It is evident that we have the relations,

$$\text{XXVII.}\ldots\ m_0 = \phi_0\psi_0 = \psi_0\phi_0;$$

and that, in a sense already explained, ϕ_0, ψ_0, and m_0 are of the *first*, *second*, and *third* dimensions, respectively.

350. After thus considering the *vector parts* of the *two quaternion constants*, q_1 and q_2, we proceed to consider their *scalar parts;* which will introduce *two new scalar constants*, m'' and m', and will lead to the employment of *two new conjugate auxiliary functions*, $\chi\rho$ and $\chi'\rho$; whence also will result the establishment of a certain *Symbolic and Cubic Equation*,

$$\text{I.}\ldots\ 0 = m - m'\phi + m''\phi^2 - \phi^3,$$

which is *satisfied by the Linear Symbol of Operation*, ϕ, and is of great importance in this whole *Theory of Linear Functions*.

(1.) Writing, then,

$$\text{II.}\ldots\ m'' = \mathrm{S}q_1, \quad \text{and} \quad \text{III.}\ldots\ m' = \mathrm{S}q_2,$$

we see first that neither of these *two new constants* changes value, when we pass from ϕ to ϕ', or from γ to $-\gamma$; because, in such a passage, it has been seen that we only change q_1 and q_2 to $\mathrm{K}q_1$ and $\mathrm{K}q_2$. Accordingly, if we denote by m'_0 and m''_0 what m' and m'' become, when ϕ is changed to ϕ_0, we easily find the expressions,

$$\text{IV.}\ldots\ m'' = m''_0; \quad \text{and} \quad \text{V.}\ldots\ m' = m'_0 - \gamma^2.$$

(2.) It may be noted that m'', or m''_0, is of the *first* dimension, but that m' and m'_0 are of the *second*, with respect to the standard form of ϕ; and accordingly, with that form we have,

$$\text{VI.}\ldots\ m'' = \mathrm{S}\alpha\beta + \mathrm{S}\alpha'\beta' + \mathrm{S}\alpha''\beta'';$$

and $$\text{VII.}\ldots\ m' = \mathrm{S}(\mathrm{V}\alpha'\alpha''.\mathrm{V}\beta''\beta' + \mathrm{V}\alpha''\alpha.\mathrm{V}\beta\beta'' + \mathrm{V}\alpha\alpha'.\mathrm{V}\beta'\beta).$$

(3.) If we introduce *two new linear functions*, $\chi\rho$ and $\chi'\rho$, such that

$$\text{VIII.}\ldots\ \chi\mathrm{V}\mu\nu = \mathrm{V}(\mu\phi'\nu - \nu\phi'\mu),$$

and $$\text{IX.}\ldots\ \chi'\mathrm{V}\mu\nu = \mathrm{V}(\mu\phi\nu - \nu\phi\mu),$$

it is easily proved that these functions are *conjugate* to each other, and that each is of the *first* dimension; in fact, with the standard form of $\phi\rho$, we have the expressions,

$$\text{X.} \ldots \chi\rho = \mathrm{V}(\alpha\mathrm{V}\beta\rho + \alpha'\mathrm{V}\beta'\rho + \alpha''\mathrm{V}\beta''\rho),$$

$$\text{XI.} \ldots \chi'\rho = \mathrm{V}(\beta\mathrm{V}\alpha\rho + \beta'\mathrm{V}\alpha'\rho + \beta''\mathrm{V}\alpha''\rho);$$

and $\mathrm{S}\,.\,\lambda\alpha\mathrm{V}\beta\rho = \mathrm{S}\,.\,\rho\beta\mathrm{V}\alpha\lambda$, &c. Also, if χ_0 be formed from ϕ_0, as χ from ϕ, it will be found that

$$\text{XII.} \ldots \chi\rho = \chi_0\rho - \mathrm{V}\gamma\rho, \quad \text{and} \quad \text{XIII.} \ldots \chi'\rho = \chi_0\rho + \mathrm{V}\gamma\rho;$$

where χ_0 is of the first dimension.

(4.) Since

$$\mathrm{S}\lambda\chi\lambda' = \mathrm{S}\,.\,\lambda(\mu\phi'\nu - \nu\phi'\mu) = \mathrm{S}(\mu'\phi'\mu + \nu'\phi'\nu),$$

the expression II. gives, by 349, V., the equation,

$$\text{XIV.} \ldots m''\mathrm{S}\lambda\lambda' = \mathrm{S}\,.\,\lambda(\phi + \chi)\lambda',$$

λ and λ' being two *arbitrary* and *independent* vectors; which can only be, by our having the *functional relation*,

$$\text{XV.} \ldots \phi\rho + \chi\rho = m''\rho;$$

or briefly and symbolically,

$$\text{XVI.} \ldots \chi + \phi = m''.$$

Accordingly it is evident that the relation XV. is verified, by the form X. of $\chi\rho$, combined with the standard form of $\phi\rho$, and with the expression VI. for the constant m''.

(5.) The formula XVI. gives,

$$\text{XVII.} \ldots \chi\phi = m''\phi - \phi^2 = \phi\chi;$$

and accordingly the identity of $\chi\phi$ and $\phi\chi$ may easily be otherwise proved, by changing μ and ν to $\psi'\mu$ and $\psi'\nu$ in the definition VIII. of χ, and remembering that

$$\mathrm{V}.\psi'\mu\psi'\nu = m\phi\mathrm{V}\mu\nu, \quad \phi'\psi' = m, \quad \text{and} \quad \mathrm{V}\mu\psi'\nu = -\phi\mathrm{V}\nu\phi'\mu;$$

for thus we have,

$$\text{XVIII.} \ldots \chi\phi\mathrm{V}\mu\nu = \mathrm{V}(\mu\psi'\nu - \nu\psi'\mu) = \phi\mathrm{V}(\mu\phi'\nu - \nu\phi'\mu) = \phi\chi\mathrm{V}\mu\nu,$$

as required.

(6.) Since, then,

$$\mathrm{S}\,.\,\lambda\phi\chi\lambda' = \mathrm{S}\,.\,\lambda(\mu\psi'\nu - \nu\psi'\mu) = \mathrm{S}(\mu'\psi'\mu + \nu'\psi'\nu),$$

the value III. of m' gives, by 349, VI., the equation,

$$\text{XIX.} \ldots m'\mathrm{S}\lambda\lambda' = \mathrm{S}\,.\,\lambda(\psi + \phi\chi)\lambda',$$

λ and λ' being independent vectors; hence,

$$\text{XX.} \ldots \psi\rho + \phi\chi\rho = m'\rho,$$

or briefly,

$$\text{XXI.} \ldots \psi + \phi\chi = m'.$$

And in fact, with the standard form of $\phi\rho$, we have

$$\text{XXII.} \ldots \phi\chi\rho = \chi\phi\rho = \mathrm{V}(\mathrm{V}\beta'\beta''.\mathrm{V}\rho\mathrm{V}\alpha'\alpha'' + \mathrm{V}\beta''\beta.\mathrm{V}\rho\mathrm{V}\alpha''\alpha + \mathrm{V}\beta\beta'.\mathrm{V}\rho\mathrm{V}\alpha\alpha');$$

which verifies the equation XX., when it is combined with the value VII. of m', and with the expression 347, XVIII. for $\psi\rho$.

(7.) *Eliminating* the *symbol* χ, between the two equations XVI. and XXI., and remembering that $\phi\psi = \psi\phi = m$, we find the symbolic expression,

$$\text{XXIII.} \ldots m\phi^{-1} = \psi = m' - m''\phi + \phi^2;$$

and thus the *symbolic and cubic equation* I. is proved.

(8.) And because the *coefficients*, m, m', m'', of that equation, have been seen to remain unaltered, in the passage from ϕ to ϕ', we may write also this *conjugate equation*,

$$\text{XXIV.} \ldots 0 = m - m'\phi' + m''\phi'^2 - \phi'^3.$$

(9.) Multiplying symbolically the equation I. by $-m^{-1}\psi^3$, and reducing by $\psi\phi = m$, we eliminate the symbol ϕ, and obtain this *cubic in* ψ,

$$\text{XXV.} \ldots 0 = m^2 - mm''\psi + m'\psi^2 - \psi^3;$$

in which ψ' may be substituted for ψ.

(10.) In general, it may be remarked, that when we change ϕ to ψ, and therefore ψ to $m\phi$, as before, we change not only m to m^2, but also m' to mm'', and m'' to m'; while χ is at the same time changed to $\phi\chi$, or to $\chi\phi$, and the quaternion q_1 is changed to q_2. Accordingly, we may thus pass from the relation XVI. to XXI.; and conversely, from the latter to the former.

(11.) And if the two new auxiliary functions, χ and χ', be considered as *defined* by the equations VIII. and IX., their *conjugate relation* (3.) to each other may be *proved*, without any reference to the *standard form* of $\phi\rho$, by reasonings similar to those which were employed in 347, (8.), to establish the corresponding conjugation of the functions ψ and ψ'.

(12.) It may be added that the relations between ϕ, ϕ', χ, χ', and m'' give the following additional transformations, which are occasionally useful:

$$\text{XXVI.} \ldots \phi'\mathrm{V}\mu\nu = \mathrm{V}(\mu\chi\nu + \nu\phi\mu) = -\mathrm{V}(\nu\chi\mu + \mu\phi\nu);$$
$$\text{XXVII.} \ldots \phi\mathrm{V}\mu\nu = \mathrm{V}(\mu\chi'\nu + \nu\phi'\mu) = -\mathrm{V}(\nu\chi'\mu + \mu\phi'\nu);$$

with others on which we cannot here delay.

351. The cubic in ϕ may be thus written:

$$\text{I.} \ldots 0 = m\rho - m'\phi\rho + m''\phi^2\rho - \phi^3\rho;$$

where ρ is an arbitrary vector. If then it happen that for some *particular* but *actual* vector, ρ, the linear function $\phi\rho$ vanishes, so that $\phi\rho = 0$, $\phi^2\rho = 0$, $\phi^3\rho = 0$, &c., the *constant* m must be *zero;* or in symbols,

$$\text{II.} \ldots \text{if } \phi\rho = 0, \quad \text{and} \quad \mathrm{T}\rho > 0, \quad \text{then} \quad m = 0.$$

Hence, by the expression 347, XXIII. for m, when the standard form for $\phi\rho$ is adopted, we must have either

$$\text{III.} \ldots \mathrm{S}\alpha\alpha'\alpha'' = 0, \quad \text{or else} \quad \text{IV.} \ldots \mathrm{S}\beta''\beta'\beta = 0;$$

so that, in *each* case, that *generally trinomial form*, 347, XV., must admit of being *reduced to a binomial*. Conversely, when we have thus a function of the *particular form*,

$$\text{V.} \ldots \phi\rho = \beta S\alpha\rho + \beta' S\alpha'\rho,$$

we have then,

$$\text{VI.} \ldots \phi V\alpha\alpha' = 0;$$

so that if α and α' be *actual* and *non-parallel* lines, the *real* and *actual vector* $V\alpha\alpha'$ will be a value of ρ, which will satisfy the equation $\phi\rho = 0$; but *no other real and actual value* of ρ, except $\rho = xV\alpha\alpha'$, will satisfy that equation, if β and β' be *actual*, and *non-parallel*. In this case V., the *operation* ϕ *reduces every other vector to the fixed plane* of β, β', which *plane* is therefore the *locus* of $\phi\rho$; and since we have also,

$$\text{VII.} \ldots \phi'\rho = \alpha S\beta\rho + \alpha' S\beta'\rho,$$

we see that the *locus of the functionally conjugate vector*, $\phi'\rho$, is *another fixed plane*, namely that of α, α'. Also, the *normal* to the *latter plane* is the *line* which is *destroyed* by the *former operation*, namely by ϕ; while the *normal* to the *former plane* is in like manner the *line*, which is *annihilated* by the *latter operation*, ϕ', since we have,

$$\text{VIII.} \ldots \phi' V\beta\beta' = 0,$$

but *not* $\phi'\rho = 0$, for any *actual* ρ, in any *direction* except that of $V\beta\beta'$, or its *opposite*, which may however, for the *present purpose*, be regarded as the *same*.*. In this case we have also *monomial forms* for $\psi\rho$ and $\psi'\rho$, namely

$$\text{IX.} \ldots \psi\rho = V\alpha\alpha' S\beta'\beta\rho, \quad \text{and} \quad \text{X.} \ldots \psi'\rho = V\beta\beta' S\alpha'\alpha\rho;$$

so that the *operation* ψ *destroys every line in the first fixed plane* (of β, β'), and the *conjugate* operation ψ' annihilates every line in the *second* fixed plane (of α, α'). On the other hand, the operation ψ *reduces* every line, which is *out of the first plane*, to the *fixed direction* of the *normal* to the *second plane;* and the operation ψ' reduces every line which is *out of* the *second* plane, to that *other fixed direction*, which is *normal* to the *first* plane. And thus it comes to pass, that whether we operate first with ψ, and then with ϕ; or first with ϕ, and then with ψ; or first with ψ' and then with ϕ'; or first with ϕ', and then with ψ'; in *all* these cases, we *arrive at last at a null line*, in conformity with the symbolic equations,

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* Accordingly, *in the present investigation*, whenever we shall speak of a "*fixed direction*," or the "*direction of a given line*," &c., we are always to be understood as meaning, "*or the opposite* of that direction."

$$\text{XI.}\ \ldots\ \phi\psi = \psi\phi = \phi'\psi' = \psi'\phi' = m = 0,$$

which belong to the case here considered.

(1.) Without recurring to the *standard form* of $\phi\rho$, the equation 348, VI., namely $\mathrm{V}.\psi'\mu\psi'\nu = m\phi\mathrm{V}\mu\nu$, and the analogous equation $\mathrm{V}.\psi\mu\psi\nu = m\phi'\mathrm{V}\mu\nu$, might have enabled us to foresee that $\psi'\rho$ and $\psi\rho$, if they do not both *constantly vanish*, must (if $m = 0$) have each a *fixed direction;* and therefore that each must be expressible by a *monome*, as above: the fixed *direction* of $\psi\rho$ being that of a line which is *annihilated* by the *operation* ϕ, and similarly for $\psi'\rho$ and ϕ'.

(2.) And because, by 347, XI. and XXV., we have

$$\psi\mathrm{V}\mu\nu = \mathrm{V}.\phi'\mu\phi'\nu, \quad\text{and}\quad \psi'\mathrm{V}\mu\nu = \mathrm{V}.\phi\mu\phi\nu,$$

so that the line $\phi'\mu$, if actual, is perpendicular to $\psi\mathrm{V}\mu\nu$, and the line $\phi\mu$ perpendicular to $\psi'\mathrm{V}\mu\nu$, we see that *each of the two lines*, $\phi'\rho$ and and $\phi\rho$, must have (in the present case) a *plane locus;* whence the *binomial forms* of the two *conjugate vector functions*, $\phi\rho$ and $\phi'\rho$, might have been foreseen: $\psi\rho$ and $\psi'\rho$ being here supposed to be *actual* vectors.

(3.) The *relations of rectangularity*, of the *two fixed lines* (or *directions*), to the *two fixed planes*, might also have been thus deduced, through the two *conjugate binomial forms*, V. and VII., without the *previous* establishment of the more *general trinomial* (or *standard*) *form* of $\phi\rho$.

(4.) The existence of a plane locus for $\phi\rho$, and of another for $\phi'\rho$, for the case when $m = 0$, might also have been foreseen from the equations,

$$\mathrm{S}.\phi\lambda\phi\mu\phi\nu = \mathrm{S}.\phi'\lambda\phi'\mu\phi'\nu = m\mathrm{S}\lambda\mu\nu;$$

and the same equations might have enabled us to foresee, that the *scalar constant* m must be *zero*, if for any *one actual vector*, such as λ, either $\phi\lambda$ or $\phi'\lambda$ becomes *null*.

(5.) And the *reducibility* of the *trinomial* to the *binomial form*, when this last condition is satisfied, might have been anticipated, without any reference to the composition of the constant m, from the simple consideration (comp. 294, (10.)), that *no actual vector* ρ can be *perpendicular*, at once, to *three diplanar lines*.

352. It may happen, that besides the recent reduction (351) of the *linear function* $\phi\rho$ to a *binomial form*, when the *relation*

$$\text{I.}\ \ldots\ m = 0$$

exists between the *constants* of that function, in which case the symbolic and *cubic equation* 350, I. reduces itself to the form,

$$\text{II.}\ \ldots\ \phi^3 - m''\phi^2 + m'\phi = 0,$$

thus losing its absolute term, or having *one root* equal to *zero*, this equation may undergo a further *reduction*, by *two* of its roots becoming *equal* to *each other;* namely either by our having

$$\text{III.}\ldots m' = 0, \quad \text{and} \quad \text{IV.}\ldots \phi^2(\phi - m'') = 0;$$

or in another way, by the existence of these other equations,

$$\text{V.}\ldots m''^2 - 4m' = 0, \quad \text{and} \quad \text{VI.}\ldots \phi(\phi - \tfrac{1}{2}m'')^2 = 0.$$

In *each* of these two cases, we shall find that certain *new geometrical relations* arise, which it may be interesting briefly to investigate; and of which the principal is the mutual *rectangularity* of *two fixed planes*, which are the *loci* (comp. 351) of certain *derived*, and *functionally conjugate vectors:* namely, in the case III. IV., the loci of $\phi\rho$ and $\phi'\rho$; and in the case V. VI., the loci of $\Phi\rho$ and $\Phi'\rho$, if

$$\text{VII.}\ldots \Phi = \phi - \tfrac{1}{2}m'', \quad \text{and} \quad \text{VIII.}\ldots \Phi' = \phi' - \tfrac{1}{2}m'',$$

so that, in this last case, the symbol Φ satisfies this *new cubic*,

$$\text{IX.}\ldots 0 = \Phi^2(\Phi + \tfrac{1}{2}m'');$$

while Φ' satisfies at the same time a cubic equation with the *same coefficients* (comp. 350, (8.)), namely

$$\text{X.}\ldots 0 = \Phi'^2(\Phi' + \tfrac{1}{2}m'').$$

(1.) We saw in 351, (1.), (2.), that when $m = 0$ the line $\psi'\rho$ has *generally* a *fixed direction*, to which that of the line $\phi\rho$ is *perpendicular;* and that in like manner the line $\psi\rho$ has then *another* fixed direction, to which $\phi'\rho$ is perpendicular. If then the *plane loci* of $\phi\rho$ and $\phi'\rho$ be at *right angles* to each other, we must also have the *fixed lines* $\psi'\lambda$ and $\psi\mu$ *rectangular*, or

$$\text{XI.}\ldots 0 = \text{S}.\psi'\lambda\psi\mu = \text{S}\lambda\psi^2\mu,$$

independently of the directions of λ and μ; whence

$$\text{XII.}\ldots 0 = \psi^2\mu, \quad \text{or} \quad \text{XIII.}\ldots \psi^2 = 0,$$

since μ is an arbitrary vector.

(2.). Now *in general*, by the functional relation 350, XXI. combined with $\psi\phi = m$, we have the transformation,

$$\text{XIV.}\ldots \psi^2 = \psi(m' - \phi\chi) = m'\psi - m\chi;$$

if then $m = 0$, as in I., the symbol ψ must satisfy the *depressed* or *quadratic equation*,

$$\text{XV.}\ldots 0 = m'\psi - \psi^2;$$

which is accordingly a *factor* of the *cubic* equation,

$$\text{XVI.}\ldots 0 = m'\psi^2 - \psi^3,$$

whereto the general equation 350, XXV. is *reduced*, by this supposition of m vanishing.

(3.) If then we have *not only* $m = 0$, as in I., but *also* $m' = 0$, as in III., the condition XIII. is satisfied, by XV.; and the *two planes*, above referred to, are generally *rectangular*.

(4.) We might indeed propose to satisfy that condition XIII., by supposing that we had always,

$$\text{XVII.} \ldots \psi = 0, \quad \text{that is,} \quad \text{XVII}'. \ldots \psi\rho = 0,$$

for *every direction* of ρ; but in *this* case, the *quaternion constant* q_2 would *vanish* (by 349, II.); and therefore the constant m', as being its *scalar part* (by 350, III.), would *still* be equal to *zero*.

(5.) The particular supposition XVII. would however *alter* completely the *geometrical character* of the question; for it would imply (comp. 351, (2.)) that the *directions* of the *lines* $\phi\rho$ and $\phi'\rho$ (when not *evanescent*) are *fixed*, instead of those lines having only certain *planes* for their *loci*, as before.

(6.) On the side of *calculation*, we should thus have, for the two *conjugate functions*, $\phi\rho$ and $\phi'\rho$, *monomial expressions* of the forms,

$$\text{XVIII.} \ldots \phi\rho = \beta S\alpha\rho, \quad \phi'\rho = \alpha S\beta\rho;$$

whence, by 347, XVIII., and 350, VII., we should recover the equations, $\psi\rho = 0$ and $m' = 0$.

(7.) We should have also, in this particular case,

$$\text{XIX.} \ldots \phi\rho = 0, \quad \text{if} \quad \rho \perp \alpha, \quad \text{and} \quad \text{XX.} \ldots \phi'\rho = 0, \quad \text{if} \quad \rho \perp \beta;$$

so that $\phi\rho$ now *vanishes*, if ρ be *any line* in the *fixed plane* perpendicular to α; and in like manner $\phi'\rho$ is a null line, if ρ be in that *other fixed plane*, which is at right angles to the *other given line*, β.

(8.) *These* two *planes*, or their *normals* α and β, or the fixed directions of the two *lines* $\phi'\rho$ and $\phi\rho$, will be *rectangular* (comp. (1.)), if we have this new equation,

$$\text{XXI.} \ldots \phi^2 = 0, \quad \text{or} \quad \text{XXI}'. \ldots \phi^2\rho = 0,$$

for every direction of ρ; and accordingly the expression XVIII. gives

$$\phi^2\rho = S\alpha\beta . \phi\rho = 0, \quad \text{if} \quad \beta \perp \alpha, \text{ and reciprocally.}$$

(9.) Without expressly introducing α and β, the equation 350, XXIII. shows that when $\psi = 0$, and therefore also $m' = 0$, as in (4.), the symbol ϕ satisfies (comp. (2.)) the *new quadratic* or *depressed equation*,

$$\text{XXII.} \ldots 0 = \phi^2 - m''\phi;$$

which is accordingly a *factor* of the *cubic* IV., but to which that cubic is *not reducible*, *unless* we have thus $\psi = 0$, as well as $m' = 0$.

(10.) The *condition*, then, of the *existence* and *rectangularity* of the *two planes* (7.), for which we have respectively $\phi\rho = 0$ and $\phi'\rho = 0$, without $\phi\rho$ *generally* vanishing (a case which it would be useless to consider), is that the four following equations should subsist:

$$\text{XXIII.} \ldots m = 0, \quad m' = 0, \quad m'' = 0, \quad \text{and} \quad \text{XVII.} \ldots \psi = 0;$$

or that the *cubic* IV., and its *quadratic factor* XXII., should reduce themselves to the very simple forms,

$$\text{XXIV.} \ldots \phi^3 = 0, \quad \text{and} \quad \text{XXV.} \ldots \phi^2 = 0;$$

the cubic in ϕ having thus its *three roots equal*, and *null*, and $\psi\rho$ *vanishing*.

(11.) We may also observe that as, when even *one* root of the general cubic 350, I. is *zero*, that is when $m = 0$, the *vector equation*

$$\text{XXVI.} \ldots \phi\rho = 0$$

was seen (in 351) to be satisfied by *one real direction* of ρ, so when we have *also* $m' = 0$, or when the cubic in ϕ has *two null roots*, or takes the form IV., then the *two vector equations*,

$$\text{XXVII.} \ldots \phi\rho = 0, \quad \psi\rho = 0,$$

are satisfied by one *common direction* of the *real* and *actual line* ρ; because we have, by 350, XVII. and XX., the *general* relation,

$$\psi\rho = m'\rho - \chi\phi\rho.$$

(12.) And because, by 350, XV., we have also the relation $\chi\rho = m''\rho - \phi\rho$, it follows that when the *three roots* of the cubic *all vanish*, or when the *three scalar equations* XXIII. are satisfied, then the *three vector equations*,

$$\text{XXVIII.} \ldots \phi\rho = 0, \quad \psi\rho = 0, \quad \chi\rho = 0,$$

have a *common* (*real* and *actual*) *vector root;* or are all satisfied by one *common direction* of ρ.

(13.) Since $m'' - \phi = \chi$, the cubic IV. may be written under any one of the following forms,

$$\text{XXIX.} \ldots 0 = \phi^2\chi = \phi\chi\phi = \chi\phi^2 = \phi \,.\, \phi\chi = \&\text{c.},$$

in which accented may be substituted for unaccented symbols: and its *geometrical signification* may be illustrated by a reference to certain *fixed lines*, and *fixed planes*, as follows.

(14.) Suppose first that m and m' both vanish, but that m'' is different from zero, so that the cubic in ϕ is reducible to the form IV., but *not* to the form XXIV.; and that the operation ψ, which is here equivalent to $-\phi\chi$, or to $-\chi\phi$, does not annihilate *every* vector ρ, so that (comp. (4.) (5.) (6.)) $\phi\rho$ and $\phi'\rho$ have *not* the directions of *two fixed lines*, but have only (comp. (1.) and (3.)) *two fixed* and *rectangular planes*, Π and Π', as their *loci;* and let the *normals* to these two planes be denoted by λ and λ', so that these two rectangular lines, λ and λ', are situated respectively in the planes Π' and Π.

(15.) Then it is easily shown (comp. 351) that the operation ϕ *destroys* the line λ' *itself*, while it *reduces*[*] every *other* line (that is, every line which is not of the form $x\lambda'$, with $\mathrm{V}x = 0$) to the *plane* $\Pi \perp \lambda$; and that it reduces every line *in* that plane to a *fixed direction*, μ, in the same plane, which is thus the *common* direction of all the lines $\phi^2\rho$, whatever the direction of ρ may be. And the symbolical equation, $\chi \cdot \phi^2 = 0$, expresses that this fixed direction μ of $\phi^2\rho$ may also be denoted by $\chi^{-1}0$; or that we have the equation,

$$\text{XXX.} \ldots 0 = \chi\mu = m''\mu - \phi\mu, \quad \text{if} \quad \mu = \phi^2\rho,$$

which can accordingly be otherwise proved: with similar results for the conjugate symbols, ϕ' and χ'.

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[*] We propose to include the case where an *operation* of this sort *destroys* a line, or reduces it to *zero*, under the case when the same operation *reduces* a line to a *fixed direction*, or to a *fixed plane*.

(16.) For example, we may represent the conditions of the present case by the following system of equations (comp. 351, V. VII. IX. X., and 350, VI. VII. X. XI):

$$\text{XXXI.}\ldots\begin{cases}\phi\rho = \beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho, \quad \phi'\rho = \alpha\mathrm{S}\beta\rho + \alpha'\mathrm{S}\beta'\rho, \\ 0 = m' = \mathrm{S}(\mathrm{V}\alpha\alpha'.\mathrm{V}\beta'\beta) = \mathrm{S}\alpha\beta\mathrm{S}\alpha'\beta' - \mathrm{S}\alpha\beta'\mathrm{S}\alpha'\beta, \\ m'' = \mathrm{S}\alpha\beta + \mathrm{S}\alpha'\beta'\,; \end{cases}$$

$$\text{XXXII.}\ldots\begin{cases}\chi\rho = \mathrm{V}(\alpha\mathrm{V}\beta\rho + \alpha'\mathrm{V}\beta'\rho) = m''\rho - \phi\rho, \\ \chi'\rho = \mathrm{V}(\beta\mathrm{V}\alpha\rho + \beta'\mathrm{V}\alpha'\rho) = m''\rho - \phi'\rho, \\ -\psi\rho = \phi\chi\rho = \chi\phi\rho = \mathrm{V}\alpha\alpha'\mathrm{S}\beta\beta'\rho, \\ -\psi'\rho = \phi'\chi'\rho = \chi'\phi'\rho = \mathrm{V}\beta\beta'\mathrm{S}\alpha\alpha'\rho\,; \end{cases}$$

and may then write (not *here* supposing $\lambda' = \mathrm{V}\mu\nu$, &c.),

$$\text{XXXIII.}\ldots\begin{cases}\lambda = \mathrm{V}\beta\beta', \quad \lambda' = \mathrm{V}\alpha\alpha', \quad \mathrm{S}\lambda\lambda' = 0, \\ \mu = \phi\beta \parallel \phi\beta', \quad \mu' = \phi'\alpha' \parallel \phi'\alpha, \quad \mathrm{S}\lambda\mu = \mathrm{S}\lambda'\mu' = 0\,; \end{cases}$$

after which we easily find that

$$\text{XXXIV.}\ldots\begin{cases}\phi\lambda' = 0, \quad \phi^2\rho \parallel \mu, \quad \phi\mu = m''\mu, \quad \chi\mu = 0\,; \\ \phi'\lambda = 0, \quad \phi'^2\rho \parallel \mu', \quad \phi'\mu' = m''\mu', \quad \chi'\mu' = 0. \end{cases}$$

(17.) Since we have thus $\chi'\mu' = 0$, where μ' is a line in the fixed direction of $\phi'^2\rho$, we have also the equation,

$$\text{XXXV.}\ldots\ 0 = \mathrm{S}\rho\chi'\mu' = \mathrm{S}\mu'\chi\rho, \quad \text{or} \quad \chi\rho \perp \mu'\,;$$

the *locus* of $\chi\rho$ is therefore a *plane* perpendicular to the line μ'; and in like manner, μ is the *normal* to a plane, which is the *locus* of the line $\chi'\rho$. And the symbolical equations, $\phi.\phi\chi = 0$, $\phi^2.\chi = 0$, may be interpreted as expressing, that the operation ϕ *reduces* every line in this *new plane* of $\chi\rho$ to the *fixed direction* of $\phi^{-1}0$, or of λ'; and that the operation ϕ^2 *destroys* every line in this plane $\perp \mu'$; with analogous results, when accented are interchanged with unaccented symbols. Accordingly we see, by XXXII., that $\phi\chi\rho$ has the fixed direction of $\mathrm{V}\alpha\alpha'$, or of λ'; and that $\phi\,.\,\phi\chi\rho = 0$, because $\phi\lambda' = 0$.

(18.) We see also, that the operation $\phi\chi$, or $\chi\phi$, destroys every line in the plane Π, to which the operation ϕ reduces every line; and that thus the symbolical equations, $\phi\chi\cdot\phi = 0$, $\chi\phi\cdot\phi = 0$, may be interpreted.

(19.) As a verification, it may be remarked that the *fixed direction* λ', of $\phi\chi\rho$ or $\chi\phi\rho$, ought to be that of the *line of intersection* of the two *fixed planes* of $\phi\rho$ and $\chi\rho$; and accordingly it is perpendicular by XXXIII. to their two *normals*, λ and μ': with similar remarks respecting the fixed direction λ, of $\phi'\chi'\rho$ or $\chi'\phi'\rho$, which is perpendicular to λ' and to μ.

(20.) Let us next suppose, that besides $m = 0$, and $m' = 0$, we have $\psi = 0$, but that m'' is still different from zero. In this case, it has been seen (6.) that the expression for $\phi\rho$ reduces itself to the *monomial form*, $\beta\mathrm{S}\alpha\rho$; and therefore that the operation ϕ *destroys* every line in a *fixed plane* ($\perp \alpha$), while it *reduces* every *other* line to a *fixed direction* ($\parallel \beta$), which is *not contained* in that *plane*, because we have *not* now $\mathrm{S}\alpha\beta = 0$.

(21.) In *this* case we have by (16.), equating α' or β' to 0, the expressions,

$$\text{XXXVI.}\ldots\begin{cases}\phi\rho = \beta\mathrm{S}\alpha\rho, \quad \phi'\rho = \alpha\mathrm{S}\beta\rho, \quad m'' = \mathrm{S}\alpha\beta \gtrless 0, \\ \chi\rho = \mathrm{V}.\alpha\mathrm{V}\beta\rho = (m'' - \phi)\rho, \quad \chi'\rho = \mathrm{V}.\beta\mathrm{V}\alpha\rho = (m'' - \phi')\rho, \end{cases}$$

so that the equations XVIII. are reproduced; and the *depressed cubic*, or the *quadratic* XXII. in ϕ, may be written under the very simple form,

$$\text{XXXVII.} \ldots 0 = \phi\chi = \chi\phi.$$

(22.) Accordingly (comp. (5.) and (7.)), the operation ϕ here reduces an arbitrary line to the fixed direction of β, while χ destroys every line in that direction; and conversely, the operation χ reduces an arbitrary line to the fixed plane perpendicular to α, and ϕ destroys every line in that fixed plane. But because we do not *here* suppose that $m'' = 0$, the *fixed direction* of $\phi\rho$ is *not contained* in the *fixed plane* of $\chi\rho$; and (comp. (8.) and (10.)) the directions of $\phi\rho$ and $\phi'\rho$ are *not rectangular* to each other.

(23.) On the other hand, if we suppose that the *three roots* of the cubic in ϕ *vanish*, or that we have $m = 0$, $m' = 0$, *and* $m'' = 0$, as in XXIII., but that the equation $\psi\rho = 0$ is *not* satisfied for *all* directions of ρ, then the *binomial forms* XXXI. of $\phi\rho$ and $\phi'\rho$ reappear, but with these *two* equations of condition between their *vector constants*, whereof only *one* had occurred before:

$$\text{XXXVIII.} \ldots 0 = S\alpha\beta S\alpha'\beta' - S\alpha\beta' S\alpha'\beta, \quad 0 = S\alpha\beta + S\alpha'\beta'.$$

(24.) We have also now the expressions,

$$\text{XXXIX.} \ldots \chi\rho = -\phi\rho, \quad \chi'\rho = -\phi'\rho;$$

and the cubic in ϕ becomes simply $\phi^3 = 0$, as in XXIV.; but it is important to observe that we have *not here* (comp. (9.)) the *depressed* or *quadratic* equation $\phi^2 = 0$, since we have *now* on the contrary the two conjugate expressions,

$$\text{XL.} \ldots \phi^2\rho = \psi\rho = V\alpha\alpha' S\beta'\beta\rho, \quad \phi'^2\rho = \psi'\rho = V\beta\beta' S\alpha'\alpha\rho,$$

which do not generally vanish. And the equation $\phi^3 = 0$ is now *interpreted*, by observing that ϕ^2 here reduces *every* line to the *fixed direction* of $\phi^{-1}0$; while ϕ reduces an arbitrary vector to that *fixed plane*, all lines in which are destroyed by ϕ^2.

(25.) In this last case (23.), in which *all* the roots of the cubic in ϕ are *equal*, and are *null*, the theorem (12.), of the existence of a *common vector root* of the three equations XXVIII., may be verified by observing that we have now,

$$\text{XLI.} \ldots \phi V\alpha\alpha' = 0, \quad \psi V\alpha\alpha' = 0, \quad \chi V\alpha\alpha' = 0;$$

the *third* of which would not have here held good, unless we had supposed $m'' = 0$.

(26.) This last condition allows us to write, by (16.),

$$\text{XLII.} \ldots \phi\mu = 0, \quad \phi'\mu' = 0, \quad V\mu\lambda' = 0, \quad V\mu'\lambda = 0, \quad S\mu\mu' = 0,$$

the lines μ' and μ thus coinciding in direction with the normals λ and λ', to the planes Π and Π'; if then we write,

$$\text{XLIII.} \ldots \nu = V\lambda\lambda' \parallel V\mu\mu', \quad \text{so that} \quad S\mu\nu = 0, \quad S\mu'\nu = 0,$$

this new vector ν will be a line in the *intersection* of those *two rectangular planes*, which were lately seen (14.) to be the *loci* of the lines $\phi\rho$ and $\phi'\rho$, and are *now* (comp. (17.)) the loci of $\chi\rho$ and $\chi'\rho$; and the *three lines* μ, μ', ν (or λ', λ, ν) will compose a *rectangular system*.

(27.) *In general*, it is easy to prove that the expressions,

$$\text{XLIV.} \ldots \begin{cases} \beta = a\beta_1 + b\beta'_1, & \beta' = a'\beta_1 + b'\beta'_1, \\ \alpha_1 = a\alpha + a'\alpha', & \alpha'_1 = b\alpha + b'\alpha', \end{cases}$$

in which α, β, α', β' may be *any four vectors*, and a, b, a', b' may be *any four scalars*, conduct to the following transformations (in which ρ may be *any vector*):

$$\text{XLV.} \ldots \mathrm{S}\alpha_1\beta_1 + \mathrm{S}\alpha'_1\beta'_1 = \mathrm{S}\alpha\beta + \mathrm{S}\alpha'\beta';$$
$$\text{XLVI.} \ldots \beta_1\mathrm{S}\alpha_1\rho + \beta'_1\mathrm{S}\alpha'_1\rho = \beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho;$$
$$\text{XLVII.} \ldots \mathrm{V}\alpha_1\alpha'_1.\mathrm{V}\beta'_1\beta_1 = \mathrm{V}\alpha\alpha'.\mathrm{V}\beta'\beta;$$

so that the *scalar*, $\mathrm{S}\alpha\beta + \mathrm{S}\alpha'\beta'$; the *vector*, $\beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho$; and the *quaternion*,* $\mathrm{V}\alpha\alpha'.\mathrm{V}\beta'\beta$, remain *unaltered* in value, when we pass from a *given system* of *four vectors* $\alpha\beta\alpha'\beta'$, to *another system* of four vectors $\alpha_1\beta_1\alpha'_1\beta'_1$, by expressions of the *forms* XLIV.

(28.) With the help of this general principle (27.), and of the remarks in (26.), it may be shown, without difficulty, that in the case (23.) the vector constants of the binomial expression $\beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho$ for $\phi\rho$ may, without any real loss of generality, be supposed subject to the *four* following conditions,

$$\text{XLVIII.} \ldots 0 = \mathrm{S}\alpha\beta = \mathrm{S}\alpha'\beta = \mathrm{S}\beta\beta' = \mathrm{S}\alpha'\beta';$$

which evidently conduct to these other expressions,

$$\text{XLIX.} \ldots \phi^2\rho = \beta\mathrm{S}\alpha\beta'\mathrm{S}\alpha'\rho, \quad \phi^3\rho = 0;$$

and thus put in evidence, in a very simple manner, the *general non-depression* of the *cubic* $\phi^3 = 0$, to the *quadratic*, $\phi^2 = 0$.

(29.) The *case*, or *sub-case*, when we have not only $m = 0$, $m' = 0$, $m'' = 0$, but *also* $\psi = 0$, and therefore $\phi^2 = 0$, as a depressed form of $\phi^3 = 0$, by the linear function $\phi\rho$ reducing itself to the monomial $\beta\mathrm{S}\alpha\rho$, with the relation $\mathrm{S}\alpha\beta = 0$ between its constants, has been already considered (in (10.)); and thus the consequences of the supposition III., that there are (at least) *two equal* but *null roots* of the cubic in ϕ, have been perhaps sufficiently discussed.

(30.) As regards the *other principal case* of *equal roots*, of the *cubic* equation in ϕ, namely that in which the vector constants are connected by the relation V., or by the equation of condition,

$$\text{L.} \ldots 0 = m''^2 - 4m' = (\mathrm{S}\alpha\beta + \mathrm{S}\alpha'\beta')^2 - 4\mathrm{S}(\mathrm{V}\alpha\alpha'.\mathrm{V}\beta'\beta)$$
$$= (\mathrm{S}\alpha\beta - \mathrm{S}\alpha'\beta')^2 + 4\mathrm{S}\alpha\beta'\mathrm{S}\alpha'\beta,$$

it may suffice to remark that it conducts, by VI., or by VII. and IX., to the symbolical equation,

$$\text{LI.} \ldots 0 = \phi\Phi^2, \quad \text{if} \quad \Phi = \phi - \tfrac{1}{2}m'';$$

and that thus its *interpretation* is precisely similar to that of the analogous equation,

$$\chi\phi^2 = 0, \quad \text{where} \quad \chi = m'' - \phi, \qquad \text{XXIX.},$$

as given in (14.), and in the following sub-articles.

353. When we have $m = 0$, but *not* $m' = 0$, *nor* $m''^2 = 4m'$, the *three roots* of the cubic in ϕ are *all unequal*, while *one* of them is still *null*, as before; and the *two roots* of the *quadratic* and *scalar equation*, with *real coefficients* (347),

$$\text{I.} \ldots 0 = c^2 + m''c + m',$$

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* We have, in these *transformations*, examples of what may be called *Quaternion Invariants*.

which is formed from the cubic by changing ϕ to $-c$, and then dividing by c, are also necessarily *unequal*, whether they be *real* or *imaginary*. We shall find that when these *two scalar roots*, c_1, c_2, are *real*, there are then *two real directions*, ρ_1 and ρ_2, in that *fixed plane* Π which is the *locus* (351, 352) of the line $\phi\rho$, possessing the property that for each of them the *homogeneous and vector equation of the second degree*,

$$\text{II.} \ldots \mathrm{V}\rho\phi\rho = 0, \quad \text{or} \quad \phi\rho \parallel \rho,$$

is satisfied, *without ρ vanishing;* namely by our having, for the *first* of these two directions, the equation

$$\text{III.} \ldots \phi\rho_1 = -c_1\rho_1, \quad \text{or} \quad \phi_1\rho_1 = 0, \quad \text{if} \quad \phi_1 = \phi + c_1;$$

and for the *second* of them the analogous equation,

$$\text{IV.} \ldots \phi\rho_2 = -c_2\rho_2, \quad \text{or} \quad \phi_2\rho_2 = 0, \quad \text{if} \quad \phi_2 = \phi + c_2:$$

but that *no other direction* of the *real* and *actual vector* ρ, satisfies the equation V., except that *third* which has already been considered (351), as satisfying the *linear* and *vector equation*,

$$\text{V.} \ldots \phi\rho = 0, \quad \text{with} \quad \mathrm{T}\rho > 0.$$

It will also be shown that these *two* directions, ρ_1, ρ_2, are not only *real*, but *rectangular*, to each other and to the *third* direction ρ, when the *linear function* $\phi\rho$ is *self-conjugate* (349, (4.)), or when the condition

$$\text{VI.} \ldots \phi'\rho = \phi\rho, \quad \text{or} \quad \text{VI}'. \ldots \mathrm{S}\lambda\phi\rho = \mathrm{S}\rho\phi\lambda,$$

is satisfied by the *given form* of ϕ, or by the *constants* which enter into the composition of that *linear symbol;* but that when this *condition of self-conjugation* is *not* satisfied, the roots of the quadratic I. *may* happen to be *imaginary:* and that in *this* case there exists *no real direction* of ρ, for which the vector equation II. of the *second degree* is satisfied, by *actual* values of ρ, except that *one* direction which has been seen before to satisfy the *linear* equation V.

(1.) The most obvious mode of seeking to satisfy II., otherwise than through V., is to assume an expression of the form, $\rho = x\beta + x'\beta'$, and to seek thereby to satisfy the equation, $(\phi + c)\rho = 0$, with $\phi\rho = \beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho$, by satisfying separately the two scalar equations,

$$\text{VII.} \ldots 0 = x(c + \mathrm{S}\alpha\beta) + x'\mathrm{S}\alpha\beta', \quad 0 = x'(c + \mathrm{S}\alpha'\beta') + x\mathrm{S}\alpha'\beta,$$

which give, by elimination of $x' : x$, the following quadratic in c,

$$\text{VIII.} \ldots (c + Sa\beta)(c + Sa'\beta') = Sa\beta' Sa'\beta,$$

which is easily seen to be only another form of I. Denoting then, as above, by c_1 and c_2, the *roots* of that quadratic I., supposed for the present to be *real*, we have these two *real directions* for ρ, in the plane Π of β, β':

$$\text{IX.} \ldots \rho_1 = \beta(c_1 + Sa'\beta') - \beta' Sa'\beta = c_1\beta + Va'V\beta'\beta;$$
$$\text{X.} \ldots \rho_2 = \beta(c_2 + Sa'\beta') - \beta' Sa'\beta = c_2\beta + Va'V\beta'\beta;$$

which satisfy the equations III. and IV. In fact, the expression IX. gives

$$\phi\rho_1 = c_1\phi\beta + m'\beta = -c_1\rho_1, \quad \text{or} \quad \phi_1\rho_1 = 0,$$

because we may write it thus,

$$\text{XI.} \ldots \rho_1 = (m'' + c_1)\beta - \phi\beta = -c_2\beta - \phi\beta = -\phi_2\beta = -\phi\beta - m'c_1^{-1}\beta;$$

and in like manner, the expression X. may be thus written,

$$\text{XII.} \ldots \rho_2 = (m'' + c_2)\beta - \phi\beta = -c_1\beta - \phi\beta = -\phi_1\beta = -\phi\beta - m'c_2^{-1}\beta,$$

and gives,

$$\phi\rho_2 = c_2\phi\beta + m'\beta = -c_2\rho, \quad \text{or} \quad \phi_2\rho_2 = 0.$$

(2.) We may also write,

$$\text{XIII.} \ldots \rho'_1 = \beta'(c_1 + Sa\beta) - \beta Sa\beta' = c_1\beta' + VaV\beta\beta' = -\phi_2\beta' \parallel \rho_1;$$
$$\text{XIV.} \ldots \rho'_2 = \beta'(c_2 + Sa\beta) - \beta Sa\beta' = c_2\beta' + VaV\beta\beta' = -\phi_1\beta' \parallel \rho_2;$$

and shall then have the equations,

$$\text{XV.} \ldots \phi_1\rho'_1 = 0, \quad \phi_2\rho'_2 = 0;$$

but the *directions* of ρ'_1 and ρ'_2 will be the *same* by VIII. as those of ρ_1 and ρ_2, and so will furnish *no new solution* of the problem just resolved.

(3.) Since we have thus,

$$\text{XVI.} \ldots \phi_2\beta' \parallel \phi_2\beta \parallel \rho_1 \parallel \phi_1^{-1}0, \quad \text{and} \quad \text{XVI}'. \ldots \phi_1\beta' \parallel \phi_1\beta \parallel \rho_2 \parallel \phi_2^{-1}0,$$

it follows that the operation ϕ_2 *reduces* every line *in the fixed plane* of $\phi\rho$ to the *fixed direction* of $\phi_1^{-1}0$; and that, in like manner, the operation ϕ_1 reduces every line, in the *same* fixed plane of $\phi\rho$, to the *other* fixed direction of $\phi_2^{-1}0$.

(4.) Hence we may write the symbolic equations,

$$\text{XVII.} \ldots \phi_1 . \phi_2\phi = 0, \quad \phi_2 . \phi_1\phi = 0,$$

in which the points may be omitted; and in fact we have the transformations,

$$\text{XVIII.} \ldots \phi_1\phi_2 = \phi_2\phi_1 = (\phi + c_1)(\phi + c_2) = \phi^2 - m''\phi + m' = \psi,$$

so that
$$\phi_1\phi_2 . \phi = \phi_2\phi_1 . \phi = \psi\phi = m = 0.$$

(5.) If we propose to form ψ_1 from ϕ_1, by the same general rule (347, XI.) by which ψ is formed from ϕ, we have

$$\text{XIX.} \ldots \psi_1 V\mu\nu = V . \phi'_1\mu\phi'_1\nu = V . (\phi'\mu + c_1\mu)(\phi'\nu + c_1\nu),$$

and therefore, by the definition 350, VIII. of χ,

$$\text{XX.} \ldots \psi_1\rho = \psi\rho + c_1\chi\rho + c_1^2\rho, \quad \text{or} \quad \text{XXI.} \ldots \psi_1 = \psi + c_1\chi + c_1^2;$$

and in like manner,

$$\text{XXII.} \ldots \psi_2 = \psi + c_2\chi + c_2^2,$$

even if m be different from *zero*, and if c_1, c_2 be *arbitrary scalars*.

(6.) Accordingly, *without* assuming that m *vanishes*, if we operate on $\psi_1\rho$ with

ϕ_1, or symbolically multiply the expression XXI. for ψ_1 by ϕ_1, we get the symbolic product,

$$\text{XXIII.}\ldots\ \phi_1\psi_1 = (\phi + c_1)(\psi + c_1\chi + c_1^2) \\ = \phi\psi + c_1(\phi\chi + \psi) + c_1^2(\phi + \chi) + c_1^3 \\ = m + c_1m' + c_1^2m'' + c_1^3 = m_1,$$

where m_1 is what the scalar m becomes, when ϕ is changed to ϕ_1, or is such that

$$\text{XXIV.}\ldots\ m_1\mathrm{S}\lambda\mu\nu = \mathrm{S}.\phi'_1\lambda\phi'_1\mu\phi'_1\nu = \mathrm{S}.(\phi'\lambda + c_1\lambda)(\phi'\mu + c_1\mu)(\phi'\nu + c_1\nu);$$

as appears by the definitions of ϕ', ψ, χ, m, m', m'', and by the relations between those symbols which have been established in recent Articles, or in the sub-articles appended to them.

(7.) Supposing now again that $m = 0$, and that c_1, c_2 are the roots of the quadratic I. in c, we have by XXIII.,

$$\text{XXV.}\ldots\ \phi_1\psi_1 = m_1 = 0; \quad \text{and in like manner} \quad \text{XXVI.}\ldots\ \phi_2\psi_2 = m_2 = 0,$$

if m_2 be formed from m_1, by changing c_1 to c_2.

(8.) Comparing XXV. with XVII., we may be led to suspect the existence of an intimate connexion existing between ψ_1 and $\phi_2\phi$, since each reduces an arbitrary vector to the fixed direction of $\phi_1^{-1}0$, or of ρ_1; and in fact these two operations are *identical*, because, by XXI., and by the known relations between the symbols, we have the transformations,

$$\text{XXVII.}\ldots\ \psi_1 = \psi + c_1\chi + c_1^2 = (m' - m''\phi + \phi^2) + c_1(m'' - \phi) + c_1^2 \\ = \phi^2 - (m'' + c_1)\phi = \phi^2 + c_2\phi = \phi\phi_2;$$

and similarly, $\text{XXVIII.}\ldots\ \psi_2 = \phi^2 + c_1\phi = \phi\phi_1;$

while $\psi = \phi_1\phi_2$, as before.

(9.) We have thus the *new symbolic equation*,

$$\text{XXIX.}\ldots\ \phi\phi_1\phi_2 = 0,$$

in which the *three symbolic factors* ϕ, ϕ_1, ϕ_2 may be in any manner *grouped* and *transposed*, so that it *includes* the two equations XVII.; and in which the subject of operation is an *arbitrary vector* ρ. Its interpretation has been already partly given; but we may add, that while ϕ *reduces* every vector *to* the *fixed plane* Π, ϕ_1 reduces every line to *another* fixed plane, Π_1, and ϕ_2 reduces to a *third* plane, Π_2; thus $\phi_1\phi_2$, or $\phi_2\phi_1$, while it *destroys two lines* ρ_1, ρ_2, and therefore *every* line *in* the plane Π, *reduces* an *arbitrary* line to the *fixed direction* of the *intersection* of the *two planes* $\Pi_1\Pi_2$, which intersection must thus have the direction of $\phi^{-1}0$; and in like manner, the fixed direction ρ_1 of $\phi_1^{-1}0$, as being that to which an arbitrary vector is reduced (8.) by the compound operation $\phi_2\phi$, or $\phi\phi_2$, must be that of the intersection of the planes $\Pi\Pi_2$; and ρ_2, or $\phi_2^{-1}0$, has the direction of the intersection of $\Pi\Pi_1$; while on the other hand $\phi\phi_2$ *destroys* every line in Π_1, and $\phi\phi_1$ every line in Π_2: so that *these three planes*, with their *three lines of intersection*, are the chief elements in the *geometrical interpretation* of the equation $\phi\phi_1\phi_2 = 0$.

(10.) The *conjugate equation*,

$$\text{XXX.}\ldots\ \phi'\phi'_1\phi'_2 = 0,$$

may be interpreted in a similar way, and so conducts to the consideration of a *conjugate system* of *planes and lines*; namely the planes Π', Π'_1, Π'_2, which are the *loci* of $\phi'\rho$, $\phi'_1\rho$, $\phi'_2\rho$, while the operations $\phi'_1\phi'_2$, $\phi'_2\phi'_1$, and $\phi'\phi'_1$ *destroy* all lines in these three planes respectively, and *reduce arbitrary* lines to the fixed directions of the *intersections*, $\Pi'_1\Pi'_2$, $\Pi'_2\Pi'$, $\Pi'\Pi'_1$, which are also those of $\phi'^{-1}0$, $\phi'^{-1}_1 0$, $\phi'^{-1}_2 0$.

(11.) It is important to observe that these *three last lines* are the *normals* to the *three first planes*, Π, Π', Π''; and that, in like manner, the three *former lines* are *perpendicular* to the three *latter planes*. To prove this, it is sufficient to observe that

$$\text{XXXI.} \ldots \quad S\rho'\phi\rho = S\rho\phi'\rho' = 0, \quad \text{if} \quad \phi'\rho' = 0, \quad \text{or that} \quad \phi\rho \perp \phi'^{-1}0;$$

and similarly, $\phi'\rho \perp \phi^{-1}0$, &c.

(12.) Instead of eliminating $x' : x$ between the two equations VII., we might have eliminated c; which would have given this *other quadratic*,

$$\text{XXXII.} \ldots \quad 0 = x^2 S\alpha'\beta + xx'(S\alpha'\beta' - S\alpha\beta) - x'^2 S\alpha\beta';$$

also, if $x'_1 : x_1$ and $x_2' : x_2$ be the two values of $x' : x$, then

$$\text{XXXIII.} \ldots \quad \rho_1 \parallel x_1\beta + x'_1\beta', \quad \rho_2 \parallel x_2\beta + x'_2\beta',$$

and $$\text{XXXIV.} \ldots \quad x_1x_2 : (x_1x'_2 + x_2x'_1) : x'_1x'_2 = -S\alpha\beta' : (S\alpha\beta - S\alpha'\beta') : S\alpha'\beta;$$

hence the *condition of rectangularity* of the *two lines* ρ_1, ρ_2, or $\phi_1^{-1}0$, $\phi_2^{-1}0$, is expressed by the equation,

$$\text{XXXV.} \ldots \quad 0 = -\beta^2 S\alpha\beta' + S\beta\beta'(S\alpha\beta - S\alpha'\beta') + \beta'^2 S\alpha'\beta = S.\beta\beta' V(\beta\alpha + \beta'\alpha');$$

and consequently it is *satisfied*, if the given function ϕ be *self-conjugate* (VI.), because we have then the relation,

$$\text{XXXVI.} \ldots \quad V\beta\alpha + V\beta'\alpha' = 0;$$

in fact the binomial form of ϕ gives (comp. 349, XXII.),

$$\text{XXXVII.} \ldots \quad \phi'\rho - \phi\rho = (\alpha S\beta\rho - \beta S\alpha\rho) + (\alpha' S\beta'\rho - \beta' S\alpha'\rho) = V.\rho V(\beta\alpha + \beta'\alpha'),$$

which cannot vanish independently of ρ, unless the constants satisfy the condition XXXVI.

(13.) With this *condition* then, of *self-conjugation* of ϕ, we have the relation of *rectangularity*,

$$\text{XXXVIII.} \ldots \quad S\rho_1\rho_2 = 0, \quad \text{or} \quad \phi_1^{-1}0 \perp \phi_2^{-1}0;$$

at least if these *directions* ρ_1 and ρ_2 be *real*, which they can easily be proved to be, as follows. The condition XXXVI. gives,

$$\text{XXXIX.} \ldots \quad 0 = S.\alpha\alpha' V(\beta\alpha + \beta'\alpha') = \alpha^2 S\alpha'\beta + S\alpha\alpha'(S\alpha'\beta' - S\alpha\beta) - \alpha'^2 S\alpha\beta';$$

hence $$(\alpha^2 S\alpha'\beta - \alpha'^2 S\alpha\beta')^2 = (S\alpha\alpha')^2 (S\alpha\beta - S\alpha'\beta')^2,$$

$$\alpha^2\alpha'^2(m''^2 - 4m') = \alpha^2\alpha'^2\{(S\alpha\beta - S\alpha'\beta')^2 + 4S\alpha\beta' S\alpha'\beta\}$$

$$= (\alpha^2\alpha'^2 - (S\alpha\alpha')^2)(\alpha\beta - S\alpha'\beta')^2 + (\alpha^2 S\alpha'\beta + \alpha'^2 S\alpha\beta')^2 > 0,$$

and $$\text{XL.} \ldots \quad (S\alpha\beta - S\alpha'\beta')^2 + 4S\alpha\beta' S\alpha'\beta = m''^2 - 4m' > 0;$$

so that each of the two quadratics, I. (or VIII.), and XXXII., has *real and unequal roots*: a conclusion which may also be otherwise derived, from the expressions $\beta = a\alpha + b\alpha'$, $\beta' = b\alpha + a'\alpha'$, which the condition allows us to substitute for β and β'.

(14.) The same condition XXXVI. shows that the *four vectors* $\alpha\beta\alpha'\beta'$ are *complanar*, or that we have the relations,

$$\text{XLI.} \ldots \quad S\alpha\beta\beta' = 0, \quad S\alpha'\beta\beta' = 0, \quad V(V\alpha\alpha'.V\beta'\beta) = 0;$$

hence $V\alpha\alpha'$, or $\phi^{-1}0$ is now *normal* to the *plane* Π; and therefore by (13.), *when the function* ϕ *is self-conjugate* (VI.), *the three directions*,

$$\text{XLII.} \ldots \rho,\ \rho_1,\ \rho_2, \quad \text{or} \quad \phi^{-1}0,\ \phi_1^{-1}0,\ \phi_2^{-1}0,$$

compose a real and rectangular system.

(15.) In the *present series* of subarticles (to 353), we suppose that the *three roots* of the *cubic* in ϕ are *all unequal*, the cases of *equal* roots (with $m = 0$) having been discussed in a preceding series (352); but it may be remarked in passing, that when a *self-conjugate function* $\phi\rho$ is reducible to the *monomial form* $\beta S \alpha\rho$, we must have the relation $V\beta\alpha = 0$; and that thus the *line* β, to the *fixed direction* of which (comp. 352, (5.) and (6.)) the operation ϕ then *reduces* an *arbitrary* vector, is *perpendicular* to the *fixed plane* (352, (7.)), every line in which is *destroyed* by that operation ϕ.

(16.) In general, if ϕ be thus self-conjugate, it is evident that the *three planes* Π', Π'_1, Π'_2, which are (comp. (10.)) the *loci* of $\phi'\rho$, $\phi_1'\rho$, $\phi'_2\rho$, *coincide* with the planes Π, Π_1, Π_2, which are the loci of $\phi\rho$, $\phi_1\rho$, $\phi_2\rho$.

(17.) When ϕ is *not* self-conjugate, so that $\phi\rho$ and $\phi'\rho$ are not generally equal, it has been remarked that the *scalar quadratic* I., and therefore also the symbolical cubic in ϕ, *may* have *imaginary roots;* and that, in *this* case, the *vector equation* II. of the *second degree* cannot be satisfied by *any real direction* of ρ, except that *one* which satisfies the *linear* equation V., or causes $\phi\rho$ itself to vanish, while ρ remains real and actual. As an *example* of such *imaginary scalars*, as *roots* of I., and of what may be called *imaginary directions*, or *imaginary vectors* (comp. 214, (4.)), which *correspond* to those scalars, and are themselves *imaginary roots* of II., we may take the very simple expressions (comp. 349, XII.),

$$\text{XLIII.} \ldots \phi\rho = V\gamma\rho, \quad \phi'\rho = -V\gamma\rho;$$

in which γ denotes some real and given vector, and which evidently do not satisfy the condition VI., the function ϕ being *here* the *negative* of its own conjugate, so that its *self-conjugate part* ϕ_0 is *zero* (comp. 349, XIII.). We have thus,

$$\text{XLIV.} \ldots m_0 = 0, \quad m'_0 = 0, \quad m''_0 = 0, \quad \phi_0 = 0, \quad \psi_0 = 0, \quad \chi_0 = 0,$$

and consequently, by the sub-articles to 349 and 350,

$$\text{XLV.} \ldots m = 0, \quad m' = -\gamma^2, \quad m'' = 0, \quad \psi\rho = -\gamma S\gamma\rho, \quad \chi\rho = -V\gamma\rho;$$

the quadratic I., and its roots c_1, c_2, become therefore,

$$\text{XLVI.} \ldots c^2 - \gamma^2 = 0, \quad c_1 = +\sqrt{-1}\,.\,T\gamma, \quad c_2 = -\sqrt{-1}\,.\,T\gamma,$$

where $\sqrt{-1}$ is the *imaginary of algebra* (comp. 214, (3.)); thus by XX. or XXI., and XXII.) we have now

$$\text{XLVII.} \ldots \psi_1\sigma = -\gamma S\gamma\sigma - c_1 V\gamma\sigma + c_1^2\sigma = (\gamma - c_1)V\gamma\sigma, \quad \psi_2\sigma = (\gamma - c_2)V\gamma\sigma;$$

hence

$$S\gamma\psi_1\sigma = 0, \quad V\gamma\psi_1\sigma = \gamma\psi_1\sigma, \text{ \&c.,}$$

and

$$\text{XLVIII.} \ldots \phi_1\psi_1\sigma = (\phi + c_1)\psi_1\sigma = (\gamma + c_1)\,(\gamma - c_1)V\gamma\sigma = (\gamma^2 - c_1^2)V\gamma\sigma = 0;$$

and in like manner $\quad \text{XLVIII}'. \ldots \phi_2\psi_2\sigma = 0;$

if then we take an *arbitrary vector* σ, and derive (or rather *conceive* as derived) from it *two (imaginary) vectors* ρ_1 and ρ_2 by the (*imaginary*) *operations* ψ_1 and ψ_2, we shall have (comp. III. and IV.) the equations,

$$\text{XLIX.} \ldots \rho_1 = \psi_1\sigma, \quad \phi_1\rho_1 = 0, \quad \phi\rho_1 = -c_1\rho_1, \quad V\rho_1\phi\rho_1 = 0,$$

and $\quad \text{L.} \ldots \rho_2 = \psi_2\sigma, \quad \phi_2\rho_2 = 0, \quad \phi\rho_2 = -c_2\rho_2, \quad V\rho_2\phi\rho_2 = 0,$

as ones which are at least *symbolically true*. We find then that the *two imaginary directions*, ρ_1 and ρ_2, *satisfy* (at least in a symbolical sense, or as far as *calculation* is concerned) the *vector equation* II., or that ρ_1 and ρ_2 are *two imaginary vector roots* of $V\rho\phi\rho = 0$; but that, because the *scalar quadratic* I. has here *imaginary roots*, this vector equation II. has (as above stated) *no real vector root* ρ, except one in the *direction* of the *given* and *real vector* γ, which satisfies the *linear* equation V., or gives $\phi\rho = 0$.

(18.) This particular example might have been more simply treated, by a less general method, as follows. We wish to satisfy the equation,

$$\text{LI.} \ldots 0 = V.\rho V\gamma\rho = \rho S\gamma\rho - \rho^2\gamma;$$

which gives, when we operate on it by $V.\gamma$ and $V.\rho$, these others,

$$\text{LII.} \ldots 0 = V\gamma\rho . S\gamma\rho, \quad 0 = \rho^2 V\gamma\rho;$$

if then we wish to *avoid* supposing $\phi\rho = V\gamma\rho = 0$, we must seek to satisfy the *two scalar equations*,

$$\text{LIII.} \ldots S\gamma\rho = 0, \quad \rho^2 = 0;$$

and conversely, if we can satisfy *these* by any (real or imaginary) ρ, we shall have satisfied (really or symbolically) the *vector* equation LI. Now the *first* equation LIII. is satisfied, when we assume the expression,

$$\text{LIV.} \ldots \rho = (c + \gamma)V\gamma\sigma = V\gamma\sigma.(c - \gamma),$$

where σ is an *arbitrary vector*, and c is *any scalar*, or *symbol* subject to the *laws of scalars*; and this expression LIV. for ρ, with its transformation just assigned, gives

$$\text{LV.} \ldots \rho^2 = (c^2 - \gamma^2)(V\gamma\sigma)^2 = 0, \quad \text{if} \quad c^2 - \gamma^2 = 0;$$

the *quadratic* XLVI. is therefore reproduced, and we have the *same imaginary roots*, and *imaginary directions*, as before.

(19.) *Geometrically*, the *imaginary character* of the recent problem, of satisfying the equation $V.\rho V\gamma\rho = 0$ by any direction of ρ except that of the given line γ, is apparent from the circumstance that $\phi\rho$, or $V\gamma\rho$, is here a vector *perpendicular* to ρ, if *both* be *actual* lines; and that therefore the one cannot be also *parallel* to the other, so long as both are *real*.*

354. In the three preceding Articles, and in the sub-articles annexed, we have supposed throughout that the *absolute term* of the cubic in ϕ is *wanting*, or that the condition $m = 0$ is satisfied; in which case we have seen (351) that it is always possible to satisfy the *linear equation* $\phi\rho = 0$, by at least *one* real and actual value of ρ (with an arbitrary scalar coefficient); or by at least *one* real direction. It will be easy now to show,

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* Accordingly *the two imaginary directions*, above found for ρ, are easily seen to be those which in modern geometry are called the directions of *lines drawn in a given plane* (perpendicular here to the given line γ), *to the circular points at infinity*: of which supposed *directions* the *imaginary character* may be said to be precisely this, that *each* is (in the given plane) *its own perpendicular*.

that although conversely (comp. 351, (4.)) the function $\phi\rho$ *cannot vanish* for *any actual* vector ρ, *unless* we have thus $m = 0$, yet there is *always at least one real direction* for which the *vector equation of the second degree*,

$$\text{I.} \ldots \mathrm{V}\rho\phi\rho = 0,$$

which has already been considered (353) in *combination* with the *condition* $m = 0$, is satisfied; and that if the function ϕ be a *self-conjugate* one, then this equation I. is *always* satisfied by *at least three real and rectangular directions*, but *not generally* by *more* directions than *three;* although, in this *case of self-conjugation*, namely when

$$\text{II.} \ldots \phi'\rho = \phi\rho, \quad \text{or} \quad \text{II}'. \ldots \mathrm{S}\lambda\phi\rho = \mathrm{S}\rho\phi\lambda,$$

for all values of the vectors ρ and λ, the equation I. may happen to become true, for *one* real direction of ρ, and for *every* direction *perpendicular* thereto: or even for *all* possible directions, according to the particular system of *constants*, which enter into the composition of the *function* $\phi\rho$. We shall show also that the *scalar* (or *algebraic*) and *cubic equation*,

$$\text{III.} \ldots 0 = m + m'c + m''c^2 + c^3,$$

which is formed from the *symbolic* and *cubic equation* 350, I., by changing ϕ to $-c$, enters importantly into this whole theory; and that if it have *one real* and *two imaginary roots*, the *quadratic* and *vector equation* I. is satisfied by *only one real direction* of ρ; but that it may *then* be said (comp. 353, (17.)) to be satisfied *also* by *two imaginary directions*, or to have *two imaginary and vector roots:* so that this *equation* I. may be said to represent *generally* a *system of three right lines*, whereof *one* at least must be *real*. For the *case* II., the *scalar roots* of III. will be proved to be *always real;* so that if m_0, m'_0, and m''_0 be formed (as in sub-articles to 349 and 350) from the *self-conjugate part* $\phi_0\rho$ of *any linear and vector function* $\phi\rho$, as m, m', and m'' are formed from that function $\phi\rho$ *itself*, then *the new cubic*,

$$\text{IV.} \ldots 0 = m_0 + m'_0c + m''_0c^2 + c^3,$$

which thus results, *can never have imaginary roots.*

(1.) If we write,

$$\text{V.} \ldots \Phi\rho = \phi\rho + c\rho, \quad \Phi'\rho = \phi'\rho + c\rho, \quad \text{or briefly,} \quad \text{V}'. \ldots \Phi = \phi + c, \quad \Phi' = \phi' + c,$$

where c is an arbitrary scalar, and if we denote by Ψ, Ψ', and M what ψ, ψ', and m become, by this change of ϕ to $\phi + c$ or Φ, the calculations in 353, (5.), (6.), show that we have the expressions,

$$\text{VI.} \ldots \Psi = \psi + c\chi + c^2, \quad \Psi' = \psi' + c\chi' + c^2,$$

and

$$\text{VII.} \ldots M = m + m'c + m''c^2 + c^3,$$

with

$$\text{VIII.} \ldots M = \Phi\Psi = \Psi\Phi = \Phi'\Psi' = \Psi'\Phi'.$$

(2.) Hence it may be inferred that the functions χ, χ', and the constants m', m'' become,

$$\text{IX.} \ldots \mathrm{X} = \mathrm{D}_c\Psi = \chi + 2c, \quad \mathrm{X}' = \mathrm{D}_c\Psi' = \chi' + 2c,$$

$$\text{X.} \ldots \begin{cases} M' = \mathrm{D}_c M = m' + 2m''c + 3c^2, \\ M'' = \frac{1}{2}\mathrm{D}_c{}^2 M = m'' + 3c\,; \end{cases}$$

with the verifications,

$$\text{XI.} \ldots \Phi + \mathrm{X} = \Phi' + \mathrm{X}' = M'', \quad \Phi\mathrm{X} + \Psi = \Phi'\mathrm{X}' + \Psi' = M',$$

as we had, by the sub-articles to 350,

$$\phi + \chi = \phi' + \chi' = m'', \quad \phi\chi + \psi = \phi'\chi' + \psi' = m'.$$

(3.) The *new linear symbol* Φ must satisfy the *new cubic*,

$$\text{XII.} \ldots 0 = M - M'\Phi + M''\Phi^2 - \Phi^3\,;$$

which accordingly can be at once derived from the *old cubic* 350, I., under the form,

$$\text{XIII.} \ldots 0 = m + m'(c - \Phi) + m''(c - \Phi)^2 + (c - \Phi)^3.$$

(4.) Now it is always possible to satisfy the condition,

$$\text{XIV.} \ldots M = 0,$$

by substituting for c a *real root* of the *scalar cubic* III.; and thereby to *reduce* the *new symbolical cubic* XII. to the *form*,

$$\text{XV.} \ldots 0 = \Phi^3 - M''\Phi^2 + M'\Phi;$$

which is precisely similar to the form,

$$0 = \phi^3 - m''\phi^2 + m'\phi, \qquad 352,\ \text{II.},$$

and conducts to analogous consequences, which need not here be developed in detail, since they can easily be supplied by any one who will take the trouble to read again the few recent series of sub-articles.

(5.) For example, unless it happen that $\Psi\rho$ *constantly vanishes*, in which case $M' = 0$, and $\Phi\rho$ (if not *identically null*) takes a *monomial form*, which is reduced to *zero* (comp. 352, (7.)) for *every direction* of ρ in a *given plane*, the operation Ψ *reduces* (comp. 351) an *arbitrary vector* to a *given direction;* and the operation Φ *destroys* every line in *that* direction: so that, *in every case*, there is *at least one real way* of satisfying the *vector equation* $\Phi\rho = 0$, and therefore also (as above asserted) the equation I., without causing ρ *itself* to vanish.

(6.) And since that equation I. may be thus written,

$$\text{XVI.} \ldots \mathrm{V}\rho\Phi\rho = 0, \quad \text{or} \quad \Phi\rho \parallel \rho,$$

we see that it can be satisfied *without* $\Phi\rho$ *vanishing*, if this *new scalar and quadratic* equation,

$$\text{XVII.} \ldots 0 = C^2 + M''C + M', \qquad \text{comp. 353, I.,}$$

have *real and unequal roots* C_1, C_2; for if we then write,

$$\text{XVIII.} \ldots \Phi_1 = \Phi + C_1, \quad \Phi_2 = \Phi + C_2,$$

the line $\Phi\rho$ will *generally* have for its *locus a given plane*, and there will be *two real and distinct directions* ρ_1 and ρ_2 in that plane, for one of which $\Phi_1\rho_1 = 0$, while $\Phi_2\rho_2 = 0$ for the other, so that *each* satisfies XVI., or I.; and these are precisely the *fixed directions* of $\Psi_1\rho$ and $\Psi_2\rho$, if Ψ_1 and Ψ_2 be formed from Ψ by changing Φ to Φ_1 and Φ_2 respectively.

(7.) Cases of *equal* and of *imaginary roots* need not be dwelt on here; but it may be remarked in passing, that if the function $\phi\rho$ have the *particular form* (g being any scalar constant),

$$\text{XIX.} \ldots \phi\rho = g\rho, \quad \text{then} \quad \text{XX.} \ldots (g - \phi)^3 = 0, \quad \text{and} \quad \text{XXI.} \ldots M = (g + c)^3;$$

the cubic XIV. or III. having thus *all* its roots *equal*, and the equation I. being satisfied by *every direction* of ρ, in this particular case.

(8.) The *general existence* of a *real* and *rectangular system* of *three directions* satisfying I., when the *condition* II. is satisfied, may be proved as in 353, (14.); and it is unnecessary to dwell on the case where, by *two roots* of the cubic becoming *equal*, *all lines* in a *given plane*, and *also* the *normal* to that plane, are *vector roots* of I., with the same condition II.

(9.) And because the *quadratic*, $0 = c^2 + m''c + m'$ (353, I.), has been proved to have always *real* roots (353, (13.)) when $\phi'\rho = \phi\rho$, the *analogous* quadratic XVII. must likewise then have real roots, C_1, C_2; whence it immediately follows (comp. XII. and XIII.), that (under the same *condition of self-conjugation*) the *cubic* III. has *three* real roots, c, $c + C_1$, $c + C_2$; and therefore that (as above stated) the *other cubic* IV., which is formed from the *self-conjugate part* ϕ_0 of the *general linear and vector function* ϕ, and which may on that account be thus denoted,

$$\text{XXII.} \ldots M_0 = 0, \textit{ has its roots always real.}$$

(10.) If we denote in like manner by Φ_0 the symbol $\phi_0 + c$, the equation $m = m_0 - \mathrm{S}\gamma\phi_0\gamma$ (349, XXVI., comp. 349, XXI.) becomes,

$$\text{XXIII.} \ldots M = M_0 - \mathrm{S}\gamma\Phi_0\gamma;$$

whence, by comparing powers of c, we recover the relations,

$$m' = m'_0 - \gamma^2, \quad \text{and} \quad m'' = m''_0, \text{ as in 350, (1.).}$$

(11.) On a similar plan, the equation $m\phi'\mathrm{V}\mu\nu = \mathrm{V}.\psi\mu\psi\nu$ becomes,

$$\text{XXIV.} \ldots M\Phi'\mathrm{V}\mu\nu = \mathrm{V}.\Psi\mu\Psi\nu, \qquad \text{comp. 348, (1.),}$$

in which μ and ν are *arbitrary vectors*, and c is an *arbitrary scalar*; or more fully,

$$\text{XXV.} \ldots (m + m'c + m''c^2 + c^3)(\phi' + c)\mathrm{V}\mu\nu = \mathrm{V}.(\psi\mu + c\chi\mu + c^2\mu)(\psi\nu + c\chi\nu + c^3\nu);$$

whence follow these new equations,

$$\text{XXVI.} \ldots (m + m'\phi')\mathrm{V}\mu\nu = \mathrm{V}(\psi\mu.\chi\nu - \psi\nu.\chi\mu),$$
$$\text{XXVII.} \ldots (m' + m''\phi')\mathrm{V}\mu\nu = \mathrm{V}(\mu\psi\nu - \nu\psi\mu + \chi\mu.\chi\nu),$$
$$\text{XXVIII.} \ldots (m'' + \phi')\mathrm{V}\mu\nu = \mathrm{V}(\mu\chi\nu - \nu\chi\mu),$$

which can all be otherwise proved, and from the last of which (by changing ϕ to ψ, &c.) we can infer this other of the same kind,

$$\text{XXIX.} \ldots (m' + \psi')\mathrm{V}\mu\nu = \mathrm{V}(\mu\phi\chi\nu - \nu\phi\chi\mu).$$

(12.) As an *example* of the existence of a *real* and *rectangular system* of *three directions* (8.), represented jointly by an equation of the form I., and of a system of *three real roots* of the *scalar cubic* III., when the *condition* II. is satisfied, let us take the form,

$$\text{XXX.} \ldots \phi\rho = g\rho + \mathrm{V}\lambda\rho\mu = \phi'\rho,$$

g being here any *real* and *given scalar*, and λ, μ any real and non-parallel *given vectors*; to which *form*, indeed, we shall soon find that *every* self-conjugate function $\phi_0\rho$ can be brought. We have now (after some reductions),

$$\text{XXXI.} \ldots \psi\rho = \mathrm{V}\lambda\rho\mu\mathrm{S}\lambda\mu - \mathrm{V}\lambda\mu\mathrm{S}\lambda\rho\mu - g(\lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho) + g^2\rho,$$

$$\text{XXXII.} \ldots \chi\rho = -(\lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho) + 2g\rho,$$

and $$\text{XXXIII.} \ldots m = (g - \mathrm{S}\lambda\mu)(g^2 - \lambda^2\mu^2), \quad m' = -\lambda^2\mu^2 - 2g\mathrm{S}\lambda\mu + 3g^2,$$
$$m'' = -\mathrm{S}\lambda\mu + 3g;$$

where the part of $\psi\rho$ which is independent of g may be put under several other forms, such as the following,

$$\text{XXXIV.} \ldots \mathrm{V}(\lambda\rho\mu\mathrm{S}\lambda\mu - \lambda\mu\mathrm{S}\lambda\rho\mu) = \lambda\rho\mu\mathrm{S}\lambda\mu - \lambda\mu\mathrm{S}\lambda\rho\mu$$
$$= \lambda(\rho\mathrm{S}\lambda\mu + \mathrm{S}\lambda\mu\rho)\mu = \tfrac{1}{2}\lambda(\lambda\mu\rho + \rho\lambda\mu)\mu = \lambda(\lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho - \lambda\rho\mu)\mu, \text{ \&c.};$$

and Φ, Ψ, X, M, M', M'' may be formed from ϕ, ψ, χ, m, m', m'', by simply changing g to $c + g$. The equation $M = 0$ has therefore here three real and unequal roots, namely the three following,

$$\text{XXXV.} \ldots c = -g + \mathrm{S}\lambda\mu, \quad c + C_1 = -g + \mathrm{T}\lambda\mu, \quad c + C_2 = -g - \mathrm{T}\lambda\mu;$$

and the corresponding forms of $\Psi\rho$ are found to be,

$$\text{XXXVI.} \ldots \Psi\rho = \mathrm{V}\lambda\mu\mathrm{S}\lambda\mu\rho, \quad \Psi_1\rho = -(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda)\mathrm{S}.\rho(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda),$$
$$\psi_2\rho = -(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda)\mathrm{S}.\rho(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda).$$

Thus $\Psi\rho$, $\Psi_1\rho$, and $\Psi_2\rho$ have in fact the three fixed and rectangular directions of $\mathrm{V}\lambda\mu$, $\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda$, and $\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda$, namely of the normal to the given plane of λ, μ, and the bisectors of the angles made by those two given lines; and *these* are accordingly the *only* directions which satisfy the vector equation of the second degree,

$$\text{XXXVII.} \ldots (\mathrm{V}\rho\phi\rho = \mathrm{V}.\rho\mathrm{V}\lambda\rho\mu =)\mathrm{V}\rho\lambda\mathrm{S}\mu\rho + \mathrm{V}\rho\mu\mathrm{S}\lambda\rho = 0;$$

so that this last *equation* represents (as was expected) a *system of three right lines*, in these three respective *directions*.

(13.) *In general*, if c_1, c_2, c_3 denote the *three roots* (real or imaginary) of the *cubic* equation $M = 0$, and if we write,

$$\text{XXXVIII.} \ldots \Phi_1 = \phi + c_1, \quad \Phi_2 = \phi + c_2, \quad \Phi_3 = \phi + c_3,$$

the corresponding values of Ψ will be (comp. VI.),

$$\text{XXXIX.} \ldots \Psi_1 = \psi + c_1\chi + c_1^2, \quad \Psi_2 = \psi + c_2\chi + c_2^2, \quad \Psi_3 = \psi + c_3\chi + c_3^2;$$

also we have the relations,

$$\text{XL.} \ldots \begin{cases} c_1 + c_2 + c_3 = -m'' = -\phi - \chi, \\ c_2c_3 + c_3c_1 + c_1c_2 = +m' = \phi\chi + \psi, \\ c_1c_2c_3 = -m = -\phi\psi; \end{cases}$$

whence it is easy to infer the expressions,

$$\text{XLI.} \ldots \Phi_1 = (c_2 - c_3)^{-1}(\Psi_3 - \Psi_2), \quad \Phi_2 = (c_3 - c_1)^{-1}(\Psi_1 - \Psi_3),$$
$$\Phi_3 = (c_1 - c_2)^{-1}(\Psi_2 - \Psi_1);$$

which enable us to express the *functions* $\Phi_1\rho$, $\Phi_2\rho$, $\Phi_3\rho$ as *binomials* (comp. 351, &c.), when $\Psi_1\rho$, $\Psi_2\rho$, $\Psi_3\rho$ have been expressed as *monomes*, and to assign the *planes* (real or imaginary), which are the *loci* of the *lines* $\Phi_1\rho$, $\Phi_2\rho$, $\Phi_3\rho$.

(14.) Accordingly, the *three operations*, Φ, Φ_1, Φ_2, by which lines in the three lately determined *directions* (12.) are *destroyed*, or reduced to *zero*, and which at first present themselves under the forms,

$$\text{XLII.} \ldots \Phi\rho = \lambda S\mu\rho + \mu S\lambda\rho, \quad \Phi_1\rho = V\lambda\rho\mu + \rho T\lambda\mu, \quad \Phi_2 = V\lambda\rho\mu - \rho T\lambda\mu,$$

are found to admit of the transformations,

$$\text{XLIII.} \ldots \Phi\rho = \frac{\Psi_2\rho - \Psi_1\rho}{2T\lambda\mu}; \quad \Phi_1\rho = \frac{\Psi_2\rho - \Psi\rho}{T\lambda\mu + S\lambda\mu}; \quad \Phi_2\rho = \frac{\Psi\rho - \Psi_1\rho}{T\lambda\mu - S\lambda\mu};$$

where Ψ, Ψ_1, Ψ_2 have the recent forms XXXVI., and the *loci* of $\Phi\rho$, $\Phi_1\rho$, $\Phi_2\rho$ compose a system of *three rectangular planes.*

(15.) In general, the relations (13.) give also (comp. 353, (8.)),

$$\text{XLIV.} \ldots \Psi_1 = \Phi_2\Phi_3, \quad \Psi_2 = \Phi_3\Phi_1, \quad \Psi_3 = \Phi_1\Phi_2,$$

and

$$\text{XLV.} \ldots \Phi_1\Psi_1 = \Phi_2\Psi_2 = \Phi_3\Psi_3 = \Phi_1\Phi_2\Phi_3 = 0,$$

whence also,

$$\text{XLVI.} \ldots \Psi_1\Psi_2 = \Psi_2\Psi_3 = \Psi_3\Psi_1 = 0,$$

the *symbols* (in any *one* system of this sort) admitting of being *transposed* and *grouped* at pleasure; if then the roots of $M = 0$ be *real* and *unequal*, there arises a system of *three real* and *distinct planes*, which are connected with the *interpretation* of the *symbolical equation*, $\Phi_1\Phi_2\Phi_3 = 0$, exactly as the three planes in 353, (9.) were connected with the analogous equation, $\phi\phi_1\phi_2 = 0$.

(16.) And when the cubic has *two imaginary roots*, it may then be said that there is *one real plane* (such as the plane $\perp \gamma$ in 353, (18.), (19.)), containing the *two imaginary directions* which then satisfy the equation I.; and *two imaginary planes*, which respectively *contain* those two directions, and *intersect* each other in *one real line* (such as the line γ in the example cited), namely the *one real vector root* of the same equation I.

355. Some additional light may be thrown upon that *vector equation* of the *second degree*, by considering the system of the *two scalar equations*,

$$\text{I.} \ldots S\lambda\rho\phi\rho = 0, \quad \text{and} \quad \text{II.} \ldots S\lambda\rho = 0,$$

and investigating the condition of the *reality* of the *two** *directions*, ρ_1 and ρ_2, by which they are generally satisfied, and for each of which the *plane* of ρ and $\phi\rho$ contains generally the *given line* λ in I., or is *normal* to the plane *locus* II. of ρ. We shall find that these two directions are *always real* and *rectangular* (except that they *may* become *indeterminate*), when the linear function ϕ is *its own conjugate;* and that *then*, if λ be a *root* ρ_0 of the *vector equation*,

$$\text{III.} \ldots V\rho\phi\rho = 0,$$

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* Geometrically, the equation I. represents a *cone of the second order*, with λ for *one side*, and with the *three lines* ρ which satisfy III. for three *other* sides; and II. represents a *plane* through the vertex, *perpendicular* to the side λ. The *two directions* sought are thus the *two sides*, in which this plane *cuts* the cone.

which has been already otherwise discussed, the *lines* ρ_1 and ρ_2 are *also roots* of that equation; the *general existence* (354) of a *system* of *three real and rectangular directions*, which satisfy this equation III. when $\phi'\rho = \phi\rho$, being thus *proved anew:* whence also will follow a *new proof* of the *reality* of the *scalar roots* of the *cubic* $M = 0$, for this *case* of *self-conjugation* of ϕ; and therefore of the *necessary reality* of the roots of that *other cubic*, $M_0 = 0$, which is formed (354, IV. or XXII.) from the *self-conjugate part* ϕ_0 of the *general* linear and vector function ϕ, as $M = 0$ was formed from ϕ.

(1.) Let λ, μ, ν be a system of *three rectangular vector units*, following in all respects the laws (182, 183), of the symbols i, j, k. Writing then,

$$\text{IV.} \ldots \rho = y\mu + z\nu, \quad \text{and therefore,} \quad \lambda\rho = y\nu - z\mu, \quad \phi\rho = y\phi\mu + z\phi\nu,$$

the equation II. is satisfied, and I. becomes,

$$\text{V.} \ldots 0 = y^2 \mathrm{S}\nu\phi\mu + yz(\mathrm{S}\nu\phi\nu - \mathrm{S}\mu\phi\mu) - z^2 \mathrm{S}\mu\phi\nu;$$

the roots of which quadratic will be real and unequal, if

$$\text{VI.} \ldots (\mathrm{S}\nu\phi\nu - \mathrm{S}\mu\phi\mu)^2 + 4\mathrm{S}\mu\phi\nu \mathrm{S}\nu\phi\mu > 0;$$

and the corresponding directions of ρ will be rectangular, if

$$\text{VII.} \ldots 0 = \mathrm{S}(y_1\mu + z_1\nu)(y_2\mu + z_2\nu) = -(y_1y_2 + z_1z_2);$$

that is, if

$$\text{VIII.} \ldots \mathrm{S}\nu\phi\mu = \mathrm{S}\mu\phi\nu,$$

at least for this particular pair of vectors, μ and ν.

(2.) Introducing now the expression, $\phi\rho = \phi_0\rho + \mathrm{V}\gamma\rho$ (349, XII.), the conditions VI. and VIII. take the forms,

$$\text{IX.} \ldots (\mathrm{S}\nu\phi_0\nu - \mathrm{S}\mu\phi_0\mu)^2 + 4\mathrm{S}(\mu\phi_0\nu)^2 > 4(\mathrm{S}\gamma\mu\nu)^2, \quad \text{and} \quad \text{X.} \ldots \mathrm{S}\gamma\mu\nu = 0;$$

which are *both* satisfied generally when $\gamma = 0$, or $\phi = \phi' = \phi_0$; the only exception being, that the quadratic V. *may* happen to become an *identity*, by all its coefficients vanishing: but the *opposite inequality* (to VI. and IX.) can *never* hold good, that is to say, the *roots* of that quadratic can never be *imaginary*, when ϕ is thus *self-conjugate*.

(3.) On the other hand, when γ is *actual*, or $\phi'\rho$ *not* generally $= \phi\rho$, the condition X. of *rectangularity* can only *accidentally* be satisfied, namely by the given or *fixed line* γ happening to be *in the assumed plane* of μ, ν; and when the *two directions* of ρ are thus *not rectangular*, or when the scalar $\mathrm{S}\gamma\mu\nu$ does not vanish, we have only to suppose that the square of this scalar becomes *large enough*, in order to render (by IX.) those directions *coincident*, or *imaginary*.

(4.) When $\phi' = \phi$, or $\gamma = 0$, we may take μ and ν for the two rectangular directions of ρ, or may reduce the quadratic to the very simple form $yz = 0$; but, for this purpose, we must establish the relations,

$$\text{XI.} \ldots \mathrm{S}\mu\phi\nu = \mathrm{S}\nu\phi\mu = 0.$$

(5.) And if, at the same time, λ satisfies the equation III., so that $\phi\lambda \parallel \lambda$, we shall have these other scalar equations,

$$\text{XII.}\ldots\ 0 = S\mu\phi\lambda = S\nu\phi\lambda = S\lambda\phi\mu = S\lambda\phi\nu;$$

whence $$\phi\mu \parallel V\nu\lambda \parallel \mu, \quad \text{and} \quad \phi\nu \parallel V\lambda\mu \parallel \nu,$$

or, $$\text{XIII.}\ldots\ 0 = V\lambda\phi\lambda = V\mu\phi\mu = V\nu\phi\nu;$$

λ, μ, ν thus forming (as above stated) a ***system***, of ***three*** real and rectangular ***roots*** of that ***vector*** equation III.

(6.) But in general, if III. be satisfied by even ***two real*** and ***distinct*** directions of ρ, the ***scalar*** and ***cubic*** equation $M = 0$ can have ***no imaginary root;*** for if those two directions give ***two unequal*** but ***real*** and ***scalar values***, c_1 and c_2, for the ***quotient*** $-\phi\rho : \rho$, then c_1 and c_2 are ***two real roots*** of the cubic, of which therefore the ***third*** root is ***also*** real; and if, on the other hand, the two directions ρ_1 and ρ_2 give one ***common*** real and scalar value, such as c_1, for that quotient, then $\phi\rho = -c_1\rho$, or $\Phi_1\rho = (\phi + c_1)\rho = 0$, for ***every line in the plane*** of ρ_1, ρ_2; so that $\phi\rho$ must be of the ***form***, $-c_1\rho + \beta S\rho_1\rho_2\rho$, and the ***cubic*** will have at least ***two equal roots***, since it will take the form,

$$\text{XIV.}\ldots\ 0 = (c - c_1)^2\,(c - c_1 + S\rho_1\rho_2\beta),$$

as is easily shown from principles and formulæ already established.

(7.) It is then proved anew, that the equation $M = 0$ has ***all*** its roots ***real***, if $\phi'\rho = \phi\rho$; and therefore that the equation $M_0 = 0$ (as above stated) can ***never*** have an ***imaginary root***.

(8.) And we see, at the same time, how the ***scalar cubic*** $M = 0$ might have been deduced from the ***symbolical cubic*** 350, I., or from the equation 351, I., as the condition for the vector equation III. being satisfied by any ***actual*** ρ; namely by observing that if $\phi\rho = -c\rho$, then $\phi^2\rho = c^2\rho$, $\phi^3\rho = -c^3\rho$, &c., and therefore $M\rho = 0$, in which ρ, by supposition, is different from zero.

(9.) Finally, as regards the ***case**** ***of indetermination***, above alluded to, when the quadratic V. ***fails*** to assign any ***definite values*** to $y : z$, or any ***definite directions*** in the given plane to ρ, this case is evidently distinguished by the condition,

$$\text{XV.}\ldots\ S\mu\phi\mu = S\nu\phi\nu,$$

in combination with the equations XI.

356. The existence of the *Symbolic and Cubic Equation* (350), which is satisfied by the *linear and vector symbol* ϕ, suggests a *Theorem*† *of Geometrical Deformation*, which may be thus enunciated:—

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[*] It will be found that this ***case*** corresponds to the ***circular sections*** of a ***surface of the second order;*** while the less particular case in which $\phi'\rho = \phi\rho$, but ***not*** $S\mu\phi\mu = S\nu\phi\nu$, so that the ***two directions*** of ρ are ***determined, real, and rectangular***, corresponds to the ***axes*** of a ***non-circular section*** of such a surface.

[†] This theorem was stated, nearly in the same way, in page 568 of the *Lectures;* and the problem of ***inversion*** of a linear and vector function was treated, in the few preceding pages (559, &c.), though with somewhat less of completeness and perhaps of simplicity than in the present Section, and with a slightly different notation. The ***general form*** of such a function which was there adopted may now be thus expressed:

$$\phi\rho = \Sigma\beta S\alpha\rho + Vr\rho, \ r \text{ being a given quaternion};$$

the resulting value of m was found to be (page 561), "*If, by any given Mode, or Law, of Linear Derivation,* of the kind above denoted by the symbol ϕ, *we pass from any assumed Vector* ρ *to a Series of Successively Derived Vectors,* ρ_1, ρ_2, ρ_3, . . . or $\phi^1\rho$, $\phi^2\rho$, $\phi^3\rho$, . .; *and if, by constructing a Parallelepiped, we decompose any Line of this Series,* such as ρ_3, *into three partial or component lines,* $m\rho$, $-m'\rho_1$, $m''\rho_2$, *in the Directions of the three which precede it,* as here of ρ, ρ_1, ρ_2; *then the Three Scalar Coefficients,* m, $-m'$, m'', *or the Three Ratios which these three Components of the Fourth Line* ρ_3 *bear to the Three Preceding Lines of the Series, will depend only on the given Mode or Law of Derivation, and will be entirely independent of the assumed Length and Direction of the Initial Vector.*"

(1.) As an *Example* of such successive *Derivation*, let us take the *law*,

$$\text{I.} \ldots \rho_1 = \phi\rho = -\mathrm{V}\beta\rho\gamma, \quad \rho_2 = \phi^2\rho = -\mathrm{V}\beta\rho_1\gamma, \text{ \&c.},$$

which answers to the *construction* in 305, (1.), &c., when we suppose that β and γ are *unit-lines*. Treating them at first as *any two given vectors*, our general method conducts to the equation,

$$\text{II.} \ldots \rho_3 = m\rho - m'\rho_1 + m''\rho_2,$$

with the following values of the coefficients,

$$\text{III.} \ldots m = -\beta^2\gamma^2\mathrm{S}\beta\gamma, \quad m' = -\beta^2\gamma^2, \quad m'' = \mathrm{S}\beta\gamma;$$

as may be seen, without any new calculation, by merely changing g, λ, and μ, in 354, XXXIII., to 0, β, and $-\gamma$.

(2.) Supposing next, for comparison with 305, that

$$\text{IV.} \ldots \beta^2 = \gamma^2 = -1, \quad \text{and} \quad \mathrm{S}\beta\gamma = -l,$$

so that β, γ are unit lines, and l is the cosine of their inclination to each other, the values III. become,

$$\text{V.} \ldots m = l, \quad m' = -1, \quad m'' = -l;$$

and the equation II., connecting *four successive lines* of the series, takes the form,

$$\text{VI.} \ldots \rho_3 = l\rho + \rho_1 - l\rho_2, \quad \text{or} \quad \text{VII.} \ldots \rho_3 - \rho_1 = -l(\rho_2 - \rho);$$

$$m = \Sigma\mathrm{S}\alpha\alpha'\alpha''\mathrm{S}\beta''\beta'\beta + \Sigma\mathrm{S}(r\mathrm{V}\alpha\alpha'.\mathrm{V}\beta'\beta) + \mathrm{S}r\Sigma\mathrm{S}\alpha\beta r - \Sigma\mathrm{S}\alpha r\mathrm{S}\beta r + \mathrm{S}r\mathrm{T}r^2;$$

and the auxiliary function which we now denote by ψ was,

$$m\phi^{-1}\sigma = \psi\sigma = \Sigma\mathrm{V}\alpha\alpha'\mathrm{S}\beta'\beta\sigma + \Sigma\mathrm{V}.\alpha\mathrm{V}(\mathrm{V}\beta\sigma.r) + (\mathrm{V}\sigma r\mathrm{S}r - \mathrm{V}r\mathrm{S}\sigma r);$$

where the sum of the two last terms of $\psi\sigma$ might have been written as $\sigma r\mathrm{S}r - r\mathrm{S}\sigma r$. A student might find it an useful exercise, to prove the correctness of these expressions by the principles of the present Section. One way of doing so would be, to treat $\Sigma\beta\mathrm{S}\alpha\rho$ and r as respectively equal to $\phi_0\rho + \mathrm{V}\gamma\rho$ and $c + \epsilon$; which would transform m and $\psi\sigma$, *as above written*, into the following,

$M_0 - \mathrm{S}(\gamma + \epsilon)(\phi_0 + c)(\gamma + \epsilon)$, and $\Psi_0\sigma - (\gamma + \epsilon)\mathrm{S}(\gamma + \epsilon)\sigma + \mathrm{V}\sigma(\phi_0 + c)(\gamma + \epsilon)$; that is, into the new values which the M and $\Psi\sigma$ of the Section assume, when $\Phi\rho$ takes the new value, $\Phi\rho = (\phi_0 + c)\rho + \mathrm{V}(\gamma + \epsilon)\rho$.

a result which agrees with 305, (2.), since we there found that if $\rho = \text{OP}$, &c., the *interval* P_1P_3 was $= -l \times \text{PP}_3$.

(3.) And as regards the *inversion* of a linear and vector function (347), or the *return* from any one line ρ_1 of such a *series* to the line ρ which *precedes* it, our general method gives, for the example I., by 354, (12.),

$$\text{VIII.} \ldots \psi\rho_1 = \tfrac{1}{2}\beta(\beta\gamma\rho_1 + \rho_1\beta\gamma)\gamma,$$

and

$$\text{IX.} \ldots \rho = \phi^{-1}\rho_1 = m^{-1}\psi\rho_1 = -\frac{\beta\rho_1\beta^{-1} + \gamma\rho_1\gamma^{-1}}{\beta\gamma + \gamma\beta};$$

a result which it is easy to verify and to interpret, on principles already explained.

357. We are now prepared to assign some new and general *Forms*, to which the *Linear* and *Vector Function* (with real constants) of a variable vector can be brought, *without* assuming its *self-conjugation;* one of the simplest of which forms is the following,

$$\text{I.} \ldots \phi\rho = \text{V}q_0\rho + \text{V}\lambda\rho\mu, \quad \text{with} \quad \text{I}'. \ldots q_0 = g + \gamma;$$

q_0 being here a *real* and *constant quaternion*, and λ, μ *two real* and *constant vectors*, which can *all* be *definitely assigned*, when the *particular form* of ϕ is *given:* except that λ and μ may be *interchanged* (by 295, VII.), and that *either* may be *multiplied* by any scalar, if the *other* be *divided* by the same. It will follow that the *scalar*, *quadratic*, and *homogeneous function of a vector*, denoted by $\text{S}\rho\phi\rho$, can always be thus expressed:

$$\text{II.} \ldots \text{S}\rho\phi\rho = g\rho^2 + \text{S}\lambda\rho\mu\rho;$$

or thus,

$$\text{II}'. \ldots \text{S}\rho\phi\rho = g'\rho^2 + 2\text{S}\lambda\rho\text{S}\mu\rho, \quad \text{if} \quad g' = g - \text{S}\lambda\mu;$$

a *general* and (as above remarked) *definite transformation*, which is found to be one of great utility in the theory of *Surfaces** of the *Second Order*.

(1.) Attending first to the *case* of *self-conjugate* functions $\phi_0\rho$, from which we can pass to the *general* case by merely adding the *term* $\text{V}\gamma\rho$, and supposing (in virtue of what precedes) that $\alpha_1\alpha_2\alpha_3$ are three *real* and *rectangular vector-units*, and $c_1c_2c_3$ three *real scalars* (the roots of the cubic $M_0 = 0$), such that

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* In the theory of such surfaces, the two constant and real vectors, λ and μ, have the directions of what are called the *cyclic normals*.

III. . . $\phi_1\alpha_1 = (\phi_0 + c_1)\alpha_1 = 0$, $\phi_2\alpha_2 = (\phi_0 + c_2)\alpha_2 = 0$, $\phi_3\alpha_3 = (\phi_0 + c_3)\alpha_3 = 0$,
we may write

$$\text{IV.} \ldots \rho = -(\alpha_1 S\alpha_1\rho + \alpha_2 S\alpha_2\rho + \alpha_3 S\alpha_3\rho),$$

and therefore

$$\text{V.} \ldots \phi_0\rho = c_1\alpha_1 S\alpha_1\rho + c_2\alpha_2 S\alpha_2\rho + c_3\alpha_3 S\alpha_3\rho;$$

so that

$$\text{VI.} \ldots \begin{cases} \phi_1\rho = (c_2 - c_1)\alpha_2 S\alpha_2\rho + (c_3 - c_1)\alpha_3 S\alpha_3\rho, \\ \phi_2\rho = (c_3 - c_2)\alpha_3 S\alpha_3\rho + (c_1 - c_2)\alpha_1 S\alpha_1\rho, \\ \phi_3\rho = (c_1 - c_3)\alpha_1 S\alpha_1\rho + (c_2 - c_3)\alpha_2 S\alpha_2\rho, \end{cases}$$

the *binomial forms* of ϕ_1, ϕ_2, ϕ_3 being thus put in evidence.

(2.) We have thus the general but *scalar* expressions:

$$\text{VII.} \ldots -\rho^2 = (S\alpha_1\rho)^2 + (S\alpha_2\rho)^2 + (S\alpha_3\rho)^2;$$

$$\begin{aligned} \text{VIII.} \ldots S\rho\phi\rho = S\rho\phi_0\rho &= c_1(S\alpha_1\rho)^2 + c_2(S\alpha_2\rho)^2 + c_3(S\alpha_3\rho)^2 \\ &= -c_1\rho^2 + (c_2 - c_1)(S\alpha_2\rho)^2 + (c_3 - c_1)(S\alpha_3\rho)^2 \\ &= -c_2\rho^2 - (c_2 - c_1)(S\alpha_1\rho)^2 + (c_3 - c_2)(S\alpha_3\rho)^2 \\ &= -c_3\rho^2 - (c_3 - c_1)(S\alpha_1\rho)^2 - (c_3 - c_2)(S\alpha_2\rho)^2 : \end{aligned}$$

in which it is in general permitted to assume that

$$\text{IX.} \ldots c_1 < c_2 < c_3, \quad \text{or that} \quad \text{X.} \ldots c_2 - c_1 = 2e^2, \quad c_3 - c_2 = 2e'^2,$$

e and e' being *real* scalars, and the numerical *coefficients* being introduced for a motive of convenience which will presently appear.

(3.) Comparing the last but one of the expressions VIII. with II'., we see that we may bring $S\rho\phi\rho$ to the proposed form II., by assuming,

$$\text{XI.} \ldots \lambda = e\alpha_1 + e'\alpha_3, \quad \mu = -e\alpha_1 + e'\alpha_3, \quad g = S\lambda\mu - c_2 = -\tfrac{1}{2}(c_1 + c_3),$$

because $S\lambda\mu = e^2 - e'^2 = c_2 - \frac{1}{2}(c_1 + c_3)$.

(4.) But in general (comp. 349, (4.)) we cannot have, for *all* values of ρ,

$$\text{XII.} \ldots S\rho\phi\rho = S\rho\phi'\rho, \quad \text{unless} \quad \text{XIII.} \ldots \phi_0\rho = \phi'_0\rho,$$

that is, unless the *self-conjugate parts* of ϕ and ϕ' be *equal*; we can therefore *infer* from II. that $\phi_0\rho = g\rho + V\lambda\rho\mu$, because $V\lambda\rho\mu = V\mu\rho\lambda$ = its own conjugate; and thus the transformation I. is proved to be *possible*, and *real*.

(5.) Accordingly, with the values XI. of λ, μ, g, the expression,

$$\text{XIV.} \ldots \phi_0\rho = g\rho + V\lambda\rho\mu = \rho(g - S\lambda\mu) + \lambda S\mu\rho + \mu S\lambda\rho,$$

becomes,

$$\begin{aligned} \text{XV.} \ldots \phi_0\rho &= -c_2\rho + (e'\alpha_3 + e\alpha_1) S(e'\alpha_3 - e\alpha_1)\rho + (e'\alpha_3 - e\alpha_1) S(e'\alpha_3 + e\alpha_1)\rho \\ &= -c_2\rho - 2e^2\alpha_1 S\alpha_1\rho + 2e'^2\alpha_3 S\alpha_3\rho; \end{aligned}$$

which agrees, by X., with VI.

(6.) Conversely if g, λ, and μ be constants such that $\phi_0\rho = g\rho + V\lambda\rho\mu$, then $\phi_0 V\lambda\mu = g'V\lambda\mu$, where $g' = g - S\lambda\mu$, as before; hence $-g'$ must be one of the three roots c_1, c_2, c_3 of the cubic $M_0 = 0$, and the normal to the plane of λ, μ must have one of the three directions of α_1, α_2, α_3; if then we assume, on trial, that this plane is that of α_1, α_3, and write accordingly,

$$\text{XVI.} \ldots \lambda = a\alpha_1 + a'\alpha_3, \quad \mu = b\alpha_1 + b'\alpha_3, \quad \phi_2\rho = \lambda S\mu\rho + \mu S\lambda\rho,$$

we are, by VI., to seek for scalars $aa'bb'$ which shall satisfy the three conditions,

$$\text{XVII.} \ldots 2ab = c_1 - c_2, \quad 2a'b' = c_3 - c_2, \quad ab' + ba' = 0;$$

but these give

$$\text{XVIII.} \ldots (2ab')^2 = (2ba')^2 = (c_3 - c_2)(c_2 - c_1),$$

so that if the transformation is to be a *real* one, we must suppose that $c_2 - c_1$ and $c_3 - c_2$ are either *both positive*, as in IX., or else *both negative*; or in other words we must so *arrange* the three real roots of the cubic, that c_2 may be (algebraically) *intermediate* in value between the other two. Adopting then the *order* IX., with the values X., we satisfy the conditions XVII. by supposing that

$$\text{XIX.} \ldots a' = b' = e', \quad a = -b = e;$$

and are thus led back from XVI. to the expressions XI., as the *only real ones* for λ, μ, and g which render possible the transformations I. and II.; except that λ and μ may be *interchanged*, &c., as before.

(7.) We see, however, that *in an imaginary sense* there exist *two other solutions* of the problem, to *transform* $\phi\rho$ and $S\rho\phi\rho$ as above; for if we retain the order IX., and equate g' in II′. to either $-c_1$ or $-c_3$, we may in each case conceive the corresponding *sum of two squares* in VIII. as being the *product* of *two imaginary* but *linear factors*; the *planes* of the two *imaginary pairs* of vectors which result being *real*, and perpendicular respectively to α_1 and α_3.

(8.) And if the real expression XIV. for $\phi_0\rho$ be *given*, and it be required to pass from it to the expression V., with the order of inequality IX., the investigation in 354, (12.) enables us at once to establish the formulæ:

$$\text{XX.} \ldots c_1 = -g - T\lambda\mu, \quad c_2 = -g + S\lambda\mu, \quad c_3 = -g + T\lambda\mu;$$
$$\text{XXI.} \ldots \alpha_1 = U(\lambda T\mu - \mu T\lambda), \quad \alpha_2 = UV\lambda\mu, \quad \alpha_3 = U(\lambda T\mu + \mu T\lambda);$$

in which however it is permitted to change the sign of any one of the three vector units. Accordingly the expressions XI. give,

$$T\lambda\mu + S\lambda\mu = 2e^2 = c_2 - c_1, \quad T\lambda\mu - S\lambda\mu = 2e'^2 = c_3 - c_2, \quad S\lambda\mu = g + c_2;$$
$$T\lambda = T\mu, \quad \lambda - \mu = 2e\alpha_1, \quad V\lambda\mu = -2ee'\alpha_3\alpha_1 = \mp 2ee'\alpha_2, \quad \lambda + \mu = 2e'\alpha_3.$$

(9.) We have also the two *identical* transformations,

$$\text{XXII.} \ldots S\lambda\rho\mu\rho = \rho^2 T\lambda\mu + \{(S\lambda\mu\rho)^2 + (S\lambda\rho T\mu + S\mu\rho T\lambda)^2\}(T\lambda\mu - S\lambda\mu)^{-1},$$
$$\text{XXIII.} \ldots S\lambda\rho\mu\rho = -\rho^2 T\lambda\mu - \{(S\lambda\mu\rho)^2 + (S\lambda\rho T\mu - S\mu\rho T\lambda)^2\}(T\lambda\mu + S\lambda\mu)^{-1},$$

which hold good for *any three vectors*, λ, μ, ρ, and may (among other ways) be deduced, through the expressions XX. and XXI., from II. and VIII.

(10.) Finally, as regards the expressions VI. for $\phi_1\rho$, &c., if we denote the corresponding forms of $\psi\rho$ by $\psi_1\rho$, &c., we have (comp. 354, (15.)) these other expressions, which are as usual (comp. 351, &c.) of *monomial form*:

$$\text{XXIV.} \ldots \begin{cases} \psi_1\rho = \phi_2\phi_3\rho = (c_2 - c_1)(c_1 - c_3)\alpha_1 S\alpha_1\rho; \\ \psi_2\rho = \phi_3\phi_1\rho = (c_3 - c_2)(c_2 - c_1)\alpha_2 S\alpha_2\rho; \\ \psi_3\rho = \phi_1\phi_2\rho = (c_1 - c_3)(c_3 - c_2)\alpha_3 S\alpha_3\rho; \end{cases}$$

and which verify the relations 354, XLI., and several other parts of the whole foregoing theory.

358. The *general linear* and *vector function* $\phi\rho$ of a *vector* has been seen (347, (1.)) to contain, at least implicitly, *nine scalar constants*; and accordingly the expression 357, I. involves that number, namely *four* in the term $Vq_0\rho$, on account of the constant *quaternion* q_0, and *five* in the other term $V\lambda\rho\mu$, *each* of the *two unit-vectors*, $U\lambda$ and $U\mu$, counting as *two scalars*, and the *tensor* $T\lambda\mu$ as *one more*. But a *self-*

conjugate linear and vector function, or the *self-conjugate part* $\phi_0\rho$ of the *general* function $\phi\rho$, involves only *six* scalar constants; either because *three* disappear with the *term* $V\gamma\rho$ of $\phi\rho$; or because the *condition of self-conjugation*, $\Sigma V\beta\alpha = 2\gamma = 0$ (comp. 349, XXII. and 353, XXXVI.), which arises when we take for $\phi\rho$ the form $\Sigma\beta S\alpha\rho$ (347, XXXI.), is equivalent to a system of *three scalar equations*, connecting the *nine constants*. And for the same reason the *general quadratic* but *scalar* function, $S\rho\phi\rho$, involves in like manner only *six* scalar constants. Accordingly there enter only six such constants into the expressions 357, II., II′., V., VIII., XIV.; c_1, c_2, c_3, for instance, being *three* such, and the rectangular unit system α_1, α_2, α_3 answering to *three others*. The following *other general transformations* of $S\rho\phi\rho$ and $\phi_0\rho$, although not quite so simple as 357, II. and XIV., involve the *same number* (*six*) of scalar constants, and deserve to be briefly considered: namely the forms,

$$\text{I.} \ldots S\rho\phi\rho = a(V\alpha\rho)^2 + b(S\beta\rho)^2;$$

$$\text{II.} \ldots \phi_0\rho = -a\alpha V\alpha\rho + b\beta S\beta\rho;$$

in which a, b are two real scalars, and α, β are two real unit-vectors. We shall merely set down the leading formulæ, leaving the reader to supply the analysis, which at this stage he cannot find difficult.

(1.) In accomplishing the reduction of the expressions,

$$S\rho\phi\rho = c_1(S\alpha_1\rho)^2 + c_2(S\alpha_2\rho)^2 + c_3(S\alpha_3\rho)^2, \qquad 357, \text{VIII.}$$

and

$$\phi_0\rho = c_1\alpha_1 S\alpha_1\rho + c_2\alpha_2 S\alpha_2\rho + c_3\alpha_3 S\alpha_3\rho, \qquad 357, \text{V.},$$

to these new forms I. and II., it is found that, if the result is to be a *real* one, $-a$ must be *that* root of the scalar cubic $M_0 = 0$, the *reciprocal* of which is algebraically *intermediate*, between the reciprocals of the other two. It is therefore convenient *here* to assume this *new condition*, respecting the *order* of the *inequalities*,

$$\text{III.} \ldots c_1^{-1} > c_2^{-1} > c_3^{-1};$$

which will indeed *coincide* with the arrangement 357, IX., if the three roots c_1, c_2, c_3, be *all positive*, but will be *incompatible* with it in every *other* case.

(2.) This being laid down (or even, if we choose, the *opposite* order being taken), the (real) values of a, b; α, β may be thus expressed:

$$\text{IV.} \ldots a = -c_2, \quad b = c_1 - c_2 + c_3;$$

$$\text{V.} \ldots \alpha = x\alpha_1 + z\alpha_3, \quad \beta = x'\alpha_1 + z'\alpha_3;$$

in which

$$\text{VI.} \ldots x^2 = \frac{c_1^{-1} - c_2^{-1}}{c_1^{-1} - c_3^{-1}}, \quad z^2 = \frac{c_2^{-1} - c_3^{-1}}{c_1^{-1} - c_3^{-1}};$$

$$\text{VII.} \ldots \frac{c_1 x}{x'} = \frac{c_3 z}{z'} = b(xx' + zz') = -bS\alpha\beta = (\text{say})\ b';$$

$$\text{VIII.}\ldots b'^2 = c_1c_2^{-1}c_3b = c_1^2x^2 + c_3^2z^2; \qquad \text{IX.}\ldots x^2 + y^2 = x'^2 + y'^2 = 1;$$

$$\text{X.}\ldots bx'z' = c_2xz;$$

$$\text{XI.}\ldots c_1x^2 + c_3z^2 = c_1c_2^{-1}c_3 = b^{-1}b'^2 = b(\mathrm{S}\alpha\beta)^2, \quad c_1c_3 = -ab(\mathrm{S}\alpha\beta)^2;$$

$$\text{XII.}\ldots b'\beta = -b\beta\mathrm{S}\alpha\beta = c_1x\alpha_1 + c_3z\alpha_3; \text{ \&c.}$$

(3.) And there result the transformations:

$$\text{XIII.}\ldots \phi_2\rho = (c_1 - c_2)\alpha_1\mathrm{S}\alpha_1\rho + (c_3 - c_2)\alpha_3\mathrm{S}\alpha_3\rho$$
$$= -c_2(x\alpha_1 + z\alpha_3)\,\mathrm{S}(x\alpha_1 + z\alpha_3)\rho + \frac{c_2}{c_1c_3}(xc_1\alpha_1 + zc_3\alpha_3)\,\mathrm{S}(xc_1\alpha_1 + zc_3\alpha_3)\rho;$$

$$\text{XIV.}\ldots \phi_0\rho = c_1\alpha_1\mathrm{S}\alpha_1\rho + c_2\alpha_2\mathrm{S}\alpha_2\rho + c_3\alpha_3\mathrm{S}\alpha_3\rho$$
$$= c_2(x\alpha_1 + z\alpha_3)\,\mathrm{V}(x\alpha_1 + z\alpha_3)\rho + \frac{c_2}{c_1c_3}(xc_1\alpha_1 + zc_3\alpha_3)\,\mathrm{S}(xc_1\alpha_1 + zc_3\alpha_3)\rho;$$

$$\text{XV.}\ldots \mathrm{S}\rho\phi\rho = -c_2(\mathrm{V}(x\alpha_1 + z\alpha_3)\rho)^2 + \frac{c_2}{c_1c_3}(\mathrm{S}(xc_1\alpha_1 + zc_3\alpha_3)\rho)^2;$$

which last, *if* c_1c_3 be *positive*, gives this *other real form*,

$$\text{XVI.}\ldots \mathrm{S}\rho\phi\rho = \frac{c_2}{c_1c_3}\mathrm{N}\{\mathrm{S}(xc_1\alpha_1 + zc_3\alpha_3)\rho + (c_1c_3)^{\frac{1}{2}}\,\mathrm{V}(x\alpha_1 + z\alpha_3)\rho\};$$

x^2 and z^2 being determined by the expressions VI.

(4.) Those expressions allow us to change the *sign* of $z : x$, and thereby to determine a *second pair* of *real unit lines*, α' and β', which may be substituted for α and β in the forms I. and II.; the order of inequalities III. (or the opposite order), and the values IV. of a and b, remaining unchanged. We have therefore the *double transformations:*

$$\text{XVII.}\ldots \mathrm{S}\rho\phi\rho = -c_2(\mathrm{V}\alpha\rho)^2 + (c_1 - c_2 + c_3)(\mathrm{S}\beta\rho)^2 = -c_2(\mathrm{V}\alpha'\rho)^2$$
$$+ (c_1 - c_2 + c_3)(\mathrm{S}\beta'\rho)^2;$$

$$\text{XVIII.}\ldots \phi_0\rho = c_2\alpha\mathrm{V}\alpha\rho + (c_1 - c_2 + c_3)\beta\mathrm{S}\beta\rho = c_2\alpha'\mathrm{V}\alpha'\rho + (c_1 - c_2 + c_3)\beta'\mathrm{S}\beta'\rho.$$

(5.) If either of the two connected *forms* I. and II. had been *given*, we might have proposed to *deduce* from it the values of $c_1c_2c_3$, and of $\alpha_1\alpha_2\alpha_3$, by the *general method* of this Section. We should thus have had the cubic,

$$\text{XIX.}\ldots 0 = M_0 = (c + a)\{c^2 + (a - b)c - ab(\mathrm{S}\alpha\beta)^2\};$$

and because the quadratic $(c + a)^{-1}M_0 = 0$ may be thus written,

$$\text{XX.}\ldots (c^{-1} + a^{-1})^2(\mathrm{S}\alpha\beta)^2 - (c^{-1} + a^{-1})(a^{-1}\mathrm{S}.(\alpha\beta)^2 + b^{-1}) + a^{-2}(\mathrm{V}\alpha\beta)^2 = 0,$$

it gives two real values of $c^{-1} + a^{-1}$, one positive and the other negative; if then we *arrange* the reciprocals of the three roots of $M_0 = 0$ in the order III., we have the expressions,

$$\text{XXI.}\ldots \begin{cases} c_1 = \frac{1}{2}(b - a) + \frac{1}{2}ab\sqrt{(a^{-2} + 2a^{-1}b^{-1}\mathrm{S}.(\alpha\beta)^2 + b^{-2})}; \quad c_2 = -a; \\ c_3 = \frac{1}{2}(b - a) - \frac{1}{2}ab\sqrt{(a^{-2} + 2a^{-1}b^{-1}\mathrm{S}.(\alpha\beta)^2 + b^{-2})}; \end{cases}$$

the signs of the radical being determined by the condition that $(c_1 - c_3) : ab(\mathrm{S}\alpha\beta)^2 = c_1^{-1} - c_3^{-1} > 0$. Accordingly these expressions for the roots agree evidently with the former results, IV. and XI., because $\mathrm{S}.(\alpha\beta)^2 = 2(\mathrm{S}\alpha\beta)^2 - 1$.

(6.) The *roots* c_1, c_2, c_3 being thus known, the same general method gives for the *directions* of α_1, α_2, α_3 the *versors* of the following expressions (or of their negatives):

$$\text{XXII.}\ldots\begin{cases}\psi_1\rho = ac_3^{-1}(c_3\alpha + b\beta\mathrm{S}\alpha\beta)\,\mathrm{S}(c_3\alpha + b\beta\mathrm{S}\alpha\beta)\rho\,; \\ \psi_2\rho = ab\mathrm{V}\alpha\beta\mathrm{S}\beta\alpha\rho\,; \\ \psi_3\rho = ac_1^{-1}(c_1\alpha + b\beta\mathrm{S}\alpha\beta)\,\mathrm{S}(c_1\alpha + b\beta\mathrm{S}\alpha\beta)\rho\,;\end{cases}$$

of which the *monomial forms* may again be noted, and which give,

$$\text{XXII}'.\ldots\ \alpha_1 = \pm\,\mathrm{U}(c_3\alpha + b\beta\mathrm{S}\alpha\beta), \quad \alpha_2 = \pm\,\mathrm{UV}\alpha\beta, \quad \alpha_3 = \pm\,\mathrm{U}(c_1\alpha + b\beta\mathrm{S}\alpha\beta).$$

(7.) Accordingly the expressions in (2.), give (if we suppose $\alpha_3\alpha_1 = +\alpha_2$),

$$\text{XXIII.}\ldots\ c_3\alpha + b\beta\mathrm{S}\alpha\beta = (c_3 - c_1)x\alpha_1, \quad \mathrm{V}\alpha\beta = (x'z - xz')\,\alpha_2, \quad c_1\alpha + b\beta\mathrm{S}\alpha\beta = (c_1 - c_3)z\alpha_3\,;$$

and as an additional verification of the *consistency* of the various parts of this whole theory, it may be observed (comp. 357, XXIV.), that

$$\text{XXIV.}\ldots -ac_3^{-1}(c_3\alpha + b\beta\mathrm{S}\alpha\beta)^2 = (c_2 - c_1)\,(c_1 - c_3), \quad ab(\mathrm{V}\alpha\beta)^2 = (c_3 - c_2)\,(c_2 - c_1), \quad -ac_1^{-1}(c_1\alpha + b\beta\mathrm{S}\alpha\beta)^2 = (c_1 - c_3)\,(c_3 - c_2).$$

(8.) As regards the *second transformations*, XVII. and XVIII., it is easy to prove that we may write,

$$\text{XXV.}\ldots\ (c_3 - c_1)\alpha' = b\beta\alpha\beta - a\alpha, \quad (c_3 - c_1)\beta' = a\alpha\beta\alpha - b\beta,$$

$$\text{XXVI.}\ldots -(c_3 - c_1)^2 = (b\beta\alpha\beta - a\alpha)^2 = (a\alpha\beta\alpha - b\beta)^2\,;$$

so that we have the following equation,

$$\text{XXVII.}\ldots\ (a(\mathrm{V}\alpha\rho)^2 + b(\mathrm{S}\beta\rho)^2)\,(a^2 + 2ab\mathrm{S}.(\alpha\beta)^2 + b^2) = a\,(\mathrm{V}(b\beta\alpha\beta - a\alpha)\rho)^2 + b\,(\mathrm{S}(a\alpha\beta\alpha - b\beta)\rho)^2,$$

which is true for any vector ρ, any two unit lines α, β, and any two scalars a, b.

(9.) Accordingly it is evident from (4.), that α_1, α_3 must be the bisectors of the angles made by α, α', and also of those made by β, β'; and the expressions XXV. may be thus written (because $b - a = c_1 + c_3$),

$$\text{XXVIII.}\ldots\ (c_3 - c_1)\alpha' = (c_3 + c_1)\alpha + 2b\beta\mathrm{S}\alpha\beta, \quad (c_1 - c_3)\beta' = (c_1 + c_3)\beta - 2a\alpha\mathrm{S}\alpha\beta\,;$$

whence, by XXIII., we may write,

$$\text{XXIX.}\ldots\ \alpha + \alpha' = 2x\alpha_1, \quad \alpha - \alpha' = 2z\alpha_3\,;$$

so that α_1 bisects the internal angle, and α_3 the external angle, of the lines α, α'.

(10.) At the same time we have these other expressions,

$$\text{XXX.}\ldots\ (c_1 - c_3)(\beta + \beta') = 2(c_1\beta - a\alpha\mathrm{S}\alpha\beta), \quad (c_3 - c_1)\,(\beta - \beta') = 2(c_3\beta - a\alpha\mathrm{S}\alpha\beta)\,;$$

which can easily be reduced to the simple forms,

$$\text{XXXI.}\ldots\ \beta + \beta' = 2x'\alpha_1, \quad \beta - \beta' = 2z'\alpha_3,$$

with the recent meanings of the coefficients x' and z'.

(11.) And although, for the sake of obtaining *real transformations*, we have supposed (comp. III.) that

$$\text{XXXII.}\ldots\ (c_1^{-1} - c_2^{-1})\,(c_2^{-1} - c_3^{-1}) > 0,$$

because the assumed relation $\alpha = x\alpha_1 + z\alpha_3$ between the three unit vectors $\alpha\alpha_1\alpha_3$, whereof the two latter are rectangular, gives $x^2 + z^2 = 1$, as in IX., so that each of the two expressions VI. involves the other, and their comparison gives the ratio,

$$\text{XXXIII.}\ldots\ x^2 : z^2 = (c_1^{-1} - c_2^{-1}) : (c_2^{-1} - c_3^{-1}),$$

yet we see that, ***without*** this inequality XXXII. existing, the foregoing transformations hold good in an ***imaginary*** (or merely ***symbolical***) ***sense*** : so that we may say, in general, that the functions $S\rho\phi\rho$ and $\phi_0\rho$ can be brought to the ***forms*** I. and II. in ***six distinct ways***, whereof ***two*** are ***real***, and the ***four others*** are ***imaginary***.

(12.) It may be added that the first equation XXII. admits of being replaced by the following,

$$\text{XXXIV.} \ldots \psi_1\rho = -bc_1^{-1}(c_1\beta - a\alpha S\alpha\beta)\,S(c_1\beta - a\alpha S\alpha\beta)\,\rho,$$

with a corresponding form for $\psi_3\rho$; and that thus, instead of XXII'., we are at liberty to write the expressions,

$$\text{XXXV.} \ldots \alpha_1 = U(c_1\beta - a\alpha S\alpha\beta), \quad \alpha_2 = UV\alpha\beta, \quad \alpha_3 = U(c_3\beta - a\alpha S\alpha\beta),$$

for the rectangular unit system, deduced from I. or II.

359. If we call, as we naturally may, the expressions

$$\text{I.} \ldots \phi_0\rho = c_1\alpha_1 S\alpha_1\rho + c_2\alpha_2 S\alpha_2\rho + c_3\alpha_3 S\alpha_3\rho, \qquad 357,\ \text{V.},$$

and

$$\text{II.} \ldots S\rho\phi\rho = c_1(S\alpha_1\rho)^2 + c_2(S\alpha_2\rho)^2 + c_3(S\alpha_3\rho)^2, \qquad 357,\ \text{VIII.},$$

the *Rectangular Transformations* of the *Functions* $\phi_0\rho$ and $S\rho\phi\rho$, then by another *geometrical analogy*, which will be seen when we come to speak briefly of the theory of *Surfaces of the Second Order*, we may call the expressions,

$$\text{III.} \ldots \phi_0\rho = g\rho + V\lambda\rho\mu, \qquad 357,\ \text{XIV.},$$

and

$$\text{IV.} \ldots S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho, \qquad 357,\ \text{II.},$$

the *Cyclic** *Transformations* of the same two functions; and may say that the two other and more recent expressions,

$$\text{V.} \ldots \phi_0\rho = -a\alpha V\alpha\rho + b\beta S\beta\rho, \qquad 358,\ \text{II.},$$

and

$$\text{VI.} \ldots S\rho\phi\rho = a(V\alpha\rho)^2 + b(S\beta\rho)^2, \qquad 358,\ \text{I.},$$

are *Focal*† *Transformations* of the same. We have already shown (357) how to exchange *rectangular forms* with *cyclic* ones; and also (358) how to pass from *rectangular* expressions to *focal* ones, and reciprocally: but it may be worth while to consider briefly the mutual relations which exist, between *cyclic* and *focal* expressions, and the modes of passing from either to the other.

(1.) To pass from IV. to VI., or from the *cyclic* to the *focal* form, we may first accomplish the *rectangular* transformation II., with the values 357, XX., and XXI., of c_1, c_2, c_3, and of α_1, α_2, α_3, the order of inequality being assumed to be

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* Compare the Note to Art. 357.

† It will be found that the ***two real vectors*** α, α', of 358, are the ***two real focal lines*** of the ***real or imaginary cone***, which is ***asymptotic*** to the ***surface of the second order***, $S\rho\phi\rho = \text{const.}$

$$\text{VII.} \ldots c_3 > c_2 > c_1, \qquad \text{as in 357, IX.};$$

and then shall have (comp. 358, XV.) the following expressions:

$$\text{VIII.} \ldots 4S\rho\phi\rho = \{S.\rho(c_1^{\frac{1}{2}}(U\lambda - U\mu) + c_3^{\frac{1}{2}}(U\lambda + U\mu))\}^2 - \{V.\rho(c_1^{\frac{1}{2}}(U\lambda + U\mu) + c_3^{\frac{1}{2}}(U\lambda - U\mu))\}^2;$$

$$\text{VIII}'. \ldots 4S\rho\phi\rho = - \{S.\rho((-c_1)^{\frac{1}{2}}(U\lambda - U\mu) + (-c_3)^{\frac{1}{2}}(U\lambda + U\mu))\}^2 + \{V.\rho((-c_1)^{\frac{1}{2}}(U\lambda + U\mu) + (-c_3)^{\frac{1}{2}}(U\lambda - U\mu))\}^2;$$

$$\text{IX.} \ldots (c_3 - c_2)^2 S\rho\phi\rho = \{V.\rho(c_3^{\frac{1}{2}} V\lambda\mu + (-c_2)^{\frac{1}{2}}(\lambda T\mu + \mu T\lambda))\}^2 + \{S.\rho((-c_2)^{\frac{1}{2}} V\lambda\mu - c_3^{\frac{1}{2}}(\lambda T\mu + \mu T\lambda))\}^2;$$

$$\text{X.} \ldots (c_2 - c_1)^2 S\rho\phi\rho = - \{V.\rho((-c_1)^{\frac{1}{2}} V\lambda\mu + c_2^{\frac{1}{2}}(\lambda T\mu - \mu T\lambda))\}^2 - \{S.\rho(-c_2^{\frac{1}{2}} V\lambda\mu + (-c_1)^{\frac{1}{2}}(\lambda T\mu - \mu T\lambda))\}^2;$$

in which it is to be remembered that (by 357, XX.),

$$\text{XI.} \ldots c_1 = -g - T\lambda\mu, \quad c_2 = -g + S\lambda\mu, \quad c_3 = -g - T\lambda\mu;$$

and of which *all* are *symbolically true*, or give (as in IV.) the *real value* $g\rho^2 + S\lambda\rho\mu\rho$ for $S\rho\phi\rho$, if g, λ, μ, ρ be *real*. And in *this* symbolical *sense*, although they have been written down as *four*, they only *count* as *three* distinct *focal transformations*, of a *given* and *real cyclic form;* because the expression VIII′. is an *immediate* consequence of VIII.; and other formulæ IX′. and X′. might in like manner be at once derived from IX. and X.

(2.) But if we wish to confine ourselves to *real focal forms*, there are then *four cases* to be considered, in *each* of which *some one* of the *four equations* VIII. VIII′. IX. X. is to be adopted, to the *exclusion* of the *other three*. Thus,

if $\text{XII.} \ldots c_3 > c_2 > c_1 > 0$, and therefore $c_1^{-1} > c_2^{-1} > c_3^{-1} > 0$,

the *form* VIII. is the *only real* one. If

$\text{XIII.} \ldots c_3 > c_2 > 0 > c_1$, $c_2^{-1} > c_3^{-1} > 0 > c_1^{-1}$, then X. is the real form.

If $\text{XIV.} \ldots c_3 > 0 > c_2 > c_1$, $c_3^{-1} > 0 > c_1^{-1} > c_2^{-1}$, the only real form is IX.

Finally if $\text{XV.} \ldots 0 > c_3 > c_2 > c_1$, $0 > c_1^{-1} > c_2^{-1} > c_3^{-1}$,

that is, if *all* the roots of the cubic $M_0 = 0$ be *negative*, then VIII′. is the form to be adopted, under the same condition of *reality*.

(3.) When *all* the roots c are *positive*, or in the case when VIII. is the *real focal form*, the unit lines α, β in VI. may be thus expressed:

$$\text{XVI.} \ldots \begin{cases} \alpha = \frac{1}{2}\left(\frac{c_3}{c_2}\right)^{\frac{1}{2}}(U\lambda - U\mu) + \frac{1}{2}\left(\frac{c_1}{c_2}\right)^{\frac{1}{2}}(U\lambda + U\mu); \\ \beta = \frac{1}{2}\left(\frac{c_1}{b}\right)^{\frac{1}{2}}(U\lambda - U\mu) + \frac{1}{2}\left(\frac{c_3}{b}\right)^{\frac{1}{2}}(U\lambda + U\mu); \end{cases}$$

with $b = c_1 - c_1 + c_3$ as before (358, IV.).

(4.) In the same case VIII., the expressions for $4S\rho\phi\rho$ may be written (comp. 358, XVI.) under either of these two *other real forms:*

$$\text{XVII.} \ldots 4S\rho\phi\rho = N\{(c_3^{\frac{1}{2}} + c_1^{\frac{1}{2}})\rho.U\lambda + (c_3^{\frac{1}{2}} - c_1^{\frac{1}{2}})U\mu.\rho\};$$

$$\text{XVII}'. \ldots 4S\rho\phi\rho = N\{(c_3^{\frac{1}{2}} + c_1^{\frac{1}{2}})U\lambda.\rho + (c_3^{\frac{1}{2}} - c_1^{\frac{1}{2}})\rho.U\mu\};$$

so that if we write, for abridgment,

$$\text{XVIII.} \ldots \iota_0 = \tfrac{1}{2}(c_3^{\frac{1}{2}} + c_1^{\frac{1}{2}})U\lambda, \quad \kappa_0 = \tfrac{1}{2}(c_3^{\frac{1}{2}} - c_1^{\frac{1}{2}})U\mu,$$

we shall have, briefly,

$$\text{XIX.} \ldots S\rho\phi\rho = N(\iota_0\rho + \rho\kappa_0) = N(\rho\iota_0 + \kappa_0\rho).$$

(5.) Or we may make

XX. . . $\iota = \frac{1}{2}(c_1^{-\frac{1}{2}} + c_3^{-\frac{1}{2}})\mathrm{U}\lambda, \quad \kappa = \frac{1}{2}(c_1^{-\frac{1}{2}} - c_3^{-\frac{1}{2}})\mathrm{U}\mu,$ whence $\kappa^2 - \iota^2 = c_1^{-\frac{1}{2}} c_3^{-\frac{1}{2}}$;

and shall then have the transformation,

$$\text{XXI.} \ldots \mathrm{S}\rho\phi\rho = \mathrm{N}\frac{\iota\rho + \rho\kappa}{\kappa^2 - \iota^2},$$

which may be compared with the equation 281, XXIX. of the *ellipsoid*, and for the *reality* of which form, or of its two *vector constants*, ι, κ, it is necessary that the roots c of the cubic should all be *positive* as above.

(6.) It was lately shown (in 358, (8.), &c.), how to pass from a *given* and *real focal form* to a *second* of the same kind, with its *new real unit lines* α', β' in the *same plane* as the two *old* or *given lines*, α, β; but we have not yet shown how to pass from a *focal* form to a *cyclic* one, although the *converse* passage has been recently discussed. Let us then now suppose that the *form* VI. is *real* and *given*, or that the two scalar constants a, b, and the two unit vectors α, β, have real and given values; and let us seek to reduce this expression VI. to the earlier form IV.

(7.) We might, for this purpose, begin by assuming that

$$\text{XXII.} \ldots c_1^{-1} > c_2^{-1} > c_3^{-1}, \text{ as in 358, III.};$$

which would give the expressions 358, XXI. and XXII., for $c_1c_2c_3$ and $\alpha_1\alpha_2\alpha_3$, and so would supply the *rectangular transformation*, from which we could pass, as before, to the *cyclic* one.

(8.) But to vary a little the analysis, let us now suppose that the *given focal form* is some one of the four following (comp. (1.)):

$$\text{XXIII.} \ldots \mathrm{S}\rho\phi\rho = (\mathrm{S}\beta_0\rho)^2 - (\mathrm{V}\alpha_0\rho)^2; \quad \text{XXIII}'. \ldots \mathrm{S}\rho\phi\rho = (\mathrm{V}\alpha_0\rho)^2 - (\mathrm{S}\beta_0\rho)^2;$$
$$\text{XXIV.} \ldots \mathrm{S}\rho\phi\rho = (\mathrm{S}\beta_0\rho)^2 + (\mathrm{V}\alpha_0\rho)^2; \quad \text{XXIV}'. \ldots \mathrm{S}\rho\phi\rho = -(\mathrm{V}\alpha_0\rho)^2 - (\mathrm{S}\beta_0\rho)^2;$$

in each of which α_0 and β_0 are conceived to be *given* and *real vectors*, but *not* generally *unit lines;* and which are in fact the *four cases* included under the *general form*, $a(\mathrm{V}\alpha\rho)^2 + b(\mathrm{S}\beta\rho)^2$, according as the scalars a and b are positive or negative. It will be sufficient to consider the two cases, XXIII. and XXIV., from which the two others will follow at once.

(9.) For the case XXIII. we easily derive the *real cyclic transformation*,

$$\begin{aligned}\text{XXV.} \ldots \mathrm{S}\rho\phi\rho &= (\mathrm{S}\beta_0\rho)^2 - (\mathrm{S}\alpha_0\rho)^2 + \alpha_0^2\rho^2 \\ &= \mathrm{S}(\beta_0 + \alpha_0)\rho \,.\, \mathrm{S}(\beta_0 - \alpha_0)\rho + \alpha_0^2\rho^2 \\ &= g\rho^2 + \mathrm{S}\lambda\rho\mu\rho = (g - \mathrm{S}\lambda\mu)\rho^2 + 2\mathrm{S}\lambda\rho\mathrm{S}\mu\rho,\end{aligned}$$

where XXVI. . . $\lambda = \beta_0 + \alpha_0, \quad \mu = \frac{1}{2}(\beta_0 - \alpha_0), \quad g = \frac{1}{2}(\alpha_0^2 + \beta_0^2)$;

and the equations 357, (9.) enable us to pass thence to the two *imaginary cyclic forms.*

(10.) For example, if the proposed function be (comp. XIX.),

$$\text{XXVII.} \ldots \mathrm{S}\rho\phi\rho = \mathrm{N}(\iota_0\rho + \rho\kappa_0) = (\mathrm{S}(\iota_0 + \kappa_0)\rho)^2 - (\mathrm{V}(\iota_0 - \kappa_0)\rho)^2,$$

we may write

$$\alpha_0 = \iota_0 - \kappa_0, \quad \beta_0 = \iota_0 + \kappa_0, \quad \lambda = 2\iota_0, \quad \mu = \kappa_0, \quad g = \iota_0^2 + \kappa_0^2;$$

and the required transformation is (comp. 336, XI.),

$$\text{XXVIII.} \ldots \mathrm{N}(\iota_0\rho + \rho\kappa_0) = (\iota_0^2 + \kappa_0^2)\rho^2 + 2\mathrm{S}\iota_0\rho\kappa_0\rho.$$

(11.) To treat the case XXIV. by our general method, we may omit for simplicity the subindices $_0$, and write simply (comp. V. and VI.) the expressions,

$$\text{XXIX.}\ldots\ \phi\rho = -\alpha V\alpha\rho + \beta S\beta\rho, \quad \text{and} \quad \text{XXX.}\ldots\ S\rho\phi\rho = (V\alpha\rho)^2 + (S\beta\rho)^2\,;$$

in which however it is to be observed that α and β, though *real vectors*, are *not now unit lines* (8.). Hence because $-\alpha V\alpha\rho = \alpha S\alpha\rho - \alpha^2\rho$, we easily form the expressions:

$$\text{XXXI.}\ldots\ m = \alpha^2(S\alpha\beta)^2, \quad m' = \alpha^2(\alpha^2 - \beta^2) - (S\alpha\beta)^2, \quad m'' = \beta^2 - 2\alpha^2\,;$$

$$\text{XXXII.}\ldots \begin{cases} \psi\rho = V\alpha\beta S\beta\alpha\rho - \alpha^2(\alpha V\alpha\rho + \beta V\beta\rho) + \alpha^4\rho \\ \qquad = V\alpha\rho\beta S\alpha\beta + \alpha(\alpha^2 - \beta^2)S\alpha\rho, \\ \chi\rho = -(\alpha S\alpha\rho + \beta S\beta\rho) + (\beta^2 - \alpha^2)\rho\,; \end{cases}$$

and therefore $\quad \text{XXXIII.}\ldots\ M = (c - \alpha^2)(c^2 + (\beta^2 - \alpha^2)c - (S\alpha\beta)^2),$

and $\quad \text{XXXIV.}\ldots\ \Psi\rho = V\alpha\rho\beta S\alpha\beta + (\beta^2 - \alpha^2)(c\rho - \alpha S\alpha\rho) - c(\alpha S\alpha\rho + \beta S\beta\rho) + c^2\rho$

$$= (\alpha(\alpha^2 - \beta^2 - c) + \beta S\alpha\beta)S\alpha\rho + (\alpha S\alpha\beta - c\beta)S\beta\rho + (c^2 + (\beta^2 - \alpha^2)c - (S\alpha\beta)^2)\rho.$$

(12.) Introducing then a real and positive scalar constant, r, such that

$$\begin{aligned} \text{XXXV.}\ldots\ r^4 &= (\alpha^2 - \beta^2)^2 + 4(S\alpha\beta)^2 = (\alpha^2 + \beta^2)^2 + 4(V\alpha\beta)^2 \\ &= \alpha^4 + (\alpha\beta)^2 + (\beta\alpha)^2 + \beta^4 = \alpha^4 + 2S.(\alpha\beta)^2 + \beta^4 \\ &= \alpha^{-2}(\alpha^3 + \beta\alpha\beta)^2 = \beta^{-2}(\beta^3 + \alpha\beta\alpha)^2 = \&c., \end{aligned}$$

in which (by 199, &c.),

$$S.(\alpha\beta)^2 = (S\alpha\beta)^2 + (V\alpha\beta)^2 = 2(S\alpha\beta)^2 - \alpha^2\beta^2 = 2(V\alpha\beta)^2 + \alpha^2\beta^2,$$

the roots of $M = 0$ admit of being expressed as follows:

$$\text{XXXVI.}\ldots\ c_1 = \tfrac{1}{2}(\alpha^2 - \beta^2 + r^2), \quad c_2 = \alpha^2, \quad c_3 = \tfrac{1}{2}(\alpha^2 - \beta^2 - r^2)\,;$$

and when they are thus arranged, we have the inequalities,

$$\text{XXXVII.}\ldots\ c_1 > 0 > c_3 > c_2, \quad c_1^{-1} > 0 > c_2^{-1} > c_3^{-1}.$$

(13.) The corresponding forms of $\Psi\rho$ are the three monomial expressions,

$$\text{XXXVIII.}\ldots \begin{cases} \psi_1\rho = c_3^{-1}(\alpha c_3 + \beta S\alpha\beta)S(\alpha c_3 + \beta S\alpha\beta)\rho, \quad \psi_2\rho = V\alpha\beta S\beta\alpha\rho, \\ \psi_3\rho = c_1^{-1}(\alpha c_1 + \beta S\alpha\beta)S(\alpha c_1 + \beta S\alpha\rho)\rho\,; \end{cases}$$

which may be variously transformed and verified, and give the three following rectangular vector units,

$$\text{XXXIX.}\ldots\ \alpha_1 = U(\alpha c_3 + \beta S\alpha\beta), \quad \alpha_2 = UV\alpha\beta, \quad \alpha_3 = U(\alpha c_1 + \beta S\alpha\beta)\,;$$

in connexion with which it is easy to prove that

$$\text{XL.}\ldots \begin{cases} T(\alpha c_3 + \beta S\alpha\beta) = (-c_3)^{\frac{1}{2}}(c_1 - c_2)^{\frac{1}{2}}(c_1 - c_3)^{\frac{1}{2}} = r(c_1 - c_2)^{\frac{1}{2}}(-c_3)^{\frac{1}{2}}, \\ TV\alpha\beta = \qquad (c_1 - c_2)^{\frac{1}{2}}(c_3 - c_2)^{\frac{1}{2}}\,; \\ T(\alpha c_1 + \beta S\alpha\beta) = c_1^{\frac{1}{2}}(c_3 - c_2)^{\frac{1}{2}}(c_1 - c_3)^{\frac{1}{2}} = r(c_3 - c_2)^{\frac{1}{2}}c_1^{\frac{1}{2}}\,; \end{cases}$$

the radicals being all real, by XXXVII.

(14.) We have thus, for the *given focal form* XXX., the *rectangular transformation*,

$$\begin{aligned} \text{XLI.}\ldots\ S\rho\phi\rho &= (V\alpha\rho)^2 + (S\beta\rho)^2 \\ &= \frac{c_1(S(\alpha c_3 + \beta S\alpha\beta)\rho)^2}{-c_3(c_1 - c_2)r^2} + \frac{c_2(S\alpha\beta\rho)^2}{(c_1 - c_2)(c_3 - c_2)} + \frac{c_3(S(\alpha c_1 + \beta S\alpha\beta)\rho)^2}{c_1(c_3 - c_2)r^2}, \end{aligned}$$

or briefly,

$$\begin{aligned} \text{XLII.}\ldots\ S\rho\phi\rho = (V\alpha\rho)^2 + (S\beta\rho)^2 &= c_1(S.\rho U(\alpha c_3 + \beta S\alpha\beta)\rho)^2 \\ &\quad + \alpha^2(S.\rho UV\alpha\beta)^2 + c_3(S.\rho U(\alpha c_1 + \beta S\alpha\beta))^2\,; \end{aligned}$$

in which the first term is positive, but the two others are negative, and c_1, c_3 are the roots of the quadratic,

$$\text{XLIII.}\ldots\ 0 = c^2 + (\beta^2 - \alpha^2)c - (S\alpha\beta)^2.$$

(15.) We have also the parallelisms,

$$\text{XLIV.} \ldots \alpha c_3 + \beta S\alpha\beta \parallel \beta c_1 - \alpha S\alpha\beta, \quad \alpha c_1 + \beta S\alpha\beta \parallel \beta c_3 - \alpha S\alpha\beta,$$

because $$c_1 c_3 = -(S\alpha\beta)^2;$$

and may therefore write,

$$\text{XLV.} \ldots S\rho\phi\rho = (V\alpha\rho)^2 + (S\beta\rho)^2 = c_1(S.\rho U(\beta c_1 - \alpha S\alpha\beta))^2 \\ + \alpha^2(S.\rho UV\alpha\beta)^2 + c_3(S.\rho U(\beta c_3 - \alpha S\alpha\beta))^2;$$

while

$$\text{XLVI.} \ldots T(\beta c_1 - \alpha S\alpha\beta) = r c_1^{\frac{1}{2}}(c_1 - c_2)^{\frac{1}{2}}, \quad T(\beta c_3 - \alpha S\alpha\beta) = r(-c_3)^{\frac{1}{2}}(c_3 - c_2)^{\frac{1}{2}},$$

and $r = (c_1 - c_3)^{\frac{1}{2}}$, with real radicals as before.

(16.) Multiplying then by $r^2(TV\alpha\beta)^2$, or by $(c_1 - c_2)(c_1 - c_3)(c_3 - c_2)$, we obtain this new equation,

$$\text{XLVII.} \ldots (c_1 - c_3)\{(TV\alpha\beta)^2((V\alpha\rho)^2 + (S\beta\rho)^2) - \alpha^2(S\alpha\beta\rho)^2\} \\ = (c_3 - \alpha^2)(c_1 S\beta\rho - S\alpha\beta S\alpha\rho)^2 - (c_1 - \alpha^2)(c_3 S\beta\rho - \alpha S\alpha\beta)^2;$$

which is only another way of expressing the same rectangular transformation as before, but has the advantage of being freed from *divisors*.

(17.) Developing the second member of XLVII., and dividing by $c_1 - c_3$, we obtain this new transformation:

$$\text{XLVIII.} \ldots (TV\alpha\beta)^2 S\rho\phi\rho = -(V\alpha\beta)^2((V\alpha\rho)^2 + (S\beta\rho)^2) \\ = \alpha^2(S\alpha\beta\rho)^2 - (S\alpha\beta)^2(S\alpha\rho)^2 + 2\alpha^2 S\alpha\beta S\alpha\rho S\beta\rho + C(S\beta\rho)^2;$$

in which we have written for abridgment,

$$\text{XLIX.} \ldots C = c_1 c_3 - \alpha^2(c_1 + c_3).$$

(18.) The expressions XXXVI. for c_1, c_3 give thus,

$$\text{L.} \ldots C = -\alpha^4 - (V\alpha\beta)^2;$$

and accordingly, when this value is substituted for C in XLVIII., that *equation* becomes an *identity*, or holds good for *all values* of the *three vectors*, α, β, ρ; as may be proved* in various ways.

(19.) Admitting this result, we see that for the mere establishment of the equation XLVII., it is *not necessary* that c_1 and c_2 should be roots of the *particular quadratic* XLIII. It is sufficient, for *this* purpose, that they should be roots of *any* quadratic,

$$\text{LI.} \ldots c^2 + Ac + B = 0, \quad \text{with the } \textit{relation} \quad \text{LII.} \ldots A\alpha^2 + B + \alpha^4 + (V\alpha\beta)^2 = 0,$$

between its coefficients. But when we *combine* with this the *condition of rectangularity*, $\alpha_3 \perp \alpha_1$, or

$$\text{LIII.} \ldots 0 = S.(c_1\beta - \alpha S\alpha\beta)(c_3\beta - \alpha S\alpha\beta) = A(S\alpha\beta)^2 + B\beta^2 + \alpha^2(S\alpha\beta)^2,$$

we obtain thus a *second* relation, which gives *definitely*, for the two coefficients, the values,

$$\text{LIV.} \ldots A = \beta^2 - \alpha^2, \quad B = -(S\alpha\beta)^2;$$

and so conducts, in a new way, to the equation XLIII.

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* Many such proofs, or verifications, as the one here alluded to, are purposely left, at this stage, as exercises, to the student.

(20.) In this manner, then, we *might* have been led to perceive the truth of the rectangular transformation XLVII., with the *quadratic* equation XLIII. of which c_1 and c_3 are roots, without having previously found the *cubic* XXXIII., of which the quadratic is a *factor*, and of which the *other* root is $c_2 = a^2$. But if we had not employed the *general method* of the present Section, which conducted us to form *first* that *cubic* equation, there would have been nothing to *suggest* the *particular form* XLVII., which could thus have only been by some sort of *chance* arrived at.

(21.) The values of $\alpha_1\alpha_2\alpha_3$ give also (comp. 357, VII.),

$$\text{LV.} \ldots -\rho^2 = (\mathrm{S}.\rho\mathrm{U}(\beta c_1 - \alpha\mathrm{S}\alpha\beta))^2 + (\mathrm{S}.\rho\mathrm{UV}\alpha\beta)^2 + (\mathrm{S}.\rho\mathrm{U}(\beta c_3 - \alpha\mathrm{S}\alpha\beta))^2;$$

that is, by XL. and XLVI.,

$$\begin{aligned}\text{LVI.} \ldots c_1c_3(c_1 - c_3)(\rho^2(\mathrm{V}\alpha\beta)^2 - (\mathrm{S}\alpha\beta\rho)^2) &= c_3(c_3 - a^2)(c_1\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2 \\ &\quad - c_1(c_1 - a^2)(c_3\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho);\end{aligned}$$

and accordingly the values XXXVI. of c_1, c_3 enable us to express each member of this last equation under the common form, $-c_1c_3(c_1 - c_3)(\alpha\mathrm{S}\beta\rho - \beta\mathrm{S}\alpha\rho)^2$.

(22.) Comparing the recent inequalities $c_1 > c_3 > c_2$ (XXXVII.) with the arrangement 357, IX., we see, by 357, (6.), that for the *real cyclic transformation* (6.) at present sought, the plane of λ, μ is to be perpendicular to α_3 (and not to α_2, as in 357, (3.), &c.). We are therefore to eliminate $(c_3\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2$ between the equations XLVII. and LVI., which gives (after a few reductions) the real transformation:

$$\begin{aligned}\text{LVII.} \ldots ((\mathrm{S}\alpha\beta)^2 - c_1\beta^2)((\mathrm{V}\alpha\rho)^2 + (\mathrm{S}\beta\rho)^2) - (c_1 - a^2)(\mathrm{S}\alpha\beta)^2\rho^2 \\ = (c_1\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2 - c_1(\mathrm{S}\alpha\beta\rho)^2 \\ = \mathrm{S}.\rho(c_1\beta - \alpha\mathrm{S}\alpha\beta + c_1^{\frac{1}{2}}\mathrm{V}\alpha\beta)\,\mathrm{S}.\rho(c_1\beta - \alpha\mathrm{S}\alpha\beta - c_1^{\frac{1}{2}}\mathrm{V}\alpha\beta);\end{aligned}$$

which is of the kind required.

(23.) Accordingly it will be found that the following equation,

$$\begin{aligned}\text{LVIII.} \ldots ((\mathrm{S}\alpha\beta)^2 - c\beta^2)(\mathrm{V}\alpha\rho)^2 + (c - a^2)(c(\mathrm{S}\beta\rho)^2 - \rho^2\mathrm{S}(\alpha\beta)^2) \\ = (c\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2 - c(\mathrm{S}\alpha\beta\rho)^2,\end{aligned}$$

is an *identity*, or that it holds good for *all values* of the scalar c, and of the vectors α, β, ρ; since, by addition of $c(\mathrm{V}\alpha\beta)^2\rho^2$ on both sides, it takes this *obviously* identical form,

$$\begin{aligned}\text{LIX.} \ldots ((\mathrm{S}\alpha\beta)^2 - c\beta^2)(\mathrm{S}\alpha\rho)^2 + c(c - a^2)(\mathrm{S}\beta\rho)^2 = (c\mathrm{S}\beta\rho - \mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2 \\ - c(\alpha\mathrm{S}\beta\rho - \beta\mathrm{S}\alpha\rho)^2;\end{aligned}$$

so that if c_1 be *either root* of the quadratic XLIII., or if $c_1(c_1 - a^2) = (\mathrm{S}\alpha\beta)^2 - c_1\beta^2$, the *transformation* LVII. is at least *symbolically valid:* but we must take, as above, the *positive root* of that quadratic for c_1, if we wish that transformation to be a *real* one, as regards the *constants* which it employs. And if we had *happened* (comp. (20.)) to perceive this *identity* LIX., and to see its transformation LVIII., we might have been in that way led to form the *quadratic* XLIII., without having previously formed the *cubic* XXXIII.

(24.) Already, then, we see how to obtain *one* of the two *imaginary cyclic transformations* of the *given focal form* XXX., namely by changing c_1 to c_3 in LVII.; and the *other* imaginary transformation is had, on principles before explained, by eliminating $(\mathrm{S}\alpha\beta\rho)^2$ between XLVII. and LVI.; a process which easily conducts to the equation,

$$\text{LX.}\ldots(\mathrm{V}\alpha\rho)^2+(\mathrm{S}\beta\rho)^2+\alpha^2\rho^2=(c_1-c_3)^{-1}\{c_1^{-1}(c\mathrm{S}\beta\rho-\mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2$$
$$-c_3^{-1}(c_3\mathrm{S}\beta\rho-\mathrm{S}\alpha\beta\mathrm{S}\alpha\rho)^2\},$$

where the second member is the *sum of two squares* (c_1 being > 0, but $c_3 < 0$), as the second expression LVII. would also become, if c_1 were replaced by c_3. Accordingly, each member of LX. is equal to $(\mathrm{S}\alpha\rho)^2+(\mathrm{S}\beta\rho)^2$, if c_1, c_3 be the roots of *any* quadratic LI., with only the *one* condition,

$$\text{LXI.}\ldots c_1c_3=B=-(\mathrm{S}\alpha\beta)^2;$$

which however, when *combined* with the *condition of rectangularity* LIII., suffices to give also $A=\beta^2-\alpha^2$, as in LIV., and so to lead us back to the quadratic XLIII., which had been deduced by the general method, as a *factor* of the *cubic* equation XXXIII.

(25.) And since the values XXXVI. of c_1, c_3 reduce, as above, the second member of LX. to the simple form $(\mathrm{S}\alpha\rho)^2+(\mathrm{S}\beta\rho)^2$, we may thus, or even without employing the *roots* c_1, c_3 at all, deduce the following expression for the last imaginary cyclic transformation:

$$\text{LXII.}\ldots \mathrm{S}\rho\phi\rho=(\mathrm{V}\alpha\rho)^2+(\mathrm{S}\beta\rho)^2=-\alpha^2\rho^2+\mathrm{S}(\alpha+\sqrt{-1}\beta)\rho\,.\,\mathrm{S}(\alpha-\sqrt{-1}\beta)\rho,$$

where $\sqrt{-1}$ is the imaginary of algebra (comp. 214, (6.)); while the *real scalar* r^4 of XXXV. may at the same time receive the connected *imaginary form*,

$$\text{LXIII.}\ldots r^4=(\alpha^2-\beta^2)^2+4(\mathrm{S}\alpha\beta)^2=(\alpha+\sqrt{-1}\beta)^2(\alpha-\sqrt{-1}\beta)^2.$$

(26.) Finally, as regards the passage from the *given form* XXX., to a *second real focal form* (comp. 358, (4.)), or the transformation,

$$\text{LXIV.}\ldots(\mathrm{V}\alpha\rho)^2+(\mathrm{S}\beta\rho)^2=(\mathrm{V}\alpha'\rho)^2+(\mathrm{S}\beta'\rho)^2,$$

in which α' and β' are real vectors, distinct from $\pm\alpha$ and $\pm\beta$, but in the same plane with them, it may be sufficient (comp. 358, (8.)), to write down the formulæ:

$$\text{LXV.}\ldots r^2\alpha'=-(\alpha^3+\beta\alpha\beta),\quad r^2\beta'=-(\beta^3+\alpha\beta\alpha),$$

with the same real value of r^2 as before; so that (by XXXV., &c.) we have the relations,

$$\text{LXVI.}\ldots \mathrm{T}\alpha'=\mathrm{T}\alpha,\quad \mathrm{T}\beta'=\mathrm{T}\beta,\quad \mathrm{S}\alpha'\beta'=\mathrm{S}\alpha\beta;$$

$$\text{LXVII.}\ldots\begin{cases}r^2(\alpha+\alpha')=\alpha(r^2-\alpha^2+\beta^2)-2\beta\mathrm{S}\alpha\beta=-2(\alpha c_3+\beta\mathrm{S}\alpha\beta)\parallel\alpha_1,\\ r^2(\alpha-\alpha')=\alpha(r^2+\alpha^2-\beta^2)+2\beta\mathrm{S}\alpha\beta=2(\alpha c_1+\beta\mathrm{S}\alpha\beta)\parallel\alpha_3;\end{cases}$$

$$\text{LXVIII.}\ldots\begin{cases}r^2(\beta+\beta')=\beta(r^2+\alpha^2-\beta^2)-2\alpha\mathrm{S}\alpha\beta=2(\beta c_1-\alpha\mathrm{S}\alpha\beta)\parallel\alpha_1,\\ r^2(\beta-\beta')=\beta(r^2-\alpha^2+\beta^2)+2\alpha\mathrm{S}\alpha\beta=-2(\beta c_3-\alpha\mathrm{S}\alpha\beta)\parallel\alpha_3.\end{cases}$$

(27.) We have then the identity,

$$\text{LXIX.}\ldots(\mathrm{V}(\alpha^3+\beta\alpha\beta)\rho)^2+(\mathrm{S}(\beta^3+\alpha\beta\alpha)\rho)^2$$
$$=(\alpha^4+2\mathrm{S}.(\alpha\beta)^2+\beta^4)((\mathrm{V}\alpha\rho)^2+(\mathrm{S}\beta\rho)^2);$$

with which may be combined this other of the same kind,

$$\text{LXX.}\ldots-(\mathrm{V}(\alpha^3-\beta\alpha\beta)\rho)^2+(\mathrm{S}(\beta^3-\alpha\beta\alpha)\rho)^2$$
$$=(\alpha^4-2\mathrm{S}.(\alpha\beta)^2+\beta^4)(-(\mathrm{V}\alpha\rho)^2+(\mathrm{S}\beta\rho)^2),$$

which enables us to pass from the focal form XXIII., to a second real focal form, with its two new lines in the same plane as the two old ones: and it may be noted that we can pass from LXIX. to LXX., by changing α to $\alpha\sqrt{-1}$.

360. Besides the rectangular, cyclic, and focal transformations of $S\rho\phi\rho$, which have been already considered, there are others, although perhaps of less importance: but we shall here mention only two of them, as specimens, whereof one may be called the *Bifocal*, and the other the *Mixed Transformation*.

(1.) The two lines α, α', of 359, LXV., being called *focal lines*,[*] an expression which shall introduce them *both* may be called on that account a *bifocal transformation*.

(2.) Retaining then the value 359, XXXV. of r^4, and introducing a new auxiliary constant e, which shall satisfy the equation,

$$\text{I.} \ldots \beta^2 - \alpha^2 = r^2 e, \quad \text{and therefore} \quad \text{II.} \ldots 4(S\alpha\beta)^2 = r^4(1 - e^2),$$

so that

$$\text{III.} \ldots 4e^2(S\alpha\beta)^2 = (1 - e^2)(\beta^2 - \alpha^2)^2,$$

the first equation 359, LXV. gives,

$$\text{IV.} \ldots r^2(e\alpha - \alpha') = 2\beta S\alpha\beta, \qquad \text{V.} \ldots r^2(eS\alpha\rho - S\alpha'\rho) = 2S\alpha\beta S\beta\rho;$$

and therefore, with the form 359, XXX. of $S\rho\phi\rho$,

$$\begin{aligned}\text{VI.} \ldots (1 - e^2)S\rho\phi\rho &= (1 - e^2)((V\alpha\rho)^2 + (S\beta\rho)^2)\\ &= (1 - e^2)(V\alpha\rho)^2 + (eS\alpha\rho - S\alpha'\rho)^2\\ &= (e^2 - 1)\alpha^2\rho^2 + (S\alpha\rho)^2 - 2eS\alpha\rho S\alpha'\rho + (S\alpha'\rho)^2;\end{aligned}$$

in which $\rho^2 = \alpha'^2$, by 359, LXVI., so that α and α' may be considered to enter *symmetrically* into this last transformation, which is of the *bifocal* kind above mentioned.

(3.) For the same reason, the expression last found for $S\rho\phi\rho$ involves again (comp. 358) *six* scalar constants; namely, e, $T\alpha (= T\alpha')$, and the four involved in the two unit lines, $U\alpha$, $U\alpha'$.

(4.) In all the foregoing transformations, the scalar and quadratic function $S\rho\phi\rho$ has been *evidently homogeneous*, or has been seen to involve no terms below the *second degree* in ρ. We may however also employ this *apparently heterogeneous* or *mixed* form,

$$\text{VII.} \ldots S\rho\phi\rho = g'(\rho - \epsilon)^2 + 2S\lambda(\rho - \zeta)S\mu(\rho - \zeta) + e;$$

in which g', λ, μ have the same significations as in 357, but e, ϵ, ζ are *three new constants*, subject to the two *conditions of homogeneity*,

$$\text{VIII.} \ldots g'\epsilon + \lambda S\mu\zeta + \mu S\lambda\zeta = 0,$$

and

$$\text{IX.} \ldots g'\epsilon^2 + 2S\lambda\zeta S\mu\zeta + e = 0,$$

in order that the expression VII. may admit of reduction to the form,

$$\text{X.} \ldots S\rho\phi\rho = g'\rho^2 + 2S\lambda\rho S\mu\rho, \text{ as in 357, II}'.$$

(5.) Other general *homogeneous* transformations of $S\rho\phi\rho$, which are themselves *real*, although *connected* with *imaginary*[†] *cyclic forms* (comp. 357, (7.)), because a *sum of two squares* of linear and scalar functions is, in an imaginary sense, a *product* of two such functions, are the two following (comp. 357, (9.)):

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[*] Compare the Note to Art. 359.

[†] $\lambda_1 \pm \sqrt{-1}\,\mu_1$, and $\lambda_2 \pm \sqrt{-1}\,\mu_2$, may here be said to be two pairs of *imaginary cyclic normals*, of that *real surface* of the second order, of which the equation is, as before, $S\rho\phi\rho = \text{const.}$ Compare the Notes to pages 468, 474.

$$\text{XI.} \ldots \mathrm{S}\rho\phi\rho = g\rho^2 + \mathrm{S}\lambda\rho\mu\rho = g_1\rho^2 + (\mathrm{S}\lambda_1\rho)^2 + (\mathrm{S}\mu_1\rho)^2;$$
$$\text{XII.} \ldots \mathrm{S}\rho\phi\rho = g\rho^2 + \mathrm{S}\lambda\rho\mu\rho = g_3\rho^2 - (\mathrm{S}\lambda_3\rho)^2 - (\mathrm{S}\mu_3\rho)^2;$$

in which (comp. 357, (2.) and (8.)),

$$\text{XIII.} \ldots g_1 = g + \mathrm{T}\lambda\mu = -e_1, \quad g_3 = g - \mathrm{T}\lambda\mu = -e_3,$$
$$\text{XIV.} \ldots \lambda_1 = \mathrm{V}\lambda\mu\,(\mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu)^{-\frac{1}{2}}, \quad \mu_1 = (\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda)\,(\mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu)^{-\frac{1}{2}},$$

and

$$\text{XV.} \ldots \lambda_3 = \mathrm{V}\lambda\mu\,(\mathrm{T}\lambda\mu + \mathrm{S}\lambda\mu)^{-\frac{1}{2}}, \quad \mu_3 = (\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda)\,(\mathrm{T}\lambda\mu + \mathrm{S}\lambda\mu)^{-\frac{1}{2}};$$

so that g_1, λ_1, μ_1, and g_3, λ_3, μ_3 are *real*, if g, λ, μ be such.

(6.) We have therefore the *two new mixed transformations* following:

$$\text{XVI.} \ldots \mathrm{S}\rho\phi\rho = g_1(\rho - \epsilon_1)^2 + (\mathrm{S}\lambda_1(\rho - \zeta_1))^2 + (\mathrm{S}\mu_1(\rho - \zeta_1))^2 + e_1;$$
$$\text{XVII.} \ldots \mathrm{S}\rho\phi\rho = g_3(\rho - \epsilon_3)^2 - (\mathrm{S}\lambda_3(\rho - \zeta_3))^2 - (\mathrm{S}\mu_3(\rho - \zeta_3))^2 + e_3;$$

with these two new pairs of equations, as *conditions* of *homogeneity*,

$$\text{XVIII.} \ldots g_1\epsilon_1 + \lambda_1\mathrm{S}\zeta_1\lambda_1 + \mu_1\mathrm{S}\zeta_1\mu_1 = 0,$$
$$\text{XIX.} \ldots g_1\epsilon_1^2 + (\mathrm{S}\zeta_1\lambda_1)^2 + (\mathrm{S}\zeta_1\mu_1)^2 + e_1 = 0,$$

and

$$\text{XX.} \ldots g_3\epsilon_3 - \lambda_3\mathrm{S}\zeta_3\lambda_3 - \mu_3\mathrm{S}\zeta_3\mu_3 = 0,$$
$$\text{XXI.} \ldots g_3\epsilon_3^2 - (\mathrm{S}\zeta_3\lambda_3)^2 - (\mathrm{S}\mu_3\zeta_3)^2 + e_3 = 0.$$

361. We saw, in the sub-articles to 336, that the *differential*, $\mathrm{d}f\rho$, of a *scalar function of a vector*, may in general be expressed under the form,

$$\text{I.} \ldots \mathrm{d}f\rho = n\mathrm{S}\nu\mathrm{d}\rho,$$

where ν is a *derived vector function*, of the same variable vector ρ, and n is a *scalar coefficient*. And we now propose to show, that if

$$\text{II.} \ldots f\rho = \mathrm{S}\rho\phi\rho,$$

$\phi\rho$ still denoting the linear and vector function which has been considered in the present Section, and of which $\phi_0\rho$ is still the self-conjugate part, we shall have the equation I. with the values,

$$\text{III.} \ldots n = 2, \quad \nu = \phi_0\rho;$$

so that the *part* $\phi_0\rho$ may thus be *deduced* from $\phi\rho$ by *operating* with $\frac{1}{2}\mathrm{dS}\,.\,\rho$, and seeking the coefficient of $\mathrm{d}\rho$ under the sign S. in the result: while there exist certain general *relations of reciprocity* (comp. 336, (6.)), between the *two vectors* ρ and ν, which are in this way *connected*, as *linear functions of each other*.

(1.) We have here, by the supposed *linear form* of $\phi\rho$, the differential equation (comp. 334, VI.),

$$\text{IV.} \ldots \mathrm{d}\phi\rho = \phi\mathrm{d}\rho;$$

also $\quad S(d\rho . \phi\rho) = S(\phi\rho . d\rho)$, and $S(\rho . \phi d\rho) = S(\phi'\rho . d\rho)$;

hence, by 349, XIII., we have, as asserted,

$$\text{V.} \ldots dS\rho\phi\rho = S(\phi\rho + \phi'\rho)d\rho = 2S . \phi_0\rho d\rho.$$

(2.) As an example of the employment of this formula, in the deduction of $\phi_0\rho$ from $\phi\rho$, let us take the expression,

$$\text{VI.} \ldots \phi\rho = \Sigma\beta S\alpha\rho, \qquad \text{347, XXXI.},$$

which gives,

$$\text{VII.} \ldots f\rho = S\rho\phi\rho = \Sigma S\alpha\rho S\beta\rho,$$

and therefore

$$\text{VIII.} \ldots df\rho = \Sigma(\beta S\alpha\rho + \alpha S\beta\rho)d\rho.$$

mparing this with the general formula,

$$\text{IX.} \ldots \tfrac{1}{2}df\rho = S\nu d\rho = S . \phi_0\rho d\rho,$$

we find that the form VI. of $\phi\rho$ has for its self-conjugate *part*,

$$\text{X.} \ldots \nu = \phi_0\rho = \tfrac{1}{2}\Sigma(\beta S\alpha\rho + \alpha S\beta\rho);$$

and in fact we saw (347, XXXII.) that this form gives, as its *conjugate*, the expression,

$$\text{XI.} \ldots \phi'\rho = \Sigma\alpha S\beta\rho.$$

(3.) Supposing now, for simplicity, that the function ϕ is *given*, or *made*, *self-conjugate*, by taking (if necessary) the semisum of itself and its own conjugate function, we may write ϕ instead of ϕ_0, and shall thus have, simply,

$$\text{XII.} \ldots \nu = \phi\rho, \qquad \text{XIII.} \ldots f\rho = S\nu\rho, \qquad \text{XIV.} \ldots df\rho = 2S\nu d\rho;$$

whence also (comp. 348, I. II.),

$$\text{XV.} \ldots \rho = \phi^{-1}\nu = m^{-1}\psi\nu, \quad \text{and} \quad \text{XVI.} \ldots S\nu d\rho = S\rho d\nu.$$

(4.) Writing, then,

$$\text{XVII.} \ldots F\nu = S\nu\phi^{-1}\nu = m^{-1}S\nu\psi\nu,$$

we shall have the equations,

$$\text{XVIII.} \ldots F\nu = f\rho, \qquad \text{XIX.} \ldots dF\nu = 2S\rho d\nu = 2S . \phi^{-1}\nu d\nu;$$

so that ρ *may be deduced from* $F\nu$, *as* ν *was deduced from* $f\rho$; and generally, as above stated, there exists a perfect *reciprocity* of relations, between the *vectors* ρ and ν, and also between their *scalar functions*, $f\rho$ and $F\nu$.

(5.) As regards the *deduction*, or *derivation*, of ν from $f\rho$, and of ρ from $F\nu$, it may occasionally be convenient to *denote* it thus:

$$\text{XX.} \ldots \nu = \tfrac{1}{2}(S . d\rho)^{-1}df\rho; \qquad \text{XXI.} \ldots \rho = \tfrac{1}{2}(S . d\nu)^{-1}dF\nu;$$

in fact, these last may be considered as only *symbolical transformations* of the expressions,

$$\text{XXII.} \ldots df\rho = 2S(d\rho . \nu), \quad dF\nu = 2S(d\nu . \rho),$$

which follow immediately from XIV. and XIX.

(6.) As an *example* of the passage from an expression such as $f\rho$, to an *equal expression* of the *reciprocal form* $F\nu$, let us resume the *cyclic form* 357, II., writing thus,

$$\text{XXIII.} \ldots f\rho = S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho,$$

and supposing that g, λ, and μ are real. Here, by what has been already shown (in sub-articles to 354 and 357), if $\phi\rho$ be supposed self-conjugate, as in (3.), we have,

$$\text{XXIV.} \ldots \nu = \phi\rho = g\rho + \mathrm{V}\lambda\rho\mu;$$

$$\text{XXV.} \ldots m = (g - \mathrm{S}\lambda\mu)\,(g^2 - \lambda^2\mu^2) = -c_1c_2c_3;$$

$$\text{XXVI.} \ldots \psi\nu = \mathrm{V}\lambda\nu\mu\mathrm{S}\lambda\mu - \mathrm{V}\lambda\mu\mathrm{S}\lambda\nu\mu - g(\lambda\mathrm{S}\mu\nu + \mu\mathrm{S}\lambda\nu) + g^2\nu;$$

and therefore

$$\begin{aligned}\text{XXVII.} \ldots mF\nu &= \mathrm{S}\nu\psi\nu \\ &= \mathrm{S}\lambda\nu\mu\nu\mathrm{S}\lambda\mu + (\mathrm{S}\lambda\nu\mu)^2 - 2g\mathrm{S}\lambda\nu\mathrm{S}\mu\nu + g^2\nu^2 \\ &= (g^2 - \lambda^2\mu^2)\,\nu^2 + \lambda^2(\mathrm{S}\mu\nu)^2 + \mu^2(\mathrm{S}\lambda\nu)^2 - 2g\mathrm{S}\lambda\nu\mathrm{S}\mu\nu;\end{aligned}$$

which last, when compared with 360, VI., is seen to be what we have called a *bifocal form:* its *focal lines* α, α' (360, (1.)) having here the directions of λ, μ, that is of what may be called the *cyclic lines** of the *form* XXIII. The *cyclic* and *bifocal transformations* are therefore *reciprocals* of each other.

(7.) As another example of this reciprocal relation between cyclic and focal lines, in the passage from $f\rho$ to $F\nu$, or conversely from the latter to the former, let us now *begin* with the *focal form*,

$$\text{XXVIII.} \ldots f\rho = \mathrm{S}\rho\phi\rho = (\mathrm{V}\alpha\rho)^2 + (\mathrm{S}\beta\rho)^2, \qquad 359,\ \text{XXX.},$$

in which α and β are supposed to be given and real vectors. We have now, by 359, (11.),

$$\text{XXIX.} \ldots \begin{cases}\nu = \phi\rho = -\alpha\mathrm{V}\alpha\rho + \beta\mathrm{S}\beta\rho, \quad m = \alpha^2(\mathrm{S}\alpha\beta)^2, \\ \psi\nu = \mathrm{V}\alpha\nu\beta\mathrm{S}\alpha\beta + \alpha(\alpha^2 - \beta^2)\mathrm{S}\alpha\nu,\end{cases}$$

and therefore,

$$\begin{aligned}\text{XXX.} \ldots mF\nu = \alpha^2(\mathrm{S}\alpha\beta)^2\,F\nu &= \mathrm{S}\nu\psi\nu \\ &= \mathrm{S}\alpha\nu\beta\nu\mathrm{S}\alpha\beta + (\alpha^2 - \beta^2)\,(\mathrm{S}\alpha\nu)^2 \\ &= -\nu^2(\mathrm{S}\alpha\beta)^2 + \mathrm{S}\alpha\nu((\alpha^2 - \beta^2)\mathrm{S}\alpha\nu + 2\mathrm{S}\alpha\beta\mathrm{S}\beta\nu) \\ &= -\nu^2(\mathrm{S}\alpha\beta)^2 + \mathrm{S}\alpha\nu\mathrm{S}(\alpha^3 + \beta\alpha\beta)\nu,\end{aligned}$$

an expression which is of *cyclic form; one* cyclic line of $F\nu$ being the *given* focal line α of $f\rho$; and the *other* cyclic line of $F\nu$ having the direction of $\pm(\alpha^3 + \beta\alpha\beta)$, and consequently (by 359, LXV.) of $\mp\alpha'$, where α' is the *second* real and focal line of $f\rho$.

(8.) And to verify the equation XVIII., or to show by an example that the *two functions* $f\rho$ and $F\nu$ are *equal in value*, although they are (generally) *different in form*, it is sufficient to substitute in XXX. the value XXIX. of ν; which, after a few reductions, will exhibit the asserted equality.

362. It is often convenient to introduce a certain *scalar and symmetric function* of *two independent vectors*, ρ and ρ', which is *linear* with respect to *each* of them, and is deduced from the linear and *self-conjugate vector function* $\phi\rho$, of a *single* vector ρ, as follows:

$$\text{I.} \ldots f(\rho, \rho') = f(\rho', \rho) = \mathrm{S}\rho'\phi\rho = \mathrm{S}\rho\phi\rho'.$$

With this notation, we have

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* They are in fact (compare the Note to page 468) the *cyclic normals*, or the normals to the *cyclic planes*, of that *surface of the second order*, which has for its equation $f\rho =$ const.; while they are, as above, the *focal lines* of that *other* or *reciprocal surface*, of which ν is the variable vector, and the equation is $F\nu =$ const.

$$\text{II.}\ldots f(\rho+\rho')=f\rho+2f(\rho,\rho')+f\rho';$$
$$\text{III.}\ldots f(\rho,\rho'+\rho'')=f(\rho,\rho')+f(\rho,\rho'');$$
$$\text{IV.}\ldots f(\rho,\rho)=f\rho;\qquad \text{V.}\ldots \mathrm{d}f\rho=2f(\rho,\mathrm{d}\rho);$$
$$\text{VI.}\ldots f(x\rho,y\rho)=xyf(\rho,\rho'),\quad \text{if}\quad \mathrm{V}x=\mathrm{V}y=0;$$

and as a verification,

$$\text{VII.}\ldots f(x\rho)=x^2f\rho,$$

a result which might have been obtained, without introducing this new function I.

(1.) It appears to be unnecessary, at this stage, to write down proofs of the foregoing consequences, II. to VI., of the definition I.; but it may be worth remarking, that we *here* depart a little, in the formula V., from a *notation* (325) which was used in some early Articles of the present Chapter, although avowedly only as a *temporary* one, and adopted merely for convenience of exposition of the *principles* of Quaternion Differentials.

(2.) In *that* provisional notation (comp. 325, IX.) we should have had, for the differentiation of the recent function $f\rho$ (361, II.), the formulæ,

$$\mathrm{d}f\rho=f(\rho,\mathrm{d}\rho),\quad f(\rho,\rho')=2\mathrm{S}\rho'\phi\rho;$$

the numerical coefficient being thus *transferred* from one of them to the other, as compared with the recent equations, I. and V. But there is a convenience *now* in adopting these last equations V. and I., namely,

$$\mathrm{d}f\rho=2f(\rho,\mathrm{d}\rho),\quad f(\rho,\rho')=\mathrm{S}\rho'\phi\rho;$$

because this *function* $\mathrm{S}\rho'\phi\rho$, or $\mathrm{S}\rho\phi\rho'$, occurs frequently in the applications of quaternions to surfaces of the second order, and not always with the *coefficient* 2.

(3.) Retaining then the recent notations, and treating $\mathrm{d}\rho$ as constant, or $\mathrm{d}^2\rho$ as null, successive differentiation of $f\rho$ gives, by IV. and V., the formulæ,

$$\text{VIII.}\ldots \mathrm{d}^2f\rho=2f(\mathrm{d}\rho);\quad \mathrm{d}^3f\rho=0;\ \&c.;$$

so that the theorem 342, I. is here verified, under the form,

$$\text{IX.}\ldots \epsilon^{\mathrm{d}}f\rho=(1+\mathrm{d}+\tfrac{1}{2}\mathrm{d}^2)f\rho$$
$$=f\rho+2f(\rho,\mathrm{d}\rho)+f\mathrm{d}\rho;$$

or briefly,

$$\text{X.}\ldots \epsilon^{\mathrm{d}}f\rho=f(\rho+\mathrm{d}\rho),$$

an equation which by II. is *rigorously exact* (comp. 339, (4.)), without any supposition whatever being made, respecting any *smallness* of the *tensor*, $\mathrm{Td}\rho$.

363. *Linear* and *vector functions* of vectors, such as those considered in the present Section, although *not generally* satisfying the condition of *self-conjugation*, present themselves generally in the *differentiation* of *non-linear* but *vector functions* of vectors. In fact, if we denote for the moment *such* a non-linear function by $\omega(\rho)$, or simply by $\omega\rho$, the general *distributive property* (326) of differential expressions allows us to write,

$$\text{I.}\ldots \mathrm{d}\omega(\rho)=\phi(\mathrm{d}\rho),\quad \text{or briefly,}\quad \text{I}'.\ldots \mathrm{d}\omega\rho=\phi\mathrm{d}\rho;$$

where ϕ has all the properties hitherto employed, including that of *not* being *generally* self-conjugate, as has been just observed. There is, however, as we shall soon see, an extensive and important *case*, in which the property of self-conjugation *exists*, for such a function ϕ; namely when the *differentiated function*, $\omega\rho$, is *itself* the result ν of the *differentiation* of a *scalar function* $f\rho$ of the variable vector ρ, although *not necessarily* a function of the *second dimension*, such as has been recently considered (361); or more fully, when it is the coefficient of $d\rho$, under the sign S., in the differential (361, I.) of that scalar function $f\rho$, whether it be multiplied or not by any *scalar constant* (such as n, in the formula last referred to). And generally (comp. 346), the *inversion* of the linear and vector function ϕ in I. corresponds to the *differentiation* of the *inverse* (or *implicit*) function ω^{-1}; in such a manner that the equation I. or I′. may be written under this other form,

$$\text{II.} \ldots \mathrm{d}\omega^{-1}\sigma = \phi^{-1}\mathrm{d}\sigma = m^{-1}\psi\mathrm{d}\sigma, \quad \text{if} \quad \sigma = \omega\rho.$$

(1.) As a very simple *example* of a non-linear but vector function, let us take the form,

$$\text{III.} \ldots \sigma = \omega(\rho) = \rho\alpha\rho, \text{ where } \alpha \text{ is a constant vector.}$$

This gives, if $\mathrm{d}\rho = \rho'$,

$$\text{IV.} \ldots \phi\rho' = \phi\mathrm{d}\rho = \mathrm{d}\omega\rho = \rho'\alpha\rho + \rho\alpha\rho' = 2\mathrm{V}\rho\alpha\rho';$$

$$\text{V.} \ldots \mathrm{S}\lambda\phi\rho' = 2\mathrm{S}\lambda\rho\alpha\rho' = \mathrm{S}\rho'\phi'\lambda;$$

$$\text{VI.} \ldots \phi'\lambda = 2\mathrm{V}\lambda\rho\alpha = 2\mathrm{V}\alpha\rho\lambda, \quad \phi'\rho' = 2\mathrm{V}\alpha\rho\rho';$$

so that $\phi\rho'$ and $\phi'\rho'$ are unequal, and the linear function $\phi\rho'$ is *not* self-conjugate.

(2.) To find its self-conjugate *part* $\phi_0\rho'$, by the method of Art. 361, we are to form the scalar expression,

$$\text{VII.} \ldots \tfrac{1}{2}f\rho' = \tfrac{1}{2}\mathrm{S}\rho'\phi\rho' = \rho'^2\mathrm{S}\alpha\rho;$$

of which the differential, taken with respect to ρ', is

$$\text{VIII.} \ldots \tfrac{1}{2}\mathrm{d}f\rho' = \mathrm{S}\,.\,\phi_0\rho'\mathrm{d}\rho' = 2\mathrm{S}\alpha\rho\mathrm{S}\rho'\mathrm{d}\rho', \quad \text{giving} \quad \text{IX.} \ldots \phi_0\rho' = 2\rho'\mathrm{S}\alpha\rho;$$

and accordingly this is equal to the semisum of the two expressions, IV. and VI., for $\phi\rho'$ and its conjugate.

(3.) On the other hand, as an *example of* the *self-conjugation* of the linear and vector function,

$$\text{X.} \ldots \mathrm{d}\nu = \mathrm{d}\omega\rho = \phi\mathrm{d}\rho, \quad \text{when} \quad \text{X}'. \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho = 2\mathrm{S}\,.\,\omega\rho\mathrm{d}\rho,$$

even if the *scalar* function $f\rho$ be of a higher dimension than the second, let this last function have the form,

$$\text{XI.} \ldots f\rho = \mathrm{S}q\rho q'\rho q''\rho, \quad q,\ q',\ q'' \text{ being three constant quaternions.}$$

Here
$$\text{XII.} \ldots \nu = \omega\rho = \tfrac{1}{2}\mathrm{V}(q\rho q'\rho q'' + q'\rho q''\rho q + q''\rho q\rho q');$$

$$\text{XIII.} \ldots \mathrm{d}\nu = \phi\mathrm{d}\rho = \phi\rho' = \tfrac{1}{2}\mathrm{V}(q\rho' q'\rho q'' + q'\rho q''\rho' q) + \tfrac{1}{2}\mathrm{V}(q'\rho' q''\rho q + q''\rho q\rho' q')$$
$$+ \tfrac{1}{2}\mathrm{V}(q''\rho' q\rho q' + q\rho q'\rho' q'');$$

and $$\text{XIV.} \ldots \mathrm{S}\lambda\phi\rho' = \tfrac{1}{2}\mathrm{S}.q'\rho q''(\lambda q\rho' + \rho' q\lambda) + \&\text{c.} = \mathrm{S}\rho'\phi\lambda\,;$$

so that $\phi' = \phi$, as asserted.

(4.) In general, if δ be used as a *second* and *independent symbol* of differentiation, we may write (comp. 345, IV.),

$$\text{XV.} \ldots \delta \mathrm{d} fq = \mathrm{d}\delta fq,$$

where fq may denote any function of a quaternion; in fact, each member is, by the principles of the present Chapter (comp. 344, I., and 345, IX.), an expression for the *limit*,*

$$\text{XVI.} \ldots \lim_{\substack{n=\infty \\ n'=\infty}} nn'\{f(q + n^{-1}\mathrm{d}q + n'^{-1}\delta q) - f(q + n^{-1}\mathrm{d}q) - f(q + n'^{-1}\delta q) + fq\}.$$

(5.) As another statement of the same theorem, we may remark that a first differentiation of fq, with each symbol separately taken, gives results of the forms,

$$\text{XVII.} \ldots \mathrm{d}fq = f(q, \mathrm{d}q), \quad \delta fq = f(q, \delta q)\,;$$

and then the assertion is, that if we differentiate the first of these with δ, and the second with d, operating only on q with each, and not on $\mathrm{d}q$ nor on δq, we obtain *equal results*, of these other forms,

$$\text{XVIII.} \ldots \delta\mathrm{d}fq = f(q, \mathrm{d}q, \delta q) = f(q, \delta q, \mathrm{d}q) = \mathrm{d}\delta fq.$$

For example, if

$$\text{XIX.} \ldots fq = qcq, \text{ where } c \text{ is a constant quaternion,}$$

the common value of these last expressions is,

$$\text{XX.} \ldots \delta\mathrm{d}fq = \mathrm{d}\delta fq = \delta q.c.\mathrm{d}q + \mathrm{d}q.c.\delta q.$$

(6.) Writing then, by X.,

$$\text{XXI.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\omega\rho\mathrm{d}\rho, \quad \delta f\rho = 2\mathrm{S}\omega\rho\delta\rho,$$

and $$\text{XXII.} \ldots \delta\omega\rho = \phi\delta\rho, \quad \text{with} \quad \mathrm{d}\omega\rho = \phi\mathrm{d}\rho, \text{ as before,}$$

we have the general equation,

$$\text{XXIII.} \ldots \mathrm{S}(\mathrm{d}\rho.\phi\delta\rho) = \mathrm{S}(\delta\rho.\phi\mathrm{d}\rho),$$

in which $\mathrm{d}\rho$ and $\delta\rho$ may represent *any two vectors;* the *linear and vector function*, ϕ, which is *thus derived* from a *scalar function* $f\rho$ by *differentiation*, is therefore (as above asserted and exemplified) *always self-conjugate.*

(7.) The equation XXIII. may be thus briefly written,

$$\text{XXIV.} \ldots \mathrm{S}\mathrm{d}\rho\delta\nu = \mathrm{S}\delta\rho\mathrm{d}\nu\,;$$

and it will be found to be virtually equivalent to the following system of three known equations, in the calculus of partial differential coefficients,

$$\text{XXV.} \ldots \mathrm{D}_x\mathrm{D}_y = \mathrm{D}_y\mathrm{D}_x, \quad \mathrm{D}_y\mathrm{D}_z = \mathrm{D}_z\mathrm{D}_y, \quad \mathrm{D}_z\mathrm{D}_x = \mathrm{D}_x\mathrm{D}_z.$$

364. At the commencement of the present Section, we *reduced* (in 347) the problem of the *inversion* (346) of a *linear* (or *distributive*) *quaternion function of a quaternion*, to the

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* We may also say that each of the two symbols XV. represents the coefficient of x^1y^1, in the development of $f(q + x\mathrm{d}q + y\delta q)$ according to ascending powers of x and y, when such development is possible.

corresponding problem for *vectors;* and, under this reduced or simplified *form*, have *resolved* it. Yet it may be interesting, and it will now be easy, to *resume* the *linear* and *quaternion equation*,

$$\text{I.} \ldots fq = r, \quad \text{with} \quad \text{II.} \ldots f(q+q') = fq + fq',$$

and to assign a *quaternion expression* for the *solution* of that equation, or for the *inverse quaternion function*,

$$\text{III.} \ldots q = f^{-1}r,$$

with the aid of notations already employed, and of results already established.

(1.) The *conjugate* of the linear and quaternion function fq being defined (comp. 347, IV.) by the equation,

$$\text{IV.} \ldots \text{S}pfq = \text{S}qf'p,$$

in which p and q are arbitrary quaternions, if we set out (comp. 347, XXXI.) with the *form*,

$$\text{V.} \ldots fq = tqs + t'qs' + \ldots = \Sigma tqs,$$

in which $s, s', \ldots$ and $t, t', \ldots$ are *arbitrary* but *constant quaternions*, and which is more than sufficiently general, we shall have (comp. 347, XXXII.) the *conjugate form*,

$$\text{VI.} \ldots f'p = spt + s'pt' + \ldots = \Sigma spt;$$

whence $\text{VII.} \ldots f1 = \Sigma ts,$ and $\text{VIII.} \ldots f'1 = \Sigma st;$

it is then possible, for each *given particular form* of the linear function fq, to assign *one scalar constant* e, and *two vector* constants, ϵ, ϵ', such that

$$\text{IX.} \ldots f1 = e + \epsilon, \quad f'1 = e + \epsilon';$$

and then we shall have the general transformations (comp. 347, I.):

$$\text{X.} \ldots \text{S}fq = \text{S}.qf'1 = e\text{S}q + \text{S}\epsilon'q;$$

$$\text{XI.} \ldots \text{V}fq = \epsilon\text{S}q + \text{V}.f\text{V}q = \epsilon\text{S}q + \phi\text{V}q;$$

and $\text{XII.} \ldots fq = (e+\epsilon)\text{S}q + \text{S}\epsilon'q + \phi\text{V}q;$

in which $\text{S}\epsilon'q = \text{S}.\epsilon'\text{V}q$, and $\phi\text{V}q$ or $\text{V}f\text{V}q$ is a *linear* and *vector function* of $\text{V}q$, of the kind already considered in this Section; being also such that, with the form V. of fq, we have

$$\text{XIII.} \ldots \phi\rho = \Sigma\text{V}t\rho s.$$

(2.) As regards the *number* of *independent* and *scalar constants* which enter, at least implicitly, into the composition of the quaternion function fq, it may in various ways be shown to be *sixteen;* and accordingly, in the expression XII., the *scalar* e is *one;* the *two vectors*, ϵ and ϵ', count *each* as *three;* and the *linear* and *vector function*, $\phi\text{V}q$, counts as *nine* (comp. 347, (1.)).

(3.) Since we already know (347, &c.) how to *invert* a function of this last kind ϕ, we may in general write,

$$\text{XIV.} \ldots r = \text{S}r + \text{V}r = \text{S}r + \phi\rho, \quad \text{where} \quad \text{XV.} \ldots \rho = \phi^{-1}\text{V}r = m^{-1}\psi\text{V}r;$$

the *scalar constant*, m, and the *auxiliary linear* and *vector function*, ψ, being deduced from the function ϕ by methods already explained. It is required then to express q, or $\mathrm{S}q$ and $\mathrm{V}q$, in terms of r, or of $\mathrm{S}r$ and ρ, so as to satisfy the linear equation,

$$\text{XVI.} \ldots (e+\epsilon)\mathrm{S}q + \mathrm{S}\epsilon'q + \phi\mathrm{V}q = \mathrm{S}r + \phi\rho\,;$$

the constants e, ϵ, ϵ', and the form of ϕ, being given.

(4.) Assuming for this purpose the expression,

$$\text{XVII.} \ldots q = q' + \rho,$$

in which q' is a new sought quaternion, we have the new equation,

$$\text{XVIII.} \ldots fq' = \mathrm{S}r + \phi\rho - f\rho = \mathrm{S}(r - \epsilon'\rho)\,;$$

whence $$\text{XIX.} \ldots q' = \mathrm{S}(r-\epsilon'\rho).f^{-1}1,$$

and $$\text{XX.} \ldots q = \rho + \mathrm{S}(r-\epsilon'\rho).f^{-1}1\,;$$

in which ρ is (by supposition) a *known vector*, and $\mathrm{S}(r-\epsilon'\rho)$ is a *known scalar;* so that it only remains to determine the *unknown* but *constant quaternion*, $f^{-1}1$, or to resolve the *particular equation*,

$$\text{XXI.} \ldots fq_0 = 1, \quad \text{in which} \quad \text{XXII.} \ldots q_0 = c + \gamma = f^{-1}1,$$

c being a *new* and *sought scalar constant*, and γ being a *new* and *sought vector constant.*

(5.) Taking scalar and vector parts, the quaternion equation XXI. breaks up into the two following (comp. X. and XI.):

$$\text{XXIII.} \ldots 1 = \mathrm{S}f(c+\gamma) = ec + \mathrm{S}\epsilon'\gamma\,; \qquad \text{XXIV.} \ldots 0 = \mathrm{V}f(c+\gamma) = \epsilon c + \phi\gamma\,;$$

which give the required values of c and γ, namely,

$$\text{XXV.} \ldots c = (e - \mathrm{S}\epsilon'\phi^{-1}\epsilon)^{-1}, \quad \text{and} \quad \text{XXVI.} \ldots \gamma = -c\phi^{-1}\epsilon\,;$$

whence $$\text{XXVII.} \ldots f^{-1}1 = \frac{1-\phi^{-1}\epsilon}{e - \mathrm{S}\epsilon'\phi^{-1}\epsilon}\,;$$

and accordingly we have, by XII., the equation,

$$\text{XXVIII.} \ldots f(1-\phi^{-1}\epsilon) = e - \mathrm{S}\epsilon'\phi^{-1}\epsilon = \mathrm{V}^{-1}0.$$

(6.) The *problem of quaternion inversion* is therefore *reduced anew* to that of *vector inversion*, and *solved* thereby; but we can *now* advance some steps further, in the *elimination* of *inverse operations*, and in the *substitution* for them of *direct* ones. Thus, if we observe, that $\phi^{-1} = m^{-1}\psi$, as before, and write for abridgment,

$$\text{XXIX.} \ldots n = me - \mathrm{S}\epsilon'\psi\epsilon = f(m - \psi\epsilon),$$

so that n is a *new* and *known scalar constant*, we shall have, by XV. XX. XXVII. XXIX.,

$$\text{XXX.} \ldots m\rho = \psi\mathrm{V}r\,; \qquad \text{XXXI.} \ldots nf^{-1}1 = m - \psi\epsilon\,;$$

and $$\text{XXXII.} \ldots mnq = n\psi\mathrm{V}r + (m\mathrm{S}r - \mathrm{S}\epsilon'\psi\mathrm{V}r).(m - \psi\epsilon),$$

an expression from which all *inverse* operations have disappeared, but which still admits of being simplified, through a division by m, as follows.

(7.) Substituting (by XXIX.), in the term $n\psi\mathrm{V}r$ of XXXII., the value $me - \mathrm{S}\epsilon'\psi\epsilon$ for n, and changing (by XXX.) $\psi\mathrm{V}r$ to $m\rho$, in the terms which are not obviously divisible by m, such a division gives,

$$\text{XXXIII.} \ldots nq = (m - \psi\epsilon)\mathrm{S}r + e\psi\mathrm{V}r - \mathrm{S}\epsilon'\psi\mathrm{V}r + \sigma,$$

where $$\text{XXXIV.} \ldots \sigma = -\rho\mathrm{S}\epsilon'\psi\epsilon + \psi\epsilon\mathrm{S}\epsilon'\rho = \mathrm{V}.\epsilon'\mathrm{V}\rho\psi\epsilon.$$

But (by 348, VII., interchanging accents) we have the transformation,

$$\text{XXXV.} \ldots \mathrm{V}\rho\psi\epsilon = -\phi'\mathrm{V}\epsilon\phi\rho = -\phi'\mathrm{V}\epsilon\mathrm{V}r,$$

because $\phi\rho = \mathrm{V}r$, by XIV. or XV.; everything *inverse* therefore *again* disappears, with this new elimination of the auxiliary vector ρ, and we have this final expression,

$$\text{XXXVI.} \ldots \; nq = nf^{-1}r = (me - \mathrm{S}\epsilon'\psi\epsilon) . f^{-1}r$$
$$= (m - \psi\epsilon)\mathrm{S}r + e\psi\mathrm{V}r - \mathrm{S}\epsilon'\psi\mathrm{V}r - \mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\mathrm{V}r,$$

in which each symbol of operation governs all that follows it, except where a point indicates the contrary, and which it appears to be impossible further to reduce, as the *formula of solution* of the *linear equation* I., with the *form* XII. of the *quaternion function*, fq.

(8.) Such having been the *analysis* of the problem, the *synthesis*, by which an *à posteriori proof* of the correctness of the resulting formula is to be given, may be simplified by using the *scalar* value XXIX. of $f(m - \psi\epsilon)$; and it is sufficient to show (denoting $\mathrm{V}r$ by ω), that for every *vector* ω the following equation holds good, with the same form XII. of f:

$$\text{XXXVII.} \ldots \; f(e\psi\omega - \mathrm{S}\epsilon'\psi\omega) - f\mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\omega = (me - \mathrm{S}\epsilon'\psi\epsilon) . \omega.$$

(9.) Accordingly, that form of f gives, with the help of the principle employed in XXXV.,

$$\text{XXXVIII.} \ldots \begin{cases} ef\psi\omega = e(\mathrm{S}\epsilon'\psi\omega + m\omega), \quad -f\mathrm{S}\epsilon'\psi\omega = -(e+\epsilon)\mathrm{S}\epsilon'\psi\omega, \\ -f\mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\omega = -\phi\mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\omega = \mathrm{V}(\mathrm{V}\epsilon\omega . \psi'\epsilon') = \epsilon\mathrm{S}\epsilon'\psi\omega - \omega\mathrm{S}\epsilon'\psi\epsilon, \end{cases}$$

because $\mathrm{S}\omega\psi'\epsilon' = \mathrm{S}\epsilon'\psi\omega$, &c.; and thus the equation XXXVI. is proved, by actually operating with f.

(10.) As an *example*, if we take the particular form,

$$\text{XXXIX.} \ldots \; r = fq = pq + qp,$$

in which $\quad \text{XL.} \ldots \; p = a + \alpha = \text{a given quaternion},$

we have then,

$$\text{XLI.} \ldots \; f1 = f'1 = 2p, \quad e = 2a, \quad \epsilon = \epsilon' = 2\alpha, \quad \phi\rho = 2a\rho;$$

whence by the theory of linear and *vector* functions,

$$\text{XLII.} \ldots \; \phi'\rho = 2a\rho, \quad \psi\rho = 4a^2\rho, \quad m = 8a^3,$$

and therefore, $\quad \text{XLIII.} \ldots \; \psi\epsilon = 8a^2\alpha, \quad m - \psi\epsilon = 8a^2(a - \alpha), \quad n = 16a^2(a^2 - \alpha^2);$

so that, dividing by $8a$, the formula XXXVI. becomes,

$$\text{XLIV.} \ldots \; 2a(a^2 - \alpha^2)q = a(a - \alpha)\mathrm{S}r + a^2\mathrm{V}r - a\mathrm{S} . \alpha\mathrm{V}r - \alpha\mathrm{V} . \alpha\mathrm{V}r,$$

or $\quad \text{XLV.} \ldots \; 2a(a + \alpha)q = a\mathrm{S}r + (a + \alpha)\mathrm{V}r - \mathrm{S}\alpha r,$

or $\quad \text{XLVI.} \ldots \; 2pq\mathrm{S}p = \mathrm{S} . r\mathrm{K}p + p\mathrm{V}r = r\mathrm{S}p + \mathrm{V}(\mathrm{V}p . \mathrm{V}r),$

or $\quad \text{XLVII.} \ldots \; 4pq\mathrm{S}p = 2r\mathrm{S}p + (pr - rp) = pr + r\mathrm{K}p;$

or finally,

$$\text{XLVIII.} \ldots \; q = f^{-1}r = \frac{r + p^{-1}r\mathrm{K}p}{4\mathrm{S}p} = \frac{r + \mathrm{K}p . rp^{-1}}{4\mathrm{S}p}.$$

Accordingly,

$$\text{XLIX.} \ldots \; (pr + r\mathrm{K}p) + (rp + \mathrm{K}p . r) = 2r(p + \mathrm{K}p) = 4r\mathrm{S}p.$$

(11.) In *so simple* an example as the last, we may with advantage avail ourselves of *special methods;* for instance (comp. 346), we may use that which was employed in 332, (6.), to *differentiate the square root of a quaternion*, and which conducted there more rapidly to a formula (332, XIX.) agreeing with the recent XLVIII.

(12.) We might also have observed, in the same case XXXIX., that

$$\text{L.}\ldots\ pr - rp = p^2q - qp^2 = 2\text{V}(\text{V}(p^2).\text{V}q) = 4\text{S}p.\text{V}(\text{V}p.\text{V}q) = 2\text{S}p.(pq - qp);$$

whence $pq - qp$, and therefore pq and qp, can be at once deduced, with the same resulting value for q, or for $f^{-1}r$, as before: and generally it is possible to *differentiate*, on a similar plan, the nth *root of a quaternion.*

365. We shall conclude this Section on *Linear Functions*, of the kinds above considered, by proving the general existence of a *Symbolic and Biquadratic Equation*, of the form,

$$\text{I.}\ldots\ 0 = n - n'f + n''f^2 - n'''f^3 + f^4,$$

which is thus *satisfied by the Symbol* (f) *of Linear and Quaternion Operation on a Quaternion*, as the *Symbolic and Cubic Equation*,

$$\text{I}'.\ldots\ 0 = m - m'\phi + m''\phi^2 - \phi^3, \qquad 350,\ \text{I.},$$

was satisfied by the *symbol* (ϕ) of *linear* and *vector operation* on a *vector;* the *four coefficients*, n, n', n'', n''', being *four scalar constants*, deduced from the function f in this extended or *quaternion theory*, as the *three* scalar coefficients m, m', m'' were constants deduced from ϕ, in the former or *vector theory*. And at the same time we shall see that there exists a *System*, of *Three Auxiliary Functions*, F, G, H, of the *Linear* and *Quaternion* kind, analogous to the *two vector functions*, ψ and χ, which have been so useful in the foregoing theory of vectors, and like them connected with each other, and with the given quaternion function f, by several simple and useful relations.

(1.) The formula of solution, 364, XXXVI., of the linear and quaternion equation $fq = r$, being denoted briefly as follows,

$$\text{II.}\ldots\ nq = nf^{-1}r = Fr,$$

so that (comp. 348, III'.) we may write, briefly and symbolically,

$$\text{III.}\ldots\ fF = Ff = n,$$

it may next be proposed to examine the changes which the scalar n and the function Fr undergo, when fr is changed to $fr + er$, or f to $f + c$, where c is any scalar constant; that is, by 364, XII., when e is changed to $e + c$, and ϕ to $\phi + c$; ϕ', ψ, and m being at the same time changed, according to the laws of the earlier theory.

(2.) Writing, then,

$$\text{IV.}\ldots\ f_c = f + c, \quad e_c = e + c, \quad \phi_c = \phi + c, \quad \phi'_c = \phi' + c,$$

and

$$\text{V.}\ldots\ \psi_c = \psi + c\chi + c^2, \quad m_c = m + m'c + m''c^2 + c^3,$$

we may represent the new form of the equation 364, XXXVI. as follows:

$$\text{VI.}\ldots\ n_cf_c^{-1}r = F_cr, \quad \text{or} \quad \text{VII.}\ldots\ f_cF_c = n_c;$$

where $\quad \text{VIII.} \ldots F_c r = (m_c - \psi_c\epsilon)\,\mathrm{S}r + e_c\psi_c\mathrm{V}r - \mathrm{S}\epsilon'\psi_c\mathrm{V}r - \mathrm{V}\epsilon'\phi'_c\mathrm{V}\epsilon\mathrm{V}r,$

and $\quad \text{IX.} \ldots n_c = e_c m_c - \mathrm{S}\epsilon'\psi_c\epsilon.$

(3.) In this manner it is seen that we may write,

$$\text{X.} \ldots F_c = F + cG + c^2H + c^3,$$

and

$$\text{XI.} \ldots n_c = n + n'c + n''c^2 + n'''c^3 + c^4\,;$$

where F, G, H, are *three functional symbols*, such that

$$\text{XII.} \ldots \begin{cases} Fr = (m - \psi\epsilon)\,\mathrm{S}r + e\psi\mathrm{V}r - \mathrm{S}\epsilon'\psi\mathrm{V}r - \mathrm{V}\epsilon'\phi'\mathrm{V}\epsilon\mathrm{V}r\,; \\ Gr = (m' - \chi\epsilon)\,\mathrm{S}r + (e\chi + \psi)\mathrm{V}r - \mathrm{S}\epsilon'\chi\mathrm{V}r - \mathrm{V}\epsilon'\mathrm{V}\epsilon\mathrm{V}r\,; \\ Hr = (m'' - \epsilon)\,\mathrm{S}r + (e + \chi)\mathrm{V}r - \mathrm{S}\epsilon' r\,; \end{cases}$$

and n, n', n'', n''' are *four scalar constants*, namely,

$$\text{XIII.} \ldots \begin{cases} n = em - \mathrm{S}\epsilon'\psi\epsilon \text{ (as in 364, XXIX)}\,; \\ n' = m + em' - \mathrm{S}\epsilon'\chi\epsilon\,; \\ n'' = m' + em'' - \mathrm{S}\epsilon'\epsilon\,; \\ n''' = m'' + e. \end{cases}$$

(4.) Developing then the symbolical equation VII., with the help of X. and XI., and comparing powers of c, we obtain these new symbolical equations (comp. 350, XVI. XXI. XXIII.):

$$\text{XIV.} \ldots \begin{cases} H = n''' - f\,; \\ G = n'' - fH = n'' - n'''f + f^2\,; \\ F = n' - fG = n' - n''f + n'''f^2 - f^3\,; \end{cases}$$

and finally,

$$\text{XV.} \ldots n = Ff = n'f - n''f^2 + n'''f^3 - f^4,$$

which is only another way of writing the *symbolic and biquadratic equation* I.

(5.) *Other functional relations* exist, between these various symbols of operation, which we cannot here delay to develope: but we may remark that, as in the theory of linear and *vector* functions, these usually introduce a mixture of functions with their *conjugates* (comp. 347, XI., &c.).

(6.) This seems however to be a proper place for observing, that if we write, as temporary notations, for *any four quaternions*, p, q, r, s, the equations,

$$\text{XVI.} \ldots [pq] = pq - qp\,; \qquad \text{XVII.} \ldots (pqr) = \mathrm{S}.p[qr]\,;$$

$$\text{XVIII.} \ldots [pqr] = (pqr) + [rq]\,\mathrm{S}p + [pr]\,\mathrm{S}q + [qp]\,\mathrm{S}r\,;$$

and

$$\text{XIX.} \ldots (pqrs) = \mathrm{S}.p[qrs],$$

so that $[pq]$ is a vector, (pqr) and $(pqrs)$ are scalars, and $[pqr]$ is a quaternion, we shall have, in the first place, the relations:

$$\text{XX.} \ldots [pq] = -[qp], \quad [pp] = 0\,;$$

$$\text{XXI.} \ldots (pqr) = -(qpr) = (qrp) = \&\text{c.}, \quad (ppr) = 0\,;$$

$$\text{XXII.} \ldots [pqr] = -[qpr] = [qrp] = \&\text{c.}, \quad [ppr] = 0\,;$$

and $\quad \text{XXIII.} \ldots (pqrs) = -(qprs) = (qrps) = -(qrsp) = \&\text{c.}, \quad (pprs) = 0.$

(7.) In the next place, if t be *any fifth quaternion*, the *quaternion equation*,

$$\text{XXIV.} \ldots 0 = p\,(qrst) + q\,(rstp) + r\,(stpq) + s\,(tpqr) + t\,(pqrs),$$

which may also be thus written,

$$\text{XXV.} \ldots q\,(prst) = p\,(qrst) + r\,(pqst) + s\,(prqt) + t\,(prsq),$$

and which is analogous to the *vector equation*,

$$\text{XXVI.} \ldots 0 = \alpha\mathrm{S}\beta\gamma\delta - \beta\mathrm{S}\gamma\delta\alpha + \gamma\mathrm{S}\delta\alpha\beta - \delta\mathrm{S}\alpha\beta\gamma,$$

or to the continually* occurring transformation (comp. 294, XIV.),

$$\text{XXVII.}\ldots\ \delta\mathrm{S}\alpha\beta\gamma = \alpha\mathrm{S}\delta\beta\gamma + \beta\mathrm{S}\alpha\delta\gamma + \gamma\mathrm{S}\alpha\beta\delta,$$

is satisfied *generally*, because it is satisfied for the *four distinct suppositions*,

$$\text{XXVIII.}\ldots\ q = p, \quad q = r, \quad q = s, \quad q = t.$$

(8.) In the third place, we have this *other general quaternion equation*,

$$\text{XXIX.}\ldots\ q(prst) = [rst]\mathrm{S}pq - [stp]\mathrm{S}rq + [tpr]\mathrm{S}sq - [prs]\mathrm{S}tq,$$

which is analogous to this *other*† *useful vector formula* (comp. 294, XV.),

$$\text{XXX.}\ldots\ \delta\mathrm{S}\alpha\beta\gamma = \mathrm{V}\beta\gamma\mathrm{S}\alpha\delta + \mathrm{V}\gamma\alpha\mathrm{S}\beta\delta + \mathrm{V}\alpha\beta\mathrm{S}\gamma\delta;$$

because the equation XXIX. gives true results, when it is operated on by the *four distinct symbols* (comp. 312),

$$\text{XXXI.}\ldots\ \mathrm{S}.p, \quad \mathrm{S}.r, \quad \mathrm{S}.s, \quad \mathrm{S}.t.$$

(9.) *Assuming* then *any four quaternions*, p, r, s, t, which are *not* connected by the relation,

$$\text{XXXII.}\ldots\ (prst) = 0,$$

and *deducing* from them *four others*, p', r', s', t', by the equations,

$$\text{XXXIII.}\ldots\ \begin{cases} p'(prst) = f[rst], & r'(prst) = -f[stp], \\ s'(prst) = f[tpr], & t'(prst) = -f[prs], \end{cases}$$

in which f is still supposed to be a symbol of linear and quaternion operation on a quaternion, the formula XXIX. allows us to write generally, as an expression for the function fq, which may here be denoted by q' (because r is now otherwise used):

$$\text{XXXIV.}\ldots\ q' = fq = p'\mathrm{S}pq + r'\mathrm{S}rq + s'\mathrm{S}sq + t'\mathrm{S}tq;$$

and its *sixteen scalar constants* (comp. 364, (2.)) are now those which are involved in its *four quaternion constants*, p', r', s', t'.

(10.) Operating on this last equation with the four symbols,

$$\text{XXXV.}\ldots\ \mathrm{S}.[r's't'], \quad \mathrm{S}.[s't'p'], \quad \mathrm{S}.[t'p'r'], \quad \mathrm{S}.[p'r's'],$$

we obtain the four following results:

$$\text{XXXVI.}\ldots\ \begin{cases} (q'r's't') = (p'r's't')\mathrm{S}pq; & (q's't'p') = (r's't'p')\mathrm{S}rq; \\ (q't'p'r') = (s't'p'r')\mathrm{S}sq; & (q'p'r's') = (t'p'r's')\mathrm{S}tq; \end{cases}$$

and when the values thus found for the four scalars,

$$\text{XXXVII.}\ldots\ \mathrm{S}pq, \quad \mathrm{S}rq, \quad \mathrm{S}sq, \quad \mathrm{S}tq,$$

are substituted in the formula XXIX., we have the following *new formula of quaternion inversion*:

$$\text{XXXVIII.}\ldots\ (p'r's't')\,(prst)q = (p'r's't')\,(prst)f^{-1}q'$$
$$= [rst]\,(q'r's't') + [stp]\,(q's't'p') + [tpr]\,(q't'p'r') + [prs]\,(q'p'r's');$$

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* The equations XXVII. and XXX., which had been proved under slightly different forms in the sub-articles to 294, have been in fact freely employed as transformations in the course of the present Chapter, and are supposed to be *familiar* to the student. Compare the Note to page 437.

† Compare the Note immediately preceding.

which shows, in a new way, how to *resolve a linear equation in quaternions*, when put under what we may call (comp. 347, (1.)) the *Standard Quadrinomial Form*, XXXIV.

(11.) Accordingly, if we operate on the formula XXXVIII. with f, attending to the equations XXXIII., and dividing by $(prst)$, we get this new equation,

$$\text{XXXIX.} \ldots (p'r's't')fq = p'(q'r's't') - r'(q's't'p') + s'(q't'p'r') - t'(q'p'r's');$$

whence

$$fq = q', \text{ by XXV.}$$

(12.) It has been remarked (9.), that p, r, s, t, in recent formulæ, may be *any four quaternions*, which do not satisfy the equation XXXII.; we may therefore assume,

$$\text{XL.} \ldots p = 1, \quad r = i, \quad s = j, \quad t = k,$$

with the laws of 182, &c., for the symbols i, j, k, because those laws give here,

$$\text{XLI.} \ldots (1ijk) = -2;$$

and then it will be found that the equations XXXIII. give simply,

$$\text{XLII.} \ldots p' = f1, \quad r' = -fi, \quad s' = -fj, \quad t' = -fk;$$

so that the *standard quadrinomial form* XXXIV. becomes, with this selection of $prst$,

$$\text{XLIII.} \ldots fq = f1 . Sq - fi . Siq - fj . Sjq - fk . Skq,$$

and admits of an immediate verification, because *any quaternion*, q, may be expressed (comp. 221) by the *quadrinomial*,

$$\text{XLIV.} \ldots q = Sq - iSiq - jSjq - kSkq.$$

(13.) Conversely, if we *set out* with the expression,

$$\text{XLV.} \ldots q = w + ix + jy + kz, \qquad 221, \text{ III.},$$

which gives,

$$\text{XLVI.} \ldots fq = wf1 + xfi + yfj + zfk,$$

or briefly,

$$\text{XLVII.} \ldots e = aw + bx + cy + dz,$$

the letters $abcde$ being *here* used to denote *five known quaternions*, while $wxyz$ are *four sought scalars*, the *problem* of *quaternion inversion* comes to be that of the *separate determination* (comp. 312) of these *four* scalars, so as to satisfy the *one* equation XLVII.; and it is *resolved* (comp. XXV.) by the system of the four following formulæ:

$$\text{XLVIII.} \ldots \begin{cases} w(abcd) = (ebcd); & x(abcd) = (aecd); \\ y(abcd) = (abed); & z(abcd) = (abce); \end{cases}$$

the notations (6.) being retained.

(14.) Finally it may be shown, as follows, that the *biquadratic equation* I., for linear functions of *quaternions*, *includes** the *cubic* I'., or 350, I., for *vectors*. Sup-

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* In like manner it may be said, that the *cubic* equation *includes* a *quadratic* one, when we confine ourselves to the consideration of *vectors* in *one plane*; for which case $m = 0$, and also $\psi\rho = 0$, if ρ be a line in the given plane: for we have then $\phi\chi = m' - \psi = m'$, or

$$\phi^2 - m''\phi + m' = 0,$$

pose, for this purpose, that the linear and quaternion function, fq, reduces itself to the last term of the general expression 364, XII., or becomes,

$$\text{XLIX.} \ldots fq = \phi \mathrm{V} q, \quad \text{so that} \quad \text{L.} \ldots e = 0, \quad \varepsilon = \varepsilon' = 0, \quad f1 = f'1 = 0;$$

the coefficients n, n', n'', n''' take then, by XIII., the values,

$$\text{LI.} \ldots n = 0, \quad n' = m, \quad n'' = m', \quad n''' = m'';$$

and the biquadratic I. becomes,

$$\text{LII.} \ldots 0 = (-m + m'f - m''f^2 + f^3)f.$$

But fq is now a *vector*, by XLIX., and it *may* be *any* vector, ρ; also the *operation* f is now equivalent to that denoted by ϕ, when the *subject* of the operation is a vector; we may therefore, in the case here considered, write this last equation LII. under the form,

$$\text{LIII.} \ldots 0 = (-m + m'\phi - m''\phi^2 + \phi^3)\rho,$$

which agrees with 351, I., and reproduces the *symbolical cubic*, when the symbol of the *operand* (ρ) is suppressed.

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CHAPTER III.

ON SOME ADDITIONAL APPLICATIONS OF QUATERNIONS, WITH SOME CONCLUDING REMARKS.

SECTION 1.—*Remarks Introductory to this Concluding Chapter.*

366. WHEN the *Third Book* of the present *Elements* was begun, it was hoped (277) that this Book might be made a much shorter one, than either of the two preceding. *That* purpose it was found impossible to accomplish, without injustice to the subject; but at least an intention was expressed (317), at the commencement of the *Second Chapter*, of rendering that Chapter the *last:* while some new *Examples* of Geo-

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with this understanding as to the operand. In fact, the *cubic* gives here (because $m = 0$),

$$(\phi^2 - m''\phi + m')\phi\rho = 0;$$

and therefore $$(\phi^2 - m''\phi + m')\sigma = 0;$$

if σ be already the result of an operation with ϕ, on any vector ρ: that is if it be, as above supposed, a line in the given plane.

metrical Applications, and some few *Specimens* of *Physical* ones, were promised.

367. The promise, thus referred to, has been perhaps already in part redeemed; for instance, by the investigations (315) respecting certain *tangents*, *normals*, *areas*, *volumes*, and *pressures*, which have served to illustrate certain portions of the theory of *differentials* and *integrals* of quaternions. But it may be admitted, that the six preceding Sections have treated chiefly of that *Theory of Quaternion Differentials*, including of course its *Principles* and *Rules;* and of the connected and scarcely less important *theory* of *Linear* or *Distributive Functions*, of Vectors and Quaternions: *Examples* and *Applications* having thus played hitherto a merely *subordinate* or *illustrative* part, in the progress of the present Volume.

368. Such was, indeed, *designed* from the outset to be, *upon the whole*, the result of the present undertaking: which was rather to *teach*, than to *apply*, the *Calculus of Quaternions*. Yet it still appears to be possible, without quite exceeding suitable limits, and accordingly we shall now endeavour, to condense into a short *Third Chapter* some *Additional Examples*, geometrical and physical, of the *application* of the *principles* and *rules* of that Calculus, supposed to be already *known*, and even to have become by this time *familiar** to the reader. And then, with a few general remarks, the work may be brought to its close.

SECTION 2.—*On Tangents and Normal Planes to Curves in Space.*

369. It was shown (100) towards the close of the First Book, that if the *equation* of a *curve in space*, whether *plane* or of *double* curvature, be *given* under the form,

$$\text{I.} \ldots \rho = \phi(t) = \phi t,$$

where t is a scalar variable, and ϕ is a functional sign, then the *derived vector*,

$$\text{II.} \ldots \mathrm{D}\rho = \mathrm{D}\phi t = \phi' t = \rho' = \mathrm{d}\rho : \mathrm{d}t,$$

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* Accordingly, even *references* to former Articles will now be supplied more sparingly than before.

represents a *line* which is, or is parallel to, the *tangent* to the curve, drawn at the extremity of the variable vector ρ. If then we suppose that T is a point situated upon the tangent thus drawn to a curve PQ at P and that U is a point in the corresponding normal plane, so that the angle TPU is right, and if we denote the vectors OP, OT, OU by ρ, τ, υ, the *equations* of the *tangent line* and *normal plane* at P may now be thus expressed:

$$\text{III.} \ldots \mathrm{V}(\tau-\rho)\rho'=0; \qquad \text{IV.} \ldots \mathrm{S}(\upsilon-\rho)\rho'=0;$$

the *vector* τ being treated as the *only variable* in III., and in like manner υ as the only variable in IV., when once the *curve* PQ is *given*, and the *point* P is *selected*.

(1.) It is permitted, however, to express these last equations under *other forms;* for example, we may replace ρ' by $\mathrm{d}\rho$, and thus write, for the same tangent line and normal plane,

$$\text{V.} \ldots \mathrm{V}(\tau-\rho)\mathrm{d}\rho=0; \qquad \text{VI.} \ldots \mathrm{S}(\upsilon-\rho)\mathrm{d}\rho=0;$$

where the *vector differential* $\mathrm{d}\rho$ may represent *any line, parallel to the tangent* to the curve at P, and is *not necessarily small* (compare again 100).

(2.) We may also write, as the equation of the tangent,

$$\text{VII.} \ldots \tau=\rho+x\rho', \text{ where } x \text{ is a scalar variable;}$$

and as the equation of the normal plane,

$$\text{VIII.} \ldots \mathrm{d}_\rho\mathrm{T}(\upsilon-\rho)=0, \quad \text{or} \quad \text{VIII}'. \ldots \mathrm{dT}(\upsilon-\rho)=0, \quad \text{if} \quad \mathrm{d}\upsilon=0;$$

because this *partial differential* of $\mathrm{T}(\upsilon-\rho)$, or of $\overline{\text{PU}}$, is (by 334, XII., &c.),

$$\text{IX.} \ldots \mathrm{dT}(\upsilon-\rho)=\mathrm{S}(\mathrm{U}(\upsilon-\rho).\mathrm{d}\rho).$$

(3.) For the *circular locus* 314, (1.), or 337, (1.), of which the equation is,

$$\text{X.} \ldots \rho=\alpha^t\beta, \quad \text{with} \quad \mathrm{T}\alpha=1, \quad \text{and} \quad \mathrm{S}\alpha\beta=0,$$

the equation of the tangent is, by VII., and by the value 337, VI. of ρ',

$$\text{XI.} \ldots \tau=\rho+y\alpha\rho, \quad \text{where} \quad y \text{ is a new scalar variable;}$$

the *perpendicularity* of the *tangent* to the *radius* being thus put in evidence.

(4.) For the *plane* but *elliptic locus*, 314, (2.), or 337, (2.), for which,

$$\text{XII.} \ldots \rho=\mathrm{V}.\alpha^t\beta, \quad \text{with} \quad \mathrm{T}\alpha=1, \quad \text{but } \textit{not} \quad \mathrm{S}\alpha\beta=0,$$

the value 337, VIII. of ρ' shows that the tangent, at the extremity of any *one* semidiameter ρ, is *parallel* to the *conjugate* semidiameter of the curve; that is, to the one obtained by altering the *excentric anomaly* (314, (2.)), by a *quadrant:* or to the value of ρ which results, when we change t to $t+1$.

(5.) For the *helix*, 314, (10.), of which the equation is,

$$\text{XIII.} \ldots \rho=ct\alpha+\alpha^t\beta, \quad \text{with} \quad \mathrm{T}\alpha=1, \quad \text{and} \quad \mathrm{S}\alpha\beta=0,$$

c being a scalar constant, we have the derived vector,

$$\text{XIV.} \ldots \rho'=c\alpha+\frac{\pi}{2}\alpha^{t+1}\beta; \quad \text{whence} \quad \text{XV.} \ldots \mathrm{S}\alpha^{-1}\rho'=c,$$

$$\text{XVI.} \ldots \mathrm{TV}\alpha^{-1}\rho'=\frac{\pi}{2}\mathrm{T}\beta, \quad \text{and} \quad \text{XVII.} \ldots (\mathrm{TV}:\mathrm{S})\alpha^{-1}\rho'=\frac{\pi\mathrm{T}\beta}{2c};$$

the *tangent line* (ρ') to the *helix* is therefore inclined to the *axis* (α) of the *cylinder* whereon that curve is traced, at a *constant angle* (a), whereof the *trigonometrical tangent* ($\tan a$) is given by this formula XVII.; and accordingly, the numerator $\pi T\beta$ of that formula represents the *semicircumference* of the cylindric *base;* while the denominator $2c$ is an expression for *half* the *interval* between two *successive spires*, measured in a direction parallel to the axis. We may then write,

$$\text{XVIII.} \ldots \pi T\beta = 2c \tan a = 2c \cot b,$$

if a thus denote the *constant inclination* of the helix to the *axis*, while b denotes the constant and complementary inclination of that curve to the *base*, or to the *circles* which it crosses on the cylinder.

(6.) In general, the *parallels* ρ' to the *tangents* to a curve of *double curvature*, which are drawn from a *fixed origin* O, have a certain *cone* for their *locus;* and for the case of the *helix*, the *equation* of this cone is given by the formula XVII., or by any legitimate transformation thereof, such as the following,

$$\text{XIX.} \ldots SU\alpha^{-1}\rho' = \pm \cos a = \pm \sin b;$$

it is therefore, in this case, a *cone of revolution*, with its semiangle $= a$.

(7.) As an example of the determination of a *normal plane* to a curve of double curvature, we may observe that the equation XIII. of the helix gives,

$$\text{XX.} \ldots \rho^2 = \beta^2 - c^2t^2, \quad \text{and therefore} \quad \text{XXI.} \ldots S\rho\rho' = -c^2t;$$

the equation IV. becomes therefore, for the case of this curve,

$$\text{XXII.} \ldots 0 = S\rho'\upsilon + c^2t, \text{ with the value XIV. of } \rho'.$$

(8.) If then it be required to assign the point U in which the normal plane to the helix *meets the axis* of the cylinder, we have only to combine this equation XXII. with the condition $\upsilon \parallel \alpha$, and we find, by XIII. and XIV.,

$$\text{XXIII.} \ldots \text{OU} = \upsilon = -c^2t\alpha : S\alpha\rho' = ct\alpha, \qquad \text{XXIV.} \ldots S\alpha(\upsilon - \rho) = 0;$$

the line PU is therefore *perpendicular* to the axis, being in fact a *normal* to the *cylinder*.

370. *Another view* of *tangents* and *normal planes* may be proposed, which shall connect them in calculation with *Taylor's Series* adapted to quaternions (342), as follows.

(1.) Writing $\quad \text{I.} \ldots \rho_t = \rho_0 + u_t t\rho'_0, \quad$ or briefly, $\quad \text{I}'. \ldots \rho_t = \rho + ut\rho'$,

the *coffiecient* u_t or u will *generally* be a *quaternion*, but its *limiting value* will be *positive unity*, when t tends to *zero* as its limit; or in symbols,

$$\text{II.} \ldots u_0 = \lim_{t=0} u = 1.$$

(2.) Admitting this, which follows either from Taylor's Series, or (in so simple a case) from the mere *definition* of the *derived vector* ρ', we may conceive that vector ρ' to be *constructed* by *some* given line PT, without *yet* supposing it to be *known* that this *line* is *tangential* at P to the *curve* PQ, of which the *variable vector* is $\text{OQ} = \rho_t$, while $\text{OP} = \rho_0 = \rho$, so that the line $\text{PQ} = ut\rho'$ is a *vector chord* from P, which *diminishes* indefinitely with the *scalar variable*, t, and is *small*, if t be small.

(3.) Conceiving next that $\omega = \text{OR} =$ the vector of some new and arbitrary point R, we may let fall a perpendicular QM on the line PR, and so *decompose* the *chord* PQ into the *two rectangular lines*, PM and MQ; which, when divided by the same chord, give *rigorously* the *two* (generally) *quaternion quotients*,

$$\text{III.} \ldots \frac{\text{PM}}{\text{PQ}} = \frac{\mathrm{S}u\rho'(\omega-\rho)}{u\rho'(\omega-\rho)}, \qquad \text{IV.} \ldots \frac{\text{MQ}}{\text{PQ}} = \frac{\mathrm{V}u\rho'(\omega-\rho)}{u\rho'(\omega-\rho)};$$

the variable t thus disappearing through the division, except so far as it enters into u, which *tends* as above to 1.

(4.) Passing then to the *limits*, we have these *other* rigorous equations,

$$\text{V.} \ldots \lim. \frac{\text{PM}}{\text{PQ}} = \frac{\mathrm{S}\rho'(\omega-\rho)}{\rho'(\omega-\rho)}, \qquad \text{VI.} \ldots \lim. \frac{\text{MQ}}{\text{PQ}} = \frac{\mathrm{V}\rho'(\omega-\rho)}{\rho'(\omega-\rho)};$$

by comparing which with 369, III. and IV., we see that those two equations represent respectively, as before stated, the *tangent* and the *normal plane* to the proposed curve at P; because, if $\mathrm{V}\rho'(\omega-\rho)=0$, the *chord* PQ *tends*, by V. or VI., to *coincide*, both in *length* and in *direction*, with its *projection* PM on the line PR; while on the other hand, if $\mathrm{S}\rho'(\omega-\rho)=0$, *that* projection tends to *vanish*, even as *compared* with the chord PQ; which chord tends *now* to coincide with its *other* projection MQ, or with the *perpendicular* to the line PR, erected so as to reach the point Q: whence PR must, in this last case, be a *normal* to the *curve* at P.

(5.) We may also investigate an equation for the *normal plane*, by considering it as the *limiting position* of the plane which *perpendicularly bisects the chord*. If R be supposed to be a point of this last plane, then, with the recent notations, the vector $\omega = \text{OR}$ must satisfy the condition,

$$\text{VII.} \ldots \mathrm{T}(\omega-\rho_t)=\mathrm{T}(\omega-\rho_0), \quad \text{or} \quad \text{VIII.} \ldots (\omega-\rho-ut\rho')^2=(\omega-\rho)^2,$$

or

$$\text{IX.} \ldots 2\mathrm{S}u\rho'(\omega-\rho)=t(u\rho')^2,$$

in which it may be noted that $u\rho'$ is a *vector* (in the direction of the *chord*, PQ), although u itself is generally a *quaternion*, as before: such then is the equation of the *bisecting plane*, with ω for its variable vector, and its *limit*,

$$\text{X.} \ldots \mathrm{S}\rho'(\omega-\rho)=0, \ \text{as before.}$$

(6.) The last process may also be presented under the form,

$$\text{XI.} \ldots 0=\lim. t^{-1}\{\mathrm{T}(\omega-\rho_t)-\mathrm{T}(\omega-\rho_0)\}=\mathrm{D}_t\mathrm{T}(\omega-\rho_t), \quad \text{when} \quad t=0;$$

and thus the equation 369, VIII. may be obtained anew.

(7.) Geometrically, if we set off on RQ a portion RS equal in length to RP, as in the annexed Figure 76, we shall have the limiting equation,

$$\text{XII.} \ldots \pm\overline{\text{SQ}}:\overline{\text{PQ}}=(\overline{\text{RQ}}-\overline{\text{RP}}):\overline{\text{PQ}}=(\textit{ultimately})-\cos \text{RPT};$$

which agrees with 369, IX.

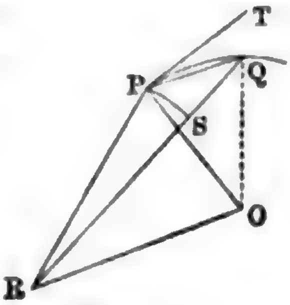

Fig. 76.

(8.) If then the point R be taken *out of* the normal plane at P, this *limit* of the *quotient*, $\overline{\text{RQ}}-\overline{\text{RP}}$ divided by $\overline{\text{PQ}}$, has a *finite value*, positive or negative; and if the *chord* PQ be called *small* of the *first order*, the *difference of distances* of its extremities from R may then be said to be *small* of the *same* (first) order. But if R be taken *in* the normal plane at P (and not coincident with that point P *itself*), this difference of distances may *then* be said to be *small*, of an *order higher than the first:* which answers to the *evanescence* of the *first differential* of the *tensor*, $\mathrm{T}(\omega-\rho)$ in XI., or $\mathrm{T}(\upsilon-\rho)$ in 369, VIII'.

371. A *curve* may occasionally be represented in quaternions, by an equation which is not of the *form*, 369, I., although it must always be conceived capable of *reduction* to that form: for instance, this new equation,

$$\text{I.} \ldots \mathrm{V}\alpha\rho.\mathrm{V}\rho\alpha' = (\mathrm{V}\alpha\alpha')^2, \quad \text{with} \quad \mathrm{TV}\alpha\alpha' > 0,$$

is not *immediately* of the form $\rho = \phi t$, but it is reducible to that form as follows,

$$\text{II.} \ldots \rho = t\alpha + t^{-1}\alpha'.$$

An equation such as I. may therefore have its *differential* or its *derivative* taken, with respect to the scalar variable t on which ρ is thus *conceived* to depend, even if the exact *law* of such dependence be unknown: and $\mathrm{d}\rho$, or ρ', may then be changed to the tangential vector $\omega - \rho$ to which it is parallel, in order to form an *equation* of the *tangent*, or a *condition* which the vector ω of a point on that sought line must satisfy.

(1.) To pass from I. to II., we may first operate with the sign V, which gives,

$$\text{III.} \ldots \rho\mathrm{S}\alpha\alpha'\rho = 0, \quad \text{or simply,} \quad \text{III}'. \ldots \mathrm{S}\alpha\alpha'\rho = 0;$$

whence, t and t' being scalars, we may write,

$$\text{IV.} \ldots \rho = t\alpha + t'\alpha', \quad \mathrm{V}\alpha\rho = t'\mathrm{V}\alpha\alpha', \quad \mathrm{V}\rho\alpha' = t\mathrm{V}\alpha\alpha', \quad tt' = 1,$$

and the required reduction is effected: while the *return* from II. to I., or the *elimination* of the scalar t, is an even easier operation.

(2.) Under the form II., it is at once seen that ρ is the vector of a *plane hyperbola*, with the origin for *centre*, and the lines α, α' for *asymptotes;* and accordingly all the properties of such a curve may be deduced from the expression II., by the rules of the present Calculus.

(3.) For example, since the derivative of that expression is,

$$\text{V.} \ldots \rho' = \alpha - t^{-2}\alpha',$$

the tangent may (comp. 369, VII.) have its equation thus written:

$$\text{VI.} \ldots \omega = (t+x)\alpha + t^{-2}(t-x)\alpha';$$

it intersects therefore the lines α, α' in the points of which the vectors are $2t\alpha$, $2t^{-1}\alpha'$; so that (as is well known) the *intercept*, upon the tangent, between the *asymptotes*, is *bisected* at the point of *contact:* and the intercepted *area* is *constant*, because $\mathrm{V}(t\alpha . t^{-1}\alpha') = \mathrm{V}\alpha\alpha'$, &c.

(4.) But we may also operate *immediately*, as above remarked, on the *form* I.; and thus arrive (by substitution of $\omega - \rho$ for $\mathrm{d}\rho$, &c.) at the *equation of conjugation*,

$$\text{VII.} \ldots \mathrm{V}\alpha\omega . \mathrm{V}\rho\alpha' + \mathrm{V}\alpha\rho . \mathrm{V}\omega\alpha' = 2(\mathrm{V}\alpha\alpha')^2,$$

which expresses (comp. 215, (13.), &c.) that if $\rho = \text{OP}$, and $\omega = \text{OR}$, as before, then either R is on the *tangent* to the curve, at the point P, or at least *each* of these two points is situated on the *polar* of the other, with respect to the same hyperbola.

(5.) Again, it is frequently convenient to consider a *curve* as the *intersection* of *two surfaces;* and, in connexion with *this* conception, to represent it by a system of *two scalar equations*, not explicitly involving any *scalar variable:* in which case, *both* equations are to be differentiated, or derivated, with reference to *such* a variable *understood*, and $d\rho$ or ρ' deduced, or replaced by $\omega - \rho$ as before.

(6.) Thus we may substitute, for the equation I., the system of the two following (whereof the first had occurred as III'.);

$$\text{VIII.} \ldots S\alpha\alpha'\rho = 0, \quad \rho^2 S\alpha\alpha' - S\alpha\rho S\alpha'\rho = (V\alpha\alpha')^2;$$

and the derivated equations corresponding are,

$$\text{IX.} \ldots S\alpha\alpha'\rho' = 0, \quad 2S\alpha\alpha' S\rho\rho' - S\alpha\rho' S\alpha'\rho - S\alpha\rho S\alpha'\rho' = 0;$$

or, with the substitution of $\omega - \rho$ for ρ', &c.,

$$\text{X.} \ldots S\alpha\alpha'\omega = 0, \quad 2S\alpha\alpha' S\rho\omega - S\alpha\omega S\alpha'\rho - S\alpha\rho S\alpha'\omega = 2(V\alpha\alpha')^2;$$

the last of which might also have been deduced from VII., by operating with S.

(7.) And it may be remarked that the two equations VIII. represent respectively in general a *plane* and an *hyperboloid*, of which the *intersection* (5.) is the *hyperbola* I. or II.; or a plane and an *hyperbolic cylinder*, if $S\alpha\alpha' = 0$.

SECTION 3.—*On Normals and Tangent Planes to Surfaces.*

372. It was early shown (100, (9.)), that when a *curved surface* is represented by an equation of the form,

$$\text{I.} \ldots \rho = \phi(x, y),$$

in which ϕ is a functional sign, and x, y are two independent and scalar variables, then either the *two partial differentials*, or the two *partial derivatives*, of the *first order*,

$$\text{II.} \ldots d_x\rho,\ d_y\rho, \quad \text{or} \quad \text{III.} \ldots D_x\rho,\ D_y\rho,$$

represent *two tangential vectors*, or at least vectors *parallel* to *two tangents* to the surface, drawn at the extremity or *term* P of ρ; so that the *plane* of these two differential vectors, or of lines parallel to them, is (or is parallel to) the *tangent plane* at that point: and the principle has been since exemplified, in 100, (11.) and (12.), and in the sub-articles to 345, &c. It follows that any *vector* ν, which is *perpendicular to both* of two such *non-parallel* differentials, or derivatives, must (comp. 345, (11.)) be a *normal vector* at P, or at least one having the *direction* of the normal to the surface at that point; so that each of the two vectors,

$$\text{IV.} \ldots V.d_x\rho d_y\rho, \qquad \text{V.} \ldots V.D_x\rho D_y\rho,$$

if *actual*, represents such a normal.

(1.) As an additional example, let us take the case of the ***ruled paraboloid***, on which a given ***gauche quadrilateral*** ABCD is ***superscribed***. The expression for the vector ρ of a variable point P of this surface, considered as a function of two independent and scalar variables, x and y, may be thus written (comp. 99, (9.)):

$$\text{VI.} \ldots \rho = xy\alpha + (1-x)y\beta + (1-x)(1-y)\gamma + x(1-y)\delta;$$

where the supposition $y = 1$ places the point P on the line AB; $x = 0$ places it on BC; $y = 0$, on CD; and $x = 1$, on DA.

(2.) We have here, by partial derivations,

$$\text{VII.} \ldots \mathrm{D}_x\rho = y(\alpha-\beta) + (1-y)(\delta-\gamma); \quad \mathrm{D}_y\rho = x(\alpha-\delta) + (1-x)(\beta-\gamma);$$

these then represent the directions of ***two*** distinct ***tangents*** to the paraboloid VI., at what may be called ***the point*** (x, y); whence it is easy to deduce the ***tangent plane*** and the ***normal*** at that point, by constructions on which we cannot here delay, except to remark that if (comp. Fig. 81, Art. 98) we draw two right lines, QS and RT, through P, so as to cut the sides AB, BC, CD, DA of the quadrilateral in points Q, R, S, T, we shall have by VI. the vectors,

$$\text{VIII.} \ldots \begin{cases} \mathrm{OQ} = x\alpha + (1-x)\beta, & \mathrm{OR} = y\beta + (1-y)\gamma, \\ \mathrm{OS} = x\delta + (1-x)\gamma, & \mathrm{OT} = y\alpha + (1-y)\delta, \end{cases}$$

and therefore, by VII.,

$$\text{IX.} \ldots \mathrm{D}_x\rho = \mathrm{RT}, \quad \mathrm{D}_y\rho = \mathrm{SQ};$$

so that ***these*** two tangents are simply the ***two generating lines*** of the surface, which pass through the proposed point P.

(3.) For example, at the point (1, 1), or A, the ***tangents*** thus found are the ***sides*** BA, DA, and the ***tangent plane*** is that of the ***angle*** BAD, as indeed is evident from geometry.

(4.) Again, the equation of the ***screw surface*** (comp. 314, XVI.),

$$\text{X.} \ldots \rho = cx\alpha + y\alpha^x\beta, \quad \text{with} \quad \mathrm{T}\alpha = 1, \quad \text{and} \quad \mathrm{S}\alpha\beta = 0,$$

gives the two tangents,

$$\text{XI.} \ldots \mathrm{D}_x\rho = c\alpha + \frac{\pi}{2}y\alpha^{x+1}\beta, \quad \mathrm{D}_y\rho = \alpha^x\beta,$$

whereof the latter is perpendicular to the former, and to the axis α of the cylinder; so that the corresponding ***normal*** to the surface X. at the point (x, y) is represented by the product,

$$\text{XII.} \ldots \nu = \mathrm{D}_x\rho \,.\, \mathrm{D}_y\rho = c\alpha^{x+1}\beta + \frac{\pi}{2}y\beta^2\alpha.$$

373. Whenever a variable vector ρ is thus expressed or even *conceived* to be expressed, as a *function* of *two* scalar variables, x and y (or s and t, &c.), if we assume any *three* diplanar vectors, such as α, β, γ (or ι, κ, λ, &c.), the *three scalar expressions*, $\mathrm{S}\alpha\rho$, $\mathrm{S}\beta\rho$, $\mathrm{S}\gamma\rho$ (or $\mathrm{S}\iota\rho$, $\mathrm{S}\kappa\rho$, $\mathrm{S}\lambda\rho$, &c.) will then be functions of the same *two* scalar variables; and will therefore be *connected* with each other by some *one* scalar *equation*, of the form,

$$\text{I.} \ldots F(\mathrm{S}\alpha\rho, \mathrm{S}\beta\rho, \mathrm{S}\gamma\rho) = 0,$$

or briefly,

$$\text{II.} \ldots f\rho = C;$$

where C is a scalar constant, introduced (instead of zero) for greater generality of expression; and F, f are used as functional but scalar signs. If then (comp. 361, XIV.) we express the *first differential* of this scalar function $f\rho$ under the form,

$$\text{III.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho,$$

in which ν is a certain *derived vector*, and is *here* considered as being (at least implicitly) a *vector function* (like ρ) of the *two scalar variables* above mentioned, we shall have the two equations,

$$\text{IV.} \ldots \mathrm{S}\nu\mathrm{d}_x\rho = 0, \quad \mathrm{S}\nu\mathrm{d}_y\rho = 0,$$

or these two other and corresponding ones,

$$\text{V.} \ldots \mathrm{S}\nu\mathrm{D}_x\rho = 0, \quad \mathrm{S}\nu\mathrm{D}_y\rho = 0;$$

from which it follows (by 372) that ν has the direction of the normal to the surface I. or II., at the point P in which the vector ρ terminates. Hence the *equation* of that *normal* (with ω for its variable vector) may, under these conditions, be thus written:

$$\text{VI.} \ldots \mathrm{V}\nu(\omega - \rho) = 0;$$

and the corresponding *equation of the tangent plane* at the same point P is,

$$\text{VII.} \ldots \mathrm{S}\nu(\omega - \rho) = 0.$$

(1.) For example, if we take the expression 308, XVIII., or 345, XII., namely

$$\text{VIII.} \ldots \rho = rk^t j^s k j^{-s} k^{-t}, \quad \text{in which} \quad kj^{-s} = j^s k, \ \&\text{c.},$$

treating the scalar r as constant, but s and t as variable, we have then (comp. 345, XIV.), the equations, α denoting any unit-vector,

$$\text{IX.} \ldots \mathrm{S}i\rho = r\mathrm{S}\,.\,\alpha^{2t}\mathrm{S}\,.\,\alpha^{2s+1}, \quad \mathrm{S}j\rho = r\mathrm{S}\,.\,\alpha^{2t-1}\mathrm{S}\,.\,\alpha^{2s+1}, \quad \mathrm{S}k\rho = r\mathrm{S}\,.\,\alpha^{2s+2};$$

between which s and t can be eliminated, by simply adding their squares, because $(\alpha^t)^2 + (\alpha^{t-1})^2 = 1$, by 315, V., if $\mathrm{T}\alpha = 1$. In this manner then we arrive at equations of the forms I. and II., namely (comp. 357, VII., and 308, (10.) and (13.)),

$$\text{X.} \ldots (\mathrm{S}i\rho)^2 + (\mathrm{S}j\rho)^2 + (\mathrm{S}k\rho)^2 - r^2 = 0,$$

and $$\text{XI.} \ldots f\rho = \rho^2 = -r^2 = \text{const.}, \quad \text{or} \quad \text{XI}'. \ldots \mathrm{T}\rho = r;$$

which last results had indeed been otherwise obtained before.

(2.) With this *form* XI. of $f\rho$, we have the *differential expression* of the first order,

$$\text{XII.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho = 2\mathrm{S}\rho\mathrm{d}\rho, \quad \text{whence} \quad \text{XIII.} \ldots \nu = \rho;$$

and if we *still conceive* that ρ is, as above, *some vector function* of *two scalar variables*, s and t, although the *particular law* VIII. of its dependence on them may now be supposed to be *unknown* (or to be forgotten), we may write also,

$$\text{XIV.} \ldots \tfrac{1}{2}\mathrm{d}f\rho = \mathrm{S}\nu\mathrm{d}\rho = \mathrm{S}\rho\mathrm{d}\rho = \mathrm{S}\rho(\mathrm{d}_s + \mathrm{d}_t)\rho = \mathrm{S}\rho\mathrm{D}_s\rho\,.\,\mathrm{d}s + \mathrm{S}\rho\mathrm{D}_t\rho\,.\,\mathrm{d}t;$$

if then the *function* $f\rho$ have (as above) a *value*, $= -r^2$, which is *constant*, or is *inde-*

pendent of *both* the variables, s and t, while *their* differentials are *arbitrary*, and are independent of *each other*, we shall thus have separately (comp. V., and 337, XIII., XVII.),

$$\text{XV.} \ldots \ \mathrm{S}\rho\mathrm{D}_s\rho = 0, \quad \mathrm{S}\rho\mathrm{D}_t\rho = 0.$$

The *radius* ρ of the *sphere* XI. is therefore in this way seen to have the direction of the *normal* at its own extremity, because it is *perpendicular to two distinct tangents*, $\mathrm{D}_s\rho$ and $\mathrm{D}_t\rho$, at that point; which are indeed, in the present case, perpendicular to *each other* also (337, (8.)).

(3.) Instead of treating the *two* scalar variables, x and y, or s and t, &c., as *both* entirely *arbitrary* and *independent*, we may conceive that *one* is an *arbitrary* (but scalar) *function* of the *other;* and *then* the vector ν, determined by the equation III., will be seen anew to be the *normal* at the extremity P of ρ, because it is *perpendicular* to the *tangent* at P to an *arbitrary curve* upon the surface, which passes through that point: or (otherwise stated) because it is a line in an *arbitrary normal plane* at P, if a normal plane to a *curve* on a surface be called (as usual) a normal plane to that *surface* also.

(4.) For example, if we conceive that s in VIII. is thus an arbitrary function of t, the last expression XIV. will take the form,

$$\text{XVI.} \ldots \ 0 = \tfrac{1}{2}\mathrm{d}f\rho = \mathrm{S}.\rho(s'\mathrm{D}_s\rho + \mathrm{D}_t\rho)\mathrm{d}t, \quad \text{if} \quad \mathrm{d}s = s'\mathrm{d}t;$$

whence, $\mathrm{d}t$ being still arbitrary, we have the *one* scalar equation,

$$\text{XVII.} \ldots \ \mathrm{S}.\rho(s'\mathrm{D}_s\rho + \mathrm{D}_t\rho) = 0, \quad \text{or} \quad \text{XVIII.} \ldots \ \rho \perp s'\mathrm{D}_s\rho + \mathrm{D}_t\rho,$$

and although, on account of the arbitrary coefficient s', this *one* equation XVII. is *equivalent* to the system of the *two* equations XV., yet it *immediately* signifies, as in XVIII., that the *directed radius* ρ, of the sphere XI., is *perpendicular* to the *arbitrary tangent*, $s'\mathrm{D}_s\rho + \mathrm{D}_t\rho$; or to the tangent to an *arbitrary spherical curve* through P, the centre O and tensor $\mathrm{T}\rho$ (or *undirected radius*, r) remaining as before.

(5.) As regards the *logic* of the subject, it may be worth while to read again the *proof* (331), of the validity of the rule for *differentiating a function of a function;* because this *rule* is virtually employed, when after thus reducing, or conceiving as reduced, the scalar function $f\rho$ of a *vector* ρ, to another scalar function such as Ft of a *scalar* t, by treating ρ as equal to some vector function ϕt of this last scalar, we *infer* that

$$\text{XIX.} \ldots \ \mathrm{d}Ft = \mathrm{d}f\phi t = 2\mathrm{S}.\nu\mathrm{d}\phi t, \quad \text{if} \quad \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho, \text{ as before.}$$

(6.) And as regards the *applications* of the formulæ VI. and VII., or of the equations given by them for the *normal* and *tangent plane* to a *surface* generally, the difficulty is only to *select*, out of a multitude of examples which might be given: yet it may not be useless to add a *few* such here, the case of the *sphere* having of course been only taken to illustrate the *theory*, because the normal property of its *radii* was manifest, independently of any calculation.

(7.) Taking then the equation of the *ellipsoid*, under the form,

$$\text{XX.} \ldots \ \mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad 282, \text{ XIX.},$$

of which the first differential may (see the sub-articles to 336) be thus written,

$$\text{XXI.} \ldots \ 0 = \mathrm{S}.\{(\iota - \kappa)^2\rho + 2(\iota\mathrm{S}\kappa\rho + \kappa\mathrm{S}\iota\rho)\}\mathrm{d}\rho = \mathrm{S}\nu\mathrm{d}\rho,$$

and introducing an auxiliary vector, ON or ξ, such that

$$\text{XXII.} \ldots \ \mathrm{ON} = \xi = -2(\iota - \kappa)^{-2}(\iota\mathrm{S}\kappa\rho + \kappa\mathrm{S}\iota\rho),$$

we have $\nu \parallel \rho - \xi$, and may write, as the equation of the normal at the extremity P of ρ, the following,

$$\text{XXIII.} \ldots \mathrm{V}.(\xi - \rho)(\omega - \rho) = 0, \quad \text{or} \quad \text{XXIV.} \ldots \omega = \rho + x(\xi - \rho),$$

in which x is a scalar variable (comp. 869, VII.); making then $x = 1$, we see that ξ is the vector of the point N in which the normal intersects the plane of the two fixed lines ι, κ, supposed to be drawn from the origin, which is here the *centre* of the ellipsoid.

(8.) If we look back on the sub-articles to 216 and 217, we shall see that these lines ι, κ have the directions of the *two real cyclic normals*, or of the normals to the *two* (real) *cyclic planes;* which planes are now represented by the two equations,

$$\text{XXV.} \ldots \mathrm{S}\iota\rho = 0, \quad \mathrm{S}\kappa\rho = 0.$$

Accordingly the equation XX. of the ellipsoid may be put (comp. 836, 357, 359) under the *cyclic forms*,

$$\begin{aligned}\text{XXVI.} \ldots \mathrm{S}\rho\phi\rho &= (\iota^2 + \kappa^2)\rho^2 + 2\mathrm{S}\iota\rho\kappa\rho \\ &= (\iota - \kappa)^2\rho^2 + 4\mathrm{S}\iota\rho\mathrm{S}\kappa\rho = (\kappa^2 - \iota^2)^2 = \text{const.};\end{aligned}$$

hence each of the *two diametral planes* XXV. cuts the surface in a *circle*, the *common radius* of these two *circular sections* being

$$\text{XXVII.} \ldots \mathrm{T}\rho = \frac{\mathrm{T}\iota^2 - \mathrm{T}\kappa^2}{\mathrm{T}(\iota - \kappa)} = b,$$

where b denotes, as in 219, (1.), the length of the *mean semiaxis* of the ellipsoid; and in fact, this value of $\mathrm{T}\rho$ can be at once obtained from the equation XX., by making either $\iota\rho = -\rho\iota$, or $\rho\kappa = -\kappa\rho$, in virtue of XXV.

(9.) By the sub-article last cited, the *greatest* and *least semiaxes* have for their *lengths*,

$$\text{XXVIII.} \ldots a = \mathrm{T}\iota + \mathrm{T}\kappa, \quad c = \mathrm{T}\iota - \mathrm{T}\kappa;$$

and the *construction* in 219, (2.) shows (by Fig. 53, annexed to 217, (4.)) that these three semiaxes a, b, c have the respective *directions* of the lines,

$$\text{XXIX.} \ldots \iota\mathrm{T}\kappa - \kappa\mathrm{T}\iota, \quad \mathrm{V}\iota\kappa, \quad \iota\mathrm{T}\kappa + \kappa\mathrm{T}\iota;$$

all which agrees with the *rectangular transformation*,

$$\begin{aligned}\text{XXX.} \ldots 1 &= \frac{\mathrm{S}\rho\phi\rho}{(\kappa^2 - \iota^2)^2} = \left(\frac{\mathrm{T}(\iota\rho + \rho\kappa)}{\kappa^2 - \iota^2}\right)^2 \\ &= \left(\frac{\mathrm{S}.\rho\mathrm{U}(\iota\mathrm{T}\kappa - \kappa\mathrm{T}\iota)}{\mathrm{T}\iota + \mathrm{T}\kappa}\right)^2 + \left(\frac{\mathrm{T}(\iota - \kappa)\mathrm{S}.\rho\mathrm{U}\mathrm{V}\iota\kappa}{\mathrm{T}\iota^2 - \mathrm{T}\kappa^2}\right)^2 + \left(\frac{\mathrm{S}.\rho\mathrm{U}(\iota\mathrm{T}\kappa + \kappa\mathrm{T}\iota)}{\mathrm{T}\iota - \mathrm{T}\kappa}\right)^2,\end{aligned}$$

in deducing which (comp. 359, (1.)) from 357, VIII., by means of the formulæ 357, XX. and XXI., we employ the values (comp. XXVI.),

$$\text{XXXI.} \ldots g = \iota^2 + \kappa^2, \quad \lambda = 2\iota, \quad \mu = \kappa.$$

(10.) The *fixed plane* (7.), of the *cyclic normals* ι and κ (8.), is therefore also the plane of the *extreme semiaxes*, a and c (9.), or that which may be called perhaps the *principal plane** of the ellipsoid: namely, the plane of the *generating tri-*

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* This *plane* may also be said to be the plane of the *principal elliptic section* (219, (9.)); or it may be distinguished (comp. the Note to page 231) as the plane of the *focal hyperbola*, of which important curve we shall soon assign the equation in quaternions.

angle (218), (1.)), in that *construction* of the surface (217, (6.) or (7.)) which is illustrated by Fig. 53, and was deduced as an *interpretation of the quaternion equation* XX., or of the somewhat less simple form 217, XVI., with the value $T\iota^2 - T\kappa^2$ of t^2.

(11.) Let n denote the length of that portion of the normal, which is intercepted between the surface and the principal plane (10.), so that, by (7.),

$$\text{XXXII.} \ldots n = \overline{\text{NP}} = T(\rho - \xi), \quad n^2 = -(\rho - \xi)^2,$$

with the value XXII. of ξ. Let $\sigma = \text{OS}$ be the vector of a point S on the surface of a new or auxiliary *sphere*, described about the point N as centre, with a radius $= n$, and therefore *tangential* to the *ellipsoid* at P; and let us inquire in what *curve* or *curves*, real or imaginary, does this sphere *cut* the ellipsoid.

(12.) The *equations* (comp. 371, (5.)) of the sought *intersection* are the two following,

$$\text{XXXIII.} \ldots (\sigma - \xi)^2 + n^2 = 0, \quad \text{and} \quad \text{XXXIV.} \ldots T(\iota\sigma + \sigma\kappa) = \kappa^2 - \iota^2;$$

whereof the first expresses that S is a point of the sphere, and the second that it is a point of the ellipsoid; while ρ or OP enters virtually into XXXIII., through ξ and n, but is *here* treated as a *constant*, the point P being now supposed to be a *given* one.

(13.) We shall *remove* (18) *the origin* to this point P of the ellipsoid, if we write,

$$\text{XXXV.} \ldots \sigma = \rho + \sigma', \quad \text{or} \quad \text{XXXV}'. \ldots \sigma' = \sigma - \rho = \text{PS};$$

and thus we obtain the new or transformed equations,

$$\text{XXXVI.} \ldots 0 = \sigma'^2 + 2S(\rho - \xi)\sigma', \qquad \text{XXXVII.} \ldots 0 = N(\iota\sigma' + \sigma'\kappa) + 2S\nu\sigma';$$

in which (as in (7.), comp. also 210, XX.),

$$\text{XXXVIII.} \ldots \nu = (\iota - \kappa)^2\rho + 2(\iota S\kappa\rho + \kappa S\iota\rho) = (\iota - \kappa)^2(\rho - \xi),$$

and

$$\text{XXXIX.} \ldots N(\iota\sigma' + \sigma'\kappa) = (\iota - \kappa)^2\sigma'^2 + 4S\iota\sigma' S\kappa\sigma'.$$

(14.) Eliminating then σ'^2, we obtain from the two equations XXXVI. and XXXVII. this other,

$$\text{XL.} \ldots S\iota\sigma' . S\kappa\sigma' = 0;$$

which like them is of the *second degree* in σ', but breaks up, as we see, into *two linear* and *scalar factors*, representing *two distinct planes*, *parallel* by XXV. to the two *diametral* and *cyclic planes* of the ellipsoid. The sought *intersection* consists then of a *pair* of (real) *circles*, upon that given surface; namely, *two circular* (but *not diametral*) *sections*, which pass *through the given point* P.

(15.) Conversely, because the equations XXXVII. XXXVIII. XXXIX. XL. give XXXVI. and XXXIII., with the foregoing values of ξ and n, it follows that these *two* plane *sections* of the ellipsoid at P are on one *common sphere*, namely that which has N for centre, and n for radius, as above; and thus we might have found, *without differentials*, that the line PN is the *normal* at P; or that this normal *crosses the principal plane* (10.), in the point determined by the formula XXII.

(16.) In general, the *cyclic form* of the equation of *any central surface of the second order*, namely the form (comp. 357, II.),

$$\text{XLI.} \ldots S\rho\phi\rho = g'\rho^2 + 2S\lambda\rho S\mu\rho = C = \text{const.},$$

shows that the *two circles* (real or imaginary) in which that surface is cut by any *two planes*,

$$\text{XLII.} \ldots S\lambda\rho = l, \quad S\mu\rho = m,$$

drawn *parallel* respectively to the two real *cyclic planes*, which are jointly represented (comp. XL., and 216, (7.)) by the one equation,

$$\text{XLIII.} \ldots \mathrm{S}\lambda\rho\mathrm{S}\mu\rho = 0,$$

are *homospherical*, being *both* on that one sphere of which the equation is,

$$\text{XLIV.} \ldots g'\rho^2 + 2(l\mathrm{S}\mu\rho + m\mathrm{S}\lambda\rho) = 2lm + C.$$

(17.) But the *centre* (say N) of this *new sphere*, has for its vector (say ξ),

$$\text{XLV.} \ldots \mathrm{ON} = \xi = -g'^{-1}(l\mu + m\lambda);$$

it is therefore situated *in the plane* of the *two real cyclic normals*, λ and μ; and if l and m in XLV. receive the values XLII., then this new ξ is the *vector of intersection* of that *plane*, with the *normal to the surface* at P: because it is (comp. 15.)) the vector of the centre of a sphere which *touches* (though also *cutting*, in the two circular sections) the surface at that point.

(18.) We can therefore thus *infer* (comp. again (15.)), *without the differential calculus*, that the line,

$$\text{XLVI.} \ldots g'(\rho - \xi) = g'\rho + \lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho = \phi\rho,$$

as having the direction of NP, is the normal at P to the surface XLI.; which agrees with, and may be considered as confirming (if confirmation were required), the conclusion otherwise obtained through the differential expression (361),

$$\text{XLVII.} \ldots \mathrm{dS}\rho\phi\rho = 2\mathrm{S}\nu\mathrm{d}\rho = 2\mathrm{S}\phi\rho\mathrm{d}\rho;$$

the linear function $\phi\rho$ being here supposed (comp. 361, (3.)) to be self-conjugate.

(19.) Hence, with the notation 362, I., the *equation of the tangent plane* to a central surface of the second order, at the same point P, may by VII. be thus written,

$$\text{XLVIII.} \ldots f(\omega, \rho) = C, \quad \text{if} \quad \mathrm{S}\rho\phi\rho = C = \text{const.};$$

in which it is to be remembered, that

$$\text{XLIX.} \ldots f(\omega, \rho) = f(\rho, \omega) = \mathrm{S}\omega\phi\rho = \mathrm{S}\rho\phi\omega.$$

(20.) And if we choose to *interpret* this equation XLVIII., which is only of the *first degree* (362) with respect to *each* separately of the *two vectors*, ρ and ω, or OP and OR, and involves them *symmetrically*, without requiring that P shall be a point *on the surface*, we may then say (comp. 215, (13.), and 316, (31.)), that the formula in question is an *equation of conjugation*, which expresses that *each* of the two points P and R, is situated in the *polar plane* of the *other*.

(21.) In general, if we suppose that the *length* and *direction* of a line ν are so adjusted as to satisfy the *two* equations (comp. 336, XII. XIII. XIV.),

$$\text{L.} \ldots \mathrm{S}\nu\rho = 1, \quad \mathrm{S}\nu\mathrm{d}\rho = 0, \quad \text{and therefore also} \quad \text{LI.} \ldots \mathrm{S}\rho\mathrm{d}\nu = 0;$$

then, because the equation VII. of the *tangent plane* to *any curved surface* may now be thus written,

$$\text{LII.} \ldots \mathrm{S}\nu(\omega - \nu^{-1}) = 0,$$

it follows that ν^{-1} represents, in length and direction, the *perpendicular from* O *on that tangent plane at* P; so that ν *itself* represents the *reciprocal* of that *perpendicular*, or what may be called (comp. 336, (8.)) the *vector of proximity*, of the tangent plane to the origin. And we see, by LI., that the *two vectors*, ρ and ν, if drawn from a *common origin*, terminate on *two surfaces* which are, in a known and important sense (comp. the sub-arts. to 361), *reciprocals*[*] of one another: the line ρ^{-1}, for instance, being the perpendicular from O on the tangent plane to the *second* surface, at the extremity of the vector ν.

374. In the two preceding Articles, we have treated the symbol $d\rho$ as representing (rigorously) a *tangent to a curve on a given surface*, and therefore also to that surface *itself*; and thus the formula $S\nu d\rho = 0$ has been considered as expressing that ν has the direction of the *normal* to that *surface*, because it is *perpendicular to two tangents* (372), and therefore generally *to every tangent* (373), which can be drawn at a given point P. But without at present introducing any *other*[†] *signification* for this *symbol* $d\rho$, we may *interpret* in another way, and with a reference to *chords* rather than to *curves*, the *differential equation*,

$$\text{I.} \ldots df\rho = 2S\nu d\rho,$$

supposed still to be a *rigorous* one (in virtue of our *definitions* of differentials, which do not require that $d\rho$ should be *small*); and may *still* deduce from it the *normal property* of the vector ν, but now with the help of *Taylor's Series* adapted to quaternions (comp. 342, 370). In fact, that series gives here a *differenced equation*, of the form,

$$\text{II.} \ldots \Delta f\rho = 2S\nu\Delta\rho + R;$$

where R is a scalar *remainder* (comp. again 342), having the property that

$$\text{III.} \ldots \lim. (R : T\Delta\rho) = 0, \quad \text{if} \quad \lim. T\Delta\rho = 0;$$

whence

$$\text{IV.} \ldots \lim. (\Delta f\rho : T\Delta\rho) = 2 \lim. S\nu U\Delta\rho,$$

whatever the ultimate direction of $\Delta\rho$ *may be.* If then we conceive that

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[*] Compare the Note to page 484.

[†] It is *permitted*, for example, by general principles above explained, to treat the *differential* $d\rho$ as denoting a *chordal vector*, or to substitute it for $\Delta\rho$, and so to represent the *differenced equation* of the surface under the form (comp. 342),

$$0 = \Delta f\rho = (\epsilon^{d} - 1)f\rho = df\rho + \tfrac{1}{2}d^2 f\rho + \&c.;$$

but *with this meaning* of the *symbol* $d\rho$, the *equation* $df\rho = 0$, or $S\nu d\rho = 0$, is *no longer rigorous*, and must (for rigour) be replaced by such an equation as the following,

$$0 = 2S\nu d\rho + Sd\nu d\rho + R, \quad \text{if} \quad df\rho = 2S\nu d\rho, \text{ as before};$$

the *remainder* R *vanishing*, when the *surface* is only of the *second order* (comp. 362, (3.)). Accordingly this last *form* is *useful* in some investigations, especially in those which relate to the *curvatures of normal sections*: but for the present it seems to be clearer to *adhere* to the recent signification of $d\rho$, and therefore to treat it as still denoting a *tangent*, which may or may not be *small*.

$\Delta\rho$ represents a *small* and *indefinitely decreasing chord* PQ of the *surface*, drawn from the extremity P of ρ, so that

$$\text{V.} \ldots \Delta f\rho = f(\rho + \Delta\rho) - f\rho = 0, \quad \text{and} \quad \lim.\,\mathrm{T}\Delta\rho = 0,$$

the equation IV. becomes simply,

$$\text{VI.} \ldots \lim.\,\mathrm{S}\nu\mathrm{U}\Delta\rho = 0;$$

and thus proves, in a *new way*, that ν is *normal to the surface at the proposed point* P, by proving that it is *ultimately perpendicular to all the chords* PQ *from that point*, when those chords become *indefinitely small*, or *tend* indefinitely to *vanish*.

(1.) For example, if

$$\text{VII.} \ldots f\rho = \rho^2, \quad \nu = \rho, \quad \text{then} \quad \text{VIII.} \ldots R = \Delta\rho^2, \quad \text{and} \quad R : \mathrm{T}\Delta\rho = -\mathrm{T}\Delta\rho;$$

thus, for *every point of space*, we have *rigorously*, with this form of $f\rho$,

$$\text{IX.} \ldots \Delta f\rho : \mathrm{T}\Delta\rho = 2\mathrm{S}\rho\mathrm{U}\Delta\rho - \mathrm{T}\Delta\rho;$$

and for *every point* Q *of the spheric surface*, $f\rho =$ *const.*, we have *with equal rigour*,

$$\text{X.} \ldots 2\mathrm{S}\rho\mathrm{U}\Delta\rho = \mathrm{T}\Delta\rho, \quad \text{or} \quad \text{XI.} \ldots \overline{\mathrm{PQ}} = 2\overline{\mathrm{OP}}.\cos \mathrm{OPQ};$$

in fact, either of these two last formulæ expresses simply, that the *projection of a diameter of a sphere, on a conterminous chord*, is *equal to that chord itself*, and of course *diminishes with it*.

(2.) Passing then to the *limit*, or conceiving the point Q of the surface to *approach* indefinitely to P, we derive the limiting equations,

$$\text{XII.} \ldots \lim.\,\mathrm{S}\rho\mathrm{U}\Delta\rho = 0; \qquad \text{XIII.} \ldots \lim.\cos \mathrm{OPQ} = 0;$$

either of which shows, in a new way, that the *radii* of a *sphere* are its *normals;* with the *analogous result* for *other surfaces*, that the *vector* ν in I. has a *normal direction*, as before: *because* its *projection on a chord* PQ tends indefinitely to *diminish with that chord.*

(3.) We may also interpret the differential equation I. as expressing, through II. and III., that the *plane* 373, VII., which is drawn through the point P in a direction perpendicular to ν, is the *tangent plane* to the surface: because the *projection of the chord* $\Delta\rho$ *on the normal* ν *to that plane*, or the *perpendicular distance*,

$$\text{XIV.} \ldots -\mathrm{S}(\mathrm{U}\nu.\Delta\rho) = \tfrac{1}{2}R.\mathrm{T}\nu^{-1},$$

of a near point Q *from the plane* thus drawn through P, is *small* of an *order higher than the first* (comp. 370, (8.)), if the *chord* PQ *itself* be considered as *small* of the *first order*.

375. This occasion may be taken (comp. 374, I. II. III.), to give a *new Enunciation of Taylor's Theorem*, in a form *adapted to Quaternions*, which has some advantages over that given (342) in the preceding Chapter. We shall therefore now express that important *Theorem* as follows:—

"*If none of the* $m + 1$ *functions,*

$$\text{I.} \ldots fq,\ \mathrm{d}fq,\ \mathrm{d}^2fq, \ldots\ \mathrm{d}^mfq, \quad \text{in which} \quad \mathrm{d}^2q = 0,$$

become infinite in the immediate vicinity of a given quaternion q, *then the quotient,*

$$\text{II.} \ldots Q = \left\{ f(q + \mathrm{d}q) - fq - \mathrm{d}fq - \frac{\mathrm{d}^2fq}{2} - \frac{\mathrm{d}^3fq}{2\,.\,3} - \&\text{c.} \right.$$

$$\left. - \frac{\mathrm{d}^mfq}{2\,.\,3\,.\,.\,m} \right\} : \frac{\mathrm{d}q^m}{2\,.\,3\,.\,.\,m},$$

can be made to tend indefinitely to zero, for any ultimate value of the versor U$\mathrm{d}q$, *by indefinitely diminishing the tensor* T$\mathrm{d}q$."

(1.) The *proof* of the theorem, as thus enunciated, can easily be supplied by an attentive reader of Articles 341, 342, and their sub-articles; a few *hints* may however here be given.

(2.) We do not *now* suppose, as in 342, that d^mfq must be *different from zero;* we only assume that it is *not infinite:* and we *add*, to the expression 342, VI. for Fx, the *term*,

$$\text{III.} \ldots \frac{-x^m\mathrm{d}^mfq}{2\,.\,3\ldots m}.$$

(3.) Hence *each* of the expressions 342, VII., for the successive *derivatives* of Fx, receives an *additional term;* the *last* of them thus becoming,

$$\text{IV.} \ldots \mathrm{D}^mFx = F^{(m)}x = \mathrm{d}^mf(q + x\mathrm{d}x) - \mathrm{d}^mfq;$$

so that we have *now* (comp. 342, X.) the values

$$\text{V.} \ldots F0 = 0, \quad F'0, \quad F''0 = 0, \ldots \quad F^{(m-1)}0 = 0, \quad F^{(m)}0 = 0.$$

(4.) Assuming therefore now (comp. 342, XII.) the new auxiliary function,

$$\text{VI.} \ldots \psi x = \frac{x^m\mathrm{d}q^m}{2\,.\,3\ldots m}, \quad \text{with} \quad \mathrm{T}\mathrm{d}q > 0,$$

which gives,

$$\text{VII.} \ldots \psi 0 = 0, \quad \psi'0 = 0, \quad \psi''0 = 0, \ldots \quad \psi^{(m-1)}0 = 0, \quad \psi^{(m)}0 = \mathrm{d}q^m,$$

we find (by 341, (8.), (9.), comp. again 342, XII.) that

$$\text{VIII.} \ldots \lim_{x=0}.\,(Fx : \psi x) = 0.$$

(5.) But these two new functions, Fx and ψx, are formed from the dividend and the divisor of the quotient Q in II., by changing $\mathrm{d}q$ to $x\mathrm{d}q$; and (comp. 342, (3.)) instead of thus *multiplying a given quaternion differential* $\mathrm{d}q$, by a *small* and indefinitely *decreasing scalar*, x, we may indefinitely *diminish the tensor*, T$\mathrm{d}q$, *without changing the versor*, U$\mathrm{d}q$.

(6.) And *even if* U$\mathrm{d}q$ *be changed*, while the differential $\mathrm{d}q$ is thus made to *tend to zero*, we can always conceive that it *tends* to *some limit;* which *limiting* or *ultimate value* of that versor U$\mathrm{d}q$ may then be treated *as if* it were a *constant* one, without affecting the *limit* of the quotient Q.

(7.) The *theorem*, as above enunciated, is therefore fully proved; and we are at liberty to *choose*, in any application, between the two forms of statement, 342 and 375, of which one is more convenient at one time, and the other at another.

SECTION 4.—*On Osculating Planes, and Absolute Normals, to Curves of Double Curvature.*

376. The variable vector ρ_t of a curve in space may in general be thus expressed, with the help of Taylor's Series (comp. 370, (1.)):

$$\text{I.} \ldots \rho_t = \rho + t\rho' + \tfrac{1}{2}t^2u\rho'', \quad \text{with} \quad u_0 = 1;$$

ρ, ρ', ρ'', u being here abridged symbols for ρ_0, ρ'_0, ρ''_0, u_t; and the product $u\rho''$ being a vector, although the factor u is generally a quaternion (comp. 370, (5.)). And the different terms of this expression I. may be thus constructed (compare the annexed Figure 77):

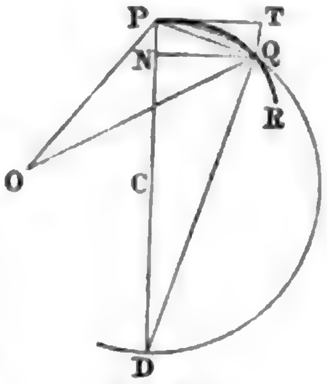

Fig. 77.

$$\text{II.} \ldots \rho = \mathrm{OP}; \quad t\rho' = \mathrm{PT}; \quad \tfrac{1}{2}t^2u\rho'' = \mathrm{TQ};$$

while $\text{III.} \ldots \rho_t = \mathrm{OQ}$, and $t\rho' + \tfrac{1}{2}t^2u\rho'' = \mathrm{PQ}$;

the line TQ, or the term $\tfrac{1}{2}t^2u\rho''$, being thus what may be called the *deflexion* of the *curve* PQR, at Q, from its *tangent* PT at P, measured in a *direction* which depends on the *law* according to which ρ_t varies with t, and on the *distance* of Q from P. The *equation of the plane* of the *triangle* PTQ is *rigorously* (by II.) the following, with ω for its variable vector,

$$\text{IV.} \ldots 0 = \mathrm{S}u\rho''\rho'(\omega - \rho);$$

this *plane* IV. then *touches* the curve at P, and (generally) *cuts* it at Q; so that if the point Q be conceived to *approach* indefinitely to P, the resulting formula,

$$\text{V.} \ldots 0 = \mathrm{S}\rho''\rho'(\omega - \rho), \quad \text{or} \quad \text{V}'. \ldots 0 = \mathrm{S}\rho'\rho''(\omega - \rho),$$

is the equation of the plane PTQ in that *limiting position*, in which it is called the *osculating plane*, or is said to *osculate to the curve* PQR, at the point P.

(1.) If the *variable vector* ρ be *immediately* given as a *function* ρ_s of a *variable scalar*, s, which is *itself* a function of the *former* scalar variable t, we shall then have (comp. 331) the expressions,

$$\text{VI.} \ldots \rho'_t = s'\mathrm{D}_s\rho_s, \quad \rho''_t = s''\mathrm{D}_s\rho_s + s'^2\mathrm{D}_s^2\rho_s, \quad \text{with} \quad s' = \mathrm{D}_ts, \quad s'' = \mathrm{D}_t^2s;$$

thus the *vector* ρ'' *may change*, even in *direction*, when we *change* the *independent scalar variable*; but ρ'' will *always* be a line, either *in* or *parallel to* the *osculating plane*; while ρ' will always represent a *tangent*, whatever scalar variable may be selected.

(2.) As an example, let us take the equation 314, XV., or 369, XIII., of the *helix*. With the independent variable t of that equation, we have (comp. 369, XIV.) the derived expressions,

$$\text{VII.}\ldots\ \rho' = c\alpha + \frac{\pi}{2}\alpha^{t+1}\beta, \qquad \rho'' = -\left(\frac{\pi}{2}\right)^2\alpha^t\beta = \left(\frac{\pi}{2}\right)^2(ct\alpha - \rho);$$

ρ'' has therefore here (comp. 369, (8.)) the direction of the *normal* to the cylinder; and consequently, the *osculating plane to the helix* is a *normal plane to the cylinder* of revolution, on which that curve is traced: a result well known, and which will soon be greatly extended.

(3.) When a curve of *double* curvature *degenerates* into a *plane* curve, its *osculating* plane becomes *constant*, and reciprocally. The *condition of planarity* of a *curve in space* may therefore be expressed by the equation,

$$\text{VIII.}\ldots\ \mathrm{UV}\rho'\rho'' = \pm \text{ a constant unit line};$$

or, by 335, II., and 338, VIII.,

$$\text{IX.}\ldots\ 0 = \mathrm{V}\frac{\mathrm{V}(\rho'\rho'')'}{\mathrm{V}\rho'\rho''} = \mathrm{V}\frac{\mathrm{V}\rho'\rho'''}{\mathrm{V}\rho'\rho''};$$

or finally, $\quad \text{X.}\ldots\ \mathrm{S}\rho'\rho''\rho''' = 0, \quad \text{or} \quad \text{XI.}\ldots\ \rho''' \parallel\mid \rho', \rho''.$

(4.) Accordingly, for a *plane curve*, if λ be a given normal to its plane, we have the three equations,

$$\text{XII.}\ldots\ \mathrm{S}\lambda\rho' = 0, \quad \mathrm{S}\lambda\rho'' = 0, \quad \mathrm{S}\lambda\rho''' = 0;$$

which conduct, by 294, (11.), to X.

(5.) For example, if we had not otherwise known that the equation 337, (2.) represented a *plane ellipse*, we might have perceived that it was the equation of *some* plane curve, because it gives the *three* successive derivatives,

$$\text{XIII.}\ldots\ \rho' = \frac{\pi}{2}\mathrm{V}\alpha^{t+1}\beta, \qquad \rho'' = -\left(\frac{\pi}{2}\right)^2\mathrm{V}\alpha^t\beta, \qquad \rho''' = -\left(\frac{\pi}{2}\right)^3\mathrm{V}\alpha^{t+1}\beta,$$

which are *complanar* lines, the third having a direction *opposite* to the first.

(6.) And generally, the formula X. enables us to assign, on *any curve of double* curvature, for which ρ is expressed as a function of t, the *points** at which it *most resembles* a *plane* curve, or *approaches most closely* to its own osculating plane.

377. An important and *characteristic property* of the *osculating plane* to a curve of double curvature, is that the *perpendiculars* let fall on it, from points of the curve near to the point of osculation, are *small* of an order *higher* than the *second*, if their *distances* from that *point* be considered as *small* of the *first* order.

(1.) To exhibit this by quaternions, let us begin by considering an *arbitrary plane*,

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* Namely, in a modern phraseology, the places of *four-point contact* with a *plane*. The equation, $\mathrm{V}\rho'\rho'' = 0$, indicates in like manner the places, if any, at which a curve has *three-point contact* with a *right line*. For curves of double curvature, these are also called points of *simple* and *double inflexion*.

$$\text{I.} \ldots \mathrm{S}\lambda(\omega-\rho)=0, \quad \text{with} \quad \mathrm{T}\lambda=1,$$

drawn through a point P of the curve. Using the expression 376, I., for the vector OQ, or ρ_t, of another point Q of the same curve, we have, for the perpendicular distance of Q from the plane I., this other rigorous expression,

$$\text{II.} \ldots \mathrm{S}\lambda(\rho_t-\rho)=t\mathrm{S}\lambda\rho'+\tfrac{1}{2}t^2\mathrm{S}\lambda u\rho'';$$

which represents, *in general*, a small quantity of the *first order*, if t be assumed to be such.

(2.) The expression II. represents however, *generally*, a small quantity of the *second* order, if the *direction* of λ satisfy the condition,

$$\text{III.} \ldots \mathrm{S}\lambda\rho'=0;$$

that is, if the *plane* I. *touch* the *curve*.

(3.) And if the condition,

$$\text{IV.} \ldots \mathrm{S}\lambda\rho''=0,$$

be *also* satisfied by λ, *then*, but *not otherwise*, the expression II. *tends* to bear an *evanescent ratio* to t^2, or is *small* of an order *higher* than the *second*.

(4.) But the combination of the *two conditions*, III. and IV., conducts to the expression,

$$\text{V.} \ldots \lambda=\pm\,\mathrm{UV}\rho'\rho'';$$

comparing which with 376, V., we see that the property above stated is one which belongs to the *osculating* plane, and to *no other*.

378. Another remarkable property* of the *osculating plane* to a *curve* is, that it is the *tangent plane* to the *cone of parallels to tangents* (369, (6.)), which has its *vertex* at the *point* of *osculation*.

(1.) In general, if $\rho=\phi x$ be (comp. 369, I.) the equation of a *curve* in space, the equation of the *cone* which has its vertex at the origin, and passes through this curve, is of the form,

$$\text{I.} \ldots \rho=y\phi x;$$

in which x and y are *two* independent and scalar variables.

(2.) We have thus the two partial derivatives,

$$\text{II.} \ldots \mathrm{D}_x\rho=y\phi' x, \quad \mathrm{D}_y\rho=\phi x;$$

and the tangent plane along the *side* (x) has for equation,

$$\text{III.} \ldots 0=\mathrm{S}(\omega.\phi x.\phi' x); \quad \text{or briefly,} \quad \text{III}'. \ldots 0=\mathrm{S}\omega\phi\phi'.$$

(3.) Changing then x, ϕ, ϕ', ω to t, ρ', ρ'', $\omega-\rho$, we see that the equation 376, V., of the *osculating plane* to the *curve* 376, I., is also that of the *tangent plane* to the *cone of parallels*, &c., as asserted.

379. Among all the *normals* to a *curve*, at any one point, there are *two* which deserve special attention; namely the one which is *in*

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* The writer does not remember seeing *this* property in print; but of course it is an easy consequence from the doctrine of *infinitesimals*, which doctrine however it has not been thought convenient to adopt, as the *basis* of the present exposition.

the osculating plane, and is called the *absolute* (or *principal*) *normal*; and the one which is *perpendicular* to that plane, and which it has been lately proposed to name the *binormal*.[*] It is easy to assign expressions, by quaternions, for these two normals, as follows.

(1.) The ***absolute normal***, as being perpendicular to ρ', but complanar with ρ' and ρ'', has a direction expressed by any one of the following formulæ (comp. 203, 334):

$$\text{I.}\ldots \mathrm{V}\rho''\rho'.\rho'^{-1}; \quad \text{or} \quad \text{II.}\ldots \mathrm{dU}\rho'; \quad \text{or} \quad \text{III.}\ldots \mathrm{dUd}\rho.$$

(2.) There is an extensive class[†] of cases, for which the following equations hold good:

$$\text{IV.}\ldots \mathrm{T}\rho' = \text{const.}; \qquad \text{V.}\ldots \rho'^2 = \text{const.}; \qquad \text{VI.}\ldots \mathrm{S}\rho'\rho'' = 0;$$

and in all ***such*** cases, the expression I. reduces itself to ρ'', which is therefore ***then*** a representative of the absolute normal.

(3.) For example, in the case of the ***helix***, with the equation several times before employed, the conditions (2.) are satisfied; and accordingly the absolute normal to that curve coincides with the normal ρ'' to the ***cylinder***, on which it is traced; the ***locus of the absolute normal*** being here that ***screw surface*** or ***Helicoid***, which has been already partially considered (comp. 314, (11.), and 372, (4.)).

(4.) And as regards the ***binormal***, it may be sufficient here to remark, that because it is perpendicular to the osculating plane, it has the ***direction*** expressed by one or other of the two symbols (comp. 377, V.),

$$\text{VII.}\ldots \mathrm{V}\rho'\rho'', \quad \text{or} \quad \text{VII}'.\ldots \mathrm{Vd}\rho\mathrm{d}^2\rho.$$

(5.) There exists, of course, a system of ***three rectangular planes***, the ***osculating*** plane being ***one***, which are connected with the system of the ***three rectangular lines***, the tangent, the absolute normal, and the binormal, and of which any one who has studied the Quaternions so far can easily form the expressions.

(6.) And a ***construction***[‡] for the ***absolute normal*** may be assigned, analogous to and including that lately given (378) for the ***osculating plane***, as an ***interpretation*** of the expression II. or III., or of the ***symbol*** $\mathrm{dU}\rho'$ or $\mathrm{dUd}\rho$. From any origin o conceive a system of unit lines ($\mathrm{U}\rho'$ or $\mathrm{Ud}\rho$) to be drawn, in the ***directions*** of the successive ***tangents*** to the ***given curve*** of double curvature; these lines will terminate on a certain *spherical curve;* and the *tangent*, say SS', to this *new* curve, at the point S which *corresponds* to the point P of the *old* one, will have the direction of the absolute normal at that old point.

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[*] By M. de Saint-Venant, as being perpendicular at once to ***two*** consecutive ***elements*** of the curve, in the infinitesimal treatment of this subject. See page 261 of the very valuable Treatise on ***Analytic Geometry of Three Dimensions*** (Hodges and Smith, Dublin), by the Rev. George Salmon, D.D., which has been published in the present year (1862), but not till after the printing of these ***Elements of Quaternions*** (begun in 1860) had been too far advanced, to allow the writer of them to profit by the study of it, so much as he would otherwise have sought to do.

[†] Namely, those in which the ***arc of the curve***, or that arc multiplied by a scalar constant, is taken as the ***independent variable***.

[‡] ***This*** construction also has not been met with by the writer in print, so far as he remembers; but it may easily have escaped his notice, even in the books which he has seen.

(7.) At the same time, the *plane* OSS' of the *great circle*, which *touches* the *new curve* upon the *unit sphere*, being the *tangent plane* to the *cone of parallels* (378), has the direction of the *osculating plane* to the old curve; and the *radius* drawn to its *pole* is parallel to the *binormal.*

(8.) As an *example* of the *auxiliary* (or *spherical*) curve, constructed as in (6.), we may take again the *helix* (369, XIII., &c.) as the *given* curve of double curvature, and observe that the expression 369, XIV., namely,

$$\text{VIII.} \ldots \rho' = c\alpha + \frac{\pi}{2}\alpha^{t+1}\beta, \text{ gives IX.} \ldots \rho'^2 = -c^2 + \frac{\pi^2\beta^2}{4} = \text{const. (comp. (3.))};$$

whence $\mathrm{T}\rho'$ is *constant* (as in IV.), and we have the equation (comp. 369, XV. XIX.),

$$\text{X.} \ldots \mathrm{S}\alpha\mathrm{U}\rho' = -c\left(c^2 - \frac{\pi^2\beta^2}{4}\right)^{-\frac{1}{2}} = -\cos a = \text{const.},$$

a being again the inclination of the helix to the axis of its cylinder; which shows that the *new* curve is in *this* case a *plane* one, namely a certain *small circle* of the unit sphere.

(9.) In general, if the *given curve* be conceived to be an *orbit* described by a *point*, which *moves* with a *constant velocity* taken for *unity*, the *auxiliary* or *spherical curve* becomes what we have proposed (100, (5.)) to call the *hodograph* of that *motion.*

(10.) And if the given curve be supposed to be described with a *variable velocity*, the *hodograph* is still *some* curve upon the *cone of parallels* to tangents.

SECTION 5.—*On Geodetic Lines, and Families of Surfaces.*

380. Adopting as the *definition* of a *geodetic line*, on any proposed curved surface, the property that it is one of which the *osculating plane* is always a *normal plane* to that surface, or that the *absolute normal* to the *curve* is also the *normal* to the *surface,* we have *two principal modes* of *expressing* by quaternions this general and *characteristic property*. For we may either write,

$$\text{I.} \ldots \mathrm{S}\nu\rho'\rho'' = 0, \quad \text{or} \quad \text{II.} \ldots \mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho = 0,$$

to express that the *normal* ν to the *surface* (comp. 373) is *perpendicular* to the *binormal* $\mathrm{V}\rho'\rho''$ or $\mathrm{Vd}\rho\mathrm{d}^2\rho$ to the *curve* (comp. 379, VII. VII'.); or else, at pleasure,

$$\text{III.} \ldots \mathrm{V}\nu(\mathrm{U}\rho')' = 0, \quad \text{or} \quad \text{IV.} \ldots \mathrm{V}\nu\mathrm{dUd}\rho = 0,$$

to express that the *same* normal ν has the *direction* of the *absolute normal* $(\mathrm{U}\rho')'$ or $\mathrm{dUd}\rho$ (comp. 379, II. III.), to the same *geodetic line.* And thus it becomes easy to deduce the known *relations* of such *lines* (or *curves*) to some important *families of surfaces*, on which they can be traced. Accordingly, after beginning for simplicity with the *sphere*, we shall proceed in the following sub-articles to determine the geodetic lines on *cylindrical* and *conical* surfaces, with *arbitrary bases;* intending afterwards to show how the corresponding lines can be investigated, upon *developable surfaces*, and surfaces of *revolution.*

(1.) On a *sphere*, with centre at the origin, we have $\nu \parallel \rho$, and the *differential equation* IV. admits of an immediate *integration ;** for it here becomes,

V. . . $0 = \mathrm{V}\rho d\mathrm{U}d\rho = d\mathrm{V}\rho\mathrm{U}d\rho$, whence VI. . . $\mathrm{V}\rho\mathrm{U}d\rho = \omega$, and VII. . . $\mathrm{S}\omega\rho = 0$,

ω being some constant vector; the curve is therefore in this case a *great circle*, as being wholly contained in *one diametral plane.*

(2.) Or we may observe that the equation,

$$\text{VIII.} \ldots \mathrm{S}\rho\rho'\rho'' = 0, \quad \text{or} \quad \text{IX.} \ldots \mathrm{S}\rho d\rho d^2\rho = 0,$$

obtained by changing ν to ρ in I. or II., has *generally* for a *first* integral (comp. 335, (1.)), whether $\mathrm{T}\rho$ be constant or variable,

$$\text{X.} \ldots \mathrm{UV}\rho\rho' = \mathrm{UV}\rho d\rho = \omega = \text{const.};$$

it expresses therefore that ρ is the vector of *some curve* (or line) *in a plane through the origin ;* which curve must consequently be here a *great circle*, as before.

(3.) Accordingly, as a verification of X., if we write

$$\text{XI.} \ldots \rho = \alpha x + \beta y, \; x \text{ and } y \text{ being scalar functions of } t,$$

where t is still some independent scalar variable, and α, β are two vector constants, we shall have the derivatives,

$$\text{XII.} \ldots \rho' = \alpha x' + \beta y', \quad \rho'' = \alpha x'' + \beta y'' \parallel\!\mid \rho, \rho';$$

so that the equation VIII. is satisfied.

(4.) For an *arbitrary cylinder*, with generating lines parallel to a fixed line α, we may write,

$$\text{XIII.} \ldots \mathrm{S}\alpha\nu = 0, \qquad \text{XIV.} \ldots \mathrm{S}\alpha d\mathrm{U}d\rho = 0, \qquad \text{XV.} \ldots \mathrm{S}\alpha\mathrm{U}d\rho = \text{const.};$$

a *geodetic* on a cylinder *crosses* therefore the *generating lines* at a *constant angle*, and consequently becomes a *right line* when the cylinder is *unfolded* into a *plane:* both which known properties are accordingly verified (comp. 369, (5.), and 376, (2.)) for the case of a cylinder of *revolution*, in which case the geodetic is a *helix.*

(5.) For an *arbitrary cone*, with vertex at the origin, we have the equations,

$$\text{XVI.} \ldots \mathrm{S}\nu\rho = 0, \qquad \text{XVII.} \ldots \mathrm{S}\rho d\mathrm{U}d\rho = 0,$$
$$\text{XVIII.} \ldots d\mathrm{S}\rho\mathrm{U}d\rho = \mathrm{S}(d\rho.\mathrm{U}d\rho) = -\mathrm{T}d\rho;$$

multiplying the last of which equations by $2\mathrm{S}\rho\mathrm{U}d\rho$, and observing that $-2\mathrm{S}\rho d\rho = -d.\rho^2$, we obtain the transformations,

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* We here assume as evident, that the *differential* of a *variable* cannot be *constantly zero* (comp. 335, (7.)); and we employ the principle (comp. 338, (5.)), that $\mathrm{V}.d\rho\,\mathrm{U}d\rho = -\mathrm{VT}d\rho = 0$.

XIX. . . $0 = \mathrm{d}\{(\mathrm{S}\rho\mathrm{U}\mathrm{d}\rho)^2 + \rho^2\} = \mathrm{d}.(\mathrm{V}\rho\mathrm{U}\mathrm{d}\rho)^2$, XX. . . $\mathrm{TV}\rho\mathrm{U}\mathrm{d}\rho = \mathrm{const.}$;

the *perpendicular from the vertex, on a tangent to any one geodetic* upon a *cone*, has therefore a *constant length;* and all *such* tangents *touch also a concentric sphere*,* or one which has its centre at the vertex of the cone.

(6.) Conceive then that at each point P or P′ of the geodetic a tangent PT or P′T′ is drawn, and that the angles OTP, OT′P′ are right; we shall have, by what has just been shown,

XXI. . . $\overline{\mathrm{OT}} = \overline{\mathrm{OT}}' = $ const. = radius of concentric sphere;

and if the *cone* be *developed* (or unfolded) into a *plane*, this constant or *common length*, of the perpendiculars from O on the tangents, will remain *unchanged*, because the length $\overline{\mathrm{OP}}$ and the angle OPT are unaltered by such development; the geodetic becomes therefore *some plane line*, with the *same property* as before; and although *this* property would belong, not only to a *right line*, but also to a *circle* with O for centre (compare the second part of the annexed Figure 78), yet we have in this result merely an effect of the *foreign factor* $\mathrm{S}\rho\mathrm{U}\mathrm{d}\rho$, which was introduced in (5.), in order to facilitate the integration of the differential equation XVIII., and which (by that very equation) cannot be constantly equal to zero. We are therefore to *exclude* the *curves* in which the *cone* is cut by *spheres* concentric with it: and there remain, as the sought *geodetic lines*, only those of which the *developments* are *rectilinear*, as in (4.).

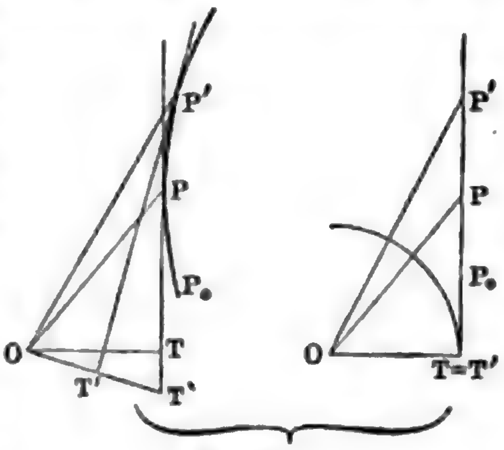

Fig. 78.

(7.) Another mode of *interpreting*, and at the same time of *integrating*, the equation XVIII., is connected with the interpretation of the *symbol* $\mathrm{T}\mathrm{d}\rho$; which can be proved, on the principles of the present Calculus, to represent *rigorously* the *differential* $\mathrm{d}s$ of the *arc* (s) of that *curve*, whatever it may be, of which ρ is the *variable vector;* so that we have the *general* and *rigorous* equation,

XXII. . . $\mathrm{T}\mathrm{d}\rho = \mathrm{d}s$, if s thus denote the *arc:*

whether that arc *itself*, or some *other* scalar, t, be taken as the *independent variable;* and whether its differential $\mathrm{d}s$ be *small* or *large*, provided that it be *positive.*

(8.) In fact if we suppose, for the sake of greater generality, that the vector ρ and the scalar s are thus *both* functions, ρ_t and s_t, of some *one* independent and scalar variable, t, our principles direct us first to take, or to conceive as taken, a *submultiple*, $n^{-1}\mathrm{d}t$, of the *finite differential* $\mathrm{d}t$, considered as an assumed and *arbitrary increment* of that *independent variable*, t; to determine next the *vector* $\rho_{t+n^{-1}\mathrm{d}t}$, and the *scalar* $s_{t+n^{-1}\mathrm{d}t}$, which correspond to the *point* $\mathrm{P}_{t+n^{-1}\mathrm{d}t}$ of the *curve* on which ρ_t terminates in P_t, and of which s_t is the *arc*, $\overset{\frown}{\mathrm{P}_0\mathrm{P}_t}$, measured to P_t from some fixed point P_0 on the same curve; to take the *differences*,

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* When the *cone* is of the *second order*, this becomes a case of a known theorem respecting *geodetic lines on a surface* of the same *second* order, the *tangents* to any *one* of which *curves* touch *also* a *confocal surface.*

$$\rho_{t+n^{-1}dt} - \rho_t, \quad \text{and} \quad s_{t+n^{-1}dt} - s_t,$$

which represent respectively the *directed chord*, and the *length*, of the *arc* $\mathrm{P}_t\mathrm{P}_{t+n^{-1}dt}$, which arc will generally be *small*, if the number n be *large*, and will *indefinitely diminish* when that *number* tends to *infinity;* to *multiply* these two decreasing differences, of ρ_t and s_t, by n; and finally to seek the *limits* to which the *products tend*, when n thus tends to ∞: such *limits* being, by our *definitions*, the values of the two sought and *simultaneous differentials*, $\mathrm{d}\rho$ and $\mathrm{d}s$, which answer to the assumed values of t and $\mathrm{d}t$. And because the *small arc*, Δs, and the *length*, $\mathrm{T}\Delta\rho$, of its *small chord*, in the foregoing construction, *tend* indefinitely to a *ratio of equality*, such must be the *rigorous ratio* of $\mathrm{d}s$ and $\mathrm{Td}\rho$, which are (comp. 320) the *limits of their equimultiples*.

(9.) Admitting then the exact *equality* XXII. of $\mathrm{Td}\rho$ and $\mathrm{d}s$, at least when the latter like the former is taken *positively*, we have only to substitute $-\mathrm{d}s$ for $-\mathrm{Td}\rho$ in the equation XVIII., which then becomes immediately integrable, and gives,

$$\text{XXIII.} \ldots s + \mathrm{S}\rho\mathrm{Ud}\rho = s - \mathrm{S}(\rho : \mathrm{Ud}\rho) = \text{const.};$$

where $\mathrm{S}(\rho : \mathrm{Ud}\rho)$ denotes the *projection* $\overline{\mathrm{TP}}$, of the vector ρ or OP, *on the tangent* to the geodetic at P, considered as a *positive scalar* when ρ makes an *acute angle* with $\mathrm{d}\rho$, that is, when the *distance* $\mathrm{T}\rho$ or $\overline{\mathrm{OP}}$ from the vertex is *increasing;* while s denotes, as above, the *length* of the *arc* $\mathrm{P}_0\mathrm{P}$ of the same curve, measured from *some fixed point* P_0 thereon, and considered as a *scalar* which *changes sign*, when the variable point P passes through the position P_0.

(10.) But the *length* of TP does not change (comp. (6.)), when the cone is *developed*, as before; we have therefore the equations (comp. again Fig. 78),

$$\text{XXIV.} \ldots \overset{\frown}{\mathrm{P}_0\mathrm{P}} - \overline{\mathrm{TP}} = \text{const.} = \overset{\frown}{\mathrm{P}_0\mathrm{P}'} - \overline{\mathrm{T'P'}}, \qquad \text{XXV.} \ldots \overset{\frown}{\mathrm{PP}'} = \overline{\mathrm{T'P'}} - \overline{\mathrm{TP}},$$

which must hold good both *before* and *after* the supposed *development* of the conical *surface;* and it is easy to see that this can only be, by the *geodetic* on the cone becoming a *right line*, as before. In fact, if OT' in the *plane* be supposed to intersect the tangent TP in a point T', and if P' be conceived to approach to P, the second member of XXV. bears a limiting ratio of equality to the first member, increased or diminished by $\overline{\mathrm{TT}'}$; which latter *line*, and therefore also the *angle* TOT' between the perpendiculars on the two near tangents, or the angle between those tangents themselves, if existing, must bear an indefinitely decreasing ratio to the *arc* $\overset{\frown}{\mathrm{PP}'}$; so that the *radius of curvature* of the *supposed curve* is *infinite*, or T' *coincides* with T, and the *development* is *rectilinear* as before.

(11.) The important and general equation, $\mathrm{Td}\rho = \mathrm{d}s$ (XXII.), conducts to many other consequences, and may be put under several other forms. For example, we may write generally,

$$\text{XXVI.} \ldots \mathrm{TD}_s\rho = 1, \qquad \text{XXVII.} \ldots (\mathrm{D}_s\rho)^2 + 1 = 0;$$

also $$\text{XXVIII.} \ldots (\mathrm{D}_t\rho)^2 + (\mathrm{D}_t s)^2 = 0, \quad \text{or} \quad \text{XXIX.} \ldots \rho'^2 + s'^2 = 0,$$

if ρ' and s' be the first derivatives of ρ and s, taken with respect to any independent scalar variable, such as t; whence, by continued derivation,

$$\text{XXX.} \ldots \mathrm{S}\rho'\rho'' + s's'' = 0, \qquad \text{XXXI.} \ldots \mathrm{S}\rho'\rho''' + \rho''^2 + s's''' + s''^2 = 0, \ \&c.$$

(12.) And if the arc s be *itself* taken as the independent variable, then (comp. 379, (2.)) the equations XXIX., &c., become,

$$\text{XXXII.} \ldots \rho'^2 + 1 = 0, \quad \mathrm{S}\rho'\rho'' = 0, \quad \mathrm{S}\rho'\rho''' + \rho''^2 = 0, \ \&c.$$

381. In general, if we conceive (comp. 372, I.) that the *vector* ρ of a *given surface* is expressed as a *given function* of *two scalar variables*, x and y, whereof *one*, suppose y, is regarded at first as an *unknown function* of the other, so that we have again,

$$\text{I.} \ldots \rho = \phi(x, y), \quad \text{but now with} \quad \text{II.} \ldots y = fx,$$

where the *form* of ϕ is *known*, but that of f is *sought;* we may then regard ρ as being *implicitly* a function of the *single* (or *independent*) *scalar variable*, x, and may consider the equation,

$$\text{III.} \ldots \rho = \phi(x, fx),$$

as being that of *some curve* on the given surface, to be determined by assigned conditions. Denoting then the *unknown total derivative* $\mathrm{D}\phi(x, fx)$ by ρ', but the *known partial derivatives* of the same first order by $\mathrm{D}_x\phi$ and $\mathrm{D}_y\phi$, with analogous notations for orders higher than the first, we have (comp. 376, VI.) the expressions,

$$\text{IV.} \ldots \rho' = \mathrm{D}_x\phi + y'\mathrm{D}_y\phi, \quad \rho'' = \mathrm{D}_x{}^2\phi + 2y'\mathrm{D}_x\mathrm{D}_y\phi + y'^2\mathrm{D}_y{}^2\phi + y''\mathrm{D}_y\phi, \ \&\text{c.};$$

in which $y' = \mathrm{D}_x y = f'x$, $y'' = \mathrm{D}_x{}^2 y = f''x$, &c. Hence, writing for the *normal* ν to the *surface* the expression,

$$\text{V.} \ldots \nu = \mathrm{V}(\mathrm{D}_x\phi \,.\, \mathrm{D}_y\phi) = \mathrm{V}\,.\,\mathrm{D}_x\phi\mathrm{D}_y\phi, \quad \text{comp. 372, V.},$$

or this vector multiplied by any scalar, the *equation* 380, I. of a *geodetic line* takes this *new form*,

$$\text{VI.} \ldots 0 = \mathrm{S}\nu\rho'\rho'' = \mathrm{S}(\mathrm{V}\,.\,\mathrm{D}_x\phi\mathrm{D}_y\phi\,.\,\mathrm{V}\rho'\rho'');$$

or, by a general transformation which has been often employed already (comp. 352, XXXI., &c.),

$$\text{VII.} \ldots 0 = \mathrm{S}\rho'\mathrm{D}_y\phi\,.\,\mathrm{S}\rho''\mathrm{D}_x\phi - \mathrm{S}\rho'\mathrm{D}_x\phi\,.\,\mathrm{S}\rho''\mathrm{D}_y\phi;$$

and thus, by substituting the expressions IV. for ρ' and ρ'', we obtain an *ordinary* (or *scalar*) *differential equation*, of the *second order*, in x and y, which is *satisfied by all the geodetics* on the given *surface*, and of which the *complete integral* (when found) expresses, with *two arbitrary* and *scalar constants*, the *form* of the *scalar function* f in II., or the *law* of the dependence of y on x, for the geodetic *curves* in question.

(1.) As an *example*, let us take the equation,

$$\text{VIII.} \ldots \rho = \phi(x, y) = y\psi x, \qquad \text{comp. 378, I.},$$

of a *cone* with its vertex at the origin; which cone becomes a *known* one, when the *form* of the vector function ψ is given, that is, when we know a *guiding curve* $\rho = \psi x$, through which the *sides* of the cone all pass. We have here the partial derivatives,

$$\text{IX.} \ldots \mathrm{D}_x\phi = y\mathrm{D}_x\psi x = y\psi', \quad \mathrm{D}_y\phi = \psi x = \psi, \qquad \text{comp. 378, II.;}$$

and

$$\text{X.} \ldots \mathrm{D}_x{}^2\phi = y\mathrm{D}_x{}^2\psi x = y\psi'', \quad \mathrm{D}_x\mathrm{D}_y\phi = \psi', \quad \mathrm{D}_y{}^2\phi = 0;$$

the expressions IV. become, then,

$$\text{XI.} \ldots \rho' = y\psi' + y'\psi, \quad \rho'' = y\psi'' + 2y'\psi' + y''\psi;$$

and since only the *direction* of the normal is important, we may divide V. by $-y$, and write,

$$\text{XII.} \ldots \nu = \mathrm{V}\psi\psi'.$$

(2.) The expressions XI. and XII. give (comp. VI. and VII.) for the *geodetics on the cone* VIII., the differential equation of the second order,

$$\begin{aligned}\text{XIII.} \ldots 0 = \mathrm{S}(\mathrm{V}\psi\psi'.\mathrm{V}\rho'\rho'') &= \mathrm{S}\rho''\psi\mathrm{S}\rho'\psi' - \mathrm{S}\rho''\psi'\mathrm{S}\rho'\psi \\ &= (y\mathrm{S}\psi\psi'' + 2y'\mathrm{S}\psi\psi' + y''\psi^2)\,(y\psi'^2 + y'\mathrm{S}\psi\psi') \\ &\quad - (y\mathrm{S}\psi'\psi'' + 2y'\psi'^2 + y''\mathrm{S}\psi\psi')\,(y\mathrm{S}\psi\psi' + y'\psi^2),\end{aligned}$$

in which ψ^2 and ψ'^2 are abridged symbols for $(\psi x)^2$ and $(\psi' x)^2$; but this equation in x and y may be greatly simplified, by some permitted suppositions.

(3.) Thus, we are allowed to suppose that the *guiding curve* (1.) is the *intersection* of the *cone* with the *concentric unit sphere*, so that

$$\text{XIV.} \ldots \mathrm{T}\psi x = 1, \quad \psi^2 = -1, \quad \mathrm{S}\psi\psi' = 0, \quad \mathrm{S}\psi\psi'' + \psi'^2 = 0;$$

and if we further assume that the *arc* of this *spherical curve* is taken as the *independent variable*, x, we have then, by 380, (12.), combined with the last equation XIV.,

$$\text{XV.} \ldots \mathrm{T}\psi' x = 1, \quad \psi'^2 = -1, \quad \mathrm{S}\psi'\psi'' = 0, \quad \mathrm{S}\psi\psi'' = -\psi'^2 = 1.$$

(4.) With these simplifications, the *differential equation* XIII. becomes,

$$\text{XVI.} \ldots 0 - (y - y'')\,(-y) - (-2y')\,(-y') = yy'' - 2y'^2 - y^2;$$

and its *complete integral* is found *by ordinary methods* to be,

$$\text{XVII.} \ldots y = b \sec (x + c),$$

in which b and c are two arbitrary but scalar constants.

(5.) To *interpret* now this *integrated* and *scalar equation* in x and y, of the *geodetics* on an *arbitrary cone*, we may observe that, by the suppositions (3.), y represents the *distance*, $\mathrm{T}\rho$ or $\overline{\mathrm{OP}}$, from the vertex O, and $x + c$ represents the *angle* AOP, in the *developed state* of cone and curve, from some *fixed line* OA in the *plane*, to the variable line OP; the *projection* of this *new* OP on that *fixed* line OA is therefore *constant* (being $= b$, by XVII.), and the *developed geodetic* is again found to be a *right line*, as before.

382. Let ABCDE ... (see the annexed Figure 79) be *any given series* of *points in space*. Draw the successive right lines, AB, BC, CD, DE, .. and prolong them to points B′, C′, D′, E′, ... the lengths of these prolongations being arbitrary; join also B′C′, C′D′, D′E′, ... We shall thus have a *series of plane triangles*, B′BC′, C′CD′, D′DE′, ... *all* generally *in different planes*; so that BCD′C′B′, CDE′D′C′, ... are generally *gauche pentagons*, while BCDE′D′C′B′ is a *gauche heptagon*, &c. But we

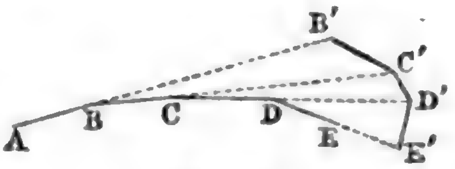

Fig. 79.

can conceive the *first triangle* B′BC′ to *turn* round its *side* BCC′, till it *comes* into the plane of the *second triangle*, C′CD′; which will transform the *first gauche pentagon* into a *plane* one, denoted still by BCD′C′B′. We can then conceive *this* plane figure to turn round its *side* CDD′, till it comes into the plane of the *third triangle*, D′DE′; whereby the *first gauche heptagon* will have *become* a plane one, denoted as before by BCDE′D′C′B′: and so we can proceed indefinitely. Passing then to the *limit*, at which the *points* ABCDE . . . are conceived to be each *indefinitely near* to the one which precedes or follows it in the series, we conclude as usual (comp. 98, (12.)) that the *locus of the tangents* to a *curve of double curvature* is a *developable surface:* or that it admits of being *unfolded* (like a cone or cylinder) *into a plane*, without any breach of continuity. It is now proposed to *translate* these *conceptions* into the *language* of *quaternions*, and to draw from them some of their consequences: especially as regards the determination of the *geodetic lines*, on such a developable surface.

(1.) Let ψ_x, or simply ψ, denote the variable vector of a point upon the *curve*, or *cusp-edge*, or *edge of regression* of the developable, to which curve the *generating lines* of that *surface* are thus *tangents*, considered as a *function* ψ of its *arc*, x, measured from some fixed point A upon it; so that while the *equation* of the surface will be of the form (comp. 100, (8.)),

$$\text{I.} \ldots \rho = \phi(x, y) = \psi_x + y\psi'_x = \psi + y\psi',$$

y being a second scalar variable, we shall have the relations (comp. 381, XV.),

$$\text{II.} \ldots \mathrm{T}\psi'_x = 1, \quad \psi'^2 = -1, \quad \mathrm{S}\psi'\psi'' = 0, \quad \mathrm{S}\psi'\psi''' = -\psi''^2 = z^2, \quad \text{if} \quad z = \mathrm{T}\psi''.$$

(2.) Hence $\quad \text{III.} \ldots \mathrm{D}_x\phi = \psi' + y\psi'', \quad \mathrm{D}_y\phi = \psi';$

$$\text{IV.} \ldots \rho' = (1 + y')\psi' + y\psi'', \quad \rho'' = y''\psi' + (1 + 2y')\psi'' + y\psi''';$$

and $\quad \text{V.} \ldots \nu = \mathrm{V}\psi'\psi'' = \psi'\psi''$, multiplied by any scalar.

(3.) The differential equation of the geodetics may therefore be thus written (comp. 381, XIII.),

$$\text{VI.} \ldots 0 = \mathrm{S}(\mathrm{V}\psi'\psi''.\mathrm{V}\rho'\rho'') = \mathrm{S}\rho'\psi''\mathrm{S}\rho''\psi' - \mathrm{S}\rho''\psi''\mathrm{S}\rho'\psi';$$

in which, by (1.) and (2.),

$$\text{VII.} \ldots \begin{cases} \mathrm{S}\rho'\psi'' = -yz^2, & \mathrm{S}\rho''\psi' = -y'' + yz^2, \\ \mathrm{S}\rho''\psi'' = -(1 + 2y')z^2 - yzz', & \mathrm{S}\rho'\psi' = -(1 + y'); \end{cases}$$

the equation becomes therefore, after division by $-z$,

$$\text{VIII.} \ldots 0 = z\{(1 + y')^2 + (yz)^2\} + (1 + y')(yz)' - y''yz,$$

or simply,

$$\text{IX.} \ldots z + v' = 0, \ \text{or IX}'. \ldots \mathrm{Td}\psi' + \mathrm{d}v = 0, \ \text{if X.} \ldots \tan v = \frac{yz}{1 + y'}, = \frac{y\mathrm{T}\psi''}{1 + y'}.$$

(4.) To *interpret* now this very simple equation IX. or IX′., we may observe that z, or $\mathrm{T}\psi''$, or $\mathrm{Td}\psi' : \mathrm{d}x$, expresses the *limiting ratio*, which the *angle* between *two near tangents* ψ' and $\psi' + \Delta\psi'$, to the *cusp-edge* (1.), bears to the *small arc* Δx

of that curve which is intercepted between their points of contact; while v is, by IV., that *other angle*, at which *such* a variable tangent, or *generating line* of the developable, *crosses the geodetic* on that surface; and therefore its *derivative*, v' or $\mathrm{d}v : \mathrm{d}x$, represents the limiting ratio, which the *change* Δv of this last angle, in passing from one generating line to another, bears to the *same small arc* Δx of the curve which those lines *touch*.

(5.) Referring then to Figure 79, in which, instead of *two continuous curves*, there were *two gauche polygons*, or at least *two systems* of *successive right lines*, connected by *prolongations* of the lines of the first system, we see that the recent formula IX. or IX′. is equivalent to this *limiting equation*,

$$\text{XI.} \ldots \lim. \frac{\mathrm{CD'C'} - \mathrm{BC'B'}}{\mathrm{C'CD'}} = -1;$$

but these three *angles* remain *unaltered*, in the *development* of the surface: the *bent line* B′C′D′ for *space* becomes therefore *ultimately* a *straight* line in the *plane*, and similarly for *all other* portions of the original polygon, or *twisted line*, B′C′D′E′..., of which B′C′D′ was a *part*.

(6.) Returning then to *curves* and *surfaces* in *space*, the quaternion analysis (3.) is found, by this simple reasoning,* to conduct to an expression for the known and *characteristic property* of the *geodetics on a developable*: namely that they *become right lines*, as those on *cylinders* (380, (4.)), and on *cones* (380, (6.) and (10.), or 381, (5.)), were lately seen to do, when the *surface* on which they are thus traced is *unfolded into a plane*.

383. This known result, respecting *geodetics on developables*, may be very simply verified, by means of a new determination of the *absolute*† *normal* (379) to a curve in space, as follows.

(1.) The arc s of any curve being taken for the independent variable, we may write (comp. 376, I.), by Taylor's Series, the following rigorous expressions,

$$\text{I.} \ldots \rho_{-s} = \rho - s\rho' + \tfrac{1}{2}s^2u_{-s}\rho'', \quad \rho_0 = \rho, \ \rho_s = \rho + s\rho' + \tfrac{1}{2}s^2u_s\rho'', \quad \text{with} \quad u_0 = 1,$$

for the vectors of three *near points*, P_{-s}, P_0, P_s, on the curve, whereof the second *bisects the arc*, $2s$, intercepted between the first and third.

(2.) If then we conceive the *parallelogram* $\mathrm{P}_{-s}\mathrm{P}_0\mathrm{P}_s\mathrm{R}_s$ to be completed, we shall have, for the *two diagonals* of this new figure these other rigorous expressions,

$$\text{II.} \ldots \mathrm{P}_{-s}\mathrm{P}_s = \rho_s - \rho_{-s} = 2s\rho' + \tfrac{1}{2}s^2(u_s - u_{-s})\rho'';$$
$$\text{III.} \ldots \mathrm{P}_0\mathrm{R}_s = \rho_s + \rho_{-s} - 2\rho_0 = \tfrac{1}{2}s^2(u_s + u_{-s})\rho'';$$

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* In the *Lectures* (page 581), nearly the same *analysis* was employed, for geodetics on a developable; but the *interpretation* of the result was made to depend on an equation which, with the recent significations of ψ and v, may be thus written, as the integral of IX′., $v + \int \mathrm{T}d\psi' = \text{const.}$; where $\int \mathrm{T}d\psi'$ represents the *finite angle* between the *extreme tangents* to the *finite arc* $\int \mathrm{T}d\psi$, or Δx, of the *cusp-edge*, when that curve is *developed* into a *plane* one.

† Called also, and perhaps more usually, the *principal normal*.

which give the limiting equations,

$$\text{IV.} \ldots \lim_{s=0} s^{-1}\mathrm{P}_{-s}\mathrm{P}_s = 2\rho'; \qquad \text{V.} \ldots \lim_{s=0} s^{-2}\mathrm{P}_0\mathrm{R}_s = \rho''.$$

(3.) But the *length* $\overline{\mathrm{P}_{-s}\mathrm{P}_s}$ of what may be called the *long diagonal*, or the *chord* of the *double arc*, $2s$, is *ultimately equal* to that double arc; we have therefore, by IV., the equation,

$$\text{VI.} \ldots \mathrm{T}\rho' = 1, \quad \text{if} \quad \rho' = \mathrm{D}_s\rho, \ \text{and if } s \text{ denote the } arc,$$

considered as the scalar variable on which the vector ρ depends: a result agreeing with what was otherwise found in 380, (12.).

(4.) At the same time, since the *ultimate direction* of the same long diagonal is evidently that of the *tangent* at P_0, we see anew that the same *first derived vector* ρ' represents what may be called the *unit-tangent** to the curve at that point.

(5.) And because the *lengths* of the *two sides* $\mathrm{P}_{-s}\mathrm{P}_0$ and $\mathrm{P}_0\mathrm{P}_s$, considered as *chords* of the two successive and *equal arcs*, s and s, are *ultimately equal* to them and to each other, it follows that the *parallelogram* (2.) is *ultimately equilateral*, and therefore that its *diagonals* are *ultimately rectangular;* but these diagonals, by IV. and V., have ultimately the directions of ρ' and ρ''; we find therefore anew the equation,

$$\text{VII.} \ldots \mathrm{S}\rho'\rho'' = 0, \ \text{if the arc be the independent variable,}$$

which had been otherwise deduced before, in 380, (12.).

(6.) But under the same condition, we saw (379, (2.)) that the *second derived vector* ρ'' has the direction of the *absolute normal* to the curve; such then is by V. the *ultimate direction* of what we may call the *short diagonal* $\mathrm{P}_0\mathrm{R}_s$, constructed as in (2.); or, *ultimately*, the direction of the *bisector* of the (obtuse) *angle* $\mathrm{P}_{-s}\mathrm{P}_0\mathrm{P}_s$, between the *two near* and *nearly equal chords* from the point P_0: while the *plane* of the *parallelogram* becomes *ultimately* the *osculating plane*.

(7.) All this is quite independent of the consideration of any *surface*, on which the *curve* may be conceived to be *traced.* But if we now conceive that this curve is formed *from a right line* B′C′D′ . . . (comp. Fig. 79), by *wrapping* round a *developable* surface a *plane* on which the *line* had been drawn, and if the successive portions B′C′, C′D′, .. of that line be supposed to have been *equal*, then because the *two right lines* C′B′ and C′D′ *originally* made *supplementary angles* with any *other* line C′C in the *plane*, the *two chords* C′B′ and C′D′ of the *curve* on the developable *tend* to make supplementary angles with the *generatrix* C′C of that surface; on which account the *bisector* (6.) of their *angle* B′C′D′ *tends* to be *perpendicular to that generating line* C′C, *as well as* to the *chord* B′D′, or *ultimately* to the *tangent to the curve* at C′, when chords and arcs *diminish* together. The *absolute normal* (6.) to the curve thus formed is therefore *perpendicular to two distinct tangents* to the surface at C′, and is consequently (comp. 372) the *normal* to that *surface* at that point; whence, by the *definition* (380), the *curve* is, as before, a *geodetic on the developable.*

(8.) As regards the asserted *rectangularity* (7.), of the *bisector* of the angle B′C′D′ to the line C′C, when the angles CC′B′ and CC′D′ are supposed to be *supplementary*, but *not in one plane*, a simple proof may be given by conceiving that the right line B′C′ is prolonged to C″, in such a manner that $\overline{C'C''} = \overline{C'D'}$; for then these two *equally long* lines from C′ make *equal angles* with the line C′C, so that the one may be formed from the other by a *rotation* round that line as an *axis;* whence C″D′, which is evidently *parallel* to the bisector of B′C′D′, is also *perpendicular* to C′C.

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* Compare the Note to page 152.

(9.) In quaternions, if α and ρ be any two vectors, and if t be any scalar, we have the equation,

$$\text{VIII.} \ldots \text{S}.\alpha(\alpha^t \rho \alpha^{-t} - \rho) = 0;$$

which is, by 308, (8.), an expression for the geometrical principle last stated.

384. The recent analysis (382) enables us to deduce with ease, by quaternions, other known and important properties of developable surfaces: for instance, the property that each such surface may be considered as the *envelope* of a *series of planes*, involving only *one* scalar and *arbitrary constant* (or *parameter*) in their *common equation;* and that *each plane* of this series *osculates* to the *cusp-edge* of the *developable.*

(1.) The equation of the developable *surface* being still,

$$\text{I.} \ldots \rho = \phi(x, y) = \psi_x + y\psi'_x = \psi + y\psi', \qquad \text{as in 382, I.,}$$

its *normal* ν is easily found to have as in 382, V., the *direction* of $\text{V}\psi'\psi''$, whether the scalar variable x be, or be not, the *arc* of the *cusp-edge*, of which curve the equation is,

$$\text{II.} \ldots \rho = \psi_x.$$

(2.) Hence, by 373, VII., the equation of the *tangent plane* takes the form,

$$\text{III.} \ldots \text{S}\omega\psi'\psi'' = \text{S}\psi\psi'\psi'',$$

from which the *second* scalar variable y thus *disappears:* this *common equation*, of all the tangent planes to the developable, involves therefore, as above stated, only *one* variable and scalar *parameter*, namely x; and the *envelope* of all these *planes* is the developable surface itself.

(3.) The plane III., for any *given value* of this parameter x, that is, for any *given point* of the cusp-edge, *touches the surface* along the *whole extent* of the *generating line*, which is the *tangent* to this last *curve*.

(4.) And by comparing its equation III. with the formula 376, V., we see at once that this plane *osculates* to the same cusp-edge, at the point of *contact* of that curve with the same *generatrix* of the developable.

385. If the *reciprocals* of the *perpendiculars*, let fall from a given origin, on the *tangent planes* to a developable surface, be considered as being themselves vectors from that origin, they terminate on a *curve*, which is connected with the *cusp-edge* of the developable by some interesting relations of *reciprocity* (comp. 373, (21.)): in such a manner that if this *new curve* be made the cusp-edge of a *new developable*, we can *return* from it to the *former* surface, and to *its* cusp-edge, by a *similar process* of construction.

(1.) In general, if ψ_x and χ_x, or briefly ψ and χ, be two vector functions of a scalar variable x, such that χ may be deduced from ψ by the three scalar equations,

$$\text{I.} \ldots \mathrm{S}\psi\chi = c, \quad \mathrm{S}\psi'\chi = 0, \quad \mathrm{S}\psi''\chi = 0,$$

in which $\mathrm{S}\psi\chi$ is written briefly for $\mathrm{S}(\psi_x . \chi_x)$, and c is any scalar constant, we have then this *reciprocal system* of three such equations,

$$\text{II.} \ldots \mathrm{S}\chi\psi = c, \quad \mathrm{S}\chi'\psi = 0, \quad \mathrm{S}\chi''\psi = 0;$$

an intermediate step being the equation,

$$\text{III.} \ldots \mathrm{S}\psi'\chi' = \mathrm{S}\chi'\psi' = 0.$$

(2.) Hence, generally,

$$\text{IV.} \ldots \text{ if } \chi = \frac{c\mathrm{V}\psi'\psi''}{\mathrm{S}\psi\psi'\psi''}, \quad \text{then} \quad \text{V.} \ldots \psi = \frac{c\mathrm{V}\chi'\chi''}{\mathrm{S}\chi\chi'\chi''}.$$

(3.) But if ρ be the variable vector of a curve in space, and ρ', ρ'' its first and second derivatives with respect to any scalar variable, then, by the equation 376, V. of the osculating plane to the curve, we have the general expression,

$$\text{VI.} \ldots \frac{\mathrm{S}\rho\rho'\rho''}{\mathrm{V}\rho'\rho''} = \textit{perpendicular from origin on osculating plane};$$

so that if ψ and χ be considered as the vectors of *two curves*, *each* vector is $c \times$ the *reciprocal of the perpendicular*, thus let fall from a common point, on the osculating plane to the *other*.

(4.) We have therefore this *Theorem*:—

If, from any assumed point, O, *there be drawn lines equal to the reciprocals of the perpendiculars from that point, on the osculating planes to a given curve of double curvature, or to those perpendiculars multiplied by any given and constant scalar; then the locus of the extremities of the lines so drawn will be a second* curve, from which we can return to the first curve by a precisely similar process.*

386. The theory of *developable surfaces*, considered as *envelopes of planes* with *one* scalar and *variable parameter* (384), may be additionally illustrated by connecting it with *Taylor's Series*, as follows.

(1.) Let α_t denote *any vector function* of a *scalar* variable t, so that

$$\text{I.} \ldots \alpha_t = \alpha_0 + tu_t\alpha'_0 = \alpha + tu\alpha', \quad \text{with} \quad u_0 = 1;$$

or, by another step in the expansion,

$$\text{II.} \ldots \alpha_t = \alpha_0 + t\alpha'_0 + \tfrac{1}{2}t^2v_t\alpha''_0 = \alpha + t\alpha' + \tfrac{1}{2}t^2v\alpha'', \quad v_0 = 1;$$

where u and v are generally *quaternions*, but $u\alpha'$ and $v\alpha''$ are *vectors*.

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* The *two curves* may be said to be *polar reciprocals*, with respect to the (real or imaginary) *sphere*, $\rho^2 = c$; and an analogous *relation of reciprocity* exists *generally*, when the *points* of one curve are the *poles* of the *osculating planes* of the other, with respect to *any surface of the second order: corresponding tangents* being then *reciprocal polars*. Compare the theory of *developables reciprocal to curves*, given in Salmon's *Analytical Geometry of Three Dimensions*, page 89; see also Chapter XI. (page 224, &c.), of the same excellent work.

(2.) Then, as the *rigorous equation* of the *variable plane*, the reciprocal of the perpendicular on which from the origin is $-\alpha_t$, we have either,

$$\text{III.} \ldots -1 = S\alpha_t\rho = S\alpha\rho + tSu\alpha'\rho,$$

or

$$\text{IV.} \ldots -1 = S\alpha\rho + tS\alpha'\rho + \tfrac{1}{2}t^2Sv\alpha''\rho,$$

according as we adopt the expression I., or the equally but not more rigorous expression II., for the variable vector α_t.

(3.) Hence, by the form III., the *line of intersection* of the *two planes*, which answer to the *two values* 0 and t of the scalar variable, or *parameter*, t, is *rigorously* represented by the system of the *two scalar equations*,

$$\text{V.} \ldots S\alpha\rho + 1 = 0, \quad Su\alpha'\rho = 0.$$

(4.) And the *limiting position* of this *right line* V., which answers to the conceived *indefinite approach* of the *second plane* to the *first*, is given with *equal rigour* by the equations,

$$\text{VI.} \ldots S\alpha\rho + 1 = 0, \quad S\alpha'\rho = 0;$$

whereof it is seen that the *second* may be formed from the *first*, by *derivating* with respect to t, and treating ρ as *constant:* although *no such rule* of *calculation* had been *previously laid down*, for the comparatively *geometrical process* which is here supposed to be adopted.

(5.) The *locus* of all the *lines* VI. is evidently *some ruled surface;* to determine the *normal* ν to which, at the extremity of the vector ρ, we may consider that vector to be a function (372) of *two* independent and scalar variables, whereof one is t, and the other may be called for the moment w; and thus we shall have the two *partial* derivatives,

$$\text{VII.} \ldots S\alpha D_t\rho = 0, \quad S\alpha D_w\rho = 0, \quad \text{giving} \quad \nu \parallel \alpha.$$

(6.) Hence the *line* α has the direction of the required *normal* ν; the *plane* $S\alpha\rho + 1 = 0$ *touches* the *surface* (comp. 384, (3.)) along the *whole extent* of the *limiting line* VI.; and the *locus* of all *such lines* is the *envelope* of all the *planes*, of the system recently considered.

(7.) The *line* VI. *cuts* generally the *plane* IV., in a *point* which is rigorously determined by the three equations,

$$\text{VIII.} \ldots S\alpha\rho + 1 = 0, \quad S\alpha'\rho = 0, \quad Sv\alpha''\rho = 0;$$

and the *limiting position* of this *intersection* is, with equal rigour, the point determined by this other system of equations,

$$\text{IX.} \ldots S\alpha\rho + 1 = 0, \quad S\alpha'\rho = 0, \quad S\alpha''\rho = 0;$$

in which it may be remarked (comp. (4.)), that the *third* is the *derivative* of the *second*, if ρ be treated as *constant*.

(8.) The *locus* of all these *points* IX. is generally *some curve* upon the *surface* (5.), which is the *locus* of the *lines* VI., and has been seen to be the *envelope* (6.) of the *planes* III. or IV.; and to find the *tangent* to this *curve*, at the point answering to a *given value* of t, or to a *given line* VI., we have by IX. the derived equations,

$$\text{X.} \ldots S\alpha\rho' = 0, \quad S\alpha'\rho' = 0, \quad \text{whence} \quad \rho' \parallel V\alpha\alpha';$$

comparing which with the equations VI. we see that *the lines* VI. *touch the curve*, which is thus *their* common *envelope*.

(9.) We see then, in a new way, that *the envelope of the planes* III., which have *one scalar parameter* (t) in their *common equation*, and may represent *any system* of planes subject to this *condition*, is a *developable surface:* because it is *in general* (comp. 382) the *locus of the tangents to a curve in space*, although this curve *may* reduce itself to a *point*, as we shall shortly see.

(10.) We may add that if α_t in III. be considered as the vector of a *given curve*, this curve is the *locus of the poles** of the *tangent planes* to the *developable*, taken with respect to the *unit sphere;* and conversely, that the developable surface is the *envelope* of the *polar planes* of the *points* of the same given curve, with respect to the same sphere.

(11.) If then it happen that this *given* curve, with α_t for vector, is a *plane* one, so that we have this new condition,

$$\text{XI.} \ldots \; S\beta\alpha_t + 1 = 0, \; \beta \text{ being any constant vector,}$$

namely the vector of the pole of the supposed plane of the given curve, the *variable plane* III., or $S\rho\alpha_t + 1 = 0$, of which the surface (5.) is the *envelope*, passes constantly through this *fixed pole;* so that the *developable* becomes in this case a *cone*, with β for the vector of its *vertex:* the equations IX. giving *now* $\rho = \beta$.

(12.) The same *degeneration*, of a developable into a *conical* surface, may also be conceived to take place in another way, by the *cusp-edge* (or at least some finite portion thereof) tending to become *indefinitely small*, while yet the *direction* of its *tangents* does not tend to become *constant*. For example, with recent notations, the *developable* which is the locus of the tangents to the *helix* may have its equation written thus:

$$\text{XII.} \ldots \; \rho = \phi(x, y) = c\left(x\alpha + \frac{2}{\pi}\tan a \,.\, \alpha^x U\beta\right) + y\alpha\left(1 + \tan a \,.\, \alpha^y U\beta\right);$$

which when the *quarter-interval*, c, between the *spires*, tends to *zero*, without their *inclination* a to the *axis* α being changed, *tends* to become a *cone of revolution* round that axis, with its *semiangle* $= a$.

387. So far, then, we may be said to have considered, in the present Section, and in connexion with *geodetic lines*, the four following *families* of *surfaces* (if the first of them may be so called). First, *spherical surfaces*, of which the characteristic *property* is expressed by the equation,

$$\text{I.} \ldots \; V\nu(\rho - \alpha) = 0, \; \text{if } \alpha \text{ be } \textit{vector of centre;}$$

second, *cylindrical* surfaces, with the property,

$$\text{II.} \ldots \; S\nu\alpha = 0, \; \text{if } \alpha \text{ be } \textit{parallel to the generating lines;}$$

third, *conical* surfaces, with the property,

$$\text{III.} \ldots \; S\nu(\rho - \alpha) = 0, \; \text{if } \alpha \text{ be } \textit{vector of vertex;}$$

and fourth, *developable* surfaces, with the distinguishing property expressed by the more general equation,

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* Compare the Note to page 525.

$$\text{IV.} \ldots \mathrm{V}\nu d\nu = 0, \text{ if } d\rho \text{ have the } \textit{direction of a generatrix};$$

ν being in each the *normal vector* to the surface, so that

$$\text{V.} \ldots \mathrm{S}\nu d\rho = 0, \text{ for } \textit{all tangential directions} \text{ of } d\rho;$$

and the *fourth* family *including* the *third*, which in its turn includes the *second*. A few additional remarks on these equations may be here made.

(1.) The *geometrical signification* of the equation I. (as regards the *radii*) is obvious; but on the side of *calculation* it may be useful to remark, that *elimination* of ν between I. and V. gives, for *spheres*,

$$\text{VI.} \ldots \mathrm{S}(\rho - \alpha)d\rho = 0, \quad \text{or} \quad \text{VII.} \ldots \mathrm{T}(\rho - \alpha) = \text{const.}$$

(2.) The equations II. and V. show that $d\rho$, and therefore $\Delta\rho$, *may* have the given direction of α; for an *arbitrary cylinder*, then, we have the *vector equation* (372),

$$\text{VIII.} \ldots \rho = \phi(x, y) = \psi_x + y\alpha,$$

where ψ_x is an *arbitrary vector function* of x.

(3.) From VIII. we can at once infer, that

$$\text{IX.} \ldots \mathrm{S}\beta\rho = \mathrm{S}\beta\psi_x, \quad \mathrm{S}\gamma\rho = \mathrm{S}\gamma\psi_x, \quad \text{if} \quad \alpha = \mathrm{V}\beta\gamma;$$

the *scalar equation* (373) of a *cylindrical surface* is therefore generally of the *form* (comp. 371, (6.), (7.)),

$$\text{X.} \ldots 0 = F(\mathrm{S}\beta\rho, \mathrm{S}\gamma\rho);$$

β and γ being two constant vectors, and the generating lines being perpendicular to both.

(4.) The equation III. may be thus written,

$$\text{XI.} \ldots \mathrm{S}\nu\mathrm{U}\alpha = \mathrm{T}\alpha^{-1}\mathrm{S}\nu\rho; \quad \text{whence} \quad \text{XII.} \ldots \mathrm{S}\nu\mathrm{U}\alpha = 0, \quad \text{if} \quad \mathrm{T}\alpha = \infty;$$

the equation for *cones* includes therefore that for *cylinders*, as was to be expected, and reduces itself thereto, when the vertex becomes infinitely distant.

(5.) The same equation III., when compared with V., shows that $d\rho$ may have the direction of $\rho - \alpha$, and therefore that $\rho - \alpha$ may be multiplied by any scalar; the *vector equation* of a *conical surface* is therefore of the form,

$$\text{XIII.} \ldots \rho = \alpha + y\psi_x, \ \psi_x \text{ being an arbitrary vector function.}$$

(6.) The *scalar equation* of a cone may be said to be the result of the *elimination* of a scalar variable t, between two equations of the forms,

$$\text{XIV.} \ldots \mathrm{S}(\rho - \alpha)\chi_t = 0, \quad \mathrm{S}(\rho - \alpha)\chi'_t = 0,$$

which express that the *cone* is the *envelope* (comp. 386, (11.)) of a *variable plane*, which passes through a *fixed point*, and involves only *one* scalar *parameter* in its equation: with a new reduction to a *cylinder*, in a case on which we need not here delay.

(7.) The equation IV. implies, that for each *point* of the surface there is a *direction* along which we may move, *without changing the tangent plane;* and therefore that the surface is an *envelope of planes*, &c., as in 386, and consequently that it is *developable*, in the sense of Art. 382.

(8.) The *vector equation* of a *general developable surface* may be written under the form,

$$\text{XV.} \ldots \rho = \phi(x, y) = \psi_x + y\mathrm{U}\psi'_x;$$

the sign of a *versor* being here introduced, for the sake of facilitating the passage, at a certain *limit*, to a *cone* (comp. 386, (12.)).

(9.) And the *scalar equation* of the same *arbitrary developable* may be represented as the result of the elimination of t, between the *two* equations,

$$\text{XVI.} \ldots \mathrm{S}\rho\chi_t + 1 = 0, \quad \mathrm{S}\rho\chi'_t = 0;$$

in which χ_t is an arbitrary vector function of t.

(10.) The envelope of a *plane* with *two* arbitrary and scalar parameters, t and u, is generally a *curved* but *undevelopable surface*, which may be represented by the system of the *three scalar equations*,

$$\text{XVII.} \ldots \mathrm{S}\rho\chi_{t,u} + 1 = 0, \qquad \mathrm{S}\rho\mathrm{D}_t\chi = 0, \qquad \mathrm{S}\rho\mathrm{D}_u\chi = 0;$$

where $-\chi$ denotes the reciprocal of the perpendicular from the origin on the tangent plane to the surface, at what may be called *the point* (t, u).

388. It remains, on the plan lately stated (380), to consider briefly *surfaces of revolution*, and to investigate the *geodetic lines*, on this additional *family* of surfaces; of which the *equation*, analogous to those marked I. II. III. IV. in 387, for spheres, cylinders, cones, and developables, is of the form,

$$\text{I.} \ldots \mathrm{S}\alpha\rho\nu = 0,$$

if α be a given line in the direction of the *axis* of revolution, supposed for simplicity to pass through the origin; but which may also be represented by either of these two other equations, not involving the normal ν,

$$\text{II.} \ldots \mathrm{T}\rho = f(\mathrm{S}\alpha\rho), \quad \text{or} \quad \text{III.} \ldots \mathrm{TV}\alpha\rho = F(\mathrm{S}\alpha\rho),$$

where f and F are used as characteristics of two *arbitrary* but *scalar functions:* between which $\mathrm{S}\alpha\rho$ may be conceived to be eliminated, and so a *third form* of the same sort obtained.

(1.) In fact, the equation I. expresses that $\nu \parallel\!| \alpha, \rho$, or that the *normal* to the surface *intersects* the *axis;* while II. expresses that the *distance* from a *fixed point* upon that axis is a *function* of its own *projection* on the same *fixed line*, or that the *sections* made by *planes perpendicular* to the axis are *circles;* and the same *circularity* of these sections is otherwise expressed by III., since that equation signifies that the *distance from the axis* depends on the *position* of the *cutting plane*, and is *constant* or *variable* with it: while the two last forms are connected with each other in *calculation*, by means of the general relation (comp. 204, XXI.),

$$\text{IV.} \ldots (\mathrm{T}\alpha\rho)^2 = (\mathrm{S}\alpha\rho)^2 + (\mathrm{TV}\alpha\rho)^2.$$

(2.) The equation I. is *analogous*, in *quaternions*, to a *partial differential equation* of the *first order*, and either of the two other equations, II. and III., is *analogous* to the *integral* of that equation, in the *usual* differential calculus of *scalars*.

(3.) To accomplish the corresponding *integration* in *quaternions*, or to pass from the form I. to II., whence III. can be deduced by IV., we may observe that the equation I. allows us to write (because $S\nu d\rho = 0$),

$$\text{V.} \ldots \nu = x\alpha + y\rho, \qquad \text{VI.} \ldots xS\alpha d\rho + yS\rho d\rho = 0,$$

so that the two scalars $S\alpha\rho$ and $T\rho$ are *together constant*, or *together variable*, and must therefore be *functions* of *each other*.

(4.) Conversely, to *eliminate* the *arbitrary function* from the form II., quaternion *differentiation* gives,

$$\text{VII.} \ldots 0 = S(U\rho . d\rho) + f'(S\alpha\rho) . S\alpha d\rho = S . (U\rho + \alpha f'S\alpha\rho) d\rho\,;$$

hence $\quad \text{VIII.} \ldots \nu \parallel U\rho + \alpha f'S\alpha\rho, \quad$ and $\quad \text{IX.} \ldots \nu \,|||\, \alpha, \rho,$ as before;

so that we can *return* in this way to the equation I., the *functional sign* f *disappearing*.

(5.) We have thus the germs of a *Calculus of Partial Differentials in Quaternions*,[*] *analogous* to that employed by Monge, in his researches respecting *families of surfaces*: but we cannot attempt to pursue the subject farther here.

(6.) But as regards the *geodetic lines* upon a surface of revolution, we have only to substitute for ν, in the recent formula I., by 380, IV., the expression $dUd\rho$, which gives at once the *differential equation*,

$$\text{X.} \ldots 0 = S\alpha\rho dUd\rho = d . S\alpha\rho Ud\rho \text{ (because } S(\alpha d\rho . Ud\rho) = -S\alpha Td\rho = 0)\,;$$

whence, by a first integration, c being a scalar constant,

$$\text{XI.} \ldots c = S\alpha\rho Ud\rho = TV\alpha\rho . SU(V\alpha\rho . d\rho).$$

(7.) The characteristic property of the sought curves is, therefore, that for each of them the *perpendicular distance from the axis of revolution varies inversely as the cosine*[†] *of the angle, at which the geodetic crosses a parallel*, or *circular section* of the surface: because, if $T\alpha = 1$, the line $V\alpha\rho$ has the *length* of the *perpendicular* let fall from a point of the curve on the axis, and has the *direction* of a *tangent* to the parallel.

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[*] The same remark was made in page 574 of the *Lectures*, in which also was given the elimination of the arbitrary function from an equation of the recent form III. It was also observed, in page 578, that *geodetics* furnish a very simple example of what may be called the *Calculus of Variations in Quaternions;* since we may write,

$$\delta\int ds = \delta\int Td\rho = \int \delta Td\rho = -\int S(Ud\rho . \delta d\rho)$$
$$= -\int S(Ud\rho . d\delta\rho) = -\Delta S(Ud\rho . \delta\rho) + \int S(dUd\rho . \delta\rho),$$

and therefore $dUd\rho \parallel \nu$, or $V\nu dUd\rho = 0$, as in 380, IV., in order that the expression under the last integral sign may vanish for all variations $\delta\rho$ consistent with the *equation of the surface:* while the evanescence of the part which is *outside* that sign $\int$ supplies the *equations of limits*, or shows that the *shortest line between two curves* on a given surface is *perpendicular to both*, as usual.

[†] Unless it happen that this cosine is *constantly zero*, in which case $c = 0$, and the *geodetic* is a *meridian* of the surface.

(8.) The equation XI. may also be thus written,

$$\text{XII.} \ldots c\mathrm{T}\rho' = \mathrm{S}\alpha\rho\rho', \quad \text{where} \quad \rho' = \mathrm{D}_t\rho\,;$$

and if the independent variable t be supposed to denote the *time*, while the *geodetic* is conceived to be a curve described by a *moving point*, then while $\mathrm{T}\rho'$ evidently represents the *linear velocity* of that point, as being $= \mathrm{d}s : \mathrm{d}t$, if s denote the *arc* (comp. 100, (5.), and 380, (7.), (11.)), it is easy to prove that $\mathrm{S}\alpha\rho\rho'$ represents the *double areal velocity, projected on a plane perpendicular to the axis;* the *one* of these two *velocities* varies therefore *directly as the other:* and in fact, it is known from mechanics, that *each* velocity would be *constant*,* if the *point* were to describe the *curve*, subject only to the *normal reaction* of the *surface*, and undisturbed by any *other force*.

(9.) As regards the *analysis*, it is to be observed that the *differential equation* X. is satisfied, *not only* by the *geodetics* on the surface of revolution, but *also* by the *parallels* on that surface: which fact of calculation is connected with the obvious geometrical property, that *every normal plane* to such a parallel *contains the axis* of revolution.

(10.) In fact if we draw the normal plane to *any curve* on the surface, at a point where it *crosses* a *parallel*, this *plane* will *intersect* the *axis*, in the point where that axis is met by the *normal* to the *surface*, drawn at the same point of crossing; but *this* construction *fails* to *determine* that normal, if the curve *coincide* with, or even *touch* a parallel, at the point where its normal plane is drawn.

SECTION 6.—*On Osculating Circles and Spheres, to Curves in Space; with some connected Constructions.*

389. Resuming the expression 376, I. for ρ_t, and referring again to Fig. 77, we see that if a *circle* PQD be described, so as to *touch* a given curve PQR, or its tangent PT, at a given point P, and to *cut* the curve at a near point Q, and if PN be the *projection* of the *chord* PQ on the *diameter* PD, or on the *radius* CP, then because we have, rigorously,

$$\text{I.} \ldots \mathrm{PQ} = t\rho' + \tfrac{1}{2}t^2u\rho'', \quad \text{with} \quad u = 1 \quad \text{for} \quad t = 0,$$

we have also

$$\text{II.} \ldots \mathrm{PN} = \tfrac{1}{2}t^2\,\mathrm{V}u\rho''\rho' : \rho',$$

and

$$\text{III.} \ldots \frac{1}{\mathrm{PC}} = \frac{2}{\mathrm{PD}} = \frac{2\mathrm{PN}}{\mathrm{PQ}^2} = \frac{\mathrm{V}u\rho''\rho'}{(\rho' + \frac{1}{2}tu\rho'')^2\rho'}.$$

Conceiving then that the near point Q *approaches indefinitely* to the given point P, in which case the *ultimate state* or *limiting position* of the *circle* PQD is said to be that of the *osculating circle* to the curve at that point P, we see that while the *plane* of this last circle is the *osculating plane* (376), the *vector* κ of its *centre* K, or of the *limiting position* of the point C, is rigorously expressed by the formula:

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* This remark is virtually made in page 443 of Professor De Morgan's *Differential and Integral Calculus* (London, 1842), which was alluded to in page 578 of the *Lectures on Quaternions*.

$$\text{IV.}\ldots\ \kappa = \rho + \frac{\rho'^3}{\mathrm{V}\rho''\rho'};$$

which may however be in many ways *transformed*, by the rules of the present Calculus.

(1.) Thus, we may write, as *transformations* of the expression IV., the following:

$$\text{V.}\ldots\ \kappa = \rho + \frac{\rho'}{\mathrm{V}\rho''\rho'^{-1}} = \rho - \frac{\mathrm{T}\rho'}{\mathrm{V}\rho''\rho'^{-1}.\mathrm{U}\rho'} = \rho - \frac{\mathrm{T}\rho'}{(\mathrm{U}\rho')'};$$

or introducing *differentials* instead of *derivatives*, but leaving still the independent variable *arbitrary*,

$$\text{VI.}\ldots\ \kappa = \rho - \frac{\mathrm{d}\rho^3}{\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho} = \rho + \frac{\mathrm{d}\rho}{\mathrm{V}\mathrm{d}^2\rho\mathrm{d}\rho^{-1}} = \rho - \frac{\mathrm{Td}\rho}{\mathrm{dU}\rho'} = \rho - \frac{\mathrm{d}s}{\mathrm{dUd}\rho},$$

if s be the *arc* of the curve; so that the last expression gives this very simple formula, for the *reciprocal of the radius of curvature*, or for the *ultimate value* of 1 : CP,

$$\text{VII.}\ldots\ (\rho - \kappa)^{-1} = \mathrm{D}_s\mathrm{U}\rho', \quad \text{where} \quad \mathrm{U}\rho' = \mathrm{Ud}\rho, \text{ as before.}$$

(2.) To *interpret* this result, we may employ again that *auxiliary* and *spherical curve*, upon the *cone of parallels to tangents*, which has already served us to *construct*, in 379, (6.) and (7.), the *osculating plane*, the *absolute normal*, and the *binormal*, to the *given curve* in space. And thus we see, that while the *semidiameter* PC has ultimately the *direction* of $\mathrm{dU}\rho'$, and therefore that of the *absolute normal* (379, II.) at P, the *length* of the same radius is ultimately equal to the *arc* PQ (or Δs) *of the given curve, divided by the corresponding arc of the auxiliary curve*; or that the *radius of curvature*, or *radius of the osculating circle* at P, is equal to the *ultimate quotient* of the *arc* PQ, divided by the *angle between the tangents*, PT and (say) QU, to that *arc* PQ *itself* at P, and to its *prolongation* QR at Q, *although* these two tangents are *generally* in *different planes*, and have *no common point*, so long as PQ remains *finite*: because we suppose that the *given* curve is *in general* one of *double curvature*, although the *formulæ*, and the *construction*, above given, are applicable to *plane curves also*.

(3.) For the *helix*, the formula IV. gives, by values already assigned for ρ, ρ', ρ'', and a, the expression,

$$\text{VIII.}\ldots\ \kappa = cta - a^t\beta\cot^2 a, \quad \text{whence} \quad \text{IX.}\ldots\ \rho - \kappa = a^t\beta\,\mathrm{cosec}^2 a,$$

a being the inclination of the given helix to the axis; the *locus of the centre* of the osculating circle is therefore in this case a *second helix*, on the *same cylinder*, if $a = \frac{\pi}{4}$, but otherwise on a *co-axal* cylinder, of which the *radius* = the given radius $\mathrm{T}\beta$, multiplied by the square of the cotangent of a; and the *radius of curvature* $= \mathrm{T}(\rho - \kappa) = \mathrm{T}\beta \times \mathrm{cosec}^2 a$, so that *this* radius always *exceeds* the radius of the cylinder, and is *cut* perpendicularly (without being *prolonged*) by the *axis*.

(4.) In general, if $\mathrm{T}\rho' = \text{const.}$, and therefore $\mathrm{S}\rho'\rho'' = 0$ (comp. 379, (2.)), the expression IV. becomes,[*]

$$\text{X.} \ldots \kappa = \rho + \frac{\rho'^2}{\rho''}; \quad \text{whence,} \quad \text{XI.} \ldots \kappa = \rho - \rho''^{-1}, \quad \text{if} \quad \mathrm{T}\rho' = 1,$$

that is, if the *arc* be taken as the *independent variable* (380, (12.)). Under this last condition, then, the formula VII. reduces itself to the following,

$$\text{XII.} \ldots (\rho - \kappa)^{-1} = \rho'' = \mathrm{D}_s{}^2\rho = \textit{ultimate reciprocal of radius } \mathrm{CP};$$

so that ρ'' (for $\mathrm{T}\rho' = 1$) may be called the *Vector of Curvature*, because its *tensor* $\mathrm{T}\rho''$ is a *numerical measure* for what is *usually* called *the curvature*[†] at the point P, and its *versor* $\mathrm{U}\rho''$ represents the *ultimate direction* of the *semidiameter* PC, of the *circle* constructed as above.

(5.) As an example of the application (2.) of the formula IV. for κ, to a *plane curve*, let us take the *ellipse*,

$$\text{XIII.} \ldots \rho = \mathrm{V}\alpha^t\beta, \quad \mathrm{T}\alpha = 1, \quad \mathrm{S}\alpha\beta \gtrless 0, \qquad 337, (2.),$$

considered as an *oblique section* (314, (4.)) of a *right cylinder*. The expressions 376, (5.) for the derivatives of ρ, combined with the expression XIII. for that vector itself, give here the relations,

$$\text{XIV.} \ldots \mathrm{V}\rho\rho'' = 0, \quad \mathrm{V}\rho'\rho''' = 0;$$

and therefore comp. (338, (5.)),

$$\text{XV.} \ldots \mathrm{V}\rho\rho' = \text{const.}^{\ddagger} = \frac{\pi}{2}\beta\gamma, \quad \mathrm{V}\rho'\rho'' = \text{const.} = \left(\frac{\pi}{2}\right)^3\beta\gamma, \quad \text{if} \quad \gamma = \mathrm{V}\alpha\beta;$$

hence for the present curve we have by IV.,

$$\text{XVI.} \ldots \kappa = \rho - \frac{\rho'^3}{\mathrm{V}\rho'\rho''} = \mathrm{V}\alpha^t\beta - (\mathrm{V}\alpha^{t+1}\beta)^3(\beta\gamma)^{-1}.$$

(6.) To *interpret* this result, we may write it as follows,

$$\text{XVII.} \ldots \kappa = \rho - \frac{\rho_1{}^2}{\mathrm{V}\rho\rho' \cdot \rho'^{-1}}, \quad \text{where} \quad \text{XVIII.} \ldots \rho_1 = \frac{2}{\pi}\rho' = \mathrm{V}\alpha^{t+1}\beta;$$

so that ρ_1 is the *conjugate semidiameter* of the ellipse (comp. 369, (4.)), and $\mathrm{V}\rho\rho' : \rho'$ is the *perpendicular from the centre* of that curve *on the tangent*. We recover then, by this simple analysis, the known result, that the *radius of curvature* of an ellipse is equal to the *square* of the *conjugate* semidiameter, *divided* by the *perpendicular*.

(7.) We may also write the equation XVI. under the form,

$$\text{XIX.} \ldots \kappa = \rho - \frac{\rho_1{}^3}{\mathrm{V}\rho\rho_1}, \quad \text{where} \quad \text{XX.} \ldots \mathrm{V}\rho\rho_1 = \beta\gamma = \text{const.};$$

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[*] The expressions X. XI. may also be easily deduced by limits, from the construction in 383, (2.).

[†] It may be remarked that the quantity z, or $\mathrm{T}\psi''$, in the investigation (382) respecting *geodetics on a developable*, represents thus the *curvature of the cusp-edge*, for any proposed value of the *arc*, x, of that curve.

[‡] These values XV. might have been obtained *without integrations*, but this seemed to be the readiest way.

and may interpret it as expressing, that the radius of curvature is equal to the *cube* of the *conjugate* semidiameter, divided by the *constant parallelogram* under *any two* such conjugates; or by the *rectangle* under the *major* and *minor semiaxes*, which are here the vectors β and γ (comp. 314, (2.)).

(8.) The expression XVI. or XIX. for κ is easily seen to *vanish*, as it ought to do, at the *limit* where the *ellipse* becomes a *circle*, by the *cylinder* being cut *perpendicularly*, or by the condition $S\alpha\beta = 0$ being satisfied; and accordingly if we write,

$$\text{XXI.} \ldots e = SU\alpha\beta = \textit{excentricity of ellipse}, \quad \text{or} \quad \text{XXII.} \ldots \gamma^2 = (1 - e^2)\beta^2,$$

we easily find the expressions,

$$\text{XXIII.} \ldots \rho = \beta S.\alpha^t + \gamma S.\alpha^{t-1}, \quad \rho_1 = -\beta S.\alpha^{t-1} + \gamma S.\alpha^t;$$

$$\text{XXIV.} \ldots \rho_1^2 = \beta^2(1 - e^2(S.\alpha^t)^2), \quad \frac{\rho_1}{V\rho\rho_1} = \frac{\rho_1}{\beta\gamma} = \beta^{-2}\left(\beta S.\alpha^t + \frac{\gamma S.\alpha^{t-1}}{1 - e^2}\right);$$

so that the formula XIX. becomes,

$$\text{XXV.} \ldots \kappa = e^2\left(\beta(S.\alpha^t)^3 - \frac{\gamma(S.\alpha^{t-1})^3}{1 - e^2}\right) = e^2\left(\beta(S.\alpha^t)^3 - \frac{\beta^2}{\gamma}(S.\alpha^{t-1})^3\right),$$

thus containing e^2 as a factor.

(9.) And it may be remarked in passing, that the expression XVI., or its recent transformation XXV., for κ as a function of t, may be considered as being in quaternions the *vector equation* (comp. 99, I., or 369, I.) *of the evolute** of the ellipse, or the equation of the *locus of centres of curvature* of that plane curve; and that the last form gives, by elimination of t (comp.† 315, (1.), and 371, (5.)), the following system of *two scalar equations* for the same evolute,

$$\text{XXVI.} \ldots \left(S\frac{\kappa}{\beta}\right)^{\frac{2}{3}} + \left(S\frac{\gamma\kappa}{\beta^2}\right)^{\frac{2}{3}} = e^{\frac{4}{3}}, \quad S\beta\gamma\kappa = 0;$$

or

$$\text{XXVI}'. \ldots (S\beta\kappa)^{\frac{2}{3}} + (S\gamma\kappa)^{\frac{2}{3}} = (e\beta)^{\frac{4}{3}}, \text{ \&c.};$$

which will be found to agree with known results.

(10.) As another example of application to a *plane* curve, we may consider the *hyperbola*,

$$\text{XXVII.} \ldots \rho = t\alpha + t^{-1}\beta, \qquad \text{comp. 371, II.,}$$

with α and β for asymptotes, and with its centre at the origin. In this case the derived vectors are,

$$\text{XXVIII.} \ldots \rho' = \alpha - t^{-2}\beta, \quad \rho'' = 2t^{-3}\beta,$$

whence

$$\text{XXIX.} \ldots V\rho''\rho' = 2t^{-3}V\beta\alpha = t^{-2}V\rho\rho',$$

and the formula IV. becomes,

$$\text{XXX.} \ldots \kappa - \rho = \frac{(t\rho')^2}{V\rho\rho' : \rho'} = \frac{\text{PT}^2}{\text{OV}},$$

where OV is the perpendicular from the centre O on the tangent to the curve at P, and PT is the portion of that tangent, intercepted between the same point P and an asymptote (comp. (6.) and 371, (3.)).

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* That is to say, of the *plane evolute;* for we shall soon have occasion to consider briefly those *evolutes of double curvature*, which have been shown by Monge to exist, *even* when the *given* curve is *plane*.

† In lately referring (373, (1.)) to the formula 315, V., that formula was inadvertently printed as $(\alpha^t)^2 + (\alpha^{t-1})^2 = 1$, the sign S. before each power being omitted.

(11.) We may also interpret the denominator in XXX. as denoting the ***projection*** of the ***semidiameter*** OP ***on the normal***, or as the line NP where N is the foot of the perpendicular from the curve on that normal line; if then K be the sought centre of the osculating circle, we have the ***geometrical*** equations,

$$\text{XXXI.} \ldots \mathrm{NP}.\mathrm{PK} = \mathrm{PT}^2, \qquad \text{XXXII.} \ldots \angle\,\mathrm{NTK} = \frac{\pi}{2};$$

whereof the last furnishes evidently an extremely simple ***construction*** for the ***centre of curvature*** of an ***hyperbola***, which we shall soon find to admit of being extended, with little modification, to a ***spherical conic**** and its ***cyclic arcs***.

(12.) The ***logarithmic spiral*** with its ***pole*** at the ***origin***,

$$\text{XXXIII.} \ldots \rho = \alpha^t\beta, \quad \mathrm{S}\alpha\beta = 0, \quad \mathrm{T}\alpha \gtrless 1, \qquad \text{comp. 314, (5.)}$$

may be taken as a ***third example*** of a ***plane curve***, for the application of the foregoing formulæ. A first derivation gives, by 333, VII.,

$$\text{XXXIV.} \ldots \rho' = (c+\gamma)\rho = \rho(c-\gamma), \ \rho'\rho^{-1} = c+\gamma, \text{ if } c = \mathrm{lT}\alpha, \text{ and } \gamma = \frac{\pi}{2}\mathrm{U}\alpha;$$

the ***constant quaternion quotient***, $\rho' : \rho$, here showing that the prolonged ***vector*** OP makes with the ***tangent*** PT a ***constant angle***, n, which is given by the formula,

$$\text{XXXV.} \ldots \tan n = (\mathrm{TV}:\mathrm{S})\,(\rho':\rho) = c^{-1}\mathrm{T}\gamma, \quad \text{or} \quad \cot n = \frac{2}{\pi}\,\mathrm{lT}\alpha;\dagger$$

and a second derivation gives next,

$$\text{XXXVI.} \ldots \rho'' = (c+\gamma)^2\rho, \quad \mathrm{V}\rho''\rho' = (c^2-\gamma^2)\,\rho^2\gamma = \rho'^2\gamma.$$

The formula IV. becomes therefore, in this case,

$$\text{XXXVII.} \ldots \kappa = \rho + \rho'\gamma^{-1} = \rho c\gamma^{-1} = -c\gamma^{-1}\rho = \frac{2\mathrm{lT}\alpha}{\pi\mathrm{T}\alpha}.\alpha^{t+1}\beta;$$

the ***evolute*** is therefore a ***second spiral***, of the same kind as the first, and the ***radius of curvature*** KP ***subtends a right angle*** at the ***common pole***. But we cannot longer here delay on ***applications within the plane***, and must resume the treatment by quaternions of curves of ***double curvature***.

390. When the *logic* by which the expression 389, IV. was obtained, for the vector κ of the centre of the osculating circle, has once been fully *understood*, the *process* may be conveniently and safely *abridged*, as follows. Referring still to Fig. 77, we may write briefly,

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[*] It was in fact for the ***spherical*** curve that the ***geometrical construction*** alluded to was ***first*** perceived by the writer, soon after the invention of the quaternions, and as a consequence of calculation with them: but it has been thought that a sub-article or two might be devoted, as above, to the ***plane case***, or ***hyperbolic limit***, which may serve at least as a verification.

[†] If r be radius vector, and θ polar angle, and if we suppose for simplicity that $\mathrm{T}\beta = 1$, the ordinary ***polar equation*** of the spiral becomes $r = a^\theta$, with $a = \mathrm{T}\alpha^{\frac{2}{\pi}}$, and $\cot n = \mathrm{l}a$, as usual.

as equations which are all *ultimately true*, or true *at the limit*, in a sense which is supposed to be now distinctly seen:

$$\text{I.} \ldots \mathrm{PT} = \mathrm{d}\rho, \quad \mathrm{TQ} = \tfrac{1}{2}\mathrm{d}^2\rho, \quad \mathrm{PN} = (\text{part of } \mathrm{PQ} \perp \mathrm{PT} =) \frac{\mathrm{Vd}^2\rho\mathrm{d}\rho}{2\mathrm{d}\rho},$$

by 203, &c.; whence, ultimately,

$$\text{II.} \ldots \kappa - \rho = \mathrm{PC} = \frac{\mathrm{PQ}^2}{2\mathrm{PN}} = \frac{\mathrm{PT}^2}{2\mathrm{PN}} = \frac{\mathrm{d}\rho^3}{\mathrm{Vd}^2\rho\mathrm{d}\rho},$$

as before: this *last* expression, in which $\mathrm{Vd}^2\rho\mathrm{d}\rho$ denotes briefly $\mathrm{V}(\mathrm{d}^2\rho.\mathrm{d}\rho)$, being *rigorous*, and permitting the choice of *any scalar*, to be used as the *independent variable*. And then, by writing,

$$\text{III.} \ldots \mathrm{d}\rho = \rho'\mathrm{d}t, \quad \mathrm{d}^2t = 0, \quad \mathrm{d}^2\rho = \rho''\mathrm{d}t^2,$$

the factor $\mathrm{d}t^3$ disappears, and we pass at once to the expression,

$$\text{IV.} \ldots \kappa - \rho = \frac{\rho'^3}{\mathrm{V}\rho''\rho'}, \qquad 389, \text{ IV.},$$

which had been otherwise found before.

(1.) When the *arc* of the curve is taken for the independent variable, then (comp. 380, (12.), &c.) the expression II. reduces itself to the following,

$$\text{V.} \ldots \kappa - \rho = \frac{\mathrm{d}\rho^2}{\mathrm{d}^2\rho}, \quad \text{because} \quad \mathrm{Sd}^2\rho\mathrm{d}\rho = 0;$$

and accordingly the *angle* PTQ in Fig. 77 is then *ultimately right* (comp. 383, (5.)), so that we may at once write, with *this choice* of the scalar variable,

$$\text{VI.} \ldots \kappa - \rho = (\textit{ult.})\,\mathrm{PC} = (\textit{ult.})\,\frac{\mathrm{PT}^2}{2\mathrm{TQ}} = \frac{\mathrm{d}\rho^2}{\mathrm{d}^2\rho}, \text{ as above.}$$

(2.) Suppose then that we have thus *geometrically* (and *very simply*) *deduced* the expression V. for $\kappa - \rho$, for this *particular choice* of the scalar variable; and let us consider how we might thence *pass*, in *calculation*, to the more *general* formula II., in which that variable is left *arbitrary*. For this purpose, we may write, by principles already stated,

$$\text{VII.} \ldots (\rho - \kappa)^{-1} = \frac{\mathrm{d}^2\rho}{(\mathrm{Td}\rho)^2} = \frac{1}{\mathrm{Td}\rho}\,\mathrm{d}\,\frac{\mathrm{d}\rho}{\mathrm{Td}\rho} = \frac{\mathrm{dUd}\rho}{\mathrm{Td}\rho} = \frac{\mathrm{Vd}^2\rho\mathrm{d}\rho^{-1}.\mathrm{Ud}\rho}{\mathrm{Td}\rho}$$
$$= -\frac{\mathrm{Vd}^2\rho\mathrm{d}\rho^{-1}}{\mathrm{d}\rho} = \frac{\mathrm{Vd}\rho\mathrm{d}^2\rho}{\mathrm{d}\rho^3};$$

and the required transformation is accomplished.

(3.) And generally, if s denote the *arc* of any curve of which ρ is the variable vector, we may establish the *symbolical equations*,

$$\text{VIII.} \ldots \mathrm{D}_s = \frac{1}{\mathrm{Td}\rho}\,\mathrm{d}; \quad \mathrm{D}_s^2 = \frac{1}{\mathrm{Td}\rho}\,\mathrm{d}\,\frac{1}{\mathrm{Td}\rho}\,\mathrm{d} = \left(\frac{1}{\mathrm{Td}\rho}\mathrm{d}\right)^2; \text{ \&c.}$$

(4.) For example (comp. 389, XII.), the *Vector of Curvature*, $\mathrm{D}_s^2\rho$, admits of being expressed *generally* under any one of the five last forms VII.

391. Instead of determining the vector κ of the centre of the osculating circle by *one vector expression*, such as 389, IV., or any of its transformations, we may determine it by a system of *three scalar equations*, such as the following,

$$\text{I.}\ldots S(\kappa-\rho)\rho'=0;\qquad \text{II.}\ldots S(\kappa-\rho)\rho''-\rho'^2=0;$$
$$\text{III.}\ldots S(\kappa-\rho)\rho'\rho''=0,$$

of which it may be observed that the second is the *derivative* of the first, if κ be treated as constant (comp. 386, (4.)); and of which the first expresses (369, IV.) that the sought *centre* is *in the normal plane* to the curve, while the third expresses (376, V.) that it is *in the osculating plane;* and the second serves to fix its position *on the absolute normal* (379), in which those two planes intersect.

(1.) Using *differentials* instead of derivatives, but leaving still the independent variable arbitrary, we may establish this equivalent system of three equations,

$$\text{IV.}\ldots S(\kappa-\rho)\,d\rho=0;\quad \text{V.}\ldots S(\kappa-\rho)\,d^2\rho-d\rho^2=0;\quad \text{VI.}\ldots S(\kappa-\rho)\,d\rho d^2\rho=0;$$

of which the second is the differential of the first, if κ be again treated as constant.

(2.) It is also permitted (comp. 369, (2.), 376, (3.), and 380, (2.)), with the same supposition respecting κ, to write these equations under the forms,

$$\text{VII.}\ldots dT(\kappa-\rho)=0;\qquad \text{VIII.}\ldots d^2T(\kappa-\rho)=0;\qquad \text{IX.}\ldots dUV(\kappa-\rho)d\rho=0;$$

and to connect them with *geometrical interpretations*.

(3.) For instance, we may say that the *centre* of the osculating circle is the point, in which the osculating *plane*, III. or VI. or IX., is *intersected* by the *axis* of that circle; namely, by the *right line* which is drawn through its centre, at right angles to its plane: and which is represented by the *two* scalar equations,

I. and II., or IV. and V., or VII. and VIII.

(4.) And we may observe (comp. 370, (8.)), that whereas for a point R taken arbitrarily *in the normal plane* to a curve at a given point P, we can only say *in general*, that if a *chord* PQ be called *small* of the *first order*, then the *difference of distances*, $\overline{RQ}-\overline{RP}$, is small of an order *higher* than the *first;* yet, if the point R be taken *on the axis* (3.) of the osculating circle, then this difference of distances is small, of an order *higher than the second*, in virtue of the equations VII. and VIII.

(5.) The *right line* I. II., or IV. V., or VII. VIII., as being the *locus* of *points* which may be called *poles* of the osculating *circle*, on all possible *spheres* passing through it, is also called the *Polar Axis* of the *curve itself*, corresponding to the given *point of osculation*.

(6.) And because the equation II. is (as above remarked) the *derivative* of I., the known theorem follows (comp. 386), that the *locus* of all such *polar axes* is a *developable surface*, namely that which is called the *Polar Developable*, or the *envelope of the normal planes* to the given curve; of which surface we shall soon have occasion to consider briefly the *cusp-edge*.

392. The following is an entirely different method of investigating, by quaternions, not merely the *radius* or the *centre* of the *osculating circle* to a *curve in space*, but the *vector equation* of that *circle itself:* and in a way which is *applicable alike*, to *plane* curves, and to curves of *double* curvature.

(1.) In general, conceive that OT $= \tau$ is a *given tangent* to a circle, at a given point which is for the moment taken as the origin; and let PP′ $= \rho'$ represent a *variable tangent*, drawn at the extremity of the variable chord OP $= \rho$: also let U be the *intersection*, OT·PP′, of these two tangents. Then the isosceles triangle OUP, combined with the formula 324, XI. for the differential of a reciprocal, gives easily the equations,

$$\text{I.} \ldots \rho' \parallel \rho\tau^{-1}\rho\,; \qquad \text{II.} \ldots \mathrm{V}\tau\rho^{-1}\rho'\rho^{-1} = -(\mathrm{V}\tau\rho^{-1})' = 0\,;$$

$$\text{III.} \ldots \mathrm{V}\tau\rho^{-1} = \text{const.} = \mathrm{V}\tau\alpha^{-1}, \text{ as in 296, IX.}'',$$

if α be the vector OA of any second *given point* A of the circumference.

(2.) The *vector equation* of the circle PQD (389) is therefore,

$$\text{IV.} \ldots \mathrm{V}\,\frac{2\rho'}{\omega - \rho} = \mathrm{V}\,\frac{2\rho'}{\rho_t - \rho} = \frac{2}{t}\,\mathrm{V}.(1 + \tfrac{1}{2}tu\rho''\rho'^{-1})^{-1} = -\mathrm{V}.u\rho''\rho'^{-1}(1 + \tfrac{1}{2}tu\rho''\rho'^{-1})^{-1}\,;$$

whence, passing to the *limit* ($t = 0$, $u = 1$), the analogous equation of the *osculating cicele* is at once found to be,

$$\text{V.} \ldots \mathrm{V}\,\frac{2\rho'}{\omega - \rho} = -\mathrm{V}\,\frac{\rho''}{\rho'}, \quad \text{or} \quad \text{VI.} \ldots \mathrm{V}\left(\frac{2\mathrm{d}\rho}{\omega - \rho} + \frac{\mathrm{d}^2\rho}{\mathrm{d}\rho}\right) = 0\,;$$

with the verification (comp. 296, (9.)), that when we suppose,

$$\text{VII.} \ldots \omega - \rho = 2(\kappa - \rho) \perp \rho',$$

the vector κ of the *centre* is seen to satisfy the equation,

$$\text{VIII.} \ldots \frac{\rho'}{\kappa - \rho} = -\mathrm{V}\,\frac{\rho''}{\rho'}, \quad \text{or} \quad \text{IX.} \ldots \frac{\mathrm{d}\rho}{\kappa - \rho} + \mathrm{V}\,\frac{\mathrm{d}^2\rho}{\mathrm{d}\rho} = 0\,;$$

which agrees with recent results (389, IV., &c.).

(3.) Instead of conceiving that a circle is described (389), so as to *touch* a given curve (Fig. 77) at P, and to *cut* it at *one* near point Q, we may conceive that a circle *cuts* the curve in the *given* point P, and *also* in *two* near points, Q and R, unconnected by any given *law*, but *both* tending together to *coincidence* with P: and may inquire what is the *limiting position* (if any) of the *circle* PQR, which thus *intersects* the curve in *three near points*, whereof *one* (P) is *given*.

(4.) In general, if α, β, ρ be *three co-initial chords*, OA, OB, OP, of any one circle, their *reciprocals* α^{-1}, β^{-1}, ρ^{-1}, if still co-initial, are *termino-collinear* (260); applying which principle, we are led to investigate the condition for the three co-initial vectors,

$$\text{X.} \ldots (\omega - \rho)^{-1}, \quad (s\rho' + \tfrac{1}{2}s^2u_s\rho'')^{-1}, \quad (t\rho' + \tfrac{1}{2}t^2u_t\rho'')^{-1},$$

with $u_0 = 1$, thus *ultimately* terminating on *one right line;* or for our having ultimately a relation of the form,

$$\text{XI.} \ldots \frac{xs + yt}{\omega - \rho} = \frac{x}{\rho' + \frac{1}{2}s\rho''} + \frac{y}{\rho' + \frac{1}{2}t\rho''}\,;$$

or
$$\text{XII.} \ldots \frac{(xs+yt)\rho'}{\omega-\rho} = \frac{x}{1+\frac{1}{2}s\rho''\rho'^{-1}} + \frac{y}{1+\frac{1}{2}t\rho''\rho'^{-1}}$$
$$= x + y - \tfrac{1}{2}(xs+yt)\rho''\rho'^{-1} + \&\text{c.}:$$
in which last equation, both members are generally *quaternions*.

(5.) The comparison of the *scalar parts* gives here no useful information, on account of the *arbitrary* character of the coefficients x and y; but *these* disappear, with the two *other* scalars, s and t, in the comparison of the *vector parts*, whence follows the *determinate* and *limiting equation*,

$$\text{XIII.} \ldots 2\mathrm{V}\rho'(\omega-\rho)^{-1} = -\mathrm{V}\rho''\rho'^{-1},$$

which evidently agrees with V.

(6.) It is then found, by this little quaternion calculation, as was of course to be expected,* that the *circle* (3.), through *any three near points* of a curve in space, coincides *ultimately* with the *osculating* circle, if the *latter* be still *defined* (389) with reference to a *given tangent*, and a *near point*, which *tends* to coincide with the *given point* of contact.

393. An osculating circle to a curve of *double* curvature does not generally meet that curve *again;* but it intersects generally a *plane* curve, of the degree n, to which it osculates, in $2n-3$ points, distinct from the point P of osculation, whereof *one* at least must be *real*, although it may happen to *coincide* with that point P: and such a circle intersects also generally a *spherical* curve of double curvature, and of the degree n, in $n-3$ other points, namely in those where the osculating *plane* to the curve meets it again. An *example* of each of these two last cases, as treated by quaternions, may be useful.

(1.) In general, if we clear the recent equation, 392, V. or XIII., of fractions, it becomes,

$$\text{I.} \ldots 0 = 2\rho'^2\mathrm{V}\rho'(\omega-\rho) + (\omega-\rho)^2\mathrm{V}\rho''\rho';$$

in which $\rho = \text{OP} =$ the vector of the given point of osculation, and ρ', ρ'' are its first and second derivatives, taken with respect to any scalar variable t, and for the particular value (whether zero or not) of that variable, which answers to the *particular point* P; while ω denotes generally the vector of *any point* upon the circle, which osculates to the given curve at that point P.

(2.) Writing then (comp. 389, (10.)),

$$\text{II.} \ldots \rho = t\alpha + t^{-1}\beta, \quad \rho' = \alpha - t^{-2}\beta, \quad \rho'' = 2t^{-3}\beta,$$
and
$$\text{III.} \ldots \omega = \text{OQ} = x\alpha + x^{-1}\beta,$$
to express that we are seeking for the *remaining intersection* Q of a *plane hyperbola*

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* This conclusion is indeed so well known, and follows so obviously from the doctrine of *infinitesimals*, that it is only deduced here as a *verification* of previous formulæ, and for the sake of *practice* in the present Calculus.

with its *osculating circle* at P, the equation I. becomes, after a few easy reductions, including a division by $\mathrm{V}\alpha\beta$, the following *biquadratic* in x,

$$\text{IV.} \ldots 0 = (x - t)^3 (t^3\alpha^2 x - \beta^2);$$

in which the *cubic factor* is to be set aside, as answering only to the point P itself.

(3.) Substituting then, in III., the remaining value IV. of x, we find the expresssion,

$$\text{V.} \ldots \omega = \mathrm{OQ} = \frac{(t\alpha)^2}{t^{-1}\beta} + \frac{(t^{-1}\beta)^2}{t\alpha} = \tfrac{1}{2}\left\{\frac{(2t\alpha)^2}{2t^{-1}\beta} + \frac{(2t^{-1}\beta)^2}{2t\alpha}\right\};$$

comparing which with 371, (3.), we see that if the tangent to the hyperbola at the given point P intersects the asymptotes in the points A, B, then the tangent at the sought point Q meets the same lines OA, OB in points A′, B′, such that

$$\text{VI.} \ldots \mathrm{OA} \,.\, \mathrm{OA}' = \mathrm{OB}^2, \quad \mathrm{OB} \,.\, \mathrm{OB}' = \mathrm{OA}^2;$$

whence Q is at once found, as the bisecting point of the line A′B′.

(4.) A still more simple construction, and one more obviously agreeing with known results, may be derived from the following expression for the *chord* PQ:

$$\text{VII.} \ldots \mathrm{PQ} = \omega - \rho = (t^2\beta^{-2} - t^{-2}\alpha^{-2})(t\alpha^2\beta - t^{-1}\alpha\beta^2)$$
$$= (t^3\beta^{-2} - t^{-1}\alpha^{-2})\alpha\rho'\beta \parallel \alpha\rho'^{-1}\beta;$$

whence it follows (comp. 226) that if this chord PQ, both ways prolonged, meets the two asymptotes OB and OA in the points R and S, we have then the *inverse similitude of triangles* (118),

$$\text{VIII.} \ldots \Delta\, \mathrm{ROS} \propto' \mathrm{AOB}.$$

(5.) As regards the *equality* of the *intercepts*, RP and QS, it can be verified *without specifying* the *second point* Q on the hyperbola, or the *second scalar*, x, by observing that the formula III., combined with the first equation II., conducts to the expressions,

$$\text{IX.} \ldots \mathrm{OR} = \frac{x\rho - t\omega}{x - t} = (x^{-1} + t^{-1})\beta, \qquad \mathrm{OS} = \frac{t\rho - x\omega}{t - x} = (x + t)\alpha;$$

which give, generally,

$$\text{X.} \ldots \mathrm{RP} = \mathrm{QS} = t\alpha - x^{-1}\beta.$$

(6.) And as regards the *general reduction*, of the determination of the osculating *circle* to a spherical curve of double curvature, to the determination of the osculating *plane*, it is sufficient to observe that when we take the centre of the sphere for the origin, and therefore write (comp. 381, XIV.),

$$\text{XI.} \ldots \rho^2 = \text{const.}, \quad \mathrm{S}\rho\rho' = 0, \quad \mathrm{S}\rho\rho'' = -\rho'^2,$$

then if we operate on the vector equation I. with the symbol $\mathrm{V} \,.\, \rho$, and divide by $-\rho'^2$, there results the scalar equation,

$$\text{XII.} \ldots 0 = 2\mathrm{S}\rho(\omega - \rho) + (\omega - \rho)^2 = \omega^2 - \rho^2,$$

which expresses that the *circle* is entirely contained on the ***same spheric***[*] ***surface*** as the curve; while the *other* scalar equation,

$$\text{XIII.} \ldots 0 = \mathrm{S}\rho''\rho'(\omega - \rho),$$

obtained by operating on I. with $\mathrm{S} \,.\, \rho''$, expresses (comp. 376, V.) that the same circle is *in the osculating plane*:* so that its *centre* K is the *foot* of the *perpendicular* let fall on that plane from the origin, and we may therefore write (comp. 385, VI.),

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[*] This conclusion is geometrically evident, but is here drawn as above, for the sake of practice in the quaternions.

$$\text{XIV.} \ldots \text{OK} = \kappa = \frac{S\rho''\rho'\rho}{V\rho''\rho'}, \text{ with the relations, XV.} \ldots S\frac{\omega}{\kappa} = S\frac{\rho}{\kappa} = 1;$$

and with the verification that the expression XIV. agrees with the general formula, 389, IV., because

$$\text{XVI.} \ldots \rho V\rho''\rho' + \rho'^3 = S\rho''\rho'\rho,$$

when the conditions XI. are satisfied.

(7.) And even if the given curve be *not* a *spherical* one, yet if we retain the *general* expression for κ,

$$\text{XVII.} \ldots \kappa = \rho + \frac{\rho'^3}{V\rho''\rho'}, \qquad 389, \text{IV.},$$

and operate on I. with $S.\rho''$ and $S.\rho''\rho'$, we find again the equation XIII. of the osculating plane, combined with a new scalar equation, which may after a few reductions be written thus,

$$\text{XVIII.} \ldots (\omega - \kappa)^2 = (\rho - \kappa)^2;$$

and which represents a *new sphere*, whereon the osculating circle to the curve is a *great circle*.

394. To give now an *example* of a *spherical curve* of double curvature, with its osculating *circle* and *plane* for any proposed point P, and with a determination of the point Q in which these meet the curve *again* (393), we may consider that *spherical conic*, or *spheroconic*, of which the equations are (comp. 357, II.),

$$\text{I.} \ldots \rho^2 + r^2 = 0, \qquad \text{II.} \ldots g\rho^2 + S\lambda\rho\mu\rho = 0;$$

namely the intersection of the *sphere*, which has its centre at the origin, and its radius $= r$, with a *cone* of the second order, which has the same origin for vertex, and has the given lines λ and μ for its two (real) cyclic normals. And thus we shall be led to some sufficiently simple *spherical constructions*, which *include*, as their *plane limits*, the analogous constructions recently assigned for the case of the common *hyperbola*.

(1.) Since $S\lambda\rho\mu\rho = 2S\lambda\rho S\mu\rho - \rho^2 S\lambda\mu$ (comp. 357, II'.), the equations I. and II. allow us to write, as their first derivatives, or at least as equations consistent therewith,

$$\text{III.} \ldots S\rho\rho' = 0, \quad S\lambda\rho' + S\lambda\rho = 0, \quad S\mu\rho' - S\mu\rho = 0,$$

because the independent variable is here arbitrary, so that we may conceive the first derived vector ρ' to be multiplied by any convenient scalar; in fact, it is only the

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* Compare the Note immediately preceding.

direction of this tangential vector ρ' which is here important, although we must *continue* the derivations consistently, and so must write, as consequences of III., the equations,

$$\text{IV.}\ldots\ S\rho\rho'' + \rho'^2 = 0,\quad S\lambda\rho'' + S\lambda\rho' = 0,\quad S\mu\rho'' - S\mu\rho' = 0.$$

(2.) Introducing then the auxiliary vectors,

$$\text{V.}\ldots\ \eta = V\lambda\mu,\quad \sigma = \lambda S\mu\rho + \mu S\lambda\rho,\quad \tau = \rho + \rho',\quad \upsilon = \rho - \rho',$$

whence

$$\text{VI.}\ldots\ 0 = S\eta\sigma = S\lambda\tau = S\mu\upsilon,\quad S\rho\sigma = 2S\lambda\rho S\mu\rho,\quad S\mu\tau = 2S\mu\rho,\quad S\lambda\upsilon = 2S\lambda\rho,$$
$$\tau^2 = \upsilon^2 = \rho^2 + \rho'^2,$$

and by new derivations,

$$\text{VII.}\ldots\ \sigma' = V\eta\rho,\quad \tau' = \rho' + \rho'',\quad \upsilon' = \rho' - \rho'',\quad S\lambda\tau' = S\mu\upsilon' = 0,\quad S\mu\tau' = S\mu\tau,$$
$$S\lambda\upsilon' = -S\lambda\upsilon,$$

we see first that τ and υ are the vectors OT and OU of the points in which the *rectilinear tangent* to the curve at P meets the two *cyclic planes*, perpendicular respectively to λ and μ; and because the *radius* OP is seen to be the *perpendicular bisector* of the *linear intercept* TU between those two planes, so that

$$\text{VIII.}\ldots\ \rho' = \mathrm{PT} = \mathrm{UP} \perp \mathrm{OP},\quad \text{we have}\quad \text{IX.}\ldots\ \mathrm{UOP} = \mathrm{POT},$$

or

$$\text{X.}\ldots\ \frown\mathrm{AP} = \frown\mathrm{PB},$$

if the *tangent arc* on the sphere, to the same conic at the same point P, meet the two *cyclic arcs* CA and CB in the points A and B: the *intercepted arc* AB being thus *bisected* at its point of *contact* P, which is a well-known property of such a curve.

(3.) Another known property of a *sphero-conic* is, that for any *one* such curve the *sum of the two spherical angles* CAB and ABC, and therefore also the *area* of the *spherical triangle* ABC, is *constant*. We can only here remark, in passing, that quaternions recognise this property, under the form (comp. II.),

$$\text{XI.}\ldots\ \cos(\mathrm{A} + \mathrm{B}) = -SU\lambda\rho\mu\rho = -g : T\lambda\mu = \text{const.}$$

(4.) The scalar equations III. and IV. give immediately the vector expressions,

$$\text{XII.}\ldots\ \rho' = \frac{V\rho(\lambda S\mu\rho + \mu S\lambda\rho)}{S\lambda\mu\rho},\qquad \text{XIII.}\ldots\ \rho'' = \rho - \frac{(\rho^2 + \rho'^2)V\lambda\mu}{S\lambda\mu\rho};$$

or by (2.),

$$\text{XIV.}\ldots\ \rho' = \frac{V\rho\sigma}{S\eta\rho},\quad \text{and}\quad \text{XV.}\ldots\ \rho'' = \rho - \xi,\quad \text{if}\quad \text{XVI.}\ldots\ \xi = \frac{\tau^2\eta}{S\eta\rho}$$
$$= \tau - \tau' = \upsilon + \upsilon',$$

the new auxiliary vector ξ being thus that of the point X, in which the osculating plane to the conic at P meets the line η of intersection of the cyclic planes: so that we have the geometrical expressions,

$$\text{XVII.}\ldots\ \rho'' = \mathrm{XP},\quad \tau' = \mathrm{XT},\quad -\upsilon' = \mathrm{XU},\quad \text{if}\quad \xi = \mathrm{OX},$$

and the lines* τ' and υ' are the traces of the osculating plane on those two cyclic planes, or of the latter on the former; while σ and σ', as being perpendicular respectively to ρ' and ρ, while each $\perp \eta$, are the traces on the plane $\lambda\mu$ of the two cyclic normals, of the normal plane to the conic at the point P, and of the tangent plane to the sphere at that point: or at least these lines have the *directions* of those *traces*.

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* We may also consider the derived vectors τ' and υ', or the lines XT and XU, as *corresponding tangents*, at the points T and U (2.), to the *two sections*, made by the *cyclic planes*, of that *developable surface* which is the *locus of the tangents* TPU to the *spherical conic* in question.

(5.) Already, from the expression XVI. for the portion OX of the radius OC (2.), or of that radius prolonged, which is cut off by the *osculating plane* at P, we can derive a simple *construction* for the position of the *spherical centre*, or *pole*, say E, of the *small circle* which osculates at that point P, to the proposed *sphero-conic*. For if we take the radius r for unity, we have the trigonometric expressions,

$$\text{XVIII.} \ldots \sec \text{CE} \cos \text{EP} = (\text{T}\xi = \text{T}\tau^2 : \text{SU}\eta^{-1}\rho =) \sec^3 \text{PB} \sec \text{CP};$$

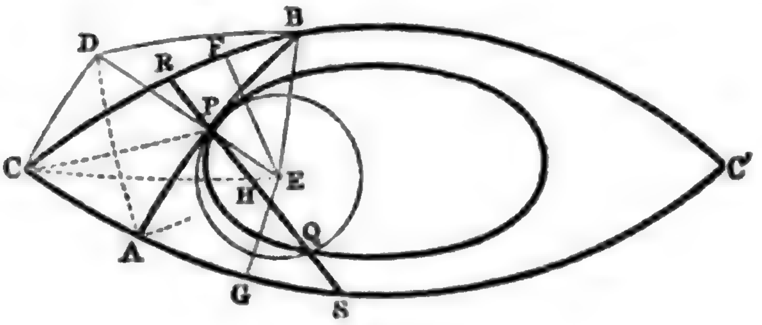

Fig. 80.

or letting fall (comp. Fig. 80) the perpendicular CD on the normal arc PE,

$$\text{XIX.} \ldots \cos \text{DE} = \cos \text{DP} \cos \text{PB} \,.\, \cos \text{PB} \cos \text{PE} = \cos \text{DB} \cos \text{BE};$$

or finally,

$$\text{XX.} \ldots \text{DBE} \ (\text{or DAE}) = \frac{\pi}{2}.$$

(6.) But although it is a perfectly *legitimate* process to *mix* thus *spherical trigonometry* with *quaternions* (since in fact the latter *include* the former), yet it may be satisfactory to deduce this last result by a more *purely* quaternionic method, which can easily be done as follows. The values (4.) of ρ' and ρ'' give,

$$\begin{aligned}\text{XXI.} \ldots \text{V}\rho'\rho''\text{S}\eta\rho = \rho\text{S}\sigma\rho'' - \sigma\text{S}\rho\rho'' &= \rho\text{S}\rho\sigma + \rho'^2\sigma \\ &= (\tau - \rho')\text{S}\sigma\tau + \sigma\text{S}\rho'\tau = \tau\text{S}\sigma\tau + \text{V}\tau\rho'\sigma \parallel\!| \tau, \quad \text{V}\tau\rho'\sigma,\end{aligned}$$

in which $\rho'\sigma$ denotes a vector $\perp \rho'$ (because $\text{S}\rho'\sigma = 0$), and $\parallel\!| \eta, \rho'$ (because $\text{S}\eta\rho'\rho'\sigma = 0$); this line $\rho'\sigma$ has therefore the direction of the *projection* of the line η on a plane perpendicular to ρ', and we are thus led to draw, through the line OC of intersection of the cyclic planes, a *plane* COD perpendicular to the normal plane to the conic at P, or to let fall (as in Fig. 80) a perpendicular *arc* CD on the normal arc PD; after which the normal to the sought osculating plane, or the *axis* OE of the osculating circle sought, as being $\parallel \text{V}\rho'\rho''$, will be contained *in the plane through the trace* τ, or OT, or OB, which is *perpendicular to the plane of* τ *and* $\rho'\sigma$, or to the plane DOB; and therefore the *spherical angle* DBE (or DAE) will be a *right* angle, as before.

(7.) We may also observe that if K be the *centre* of the osculating circle, considered *in its own plane*, or the *foot of the perpendicular* on that plane from O, then by XXI.,

$$\text{XXII.} \ldots \mathrm{OK} = \kappa = \frac{\mathrm{S}\rho\rho'\rho''}{\mathrm{V}\rho'\rho''} = \frac{\tau^2\mathrm{S}\rho\sigma}{\rho\mathrm{S}\rho\sigma + \rho'^2\sigma}, \quad \mathrm{KP} = \rho - \kappa = \frac{\rho'^2\mathrm{V}\rho\sigma}{\rho\mathrm{S}\rho\sigma + \rho'^2\sigma},$$

and therefore

$$\text{XXIII.} \ldots \frac{\mathrm{KP}}{\mathrm{OK}} = \frac{\rho - \kappa}{\kappa} = \frac{\rho'^2}{\tau^2}\frac{\mathrm{V}}{\mathrm{S}}\rho\sigma, \quad \text{XXIV.} \ldots \tan \mathrm{EP} = \sin^2 \mathrm{PB} \cot \mathrm{PD},$$

which gives again the angular relation XX.; the quotient XXIII. being thus a *vector*, as it ought by 393, XV. to be; and the *trigonometric* formula XXIV. being obtained from its expression, by observing that

$$\text{XXV.} \ldots \mathrm{T}\rho'\tau^{-1} = \overline{\mathrm{PT}} : \overline{\mathrm{OT}} = \sin \mathrm{POT} = \sin \mathrm{PB}, \quad \text{and} \quad (\mathrm{V} : \mathrm{S})\rho\sigma = \mathrm{U}\rho' . \cot \mathrm{PD},$$

because $\sigma \perp \rho'\sigma$, but $||| \rho, \rho'\sigma$, or $\rho'\sigma \perp \sigma$, but $||| \rho, \sigma$.

(8.) The rectangularity of the planes of τ, κ and τ, $\rho'\sigma$ is also expressed by the equation,

$$\text{XXVI.} \ldots 0 = \mathrm{S}(\mathrm{V}\kappa\tau . \mathrm{V}\rho'\sigma\tau) = \mathrm{S}\kappa\tau\mathrm{S}\rho'\sigma\tau - \tau^2\mathrm{S}\rho'\sigma\kappa;$$

in proving which we may employ the values,

$$\text{XXVII.} \ldots \mathrm{S}\tau\kappa^{-1} = 1, \quad \mathrm{S}\rho'\sigma\kappa^{-1} = (-\tau^{-2}\rho'^2\mathrm{S}\eta\rho =)\ \mathrm{S}\rho'\sigma\tau^{-1}.$$

(9.) We may also interpret these equations XXVII., as expressing the system of the two relations,

$$\text{XXVIII.} \ldots \kappa^{-1} - \tau^{-1} \perp \tau, \quad \kappa^{-1} - \tau^{-1} \perp \rho'\sigma;$$

from which it follows that κ^{-1}, and therefore also that κ, is a line in the plane so drawn through τ, as to be perpendicular to the plane through τ and $\rho'\sigma$, as before.

(10.) And the two relations XXVIII. are both included in the following expression,

$$\text{XXIX.} \ldots \kappa^{-1} - \tau^{-1} = \mathrm{V}\tau^{-1}\rho'\sigma : \mathrm{S}\rho\sigma.$$

(11.) We may also easily deduce, from the foregoing *spherical construction*, the following *trigonometric expressions*, for the *arcual radius* $r = \mathrm{EP}$ of the *osculating small circle* (5.), and for the *angle* $a = \mathrm{PAE} = \mathrm{EBP}$ which it subtends at A or at B:

$$\text{XXX.} \ldots \tan r = \sin\frac{c}{2}\tan a; \qquad \text{XXXI.} \ldots \tan a = \tfrac{1}{2}(\cot \mathrm{A} + \cot \mathrm{B});$$

A and B *here* denoting, as in XI., the *base angles* of the triangle ABC with C for vertex, and c denoting as usual the *base* AB, namely the portion of the *arcual tangent* (2.) to the conic, which is intercepted between the cyclic arcs.

(12.) The osculating *plane* and *circle* at P being thus fully and in various ways determined, we may next inquire (393) *in what point* Q do they *meet the conic again*. In symbols, denoting by ω the vector of this point, we have the *three scalar equations*,

$$\text{XXXII.} \ldots \mathrm{S}\kappa\omega = \mathrm{S}\kappa\rho, \quad \mathrm{S}\lambda\omega\mathrm{S}\mu\omega = \mathrm{S}\lambda\rho\mathrm{S}\mu\rho, \quad \omega^2 = \rho^2,$$

which are all evidently *satisfied* by the value $\omega = \rho$, but can in general be satisfied *also* by *one other vector value*, which it is the object of the problem to assign.

(13.) We satisfy the *two first* of these three equations XXXII., by assuming the expression,

$$\text{XXXIII.} \ldots \omega = \xi + \tfrac{1}{2}(x^{-1}\tau' - x\upsilon'),$$

in which x is any scalar; in fact we have the relations,

$$\text{XXXIV.} \ldots \mathrm{S}\kappa\xi = \mathrm{S}\kappa\rho, \quad \mathrm{S}\lambda\upsilon' = -2\mathrm{S}\lambda\rho, \quad \mathrm{S}\mu\tau' = 2\mathrm{S}\mu\rho,$$
$$0 = \mathrm{S}\lambda\xi = \mathrm{S}\mu\xi = \mathrm{S}\lambda\tau' = \mathrm{S}\mu\upsilon' = \mathrm{S}\kappa\tau' = \mathrm{S}\kappa\upsilon',$$

whence XXXIII. gives, $\text{XXXV.} \ldots \mathrm{S}\lambda\omega = x\mathrm{S}\lambda\rho, \quad \mathrm{S}\mu\omega = x^{-1}\mathrm{S}\mu\rho$, &c.

And because

$$\text{XXXVI.} \ldots \rho = \xi + \tfrac{1}{2}(\tau' - \upsilon'),$$

we shall satisfy also the *third* equation XXXII., if we adopt for x *any root* of that new scalar equation, which is obtained by equating the square of the expression XXXIII. for ω, to what that square becomes when x is changed to 1.

(14.) To facilitate the formation of this new equation, we may observe that the relations,

$$\xi = \rho - \rho'', \quad \tau' = \rho' + \rho'', \quad \upsilon' = \rho' - \rho'', \quad S\rho\rho' = 0, \quad S\rho\rho'' = -\rho'^2,$$

which have all occurred before, give

$$\text{XXXVII.} \ldots -4S\xi\tau' = 3\tau'^2 + \upsilon'^2, \quad 4S\xi\upsilon' = \tau'^2 + 3\upsilon'^2;$$

the resulting equation is therefore, after a few slight reductions, the following *biquadratic in* x,

$$\text{XXXVIII.} \ldots 0 = (x-1)^3 (\upsilon'^2 x - \tau'^2);$$

of which the *cubic factor* is to be rejected (comp. 393, (2.)), as answering only to the point P itself.

(15.) We have then the values,

$$\text{XXXIX.} \ldots x = \tau'^2 \upsilon'^{-2}, \quad \text{and} \quad \text{XL.} \ldots \mathrm{OQ} = \omega = \xi + \tfrac{1}{2}\left(\frac{\upsilon'^2}{\tau'} - \frac{\tau'^2}{\upsilon'}\right);$$

comparing which last expression with the formulæ XVII., we see that the required point of intersection Q, of the sphero-conic with its osculating circle, can be *constructed* by the following rule. On the traces (4.), of the osculating plane on the two cyclic planes, determine two points T_1 and U_1, by the conditions,

$$\text{XLI.} \ldots \mathrm{XT}.\mathrm{XT}_1 = \mathrm{XU}^2, \quad \mathrm{XU}.\mathrm{XU}_1 = \mathrm{XT}^2; \quad \text{then} \quad \text{XLII.} \ldots \mathrm{T}_1\mathrm{Q} = \mathrm{QU}_1,$$

or in words, *the right line* $\mathrm{T}_1\mathrm{U}_1$ *is bisected by the sought point* Q.

(16.) But a still more simple or more *graphic* construction may be obtained, by investigating (comp. 393, (4.)) the *direction of the chord* PQ. The *vector value* of this *rectilinear chord* is, by XXXVI. and XL.,

$$\text{XLIII.} \ldots \mathrm{PQ} = \omega - \rho = \tfrac{1}{2}(\upsilon'^2 - \tau'^2)(\upsilon'^{-1} + \tau'^{-1}) = \tfrac{1}{2}(\tau'^{-2} - \upsilon'^{-2})\,\tau'(\tau' + \upsilon')\,\upsilon'$$
$$= \left(\frac{\rho'^2}{\tau'^2} - \frac{\rho'^2}{\upsilon'^2}\right)\tau'\rho'^{-1}\upsilon', \text{ because } \rho' = \tfrac{1}{2}(\tau' + \upsilon');$$

the chord PQ has therefore the direction (or its opposite) of the *fourth proportional* (226) to the *three vectors*, ρ', τ', and $-\upsilon'$, or PT, XT, and XU; if then we conceive this chord or its prolongations to meet the traces XT, XU in two new points T_2, U_2, we shall have (comp. 393, VIII.) the two *inversely similar triangles* (118),

$$\text{XLIV.} \ldots \Delta\, \mathrm{T}_2\mathrm{XU}_2 \propto' \mathrm{UXT}.$$

(17.) To deduce hence a *spherical construction* for Q, we may conceive *four planes*, through the *axis* OKE, *perpendicular* respectively to the *four* following *right lines* in the *osculating plane :*

$$\text{XLV.} \ldots \tau', \; -\upsilon', \; \rho', \; \omega - \rho, \quad \text{or} \quad \mathrm{XT}, \; \mathrm{XU}, \; \mathrm{PT}, \; \mathrm{PQ};$$

which planes will cut the *sphere* in *four great circles*, whereof the four *arcs*,

$$\text{XLVI.} \ldots \mathrm{EF}, \; \mathrm{EG}, \; \mathrm{EP}, \; \mathrm{EH},$$

are *parts*, if F, G, H (see again Fig. 80) be the feet of the three *arcual perpendiculars* from the *pole* E of the osculating circle on the two cyclic arcs CB, CA, and on the arcual chord PQ.

(18.) *These four arcs* XLVI. are therefore connected by the *same angular relation* as the *four lines* XLV.; and we have thus the very simple formula,

$$\text{XLVII.} \ldots \mathrm{GEH} = \mathrm{PEF},$$

expressing an equality between *two spherical angles* at the pole E, which serves to determine the *direction* of the *arc* EH, and therefore also the *positions* of the *points* H and Q, by means of the relations,

$$\text{XLVIII.} \ldots \mathrm{PHE} = \frac{\pi}{2}, \quad \cap\,\mathrm{PH} = \cap\,\mathrm{HQ}.$$

(19.) If the arcual chord PQ, both ways prolonged, or any chord of the conic, cut the cyclic arcs CB and CA in the points R and S (Fig. 80), it is well known that there exists the *equality of intercepts* (comp. 270, (2.)),

$$\text{XLIX.} \ldots \cap\,\mathrm{RP} = \cap\,\mathrm{QS};$$

and conversely this equation, combined with the formulæ (11.), or with the trigonometric expression,

$$\text{L.} \ldots \tan \mathrm{PE} = \tan r = \tfrac{1}{2} \sin \frac{c}{2} (\cot \mathrm{A} + \cot \mathrm{B}),$$

for the tangent of the *arcual radius* of the osculating circle, enables us to determine what may be called perhaps the arcual *chord of osculation* PQ, by determining the spherical angle RPB, or simply P, from principles of *spherical trigonometry alone*, in a way which may serve as a verification of the results above deduced from *quaternions*.

(20.) Denoting by t the semitransversal RH = HS, and by s the semichord PH = HQ, the oblique-angled triangles RPB, SPA give the equations,

$$\text{LI.} \ldots \begin{cases} \cot (t - s) \sin \dfrac{c}{2} = \cos \mathrm{P} \cos \dfrac{c}{2} + \sin \mathrm{P} \cot \mathrm{B}, \\[2ex] \cot (t + s) \sin \dfrac{c}{2} = \cos \mathrm{P} \cos \dfrac{c}{2} - \sin \mathrm{P} \cot \mathrm{A}; \end{cases}$$

while the right angled triangle PHE gives,

$$\text{LII.} \ldots \tan s = \sin \mathrm{P} \tan r.$$

Equating then the values of cot $2s$, deduced from LI. and LII., we eliminate s and t, and obtain a quadratic in tan P, of which one root is zero, when tan r has the value L.; such then might in this new way be inferred to be the tangent of the arcual radius of curvature of the conic, and the remaining root of the equation is then,

$$\text{LIII.} \ldots \tan \mathrm{P} = \frac{\cos \dfrac{c}{2} (\cot \mathrm{B} - \cot \mathrm{A})}{\cot \mathrm{A} \cot \mathrm{B} + \cos^2 \dfrac{c}{2} - \tan^2 r};$$

a formula which ought to determine the inclination P, or RPB, or QPA, of the chord PQ to the tangent PA, but which does not appear at first sight to admit of any simple *interpretation*.*

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* We might however at once see from this formula, that P = A − B at the *plane limit*; which agrees with the known construction 393, (4.), for the corresponding chord PQ in the case of the plane *hyperbola*.

(21.) On the other hand, the *construction* (17.) (18.), to which the *quaternion analysis* led us, gives

$$\text{LIV.} \ldots \text{HEP} = \text{GEP} - \text{GEH} = \text{GEP} - \text{PEF} = \text{FEB} + \text{GEA},$$

and therefore, by the four right-angled triangles, PHE, BFE, AGE, and BPE or EPA, conducts to this other formula,

$$\text{LV.} \ldots \cot^{-1}(\cos r \cot \text{P}) = \cot^{-1}\left(\cos r \cos \frac{c}{2} \tan(\text{B} + a)\right) - \cot^{-1}\left(\cos r \cos \frac{c}{2} \tan(\text{A} + a)\right),$$

in which a is the same auxiliary angle as in XXXI.; we ought therefore to find, as the proposed *verification* (19.), that this last equation LV. expresses virtually the *same relation* between A, B, c, and P, as the formula LIII., although there seems at first to be no connexion between them; and such agreement can accordingly be proved to exist, by a chain of ordinary *trigonometric transformations*, which it may be left to the reader to investigate.

(22.) A *geometrical proof* of the validity of the construction (17.) (18.) may be derived in the following way. The *product* of the *sines* of the *arcual perpendiculars*, from a point of a given *sphero-conic* on its two *cyclic arcs*, is well known to be *constant*; hence also the *rectangle* under the *distances* of the same variable point from the two *cyclic planes* is constant, and the curve is therefore the *intersection* of the *sphere* with an *hyperbolic cylinder*, to which those planes are *asymptotic*. It may then be considered to be thus geometrically evident, that the *circle* which *osculates* to the spherical curve, at any given point P, osculates *also* to the *hyperbola*, which is the *section* of that *cylinder*, made by the *osculating plane* at this point; and that the point Q, of recent investigations, is the point in which this hyperbola is met *again*, by *its own* osculating circle at P. But the determination 393, (4.) of *such* a point of intersection, although above deduced (for practice) by quaternions, is a *plane problem* of which the solution was *known*; we may then be considered to have *reduced*, to this known and plane problem, the corresponding *spherical problem* (12.); and thus the *inverse similarity* of the *two plane triangles* XLIV., although *found* by the quaternion analysis, may be said to be *geometrically explained*, or accounted for: the *traces* XT and XU, or τ' and $-\upsilon'$, of the osculating plane to the conic on the two cyclic planes (4.), being evidently the *asymptotes* of the hyperbola in question.

(23.) In quaternions, the constant *product of sines*, &c., is expressed by this form of the equation II. of the *cone*,

$$\text{LVI.} \ldots \text{SU}\lambda\rho . \text{SU}\mu\rho = (g - \text{S}\lambda\mu) : 2\text{T}\lambda\mu = \text{const.};$$

and the scalar equation of the hyperbolic *cylinder*, obtained by eliminating ρ^2 between I. and II., after the first substitution (1.), is

$$\text{LVII.} \ldots \text{S}\lambda\rho\text{S}\mu\rho = \tfrac{1}{2}r^2(g - \text{S}\lambda\mu) = \text{const.};$$

while the expression XXXIII. for ω may be considered as the vector equation of the *hyperbola*, of which the intersection Q with the *circle*, or with the *sphere*, is determined by combining that equation with the condition $\omega^2 = \rho^2 (= -r^2)$.

(24.) In the foregoing investigation, we have treated a *sphero-conic* in connexion with its *cyclic arcs* (2.); but it would have been about equally easy to have treated the same curve, with reference to its *focal points:* or to the *focal lines* of the *cone*, of which it is the *intersection* with a concentric *sphere*. (Compare what has been called the *bifocal transformation*, in 360, (2.)).

(25.) We can however only state generally here the *result* of such an application of quaternions, as regards the construction of the osculating small circle to a spherical conic, considered relatively to its *foci:* which *construction** can indeed be also *geometrically* deduced, as a certain *polar reciprocal* of the one given above. Two focal points (not mutually opposite) being called F and G, let PN be the *normal arc* at P, which is thus *equally inclined*, by a well-known principle, to the two *vector arcs*, FP, GP; so that if the focus G be suitably distinguished from its own opposite, the spherical angle FPG is *bisected* by the arc PN, which is here supposed to *terminate* on the *given arc* FG. *At* N *erect an arc* QNR, *perpendicular to* PN, *and terminating in* Q *and* R *on the two vector arcs. Perpendiculars,* QE, RE, *to these last arcs, will meet on the normal arc* PN, *in the sought pole* (or *spherical centre*) E, *of the sought small circle, which osculates to the conic at the given point* P.

(26.) The two *focal* and arcual *chords of curvature* from P, which pass through F and G, and terminate on the osculating circle, are evidently *bisected* at Q and R, in virtue of the foregoing *construction*, which may therefore be thus enunciated :—

The great circle QR, *which is the common bisector of the two focal and arcual chords of curvature from a given point* P, *intersects the normal arc* PN *on the fixed arc* FG, *connecting the two foci;* that is, *on the arcual major axis of the conic.*

(27.) The construction (5.) fails to determine the position of the auxiliary point D in Fig. 80, for the case when the given point P is *on the minor axis* of the conic; and in fact the expressions (4.) for ρ' and ρ'' become infinite, when the denominator $S\lambda\mu\rho$ is zero. But it is easy to see that the auxiliary vector σ, which represents *generally* the *trace* of the *normal plane* to the curve on the plane of the two *cyclic normals*, becomes at the *limit* here considered the required *axis* of the osculating circle; and accordingly, if we assume simply (comp. (1.) and (2.)),

$$\text{LVIII.} \ldots \rho' = V\rho\sigma, \quad \text{and therefore} \quad \rho'' = V\rho'\sigma + V\rho\sigma',$$

we have

$$\text{LIX.} \ldots \sigma' = 0, \quad \text{and} \quad V\rho'\rho'' \parallel \sigma, \quad \text{when} \quad S\lambda\mu\rho = 0.$$

(28.) In general, if we determine three points L, M, S in the plane of $\lambda\mu$, by the formulæ (comp. again (2.)),

$$\text{LX.} \ldots \text{OL} = \frac{\lambda\rho^2}{S\lambda\rho}, \quad \text{OM} = \frac{\mu\rho^2}{S\mu\rho}, \quad \text{OS} = \frac{\sigma\rho^2}{S\sigma\rho} = \tfrac{1}{2}(\text{OL} + \text{OM}),$$

then L and M will be the intersections of the cyclic normals λ, μ with the tangent plane to the sphere at P, and the normal plane to the curve at the same point will bisect the right line LM in the point S; we shall also have this proportion of sines,

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* The reader can easily draw the Figure for himself. As regards the *known rule*, lately alluded to (in 393, (4.), and 394, (22.)), for determining the *chord of intersection* of a *plane conic* with its *osculating circle*, it will be found (for instance) in page 194 of *Hamilton's Conic Sections* (in Latin, London, 1758). The *two spherical constructions*, for the *small circle* osculating to a *spherical conic*, were early deduced and published by the present writer, as consequences of quaternion calculations. Compare the first Note to page 535.

$$\text{LXI.} \ldots \sin \text{LOS} : \sin \text{SOM} = \mathrm{SU}\lambda\rho : \mathrm{SU}\mu\rho$$
$$= \cos \text{LOP} : \cos \text{POM} = \sin \text{PP}_1 : \sin \text{PP}_2, \qquad \text{comp. (23.)},$$

if PP_1, PP_2 be the arcual perpendiculars from the point P of the conic on the two cyclic arcs; and this *general rule* for determining the position of the line OS, or σ, applies even to the *limiting case* (27.), when that variable line becomes the *axis* of the osculating circle, at a *minor summit* of the curve.

(29.) As an *example*, let us suppose that the constants g, λ, μ in the equation II. are connected by the relation,

$$\text{LXII.} \ldots g = -\mathrm{S}\lambda\mu, \quad \text{whence} \quad \text{LXIII.} \ldots \mathrm{S}(\mathrm{V}\lambda\rho.\mathrm{V}\mu\rho) = 0;$$

the *cyclic normals* are therefore in *this* case *sides* of the cone, and the *two planes* which connect them with *any third side* are mutually *rectangular;* so that the *conic* is now the *locus of the vertex* of a *right-angled spherical triangle*, of which the *hypotenuse* is *given*. And by applying either the formula LXI., or the construction (28.) which it represents, we find that the trigonometric *tangent* of the *arcual radius* of the osculating small circle to *such* a conic, at either end of the given hypotenuse, is equal to *half** *the tangent* of that *hypotenuse itself*.

(30.) It is obvious that every determination, of an *osculating circle* to a *spherical curve*, is at the same time the determination of what may be (and is) called an *osculating right cone* (or cone of *revolution*), to the *cone* which *rests* upon that curve, and has its *vertex* at the *centre* of the sphere. Applying this remark to the last example (29.), we arrive at the following theorem, which can however be otherwise deduced:—

If a cone be cut in a circle by a plane perpendicular to a side, the axis of the right cone which osculates to it along that side passes through the centre of the section.

395. When a given curve of double curvature is *not* a *spherical curve*, we may propose to investigate the *spheric surface* which approaches to it *most closely*, at any assigned point. An *osculating circle* has been defined (389) to be the *limit* of a circle, which *touches* a given curve, or its tangent PT, at a *given point* P, and *cuts* the same curve at a *near* point Q; while the *tangent* PT itself had been regarded (100) as the *limit* of a rectilinear *secant*, or as the *ultimate position* of the *small chord* PQ. It is natural then to *define* the *osculating sphere*, as being the *limit of a spheric surface*, which passes *through* the osculating *circle*, at a given point P of a curve, and also *cuts* that *curve* in a point Q, which is supposed to *approach* indefinitely to P, and *ultimately* to *coincide* with it. Accordingly we shall find that this *definition* conducts by quaternions to *formulæ* sufficiently sim-

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* This may also be inferred by limits from the formulæ (11.); in which r and a were used, provisionally, to denote a certain spherical *arc* and *angle*.

ple; and that their geometrical *interpretations* are consistent with known results: for example, the *centre of spherical curvature*, or the *centre of the osculating sphere*, will thus be shown to be, as usual, the point in which the *polar axis* (391, (5.)) *touches the cusp-edge* of the *polar developable* (391, (6.)). It will also be seen, that whereas *in general*, if R be a point *in the normal plane* (370, (8.)) to a given curve at P, we can only say that the *difference of distances*, $\overline{RQ} - \overline{RP}$, is small of an order *higher than the first*, if the *chord* PR be small of the *first order*; and whereas, even if R be *on the polar axis* (391, (4.)), we can only say generally that this difference of distances is small, of an order *higher than the second*; *yet*, if R be placed *at the centre* S *of spherical curvature*, the difference $\overline{SQ} - \overline{SP}$ is small, of an order *higher than the third*: so that *the distance of a near point* Q, *from the osculating sphere at the given point* P, *is generally small of the fourth order*, the *chord* being still small of the *first*.

(1.) Operating with S.λ, where λ is an arbitrary line, on the vector equation 392, V. of the osculating circle, we obtain the scalar equation of a sphere through that circle under the form,

$$\text{I.} \ldots 0 = 2\mathrm{S}\frac{\lambda\rho'}{\omega - \rho} + \mathrm{S}\frac{\lambda\rho''}{\rho'};$$

which may however, by 393, (7.), be brought to this other form, better suited to our present purpose,

$$\text{II.} \ldots (\omega - \kappa)^2 = (\rho - \kappa)^2 + 2c\mathrm{S}\rho''\rho'(\omega - \rho);$$

c being any scalar constant, while κ is still the vector of the centre K of the circle: and the vector σ of the centre S of the sphere is given by the formula,

$$\text{III.} \ldots \sigma = \kappa + c\mathrm{V}\rho''\rho',$$

which evidently expresses that this last centre is on the polar axis.

(2.) To express now that this sphere cuts the curve in a near point Q, we are to substitute for ω the expression,

$$\text{IV.} \ldots \omega = \rho_t = \rho + t\rho' + \tfrac{1}{2}t^2\rho'' + \tfrac{1}{6}t^3u_t\rho''', \quad \text{with} \quad u_0 = 1;$$

but κ has been seen (in 391) to satisfy the three equations,

$$\text{V.} \ldots 0 = \mathrm{S}\rho'(\kappa - \rho), \quad 0 = \mathrm{S}\rho''(\kappa - \rho) - \rho'^2, \quad 0 = \mathrm{S}\rho''\rho'(\kappa - \rho);$$

reducing then, dividing by $\frac{1}{6}t^3$, and passing to the limit, we find for the *osculating sphere* the condition,

$$\text{VI.} \ldots \mathrm{S}\rho'''(\rho - \kappa) + 3\mathrm{S}\rho'\rho'' = c\mathrm{S}\rho'''\rho''\rho';$$

so that finally the vector σ satisfies the three scalar equations,

$$\text{VII.} \ldots 0 = \mathrm{S}\rho'(\sigma - \rho), \quad 0 = \mathrm{S}\rho''(\sigma - \rho) - \rho'^2, \quad 0 = \mathrm{S}\rho'''(\sigma - \rho) - 3\mathrm{S}\rho'\rho'',$$

by which it is completely determined, and of which the two last are seen to be the successive derivatives of the first, while that first is the equation of the normal plane:

whence the *centre* S of this *sphere* is (by the sub-arts. to 386, comp. 391, (6.)) the point where the *polar axis* KS *touches the cusp-edge* of the polar developable.

(3.) Differentials may be substituted for derivatives in the equations VII., which may also be thus written (comp. 391, (4.)),

$$\text{VIII.}\ldots\ 0=\mathrm{dT}(\rho-\sigma),\quad 0=\mathrm{d}^2\mathrm{T}(\rho-\sigma),\quad 0=\mathrm{d}^3\mathrm{T}(\rho-\sigma),\quad \text{if}\quad \mathrm{d}\sigma=0;$$

the *distance* of a near point Q of the given curve from the osculating *sphere* is therefore *small* (as above said), of an order *higher than the third*, if the *chord* PQ be small of the *first* order.

(4.) The two first equations VII., combined with V., give also

$$\text{IX.}\ldots\ 0=\mathrm{S}\rho'(\sigma-\kappa),\quad 0=\mathrm{S}\rho''(\sigma-\kappa),\quad 0=\mathrm{S}(\kappa-\rho)(\sigma-\kappa);$$

which express that the line KS is perpendicular to the osculating plane and absolute normal at P, as it ought to be, because it is part of the polar axis.

(5.) Conceiving the *three points* P, K, S, or their *vectors* ρ, κ, σ, to *vary together*, the equations V. and VII., combined with their own derivatives, give among other results the following:

$$\text{X.}\ldots\ 0=\mathrm{S}\kappa'\rho'=\mathrm{S}\sigma'\rho'=\mathrm{S}\sigma'\rho''=\mathrm{S}\sigma'(\kappa-\rho)=\mathrm{S}\sigma''\rho';$$

of which the geometrical interpretations are easily perceived.

(6.) Another easy combination is the following,

$$\text{XI.}\ldots\ 0=\mathrm{S}\kappa'(\sigma+\rho-2\kappa),$$

as appears by derivating the last equation IX., with attention to other relations; but $2\kappa-\rho$ is the vector of the extremity, say M, of the *diameter* of the osculating *circle*, drawn from the given point P: we have therefore this construction:—

On the tangent KK′ *to the locus of the centre of the osculating circle, let fall a perpendicular from the extremity* M *of the diameter drawn from the given point* P; *this perpendicular prolonged will intersect the polar axis, in the centre* S *of the osculating sphere to the given curve at* P.

(7.) In general, the three scalar equations VII. conduct to the vector expression,

$$\text{XII.}\ldots\ \sigma=\rho+\frac{3\mathrm{V}\rho'\rho''\mathrm{S}\rho'\rho''+\rho'^2\mathrm{V}\rho'''\rho'}{\mathrm{S}\rho'\rho''\rho'''};$$

or with differentials,

$$\text{XIII.}\ldots\ \sigma=\rho+\frac{3\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho\mathrm{S}\mathrm{d}\rho\mathrm{d}^2\rho+\mathrm{d}\rho^2\mathrm{V}\mathrm{d}^3\rho\mathrm{d}\rho}{\mathrm{S}\mathrm{d}\rho\mathrm{d}^2\rho\mathrm{d}^3\rho};$$

the scalar variable being still left arbitrary.

(8.) And if, as an *example*, we introduce the values for the *helix*,

$$\text{XIV.}\ldots\ \rho=ct\alpha+a^t\beta,\quad \rho'=c\alpha+\frac{\pi}{2}a^{t+1}\beta,\quad \rho''=-\left(\frac{\pi}{2}\right)^2a^t\beta,$$

$$\rho'''=-\left(\frac{\pi}{2}\right)^3a^{t+1}\beta,$$

whereof the three first occurred before, we find after some slight reductions the expression, in which a denotes again the constant inclination of the curve to the axis of the cylinder,

$$\text{XV.}\ldots\ \sigma=\rho-a^t\beta\,\mathrm{cosec}^2\,a=ct\alpha-a^t\beta\cot^2 a;$$

but this is precisely what we found for κ, in 389, VIII.; *for the helix*, then, the *two centres*, K and S, of *absolute and spherical curvature, coincide.*

(9.) This known result is a consequence, and may serve as an illustration, of the general *construction* (6.); because it is easy to infer, from what was shown in 389, (3.), respecting the *locus* of the *centre* K of the *osculating circle* to the helix, as being *another helix* on a *co-axal cylinder*, that the *tangent* KK′ to *this* locus is perpendicular to the radius of curvature KP, while the same tangent (KK′ or κ') is *always* perpendicular (X.) to the tangent (PP′ or ρ') to the *curve*; KK′ is therefore *here* at right angles to the *osculating plane* of the *given* helix, or coincides with its *polar axis*: so that the perpendicular on it from the extremity M of the *diameter of curvature* falls at the point K *itself*, with which consequently the point S in the present case *coincides*, as found by calculation in (8.).

(10.) In general, if we introduce the expressions 376, VI., or the following,

$$\text{XVI.} \ldots \rho' = s'\mathrm{D}_s\rho, \quad \rho'' = s'^2\mathrm{D}_s^2\rho + s''\mathrm{D}_s\rho, \quad \rho''' = s'^3\mathrm{D}_s^3\rho + 3s's''\mathrm{D}_s^2\rho + s'''\mathrm{D}_s\rho,$$

in which s denotes the *arc* of the curve, but the accents still indicate derivations with respect to an *arbitrary* scalar t; and if we observe (comp. 380, (12.)) that the relations,

$$\text{XVII.} \ldots \mathrm{D}_s\rho^2 = -1, \quad \mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s^2\rho = 0, \quad \mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s^3\rho + \mathrm{D}_s^2\rho^2 = 0,$$

in which $\mathrm{D}_s\rho^2$ and $\mathrm{D}_s^2\rho^2$ denote the squares of $\mathrm{D}_s\rho$ and $\mathrm{D}_s^2\rho$, and $\mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s^2\rho$ denotes $\mathrm{S}(\mathrm{D}_s\rho\,.\,\mathrm{D}_s^2\rho)$, &c., exist independently of the *form* of the curve; we find that s'' and s''' disappear from the numerator and denominator of the expression XII. for $\sigma - \rho$, and that they have s'^6 for a common factor: setting aside which, we have thus the simpler formulæ,

$$\text{XVIII.} \ldots \sigma - \rho = \frac{\mathrm{V}\,.\,\mathrm{D}_s\rho\mathrm{D}_s^3\rho}{\mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s^2\rho\mathrm{D}_s^3\rho} = \frac{\mathrm{D}_s\,.\,\mathrm{D}_s\rho\mathrm{D}_s^2\rho}{\mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s^2\rho\mathrm{D}_s^3\rho}.$$

And accordingly the three scalar equations VII., which determine the centre of the osculating sphere, may now be written thus,

$$\text{XIX.} \ldots \mathrm{S}(\sigma - \rho)\mathrm{D}_s\rho = 0, \quad \mathrm{S}(\sigma - \rho)\mathrm{D}_s^2\rho + 1 = 0, \quad \mathrm{S}(\sigma - \rho)\mathrm{D}_s^3\rho = 0.$$

(11.) Conversely, when we have any formula involving thus the successive derivatives of the vector ρ taken with respect to the *arc*, s, we can always and easily *generalize* the *expression*, and introduce an *arbitrary* variable t, by inverting the equations XVI.; or by writing (comp. 390, VIII.),

$$\text{XX.} \ldots \mathrm{D}_s\rho = s'^{-1}\rho', \quad \mathrm{D}_s^2\rho = s'^{-1}(s'^{-1}\rho')' = s'^{-2}\rho'' - s'^{-3}s''\rho', \ \text{\&c.}$$

(12.) It may happen (comp. 379, (2.)) that the independent variable t is only *proportional* to s, without being *equal* thereto; but as we have the general relation,

$$\text{XXI.} \ldots \mathrm{D}_t^n\rho = s'^n\mathrm{D}_s^n\rho, \quad \text{if} \quad s' = \mathrm{D}_t s = \mathrm{T}\rho' = \text{const.},$$

it is nearly or quite as easy to effect the transformations (10.) and (11.) in the case here supposed, or to pass from t to s and reciprocally, as if we had $s' = 1$.

(13.) If the vector σ be treated as *constant* in the derivations, or if we consider for a moment the *centre* S of the *sphere* as a *fixed point*, and attend only to the *variations of distance* of a point on the *curve* from *it*, then (remembering that $\mathrm{T}(\rho - \sigma)^2 = -(\rho - \sigma)^2$) we not only easily put (comp. VIII.) the three equations XIX. under the forms,

$$\text{XXII.} \ldots 0 = \mathrm{D}_s\mathrm{T}(\rho - \sigma) = \mathrm{D}_s^2\mathrm{T}(\rho - \sigma) = \mathrm{D}_s^3\mathrm{T}(\rho - \sigma),$$

but also obtain by XVII. this *fourth* equation,

$$\text{XXIII.} \ldots \mathrm{T}(\rho - \sigma)\,\mathrm{D}_s^4\mathrm{T}(\rho - \sigma) = \mathrm{S}\,.\,(\sigma - \rho)\mathrm{D}_s^4\rho + \mathrm{D}_s^2\rho^2.$$

(14.) If then we write, for abridgment,

$$\text{XXIV.}\ldots r = \mathrm{T}(\kappa - \rho) = \mathrm{T}\mathrm{D}_s^2\rho^{-1} = \textit{radius of osculating circle};$$

$$\text{XXV.}\ldots R = \mathrm{T}(\sigma - \rho) = \textit{radius of osculating sphere};$$

and

$$\text{XXVI.}\ldots S = \frac{\mathrm{S}(\sigma - \rho)\mathrm{D}_s^4\rho}{-\mathrm{D}_s^2\rho^2} = \frac{\mathrm{S}\,.\,\mathrm{D}_s\rho^3\mathrm{D}_s^3\rho\mathrm{D}_s^4\rho}{\mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s^2\rho^3\mathrm{D}_s^3\rho},$$

we see that *this scalar*, S, *must be constantly equal to unity*, for *every spherical curve*; but that for a curve which is *non-spherical*, the *distance* $\overline{\mathrm{SQ}}$ of a *near point* Q, from the *centre* S of the *osculating sphere* at P, is generally given by an expression of the form,

$$\text{XXVII.}\ldots \overline{\mathrm{SQ}} = R + \frac{(S-1)u_s s^4}{24r^2R}, \quad \text{with} \quad u_0 = 1;$$

so that, at least for *near points* Q, on *each side* of the *given point* P, the *curve* lies *without* or *within* the *sphere* which *osculates* at that given point, according as the *scalar*, S, determined as above, is *greater* or *less* than *unity*.

(15.) In the case (12.), the formula XXVI. may be thus written,

$$\text{XXVIII.}\ldots S = \frac{\mathrm{S}\,.\,\rho'^3\rho'''\rho^{\mathrm{iv}}}{\mathrm{S}\,.\,\rho'\rho''^3\rho'''};$$

whence, by carrying the derivations one step farther than in (8.), we find for the *helix*,

$$\text{XXIX.}\ldots S = \operatorname{cosec}^2 a > 1, \quad \text{or} \quad \text{XXIX}'.\ldots S - 1 = \cot^2 a > 0;$$

and accordingly it is easy to prove that *this* curve lies *wholly without* its osculating sphere, except at the point of osculation.

(16.) In general, the *scalar* $S - 1$, which vanishes (14.) for *all spherical curves*, and which enters as a *coefficient* into the expression XXVII. for the *deviation* $\overline{\mathrm{SQ}} - \overline{\mathrm{SP}}$ of a *near point* of any *other* curve from its own osculating sphere, may be called the *Coefficient of Non-Sphericity*; and if QT be the *perpendicular* from that near point Q *on the tangent* PT to the curve at the given point P, we have then this *limiting equation*, by which the value of that coefficient may be expressed,

$$\text{XXX.}\ldots S - 1 = \lim.\ 8\left(\frac{\overline{\mathrm{SQ}}^2 - \overline{\mathrm{SP}}^2}{\overline{\mathrm{QT}}^2}\right).$$

(17.) Besides the forms XVIII., other transformations of the expressions XII. XIII. for the vector σ of the centre of an osculating sphere might be assigned; but it seems sufficient here to suggest that some useful practice may be had, in proving that those expressions for σ reduce themselves generally to zero, when the condition,

$$\text{XXXI.}\ldots \mathrm{T}\rho = \text{const.}$$

is satisfied.

(18.) It may just be remarked, that as r^{-1} is often called (comp. 389, (4.)) the *absolute curvature*, or simply *the curvature*, of the curve in space which is considered, so R^{-1} is sometimes called the *spherical curvature* of that curve: while r and R are called the *radii*[*] of those two curvatures respectively.

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[*] We shall soon have occasion to consider *another scalar radius*, which we propose to denote by the small roman letter r, of what is not uncommonly called the *torsion*, or the *second curvature*, of the same curve in space.

396. When the *arc* (s) of the curve is made the independent variable, the *calculations* (as we have seen) become considerably simplified, while no essential *generality* is lost, because the transformations requisite for the introduction of an *arbitrary* scalar variable (t) follow a simple and uniform *law* (395, (11.), &c.). Adopting then the expression (comp. 395, IV.),

$$\text{I.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3u_s\tau'', \quad \text{with} \quad u_0 = 1,$$

in which

$$\text{II.} \ldots \tau = \mathrm{D}_s\rho, \quad \tau' = \mathrm{D}_s^2\rho, \quad \tau'' = \mathrm{D}_s^3\rho,$$

and therefore

$$\text{III.} \ldots \tau^2 + 1 = 0, \quad \mathrm{S}\tau\tau' = 0, \quad \mathrm{S}\tau\tau'' + \tau'^2 = 0,$$

we shall proceed to deduce some *other affections* of the *curve*, besides its *spherical curvature* (395, (18.)), which do not involve the consideration of the *fourth power* of the *arc* (or chord). In particular, we shall determine expressions for that known *Second Curvature* (or *torsion*), which depends on the *change of the osculating plane*, and is measured by the *ultimate ratio* of that change, expressed as an *angle*, to the *arc* of the curve itself; and shall assign the quaternion equations of the known *Rectifying Plane*, and *Rectifying Line*, which are respectively the *tangent plane*, and the *generating line*, of that known *Rectifying Developable*, whereon the proposed curve is a *geodetic* (382): so that it would *become* a *right line*, by the *unfolding* of this last *surface* into a *plane*. But first it may be well to express, in this new notation, the principal affections or properties of the curve, which depend only on the *three first terms* of the expansion I., or on the *three initial vectors* ρ, τ, τ', or rather on the *two last* of these; and which *include*, as we shall see, the *rectifying plane*, but *not* the rectifying *line*: *nor* what has been called above the *second* curvature.*

(1.) Using then first, instead of I., this less expanded but still rigorous expression (comp. 376, I.),

$$\text{IV.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2u_s\tau', \quad \text{with} \quad u_0 = 1,$$

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* In a Note to a very able and interesting Memoir, "*Sur les lignes courbes non planes*" (referred to by Dr. Salmon in the Note to page 277 of his already cited Treatise, and published in *Cahier* XXX. of the *Journal de l'Ecole Polytechnique*), M. de Saint-Venant brings forward several objections to the use of this appellation, and also to the phrases *torsion*, *flexion*, &c., instead of which he proposes to introduce the new name, "*cambrure*:" but the expression "*second curvature*" may serve us for the present, as being at least not unusual, and appearing to be sufficiently suggestive

and with the relations II. and III., we have at once the following system of *three rectangular lines*, which are conceived to be all drawn from the given point P of the curve :

$$\text{V.} \ldots \tau = \textit{unit tangent}\,; \qquad \text{VI.} \ldots \tau' = \textit{vector of curvature}\ (389,\ (4.);$$

and $$\text{VII.} \ldots \nu = \tau\tau' = -\tau'\tau = \tau'\tau^{-1} = \textit{binormal}\ (\text{comp. } 379,\ (4.))\,;$$

τ being a line drawn in the direction of a conceived *motion* along the curve, in virtue of which the *arc* (s) *increases ;* while τ' is directed *towards the centre of curvature*, or of the *osculating circle*, of which centre K the vector is now,

$$\text{VIII.} \ldots \mathrm{OK} = \kappa = \rho - \tau'^{-1} = \rho + r^2\tau' = \rho + r\mathrm{U}\tau',$$

if $\text{IX.} \ldots r^{-1} = \mathrm{T}\tau' = \textit{curvature}$ at P, or $\text{IX}'. \ldots r = \mathrm{T}\tau'^{-1} = \textit{radius of curvature}\,;$

and the *third* line ν (which is *normal* at P to the *surface of tangents* to the curve) has the *same length* ($\mathrm{T}\nu = r^{-1}$) as τ', and is *directed* so that the rotation round it from τ to τ' is *positive*.

(2.) At the same time, we have evidently a system of *three rectangular vector units* from the same point P, which may be called respectively the *tangent unit*, the *normal unit*, and the *binormal unit*, namely the three lines,

$$\text{X.} \ldots \mathrm{U}\tau = \tau, \quad \mathrm{U}\tau' = r\tau', \quad \mathrm{U}\nu = r\tau\tau'\,;$$

the *normal unit* being thus directed (like τ') *towards the centre* of *curvature*.

(3.) The *vector equation* (comp. 392, (2.)) of the *circle of curvature* takes now the form,

$$\text{XI.} \ldots \mathrm{V}\frac{2\tau}{\omega - \rho} = -\nu\,;$$

with the verification that it is satisfied by the value,

$$\text{XII.} \ldots \omega = \mu = 2\kappa - \rho = \rho - 2\tau'^{-1},$$

in which μ (comp. 395. (6.)) is the vector OM of the extremity of the *diameter of curvature* PM.

(4.) The *normal plane*, the *rectifying plane*, and the *osculating plane*, to the curve at the given point, form a *rectangular system of planes* (comp. 379, (5.)), *perpendicular* respectively to the *three lines* (1.) ; so that their scalar equations are, in the present notation,

$$\text{XIII.} \ldots \mathrm{S}\tau(\omega - \rho) = 0\,; \qquad \text{XIV.} \ldots \mathrm{S}\tau'(\omega - \rho) = 0\,; \qquad \text{XV.} \ldots \mathrm{S}\nu(\omega - \rho) = 0\,;$$

by *pairing* which we can represent the *tangent*, *normal*, and *binormal* to the curve, regarded *as indefinite right lines ;* or by the three vector equations,

$$\text{XVI.} \ldots \mathrm{V}\tau(\omega - \rho) = 0\,; \quad \text{XVII.} \ldots \mathrm{V}\tau'(\omega - \rho) = 0\,; \quad \text{XVIII.} \ldots \mathrm{V}\nu(\omega - \rho) = 0.$$

(5.) In general, if the two vector equations,

$$\text{XIX.} \ldots \mathrm{V}\eta(\omega - \rho) = 0, \quad \text{and} \quad \text{XIX}'. \ldots \mathrm{V}\eta_s(\omega_s - \rho_s) = 0,$$

represent *two right lines*, PH and $\mathrm{P}_s\mathrm{H}_s$, which are conceived to *emanate* according to *any given law* from *any given curve* in space, the *identical formula*,*

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* It is obvious that we have thus an easy quaternion solution of the problem, *to draw a common perpendicular to any two right lines in space.*

$$\text{XX.} \ldots \rho_s - \rho + \mathrm{V}\left(\mathrm{V}\eta\eta_s . \mathrm{V}\frac{\rho_s - \rho}{\mathrm{V}\eta\eta_s}\right) = \frac{\mathrm{S}\eta\eta_s(\rho_s - \rho)}{\mathrm{V}\eta\eta_s},$$

shows that the *common perpendicular* to these *two emanants*, which *as a vector* is represented by *either member* of this formula XX., *intersects* the *two lines* in the *two points* of which the vectors are,

$$\text{XXI.} \ldots \omega = \rho + \eta\mathrm{S}\frac{(\rho_s - \rho)\eta_s}{\mathrm{V}\eta\eta_s}; \qquad \text{XXI}'. \ldots \omega_s = \rho_s + \eta_s\mathrm{S}\frac{(\rho_s - \rho)\eta}{\mathrm{V}\eta\eta_s}.$$

(6.) In *general* also, the *passage of a right line* from any *one* given position in space to any *other* may be conceived to be accomplished by a sort of *screw motion*, with the *common perpendicular* for the *axis* of the screw, and with *two proportional velocities*, of *translation along*, and of *rotation round* that axis: the *locus* of the *two given* and of *all* the *intermediate positions* of the *line* (when *thus interpolated*) being a *Screw Surface*, such as that of which the vector equation was assigned in 314, (11.), and was used in 372, (4.).

(7.) Again, for *any quaternion*, q, we have (by 316, XX. and XXIII.*) the two equations,

$$\text{XXII.} \ldots \mathrm{lU}q = \angle q . \mathrm{UV}q, \qquad \text{XXII}'. \ldots \mathrm{VU}q = \sin \angle q . \mathrm{UV}q;$$

comparing which we see that

$$\text{XXIII.} \ldots \mathrm{VU}q : \mathrm{lU}q = \sin \angle q : \angle q = (\text{very nearly})\ 1,$$

if the *angle* of the quaternion be *small;* so that the *logarithm* and the *vector* of the *versor* of a *small-angled quaternion* are very nearly *equal* to each other, and we may write the following general *approximate formula* for such a *versor:*

$$\text{XXIV.} \ldots \mathrm{U}q = (\epsilon^{\mathrm{lU}q} =)\, \epsilon^{\mathrm{VU}q}, \textit{ nearly}, \text{ if } \angle q \text{ be } \textit{small};$$

the *error* of this last formula being in fact small of the *third order*, if the *angle* be small of the *first*.

(8.) And thus or otherwise (comp. 334, XIII. and XV.), we may perceive that if the quaternion q have the form (comp. (5.)),

$$\text{XXV.} \ldots q = \eta_s\eta^{-1}, \quad \text{with} \quad \text{XXVI.} \ldots \eta_s = \eta + s\eta' + \ldots,$$

and if we write for abridgment,

$$\text{XXVII.} \ldots \theta = \mathrm{V}\frac{\eta'}{\eta}, \quad \text{and} \quad \text{XXVIII.} \ldots h = \mathrm{S}\frac{\eta'}{\eta},$$

we shall then have nearly, if s be small, the expressions,

$$\text{XXIX.} \ldots \mathrm{U}q = \mathrm{U}\frac{\eta_s}{\eta} = \epsilon^{s\theta}, \quad \text{and} \quad \text{XXX.} \ldots \mathrm{T}q = \mathrm{T}\frac{\eta_s}{\eta} = 1 + sh;$$

or, neglecting s^2,

$$\text{XXXI.} \ldots \eta_s = (1 + sh)\epsilon^{s\theta}\eta = \epsilon^{s\theta}\eta + sh\eta,$$

in which last binomial, the *first* (or *exponential*) *term* alone influences the *direction* of the *near emanant line* (5.).

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* Although the expression XXII'. for $\mathrm{VU}q$ is here deduced from 316, XXIII., yet it might have been introduced at a much earlier stage of these *Elements;* for instance, in connexion with the formula 204, XIX., namely $\mathrm{TVU}q = \sin \angle q$.

(9.) At the same time, by supposing s to tend to 0, the formula XXI. gives, as a limit,

$$\text{XXXII.} \ldots \text{OH} = \omega_0 = \rho + \eta \mathrm{S} \frac{\tau\eta}{\mathrm{V}\eta\eta'} = \rho - \eta \mathrm{S} \frac{\tau}{\theta\eta},$$

for the vector of the *point*, say H, *on the given emanant* PH, in which that given line is *ultimately intersected* by the *common perpendicular* (5.), or by the *axis* of the *screw rotation* (6.); but the *direction* of that axis is represented by the *versor* $\mathrm{U}\theta$, and the *angular velocity* of that rotation is represented by the *tensor* $\mathrm{T}\theta$, if the velocity of *motion* (1.) along the *given curve* be taken as *unity:* we may therefore say that the *vector θ itself*, or the *factor* which *multiplies the arc*, s, in the *exponential term* XXXI., if set off *from the point* H determined by XXXII., is the *Vector of Rotation of the Emanant*, whatever the *law* (5.) of the *emanation* may be.

(10.) And as regards the *screw translation* (6.), its *linear velocity* is in like manner represented, in length and in direction, by the following expression (obtained by limits from XX.),

$$\text{XXXIII.} \ldots \iota = \theta \mathrm{S} \frac{\tau}{\theta} \text{ (set off from H)} = \textit{Vector of Translation of Emanant},$$

$$= \textit{projection of unit-tangent on screw-axis} \text{ (or of } \tau \text{ on } \theta).$$

And the *indefinite right line* through the point H, of which this line ι is a *part*, may be called the *Axis of Displacement of the Emanant.*

(11.) It is easy in this manner to assign what may be called the *Osculating Screw Surface* to the (*generally gauche*) *Surface of Emanants*, or indeed to any proposed *skew surface;* namely, the *screw* surface which has the *given emanant* (or other) *line* for *one* of its *generatrices*, and *touches the skew surface* in the *whole extent* of that right line.

(12.) It is however more important here to observe, that in the *case* when the surface of emanants is *developable*, the *vector* ι of *translation vanishes;* and that conversely this vector ι cannot be *constantly zero*, if that surface be *undevelopable.* The *Condition of Developability* of the *Surface of Emanants* is therefore expressed by the equation,

$$\text{XXXIV.} \ldots \iota = 0, \quad \text{or} \quad \mathrm{S}\tau\theta = 0, \quad \text{or} \quad \text{XXXIV}'. \ldots \mathrm{S}\eta\eta'\tau = 0;$$

and accordingly this condition is *satisfied* (as was to be expected) when $\eta = \tau$, that is, for the *surface of tangents.*

(13.) In the same case, of $\eta = \text{or} \parallel \tau$, the vector θ of *rotation* becomes equal (by XXVII. and VII.) to the *binormal* ν; and the expression XXXII., for the vector ω_0 of the foot H of the *axis* reduces itself to ρ; and thus we might be led to see (what indeed is otherwise evident), that the *passage from a given tangent to a near one* may be approximately made, by a *rotation round the binormal, through the small angle*, $s\mathrm{T}\nu = sr^{-1}$ = *arc divided by radius of curvature.*

(14.) Instead of emanating *lines*, we may consider a system of *emanating planes*, which are respectively *perpendicular* to those lines, and pass through the *same points* of the given curve. It may be sufficient here to remark, that the *passage* from one to another of two such near emanant planes, represented by the equations,

$$\text{XXXV.} \ldots \mathrm{S}\eta(\omega - \rho) = 0, \qquad \text{XXXV}'. \ldots \mathrm{S}\eta_s(\omega - \rho) = 0,$$

may be conceived to be made by a *rotation through an angle* $= s\mathrm{T}\theta$, *round the right line*,

$$\text{XXXVI.}\ldots\ \mathrm{S}\eta(\omega-\rho)=0,\quad \mathrm{S}\eta'(\omega-\rho)-\mathrm{S}\eta\tau=0,$$

or

$$\text{XXXVI}'.\ldots\ \mathrm{V}\theta(\omega-\rho)+\eta^{-1}\mathrm{S}\eta\tau=0,$$

in which the *plane* XXXV. *touches its developable envelope*, and which is *parallel* to the recent vector θ, or to the *vector of rotation* (9.) of the *emanant line;* so that if an *equal vector* be set off *on this new line* XXXVI., it may be said to be *the Vector Axis of Rotation of the Emanant Plane.*

(15.) For example, if we again make $\eta=\tau$, so that the equation XXXV. represents now the *normal plane* to the curve, we are led to combine the equation XIII. of that plane with its *derived* equation, and so to form the system of the *two* scalar equations,

$$\text{XXXVII.}\ldots\ \mathrm{S}\tau(\omega-\rho)=0,\quad \mathrm{S}\tau'(\omega-\rho)+1=0,$$

whereof the second represents a plane *parallel* to the *rectifying plane* XIV., and *drawn through the centre of curvature* VIII.; and which jointly represent the *polar axis* (391, (5.)), considered as an *indefinite* right line, which is represented otherwise by the *one* vector equation,

$$\text{XXXVIII.}\ldots\ \mathrm{V}\nu(\omega-\kappa)=0,\quad\text{or}\quad \text{XXXVIII}'.\ldots\ \mathrm{V}\nu(\omega-\rho)=-\tau.$$

(16.) And if, *on this indefinite line*, we set off a portion *equal to the binormal* ν, such *portion* (which may conveniently be measured *from the centre* κ) may be said, by (14.), to be *the Vector Axis of Rotation of the Normal Plane;* or briefly, *the Polar Axis*, considered as representing not only the *direction* but also the *velocity* of that rotation, which velocity $=\mathrm{T}\nu=r^{-1}=$ the *curvature* (IX.) of the given curve: while *another portion* $=\mathrm{U}\nu=$ the *binormal unit* (2.), set off on the *same axis* from the *same centre* of curvature, may be called the *Polar Unit.*

(17.) This suggests a *new way* of representing the *osculating circle* by a *vector equation* (comp. (3.), and 316), as follows:

$$\begin{aligned}\text{XXXIX.}\ldots\ \omega_s&=\kappa+\epsilon^{s\nu}(\rho-\kappa)=\rho+(\epsilon^{s\nu}-1)\tau'^{-1}\\ &=\rho+s\tau+(\epsilon^{s\nu}-1-s\nu)\tau'^{-1}\\ &=\rho+s\tau+\tfrac{1}{2}s^2\tau'+(\epsilon^{s\nu}-1-s\nu-\tfrac{1}{2}s^2\nu^2)\tau'^{-1};\end{aligned}$$

which agrees, as we see, with the expression I. or IV., if s^3 be neglected; and of which, when the expansion is continued, the *next term* is,

$$\text{XL.}\ldots\ \tfrac{1}{6}s^3\nu^3\tau'^{-1}=\tfrac{1}{6}s^3\nu\tau'=-\frac{s^3\tau}{6r^2}.$$

(18.) The *complete expansion* of the *exponential form* XXXIX., for the variable *vector of the osculating circle*, may be briefly summed up in the following *trigonometric* (but *vector*) *expression:*

$$\text{XLI.}\ldots\ \omega_s=\kappa+\left(\cos\frac{s}{r}+\mathrm{U}\nu.\sin\frac{s}{r}\right)(\rho-\kappa),$$

in which, $\quad\text{XLII.}\ldots\ \rho-\kappa=-r^2\tau',\quad$ and $\quad\mathrm{U}\nu.(\rho-\kappa)=r\nu\tau'^{-1}=r\tau;$

so that we may also write, *neglecting no power of* s,

$$\text{XLIII.}\ldots\ \omega_s=\rho+r\tau\sin\frac{s}{r}+r^2\tau'\operatorname{vers}\frac{s}{r};$$

and if *this* be subtracted from the *full* expression for the vector ρ_s, the *remainder* may be called the *deviation of the given curve in space, from its own circle of curvature:* which *deviation*, as we already see, is *small* of the *third order*, and will soon be decomposed into its *two principal parts*, or *terms*, of *that* order, in the directions of the *normal* and the *binormal* respectively.

(19.) Meantime we may remark, that if we only neglect terms of the *fourth order*, the expansion I. gives, by III. and IX., for the *length* of a *small chord* PP_s, the formula:

$$\text{XLIV.} \ldots \overline{\mathrm{PP}_s} = \mathrm{T}(\rho_s - \rho) = \mathrm{T}(s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'')$$
$$= \sqrt{\{-(s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'')^2\}}$$
$$= \sqrt{\{s^2 + s^4\tau'^2(\tfrac{1}{3} - \tfrac{1}{4})\}}$$
$$= \sqrt{\left(s^2 - \frac{s^4}{12r^2}\right)} = s - \frac{s^3}{24r^2} = 2r \sin \frac{s}{2r};$$

this *length* then is the *same* (to this degree of approximation), as that of the *chord of an equally long arc* of the *osculating circle:* and although the *chord* of even a *small arc* of a *curve* is *always shorter* than that *arc itself*, yet we see that the *difference* is *generally* a small quantity of the *third* order*, if the arc be small of the *first*.

397. Resuming now the expression 396, I., but suppressing here the coefficient $u_{,,}$ of which the limit is unity, and therefore writing simply,

$$\text{I.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'',$$

with the relations,

$$\text{II.} \ldots \tau^2 = -1, \quad \mathrm{S}\tau\tau' = 0, \quad \mathrm{S}\tau\tau'' = -\tau'^2 = r^{-2}, \quad \mathrm{S}\tau'\tau'' = r^{-3}r',$$

if s = *arc*, and $r^{-1} = \mathrm{T}\tau'$ = *curvature*,† as before, or r = *radius* of curvature (> 0), while $r' = \mathrm{D}_s r$; and introducing the *new scalar*,

$$\text{III.} \ldots \mathrm{r}^{-1} = \mathrm{S}\frac{\tau''}{\tau\tau'} = \tau^{-1}\mathrm{V}\frac{\nu'}{\nu} = \textit{Second‡ Curvature},$$

with $\nu = \tau\tau'$ = *binormal*, or the *new vector*,

$$\text{IV.} \ldots \mathrm{r}^{-1}\tau = \tau\mathrm{S}\frac{\tau''}{\tau\tau'} = \mathrm{V}\frac{\nu'}{\nu} = \textit{Vector of Second Curvature},$$

supposed to be set off tangentially from the given point P of the curve, or finally this *other* new scalar ($>$ or < 0),

$$\text{V.} \ldots \mathrm{r} = \left(\mathrm{S}\frac{\tau''}{\tau\tau'}\right)^{-1} = \textit{Radius of Second Curvature},$$

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* This ought to have been expressly stated in the reasoning of 383, (5), for which it was not sufficient to observe that the *arc* and *chord tend* to bear to *each other* a *ratio of equality*, without showing (or at least mentioning) that their *difference tends* to *vanish*, *even* as *compared* with a line which is ultimately of the same order as the *square* of either.

† Whenever this word *curvature* is thus used, without any qualifying adjective, it is always to be understood as denoting the *absolute* (or *first*) curvature of the curve in space.

‡ Compare the Note to page 554.

which gives the expression,

$$\text{VI.} \ldots \tau'' = -r^{-2}\tau - r^{-1}r'\tau' + \mathrm{r}^{-1}\tau\tau' \quad .$$
$$= -r^{-2}\mathrm{U}\tau + (r^{-1})'\mathrm{U}\tau' + (r\mathrm{r})^{-1}\mathrm{U}\nu;$$

we proceed to deduce some of the chief affections of a curve in space, which depend on the *third power* of the *arc* or chord. In doing this, although everything *new* can be *ultimately* reduced to a dependence on the *two new scalars*, r' and r, or on the *one* new *vector* τ'', or even on $\nu' = \mathrm{V}\tau\tau''$, yet some *auxiliary symbols* will be found useful, and almost necessary. Retaining then the symbols ν, κ, σ, R, as well as τ, τ', r, and therefore writing as before (comp. 396, VIII.),

$$\text{VII.} \ldots \mathrm{OK} = \kappa = \rho - \tau'^{-1} = \rho + r\mathrm{U}\tau' = \rho + r^2\tau',$$

$$\text{VIII.} \ldots (\rho - \kappa)^{-1} = r^{-1}\mathrm{U}(\kappa - \rho) = \tau' = \mathrm{D}_s^2\rho = \textit{Vector of Curvature},$$

we may now write also, by 395, XVIII.,

$$\text{IX.} \ldots \mathrm{OS} = \sigma = \rho - \frac{\nu'}{\mathrm{S}\tau'\nu'} = \kappa + rr'\mathrm{r}\nu = \kappa + r'\mathrm{r}\mathrm{U}\nu,$$

and

$$\text{X.} \ldots (\rho - \sigma)^{-1} = R^{-1}\mathrm{U}(\sigma - \rho) = \nu'^{-1}\mathrm{S}\tau'\nu' = \textit{Vector of Spherical Curvature},$$
$$= \textit{projection of vector } (\tau') \textit{ of curvature on radius } (R) \textit{ of osculating sphere};$$

because we have now, by VI.,

$$\text{XI.} \ldots \nu' = (\tau\tau')' = \mathrm{V}\tau\tau'' = -\mathrm{r}^{-1}\tau' - r^{-1}r'\nu,$$

or

$$\text{XI}'. \ldots (\mathrm{U}\nu)' = (r\nu)' = -r\mathrm{r}^{-1}\tau' = -\mathrm{r}^{-1}\mathrm{U}\tau',$$

and

$$\text{XII.} \ldots \mathrm{S}\tau'\nu' = -\mathrm{S}\tau\tau'\tau'' = -\mathrm{r}^{-1}\tau'^2 = r^{-2}\mathrm{r}^{-1}.$$

If then we denote by p and P the *linear* and *angular elevations*, of the *centre* S of the osculating *sphere* above the osculating *plane*, we shall have these *two* new auxiliary *scalars*, which are positive or negative together, according as the linear *height* KS has the direction of $+\nu$ or of $-\nu$:

$$\text{XIII.} \ldots p = \frac{\sigma - \kappa}{\mathrm{U}\nu} = r'\mathrm{r}; \quad \text{XIV.} \ldots P = \mathrm{KPS} = \tan^{-1}\frac{p}{r} = \sin^{-1}\frac{p}{R} = \cos^{-1}\frac{r}{R};$$

while

$$\text{XV.} \ldots R = \mathrm{T}(\sigma - \rho) = \sqrt{(r^2 + p^2)} = \sqrt{(r^2 + r'^2\mathrm{r}^2)};$$

the *angle* P being treated as generally *acute*. Another important *line*, and an accompanying *angle* of elevation, are given by the formulæ,

$$\text{XVI.} \ldots \lambda = \mathrm{V}\frac{\tau''}{\tau'} = r^2\mathrm{V}\tau'\tau'' = \mathrm{r}^{-1}\tau + \tau\tau' = \mathrm{r}^{-1}\mathrm{U}\tau + r^{-1}\mathrm{U}\nu$$
$$= \mathrm{V}\nu'\nu^{-1} + \nu = \textit{Rectifying Vector} \text{ (set off from given point P)},$$
$$= \textit{Vector of Second Curvature plus Binormal};$$

$$\text{XVII.}\ldots H = \angle\frac{\lambda}{\tau} = \tan^{-1}\frac{\mathrm{r}}{r} = \textit{Elevation of Rectifying Line}\ (>0, <\pi),$$

= the *angle* (acute or obtuse, but here regarded as positive), which that known and important *line* (396) makes with the *tangent* to the curve; so that (by XIII., XIV.) these *two auxiliary angles*,* H and P, from which (instead of deducing them from r' and r) all the affections of the curve depending on s^3 can be deduced, are connected with each other and with r' by the relation,

$$\text{XVIII.}\ldots \tan P = r' \tan H.$$

Many other combinations of the symbols offer themselves easily, by the rules of the present calculus; for instance, the vector σ may be determined by the three scalar equations (comp. 395, XIX.),

$$\text{XIX.}\ldots \mathrm{S}\tau(\sigma-\rho)=0,\quad \mathrm{S}\tau'(\sigma-\rho)=-1,\quad \mathrm{S}\tau''(\sigma-\rho)=0,$$

whence, by XVI.,

$$\text{XX.}\ldots r^2\tau'' = r^3\mathrm{V}(\mathrm{V}\tau'\tau''.(\sigma-\rho)) = \mathrm{V}\lambda(\sigma-\rho),$$

a result which also follows from the expressions,

$$\text{XXI.}\ldots \tau'' = \left(\mathrm{V}\frac{\tau''}{\tau'} + \mathrm{S}\frac{\tau''}{\tau'}\right)\tau' = (\lambda - r^{-1}r')\tau',$$

and $$\text{XXII.}\ldots \sigma-\rho = r^2\tau' + rp\nu = r\mathrm{U}\tau' + p\mathrm{U}\nu,$$

because $$\text{XXIII.}\ldots rp\mathrm{V}\lambda\nu = -rp\mathrm{r}^{-1}\tau' = -rr'\tau';$$

we may therefore replace the formula I. for the vector of the curve by the following, which is true to the same order of approximation,†

$$\text{XXIV.}\ldots \rho_s = \rho + s\tau + \frac{s^2}{2r^2}(\kappa-\rho) + \frac{s^3}{6r^3}\mathrm{V}\lambda(\sigma-\rho):$$

and may thus exhibit, even to the eye, the dependence of all affections connected with s^3, on the *two new lines*, λ and $\sigma-\rho$, which were not required when s^3 was neglected, but can now be determined by the *two scalars* r and p (or r and r', or H and P as before). The *geometrical signification* of the scalar p is evident from what precedes, namely, the *height* (KS) of the *centre* of the osculating *sphere* above that of the osculating *circle*, divided by the *binormal unit* ($\mathrm{U}\nu$); and as regards what has been called the *radius* r of *second curvature* (V.), we shall see that this is in fact the *geometrical radius* of a *second circle*, which *osculates*, at the extremity of the tangential vector $r\tau$, to the *principal normal section* of the developable *Surface of Tangents;* and thereby determines an *osculating oblique cone* to that important surface, and *also* an *osculating right cone** thereto, of which *latter* cone the *semiangle* is H, and the *rectifying line* λ is the *axis of revolution:* being also a *side* of an *osculating right cylinder*, on which is traced what is called the *osculating helix.* We shall assign the quaternion equations of these two cones, and of this cylinder, and helix; and shall show that although the *helix* has *not* generally *complete contact* of the *third order* with the *given* curve, yet it approaches *more nearly* to that curve (supposed to be of *double* curvature), than does the osculating *circle.* But an *osculating parabola* will also be assigned, namely, the parabola which osculates to the *projection* of the curve, *on its own osculating plane:* and it will be shown that this *parabola* represents or *constructs one* of the *two* principal and *rectangular components* (396, (18.)), of the *deviation* of the *curve* from its osculating *circle*, in a direction which is (ultimately) *tangential to the osculating sphere*, while the *helix constructs* the *other component.* An *osculating right cone* to the *cone of chords*, drawn *from a given point* of the curve, will also be assigned by quaternions: and will be shown to have in general a *smaller acute semiangle* C (or $\pi - C$), than the acute semiangle H (or $\pi - H$), of the osculating right cone (above mentioned) to the *surface of tangents*, or (as will be seen) to the *cone of parallels* to tangents (369, (6.), &c.): the *relation* between these *two semiangles*, of *two* osculating *right cones*, being rigorously expressed by the formula,

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* The angle H appears to have been first considered by Lancret, in connexion with his theory of rectifying lines, planes, and surfaces: but the angle here called P was virtually included in the earlier results of Monge.

† As regards the *homogeneity* of such expressions, if we treat the four vectors ρ_s, ρ, κ, and σ, and the five scalars s, r, R, p, and r, as being each of the *first dimension*, we are then to regard the dimensions of τ, r', κ', H, and P as being each *zero*; those of τ', ν, and λ as each equal to -1; and that of either τ'' or ν' as being $= -2$.

$$\text{XXV.} \ldots \tan C = \tfrac{3}{4} \tan H.$$

A *new oblique cone* of the second order will be assigned, which has contact of the *same order* with the *cone of chords*, as the *second right cone* (C), while the latter *osculates* to *both* of them; and also an *osculating parabolic cylinder*, which rests upon the *osculating parabola*, and is *cut perpendicularly* in that auxiliary curve by the *osculating plane* to the given curve. And the *intersection* of these *two last surfaces* of the second order (oblique cone and parabolic cylinder) will be found to consist partly of the *binormal* at the given point, and partly of a certain *twisted cubic** (or *gauche curve* of the *third degree*), which latter *curve* has *complete contact* of the *third order* with the *given* curve in space. *Constructions* (comp. 395, (6.)) will be assigned, which will connect, more closely than before, the *tangent* to the *locus of centres of curvature*, with *other* properties or affections of that given curve. And finally we shall prove, by a very simple quaternion analysis, as a consequence of the formula XI′., the known theorem,† that *when the ratio of the two curvatures is constant, the curve is a geodetic on a cylinder.*

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* *These two osculating cones*, oblique and right, to the *surface of tangents*, appear to have been first assigned, in the Memoir already cited, by M. de Saint Venant: the osculating (circular) *helix*, and the osculating (circular) *cylinder*, having been previously considered by M. Olivier.

(1.) The scalar expression III., for the *second curvature* of a curve in space, as *defined* in 396, may be deduced from the formulæ (396, (5.), &c.) of the recent theory of *emanants*, which give,

$$\text{XXVI.} \ldots \theta = \mathrm{V}\nu'\nu^{-1} = \mathrm{r}^{-1}\tau, \quad \omega_0 = \rho, \quad \iota = \tau, \quad \text{if} \quad \eta = \nu,$$

while the *line of contact* (396, (14.)), of the emanant *plane* with its envelope, coincides in position with the *tangent* to the *curve;* in passing, then, from the given point P to the *near* point P_s, the *binormal* (ν) and the *osculating plane* ($\perp \nu$) have (nearly) *revolved together, round that tangent* (τ) as a *common axis*, through a *small angle* $= \mathrm{r}^{-1}s$, and therefore with a *velocity* $= \mathrm{r}^{-1}$, *if this symbol have the value* assigned by III., or by the following extended expression, in which the *scalar variable* (t) is *arbitrary* (comp. 395, (11.), &c.),

$$\text{XXVII.} \ldots \mathrm{r}^{-1} = \mathrm{S}\frac{\rho'''}{\mathrm{V}\rho'\rho''} = \mathrm{S}\frac{\mathrm{d}^3\rho}{\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho} = \textit{Second Curvature:}$$

while the *binormal* has at the same time been *translated* (nearly), in a direction perpendicular to the tangent τ, through the *small interval* $\iota s = s\tau$, which (in the present order of approximation) represents the *small chord* PP_s.

(2.) As an *example*, if we take this *new form* of the equation of the *helix*,

$$\text{XXVIII.} \ldots \rho_t = b(\alpha t \cot a + \epsilon^{\alpha t}\beta), \quad \text{with} \quad \mathrm{T}\alpha = \mathrm{T}\beta = 1, \quad \text{and} \quad \mathrm{S}\alpha\beta = 0,$$

which gives the derived vectors,

$$\text{XXIX.} \ldots \rho_t' = b\alpha(\cot a + \epsilon^{\alpha t}\beta), \quad \rho_t'' = -b\epsilon^{\alpha t}\beta, \quad \rho_t''' = \alpha\rho_t'',$$

and this expression for the *arc* s (supposed to begin with t),

$$\text{XXX.} \ldots s = s't, \quad \text{where} \quad s' = \mathrm{T}\rho' = b \operatorname{cosec} a = \text{const.},$$

we easily find (after a few reductions) the following values for the *two curvatures:*

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* This convenient appellation (of *twisted cubic*) has been proposed by Dr. Salmon, for a curve of the kind here considered: see pages 241, &c., of his already cited Treatise. The *osculating twisted cubic* will be considered somewhat later.

† This theorem was established, on sufficient grounds, in the cited Memoir of M. de Saint Venant (page 26); but it has also been otherwise deduced by M. Serret, in the Additions to M. Liouville's Edition of Monge (Paris, 1850, page 561, &c.).

$$\text{XXXI.} \ldots r^{-1} = b^{-1}\sin^2 a, \quad \mathrm{r}^{-1} = b^{-1}\sin a\cos a\,;$$

while the *common centre* (395), of the osculating *circle* and *sphere*, has now for its vector (comp. 389, (3.)),

$$\text{XXXII.} \ldots \kappa = \sigma = \rho_t - b\iota^{\alpha t}\beta\,\mathrm{cosec}^2 a = b\cot a\,(\alpha t - \iota^{\alpha t}\beta\cot a)\,;$$

b being here the *radius* of the *cylinder*, but *a* denoting still the constant *inclination* of the *tangent* (ρ') to the *axis* (α).

(3.) The *rectifying line* (396), considered merely as to its *position*, being the *line of contact* of the *rectifying plane* (396, XIV.) with its own envelope, is represented by the equations,

$$\text{XXXIII.} \ldots 0 = \mathrm{S}\tau'(\omega - \rho) = \mathrm{S}\tau''(\omega - \rho), \quad \text{or} \quad \text{XXXIII}'. \ldots 0 = \mathrm{V}\lambda(\omega - \rho),$$

with the signification XVI. of λ; and accordingly, if we treat the rectifying planes as *emanants*, or change η to τ', we find the value $\theta = \mathrm{V}\tau''\tau'^{-1} = \lambda$, which shows also that in the passage from P to P$_s$ the *rectifying plane turns* (nearly) *round the rectifying line, through a small angle* $= s\mathrm{T}\lambda$, or with a *velocity* of rotation represented by the tensor,

$$\text{XXXIV.} \ldots \mathrm{T}\lambda = \sqrt{(r^{-2} + \mathrm{r}^{-2})} = r^{-1}\,\mathrm{cosec}\,H = \mathrm{r}^{-1}\sec H\,;$$

so that what we have called the *rectifying vector*, λ, coincides in fact (by the general theory of emanants) with the *vector axis* (396, (14.)) of this *rotation of the rectifying plane*: as the *vector of second curvature* ($\mathrm{r}^{-1}\tau$) has been seen to be, in the *same full sense* (comp. (1.)), the *vector axis* of rotation of the *osculating plane*, when *velocity*, *direction*, and *position* are *all* taken into account.

(4.) When the derivative s' of the *arc* is only *constant*, without being equal to *unity* (comp. 395, (12.)), the expression XVI. may be put under this slightly more general form,

$$\text{XXXV.} \ldots \lambda = \mathrm{V}\frac{\rho'''}{s'\rho''} = \mathrm{V}\frac{\mathrm{d}^3\rho}{\mathrm{d}s\,\mathrm{d}^2\rho} = \textit{Rectifying Vector}\,;$$

and accordingly for the *helix* (2.) we have thus the values,

$$\text{XXXVI.} \ldots \lambda = \alpha s'^{-1} = \alpha b^{-1}\sin a = \alpha r^{-1}\,\mathrm{cosec}\,a, \quad \mathrm{U}\lambda = \alpha\,;$$

the *rectifying line* is therefore, for this curve, *parallel* to the *axis*, and *coincides* with the *generating line* of the *cylinder*, as is otherwise evident from geometry. The value, $\mathrm{T}\lambda = b^{-1}\sin a$, of the *velocity of rotation* of the *rectifying plane*, which is here the *tangent plane* to the cylinder, when compared with a conceived velocity of *motion along the curve*, is also easily interpreted; and the formulæ XVII., XVIII. give, for the same helix (by XXXI.), the values,

$$\text{XXXVII.} \ldots r' = 0, \quad H = a, \quad P = 0.$$

(5.) The *normal* (or the radius of curvature), as being *perpendicular* to the rectifying plane, revolves with the *same velocity*, and round a *parallel line*; to determine the *position* of which new line, or the *point* H in which it *cuts* the normal, we have only to change η to τ' in the formula 396, XXXII., which then becomes,

$$\begin{aligned}\text{XXXVIII.} \ldots \mathrm{OH} = \omega_0 &= \rho - \tau'\mathrm{S}\frac{\tau}{\lambda\tau'} = \rho - \lambda^{-2}\tau' \\ &= \rho + \frac{r^{-2}(\kappa - \rho)}{r^{-2} + \mathrm{r}^{-2}} = \frac{\mathrm{r}^2\rho + r^2\kappa}{r^2 + \mathrm{r}^2} \\ &= \rho\cos^2 H + \kappa\sin^2 H\,;\end{aligned}$$

the *vector of rotation* (396, (9.)) of the *normal* is therefore a line ∥ and $= \lambda$, which *divides* (internally) the *radius* (r) of *curvature* into the *two segments*,*

$$\text{XXXIX.} \ldots \overline{\text{PH}} = r \sin^2 H, \quad \overline{\text{HK}} = r \cos^2 H;$$

namely, into segments which are *proportional to the squares* (r^{-2} and r^{-2}) of the *first* and *second curvatures*.

(6.) At the same time, what we have called generally the *vector of translation* of an emanant line becomes, for the *normal* (by 396, (10.), changing θ to λ), the line

$$\text{XL.} \ldots \iota = \lambda \text{S} \frac{\tau}{\lambda} = \text{U}\lambda \cos H = -\text{r}^{-1}\lambda^{-1}, \text{ set off from the same point H};$$

and the *indefinite right line*, or *axis*, through that point H,

$$\text{XLI.} \ldots 0 = \text{V}\lambda(\omega - \omega_0), \quad \text{or} \quad \text{XLI}'. \ldots 0 = \text{V}\lambda(\omega - \rho\cos^2 H - \kappa \sin^2 H),$$

along which axis the normal *moves*, through the *small line* $s\iota$, while it *turns round* the *same* axis (as before) through the *small angle* $s\text{T}\lambda$, may be called (comp. again 396, (10.)) the *Axis of Displacement of the Normal* (or of the radius of curvature).

(7.) As a verification, for the *helix* (2.) we have thus the values,

$$\text{XLII.} \ldots \overline{\text{PH}} = b, \quad \omega_0 = \rho_t - b\epsilon^{\alpha t}\beta = b\alpha t \cot a, \quad \iota = \alpha \cos a;$$

so that the *axis* of *displacement* (6.) coincides with the *axis* (α) of the *cylinder*, as was of course to be expected.

(8.) When the given curve is *not* a helix, the values VI., XVI., XXXVIII., and XL., of τ'', λ, ω_0, and ι, enable us to put the expression I. for ρ_s under the form,

$$\text{XLIII.} \ldots \rho_s = \omega_0 + s\iota + \epsilon^{s\lambda}(\rho - \omega_0) - \frac{s^3 r' \tau'}{6r};$$

the *curve* therefore generally *deviates*, by this last *small vector* of the *third order*, namely by that *part* of the *term* $\frac{1}{6}s^3\tau''$ which has the direction of the *normal* τ', or of $-\tau'$, and which depends on r', *from the osculating helix*,

$$\text{XLIV.} \ldots \omega_s = \omega_0 + s\iota + \epsilon^{s\lambda}(\rho - \omega_0),$$

and from the *osculating right cylinder*,

$$\text{XLV.} \ldots \text{TV}\lambda(\omega - \omega_0) = \sin H,$$

whereon that *helix* is traced, and of which the *rectifying line* (XXXIII.) is a *side*, while its *axis of revolution* (comp. (7.)) is the *axis of displacement* (XLI.) of the normal.

(9.) *Another general transformation*, of the expression I. for the vector of the curve, is had by the substitution,

$$\text{XLVI.} \ldots s = t + \frac{t^2 r'}{6r} + \frac{t^3}{6r^2},$$

in which t is a new scalar variable; for this gives the new form,

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* This law of *division* of a *radius* of curvature into segments, by the *common perpendicular* to that radius and to its *consecutive*, has been otherwise deduced by M. de Saint Venant, in the Memoir already referred to.

$$\text{XLVII.} \ldots \rho_t = \rho + t\tau + \tfrac{1}{2}t^2\left(\tau' + \frac{r'\tau}{3r}\right) + \tfrac{1}{6}t^3\mathrm{r}^{-1}\nu,$$

and therefore shows that the *curve deviates*, by this *other small vector* of the *third order*,

$$\text{XLVIII.} \ldots \tfrac{1}{6}t^3\mathrm{r}^{-1}\nu = \tfrac{1}{6}s^3\mathrm{r}^{-1}\tau\tau',$$

that is, by the *part* of the *term* $\frac{1}{6}s^3\tau''$ which has the direction of the *binormal* ν, and which depends on r, from what we propose to call the *Osculating Parabola*, namely that new auxiliary curve of which the equation is,

$$\text{XLIX.} \ldots \omega_t = \rho + t\tau + \tfrac{1}{2}t^2\left(\tau' + \frac{r'\tau}{3r}\right):$$

or from the *parabola* which *osculates* at the given point P, to the *projection* of the given curve *on its own osculating plane*.

(10.) And because the small *deviation* XLVIII. of the *curve* from the *parabola* is also the deviation of the same curve from this last *plane*, if we conceive that a *near point* Q of the curve is *projected* into *three* new points Q_1, Q_2, Q_3, on the *tangent*, *normal*, and *binormal* respectively, we shall have the limiting equation,

$$\text{L.} \ldots \lim. \frac{3\mathrm{PQ}_3}{\mathrm{PQ}_1 \cdot \mathrm{PQ}_2} = \mathrm{r}^{-1} = \textit{Second Curvature};$$

the *sign* of this scalar *quotient* being determined by the rules of quaternions.

(11.) But we may also (comp. 396, (17.), (18.)) employ this *third general transformation* of I., *analogous* to the forms XLIII. and XLVII.,

$$\text{LI.} \ldots \rho_s = \kappa + \varepsilon^{s\nu}(\rho - \kappa) + \frac{s^3}{6}\nu'\tau,$$

with the value XI. of ν'; in which the *sum* of the *two first terms* gives the vector of the point of the *osculating circle*, which is distant from the given point PP, by an *arc* of that *circle* equal to the arc s of the given *curve*; and the *third term*,

$$\text{LII.} \ldots \tfrac{1}{6}s^3\nu'\tau = \tfrac{1}{6}s^2(\tau'' + r^{-2}\tau) = -\tfrac{1}{6}s^3r^{-1}r'\tau' + \tfrac{1}{6}s^3\mathrm{r}^{-1}\nu,$$

which represents the *deviation* from the *same circle*, measured in a direction (comp. IX. or X.) *tangential* to the osculating *sphere*, is (as we see) the *vector sum* of *two rectangular components*, which represent respectively the deviations of the curve, from the osculating *helix* (8.), and from the osculating *parabola* (9.).

(12.) It follows, then, that although *neither helix nor parabola* has in general *complete contact* of the *third order* with a given *curve in space*, since the deviation from *each* is generally a small vector of *that* (third) *order*, yet *each* of these *two auxiliary curves*, one on a *right cylinder* XLV., and the other on the osculating *plane*, approaches in general *more closely* to the *given* curve, than does the *osculating circle*: while circle, helix, and parabola have, *all three*, complete contact of the *second** *order* with the curve, and with each other.

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* It appears then that we may say that the *helix* and *parabola* have each a *contact* with the *curve* in space, which is *intermediate between the second and third orders*: or that the *exponent* of the *order* of *each* contact is the *fractional index*, $2\frac{1}{2}$. But it must be left to mathematicians to judge, whether this phraseology can properly be adopted.

(13.) As regards the *geometrical signification* of the *new variable scalar*, t, in the equation XLIX. of the parabola, that equation gives,

$$\text{LIII.} \ldots \mathrm{T}\omega'_t = \mathrm{T}\left\{\left(1 + \frac{r't}{3r}\right)\tau + t\tau'\right\} = 1 + \frac{r't}{3r} + \frac{t^2}{2r^2} \ldots,$$

and therefore (to the present order of approximation),

$$\text{LIV.} \ldots \textit{Arc of Osculating Parabola} \text{ (from } \omega_0 \text{ to } \omega_t)$$
$$= \int_0^t \mathrm{T}\omega'_t dt = t + \frac{r't^2}{6r} + \frac{t^3}{6r^2} = s \text{ (by XLVI.)}$$
$$= \textit{Arc of Curve in Space} \text{ (from } \rho_0 \text{ to } \rho_s);$$

if then an *arc* $= s$ be thus set off *upon the parabola*, with the same initial point P, and the same initial direction, and if this *parabolic arc*, or its *chord* $\omega_t - \omega_0$, be *obliquely projected* on the *initial tangent* τ, by drawing a *diameter* of the parabola through its final point, the *oblique tangential projection* so obtained will be $= t\tau$ by XLIX.; and its *length*, or the *ordinate to that diameter*, will be the scalar t.

(14.) And as regards the *direction* of the *diameter* of the osculating parabola, drawn as we may suppose from P, if we denote for a moment by D its inclination to the normal $+\tau'$, regarded as positive when towards the tangent $+\tau$, we have (by XLIX. and XVIII.) the formula,

$$\text{LV.} \ldots \tan D = \frac{r'}{3} = \tfrac{1}{3} \tan P \cot H:$$

which is an instance of the reducibility, above mentioned, of *all affections* of the curve depending on s^3, to a dependence on the *two angles*, H and P.

(15.) *Some* of these affections, besides the *direction* of the *rectifying line* λ, can be deduced from the angle *H alone*. As an example, we may observe that the vector equation of the *surface of tangents* is of the form,

$$\text{LVI.} \ldots \omega_{s,t} = \rho_s + t\rho'_s = \rho_s + t\tau_s,$$

in which s and t are *two* independent and scalar variables, and

$$\text{LVII.} \ldots \tau_s = \tau + s\tau' + \frac{s^2}{2}\tau'',$$

+ terms depending on s^4 in ρ_s. If then we *cut* this *developable* LVI. by the *plane*,

$$\text{LVIII.} \ldots \mathrm{S}\tau(\omega - \rho) = -c = \text{any given scalar constant,}$$

which is, relatively to the *surface*, a *normal plane* at the extremity of the tangential vector $c\tau$ from P, while this *tangent* is also a *generating line*, we get thus a *principal** *normal section*, of which the variable vector has for its approximate expression,

$$\text{LIX.} \ldots \omega_s = (\rho + c\tau) + (cs + ..)\tau' + (\tfrac{1}{2}cs^2r^{-1} + ..)\nu;$$

the terms suppressed being of higher orders than the terms retained, and having no influence on the *curvature* of the section. We find then thus, that the *vector of the centre* of the *osculating circle* to this *normal section of the surface of tangents* to the given curve is, *rigorously*,

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* Some *general* acquaintance with the known theory of *sections of surfaces* is here supposed, although that subject will soon be briefly treated by quaternions.

$$\text{LX.} \ldots \rho + c\tau + \frac{(cs\tau')^2}{cs^2\mathrm{r}^{-1}\nu} = \rho + c(\tau + \mathrm{r}\nu) = \rho + c\mathrm{r}\lambda\,;$$

so that the *locus* of all *such centres* is the *rectifying line* XXXIII′. And if, in particular, we make $c = r$, or cut the developable at the extremity of the tangential vector $r\tau$, the expression LX. becomes then $\rho + r\tau + \mathrm{r}\mathrm{U}\nu$; which expresses that the *radius* of the *circle of curvature* of *this* normal section of the surface is precisely what has been called the *Radius* (r) *of Second Curvature*, of the given curve in space. But *this* radius ($\mathrm{r} = r\tan H$) depends only on the angle H, when the radius (r) of (absolute) curvature is given, or has been previously determined.

(16.) The *cone of the second order*, represented by the quaternion equation,

$$\text{LXI.} \ldots 0 = 2\mathrm{r}\mathrm{S}\tau(\omega - \rho)\,\mathrm{S}\nu(\omega - \rho) + (\mathrm{V}\tau(\omega - \rho))^2,$$

has its *vertex* at the given point P, and *rests* upon the *circle* last determined; it is then the *locus* of *all* the circles lately mentioned (15.), and is therefore (in a known sense) an *osculating oblique cone* to the developable *surface of tangents*: its *cyclic normals* (comp. 357, &c.) being τ and $\tau + 2\mathrm{r}\nu$, or τ and $r\tau + 2\mathrm{r}\mathrm{U}\nu$. But, by 394, (30.), the osculating *right cone* to *this* cone LXI., and therefore also (in a sense likewise known) to the *surface* of tangents *itself*, is one which has the recent *locus of centres* (15.), namely the *rectifying line* (λ), for its *axis of revolution*, while the *tangent* (τ) to the curve is one of its *sides*: its *semiangle* is therefore $= H$, and a form of the quaternion equation of this *osculating right cone* is the following (comp. XLV.),

$$\text{LXII.} \ldots \mathrm{TVU}\lambda(\omega - \rho) = \sin H.$$

(17.) The *right cone* LXII., which thus osculates to the developable *surface of tangents* LVI., along the given tangent τ, osculates *also* along that tangential line to the *cone of parallels to tangents*, which has its vertex at the given point P; as is at once seen (comp. 394, (30.)), by changing ρ' and ρ'' to τ' and τ'', in the general expression $\mathrm{V}\rho'\rho''$ (393, (6.), or 394, (6.)), for a line in the direction of the *axis* of the osculating *circle* to a curve upon a *sphere*. And the *axis* of the right *cone* thus determined, namely (again) the *rectifying line* (λ), *intersects* the *plane* of the *great circle* of the *osculating sphere*, which is *parallel* to the osculating plane, in a point L of which the vector is,

$$\text{LXIII.} \ldots \mathrm{OL} = \rho + rp\lambda = \rho + rr'\tau + rp\nu.$$

(18.) We have thus, *in general*, a *gauche quadrilateral*, PKSL, right-angled except at L, with the help of which *one* figure *all* affections of the curve, not depending on s^4, can be geometrically represented or *constructed*: although it must be observed that when $r' = 0$, which happens for the *helix* (XXXVII.), the *osculating* circle is then *itself* a *great* circle of the osculating *sphere*, and the points P and L, like the points K and S, *coincide*.

(19.) In the general case, it may assist the conceptions to suppose lines set off, from the given point P, on the tangent and binormal, as follows:

$$\text{LXIV.} \ldots \mathrm{PT} = \mathrm{BL} = rr'\tau\,; \quad \mathrm{PB} = \mathrm{TL} = \mathrm{KS} = rp\nu\,;$$

for thus we shall have a *right triangular prism*, with the two right-angled triangles, TPK and LBS, in the osculating plane and in the parallel plane (17.), for two of its faces, while the three others are the rectangles, PKSB, PBLT, KSLT, whereof the two first are situated respectively in the normal and rectifying planes.

(20.) *All scalar properties* of this auxiliary *prism* may be deduced, by our general methods, from the *three scalars*, r, r, r', or r, H, P; and *all vector properties* of the same prism can in like manner be deduced from the *three vectors* τ, τ', τ'', or from τ, ν, ν', which (as we have seen) are not entirely arbitrary, but are subject to certain conditions.

(21.) As an example of such deduction (compare the annexed Figure 81), the *equation of the diagonal plane* SPL, which contains the radius (R) of spherical curvature and the rectifying line (λ), and the *equation of the trace*, say PU, of that plane on the osculating plane, which *trace* is evidently *parallel* (by the construction) to the *edges* LS, TK of the *prism*, are in the recent notations (comp. XX.),

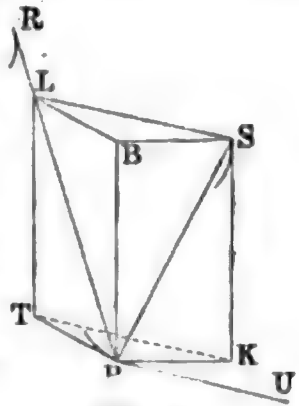

Fig. 81.

$$\text{LXV.}\ldots\ 0 = \mathrm{S}\tau''(\omega-\rho); \qquad \text{LXVI.}\ldots\ 0 = \mathrm{V}(r^{-1}\tau)'(\omega-\rho);$$

with the verification that $r\mathrm{S}\tau'\tau'' = r'\mathrm{S}\tau\tau'' = r^{-2}r'$, by II.

(22.) In general, by 204, (22.), if α and β be any two vectors, we have the expressions,

$$\text{LXVII.}\ldots\ \tan\angle\frac{\beta}{\alpha} = \tan\angle\frac{\alpha}{\beta} = -\tan\angle\beta\alpha = -\tan\angle\alpha\beta$$

$$= \mathrm{TV}\frac{\beta}{\alpha} : \mathrm{S}\frac{\beta}{\alpha} = \frac{\mathrm{TV}}{\mathrm{S}}\cdot\frac{\beta}{\alpha} = -(\mathrm{TV}:\mathrm{S})\,\alpha\beta,$$

the *angles of quaternions* here considered being supposed as usual (comp. 130) to be generally > 0, but $< \pi$; for example, we have thus,

$$\text{LXVIII.}\ldots\ \tan H = \tan\angle\frac{\lambda}{\tau} = (\mathrm{TV}:\mathrm{S})\,\lambda\tau^{-1} = (\mathrm{TV}:\mathrm{S})\,(\mathrm{r}^{-1}-\tau') = \mathrm{r}\mathrm{T}\tau' = \mathrm{r}r^{-1},$$

as in XVII.; and in like manner we have generally, by principles already explained (comp. 196, XVI.),

$$\text{LXIX.}\ldots\ \cos\angle\frac{\beta}{\alpha} = \cos\angle\frac{\alpha}{\beta} = -\cos\angle\beta\alpha = -\cos\angle\alpha\beta$$

$$= \mathrm{S}\frac{\beta}{\alpha} : \mathrm{T}\frac{\beta}{\alpha} = \mathrm{SU}\frac{\beta}{\alpha} = -\mathrm{SU}\alpha\beta.$$

(23.) Applying these principles to investigate the inclinations of the vector τ'', which is perpendicular to the diagonal plane LXV. of the prism, to the three rectangular lines τ, τ', ν, or the inclinations of that diagonal plane itself to the normal, rectifying, and osculating planes, with the help of the expressions deduced from VI. for the three products,* $\tau\tau''$, $\tau'\tau''$, $\nu\tau''$, we arrive easily at the following results:

---

* A student, who should be inclined to pursue this subject, might find it useful to form for himself a *table* of all the *binary products* of the nine vectors,

$$\tau,\ \tau',\ \tau'',\ \nu,\ \nu',\ \lambda,\ \sigma-\rho,\ \sigma-\mu,\ \text{and}\ \kappa',$$

considered as so many *quaternions*, and reduced to the *common quadrinomial form*, $a + b\tau + c\tau' + e\nu$, in which a, b, c, e are *scalars*, whereof some may vanish, but which are generally functions of r, r, and r'.

$$\text{LXX.}\ldots \cos\angle\frac{\tau''}{\tau}=\frac{-r^{-3}}{\mathrm{T}\tau''};\quad \cos\angle\frac{\tau''}{\tau'}=-\frac{r^{-2}r'}{\mathrm{T}\tau''};\quad \cos\angle\frac{\tau''}{\nu}=\frac{r^{-1}\mathrm{r}^{-1}}{\mathrm{T}\tau''};$$

with the verification, that the sum of the squares of these three cosines is unity, because

$$\text{LXXI.}\ldots r^2\mathrm{T}\tau''=\surd(1+\mathrm{r}^{-2}R^2)=\surd(1+r'^2+r^2\mathrm{r}^{-2});$$

or

$$\text{LXXI}'.\ldots r\mathrm{T}\tau''=\surd(r^{-2}r'^2+\mathrm{T}\lambda^2),\quad \mathrm{T}\tau''=\surd(r^{-4}+\mathrm{T}\nu'^2).$$

(24.) Or we may write, on the same general plan,

$$\text{LXXII.}\ldots \tan\angle\frac{\tau''}{\tau}=\frac{-R}{\mathrm{Tr}};\quad \tan\angle\frac{\tau''}{\tau'}=\frac{-r\mathrm{T}\lambda}{r'};\quad \tan\angle\frac{\tau''}{\nu}=\frac{\mathrm{r}}{r}\surd(1+r'^2);$$

or

$$\text{LXXIII.}\ldots \tan\angle\tau\tau''=R\mathrm{Tr}^{-1};\quad \tan\angle\tau'\tau''=rr'^{-1}\mathrm{T}\lambda;\quad \tan\angle\nu\tau''=-\mathrm{r}r^{-1}\surd(1+r'^2);$$

and may modify the expressions, by introducing the auxiliary angles H and P, with which may be combined, if we think fit, the following *angle of the prism*,

$$\text{LXXIV.}\ldots \mathrm{PKT}=\mathrm{BSL}=\tan^{-1}r'.$$

(25.) Instead of thus comparing the *plane* SPL with the three rectangular planes (379, (5.)) of the construction, we may inquire what is the value of the *angle* SPL, which the radius (R) of spherical curvature makes with the rectifying line (λ); and we find, on the same plan, by quaternions, the following very simple expression for the cosine of this angle, which may however be deduced by spherical trigonometry also,

$$\text{LXXV.}\ldots \cos\mathrm{SPL}=-\mathrm{SU}\lambda(\sigma-\rho)=\frac{pr^{-1}}{R\mathrm{T}\lambda}=\sin P\sin H;$$

or

$$\text{LXXV}'.\ldots \cos\mathrm{SPL}=\cos\mathrm{SPB}\cos\mathrm{BPL}.$$

(26.) In general, it is easy to form, by methods already explained, the quaternion equation of a *cone* which has a *given vertex*, and *rests* on a *given curve* in space; and also to determine the *right cone* which *osculates* (394, (30.)) to this *general cone*, along any *given side* of it.

(27.) But if we merely wish to assign the *osculating right cone* to the *cone of chords* from P, or to the *locus* of the *line* PP_s, we may imitate a recent process: and may observe that if this *new cone* be *cut* by the *normal plane* LVIII., the *vector* of the *section* has the following approximate expression, analogous to LIX., and like it sufficient for our purpose,

$$\text{LXXVI.}\ldots \omega_s=\rho+c\tau+\tfrac{1}{3}cs\tau'+\tfrac{1}{6}cs^2\mathrm{r}^{-1}\nu;$$

from which it may be inferred (comp. (15.), (16.)), that the *axis of revolution* of the *new right cone* has for equation,

$$\text{LXXVII.}\ldots 0=\mathrm{V}(\mathrm{r}^{-1}\tau+\tfrac{3}{4}\nu)(\omega-\rho).$$

This *axis* is therefore situated *in the rectifying plane*, *between the rectifying line* (λ or $\mathrm{r}^{-1}\tau+\nu$), *and the tangential vector* (IV.) *of second curvature* ($\mathrm{r}^{-1}\tau$): while the *semiangle* C of the same new cone (measured like H from $+\tau$ towards $+\nu$) has the value already assigned by anticipation in the formula XXV., and is therefore *less* than the semiangle H if both be *acute*, but *greater* than H if both be *obtuse;* so that, *in each case*, the *new right cone* (C) is *sharper* than the *old* right cone (H).

(28.) The same result may be otherwise obtained, by observing that an unit-vector in the direction of the chord PP$_s$ has (by 396, XLIV., and 397, I.) the approximate expression,

$$\text{LXXVIII.}\ldots\ \chi_s = \mathrm{U}(\rho_s - \rho) = \left(1 + \frac{s^2}{24r^2}\right)\left(\tau + \frac{s\tau'}{2} + \frac{s^2\tau''}{6}\right)$$
$$= \tau + \frac{s\tau'}{2} + \frac{s^2}{6}\left(\tau'' + \frac{r^{-2}\tau}{4}\right);$$

whence the axis of the osculating right cone to the cone of chords (27.) has rigorously the direction of the line $\mathrm{V}\chi'\chi''$ (for $s = 0$), or of the vector,

$$\text{LXXIX.}\ldots\ \xi = \mathrm{V}\tau'(r^2\tau'' + \tfrac{1}{4}\tau) = \lambda - \tfrac{1}{4}\nu = \mathrm{r}^{-1}\tau + \tfrac{3}{4}\nu, \text{ as before.}$$

(29.) *This axis* ξ makes (if we neglect s^3) the *same angle* C, with the *chord* PP$_s$, as with the *tangent* τ; whereas the *former axis* λ makes *unequal angles* with those two lines, within the same *order* (or degree) of approximation: for our methods conduct to the expression,

$$\text{LXXX.}\ldots\ \angle\frac{\rho_s - \rho}{\lambda} = H - \frac{s^2}{24r\mathrm{r}},$$

from which the *relation* XXV., between the *two right cones*, may easily be deduced anew.

(30.) Neglecting only s^4, and employing the substitution XLVI., the expression XLVII. for the vector of the given curve becomes,

$$\text{LXXXI.}\ldots\ \rho_t = \rho + t\tau + \tfrac{1}{2}t^2 v + \tfrac{1}{6}t^3\mathrm{r}^{-1}\nu, \quad \text{if} \quad \text{LXXXII.}\ldots\ v = \tau' + \frac{r'\tau}{3r};$$

where the *variable scalar* t denotes, by (13.), the *ordinate of the osculating parabola*, and the *constant vector* v has the direction, by (14.), of the *diameter* of that parabola.

(31.) In the *present order of approximation*, then, the proposed *curve in space* may be considered to be the *common intersection* of the *three* following *surfaces of the second order*, all passing through the given point P:

$$\text{LXXXIII.}\ldots\ 2(\mathrm{S}\tau'(\omega - \rho))^2 = 3\mathrm{r}\mathrm{S}\nu(\omega - \rho)\mathrm{S}v\nu(\omega - \rho);$$
$$\text{LXXXIV.}\ldots\ 2\mathrm{S}\tau'(\omega - \rho) = -r^2(\mathrm{S}v\nu(\omega - \rho))^2;$$
$$\text{LXXXV.}\ldots\ 3\mathrm{r}\mathrm{S}\nu(\omega - \rho) = -r^2\mathrm{S}\tau'(\omega - \rho)\mathrm{S}v\nu(\omega - \rho);$$

whereof the *first* represents a *new osculating oblique cone*, which has a *contact* of the *same* (*second*) *order* with the *cone of chords*, as the *osculating right cone* (27.); the *second* represents an *osculating parabolic cylinder*, which is *cut perpendicularly* in the *osculating parabola* (9.), by the *osculating plane* to the curve; and the *third* represents a certain *osculating hyperbolic* (or *ruled*) *paraboloid*, whereof the *tangent* (τ) is *one* of the *generating lines*, while the *diameter* (v) of the osculating *parabola* is *another*.

(32.) *Each* of these *three surfaces* (31.) has in fact generally a *contact of the third order* with the *given curve;* or has its *equation satisfied*, not only (as is obvious on inspection) by the *point* P *itself*, but also when we *derivate* successively with respect to the scalar variable t, and then substitute the values (comp. LXXXI.),

$$\text{LXXXVI.}\ldots\ \omega = \rho_0 = \rho, \quad \omega' = \rho_0' = \tau, \quad \omega'' = \rho_0'' = v, \quad \omega''' = \rho_0''' = \mathrm{r}^{-1}\nu;$$

r, r, ρ, τ', ν, and v being treated as *constants* of the *equation*, or of the *surface*, in each of *these* derivations.

(33.) The *cone* LXXXIII., and the *cylinder* LXXXIV., have a *common generatrix*, namely the *binormal** (ν); and in like manner, *another* generating line of the *same cone*, namely the *tangent* (τ) to the curve, has just been seen (31.) to be a line *on the paraboloid* LXXXV.: and although the *cylinder and paraboloid* have *no finitely distant right line common*, yet *each* may be said to contain the *line at infinity, in the diametral plane* of the cylinder, namely in the plane of ν and v, of which plane the quaternion equation is (comp. (14.)),

$$\text{LXXXVII.} \ldots 0 = \mathrm{S}\nu v(\omega - \rho), \quad \text{or} \quad \text{LXXXVII}'. \ldots 0 = \mathrm{S}(rr'\tau' - 3\tau)(\omega - \rho);$$

or the line in which this *diametral* meets the parallel *axial plane.*

(34.) On the whole, then, it is clear, from the known theory of intersections of surfaces of the second order having a common generating line, that *the given curve of double curvature* (whatever it may be) *has contact of the third order with the twisted cubic,*† or *gauche curve of the third degree,* which is represented without ambiguity by the system of the *two scalar equations,*

$$\text{LXXXVIII.} \ldots y = x^2, \quad z = x^3,$$

if we write for abridgment,

$$\text{LXXXIX.} \ldots \begin{cases} x = (t\ =) - r^2\mathrm{S}\nu\nu(\omega - \rho), \\ y = (t^2 =) - 2r^2\mathrm{S}\tau'(\omega - \rho), \\ z = (t^3 =) - 6r^2r\mathrm{S}\nu(\omega - \rho). \end{cases}$$

(35.) As another geometrical connexion between the elements of the present theory, it may be observed that while the *osculating plane* to the *curve*, of which plane the equation is,

$$\text{XC.} \ldots \mathrm{S}\nu(\omega - \rho) = 0, \quad \text{as in 396, XV.,}$$

touches the oblique cone LXXXIII., *along the tangent* τ to the same curve, the *diametral plane* LXXXVII. *touches the same cone along the binormal* ν, which was lately seen (33.) to be, as well as τ, a *side* of that oblique cone; but these *two sides of contact*, τ and ν, are both *in the rectifying plane* (396, XIV.), and the *two tangent planes* corresponding *intersect* in the *diameter* v of the *parabola* (9.); we have therefore this *theorem*:—

The diameter of the osculating parabola to a curve of double curvature is the polar of the rectifying plane, with respect to the osculating oblique cone LXXXIII.; that is, with respect to a certain cone of the *second order*, which has been above deduced from the expression LXXXI. for the vector ρ_t of the curve, as one naturally suggested thereby, and as having a *contact of the third order with the curve* at P, and therefore also a contact of the *second order* with the *cone of chords* from that point.

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* The *geometrical reason*, for the *osculating cone* LXXXIII. to the *cone of chords* containing the *binormal* (ν), is that *if* the expression LXXXI. for ρ_t were *rigorous*, and if the variable t were supposed to *increase indefinitely*, the *ultimate direction* of the *chord* PP_t would be *perpendicular to the osculating plane.* And the same binormal is a generating line of the *parabolic cylinder* also, because that cylinder passes through P, and *all* its generating lines are perpendicular to the last mentioned plane. It is sufficient however to observe, on the side of calculation, that the *equations* LXXXIII. and LXXXIV. are *satisfied*, when we suppose $\omega - \rho \parallel \nu$.

† Compare again page 241, already cited, of Dr. Salmon's Treatise; also Art. 285, in page 225 of the same work.

(36.) Conversely, *this particular cone* LXXXIII. is *geometrically distinguished* from all *other** cones of the *same* (*second*) *order*, which have their *vertices* at the *given point* P, and have each a *contact* of the same *second* order, with the *given cone* of *chords* from that point, or of the *third* order with the given *curve*, by the *condition* that it is *touched* (as above), *along the binormal* (ν), by the *diametral plane* (νv) of the *osculating parabolic cylinder* LXXXIV.

(37.) We have already considered, in 395, (5.), the simultaneous variations of the points P and K, or of the vectors ρ and κ. With recent notations, including the expression $\mu = 2\kappa - \rho$, we have the following among other transformations, for the *first derivative* of the latter vector, and therefore for the *tangent* KK$'$ to the *locus of centres of curvature*, of a given curve in space:

$$\begin{aligned}\text{XCI.} \ldots \mathrm{KK}' = \mathrm{D}_s\kappa = \kappa' &= (\rho - \tau'^{-1})' = \tau + \tau'^{-1}\tau''\tau'^{-1}\\ &= (\rho + r^2\tau')' = \tau + r^2\tau'' + 2rr'\tau'\\ &= rr'\tau' + r^2\mathrm{r}^{-1}\nu = rr'(\tau' + p^{-1}r\nu) = r\mathrm{r}^{-1}(p\tau' + r\nu)\\ &= \frac{rr'}{\rho - \kappa} - \frac{rr'}{\sigma - \kappa} = \frac{rr'(\sigma - \mu)}{(\sigma - \kappa)(\kappa - \rho)} = \mathrm{r}^{-1}(\sigma - \mu)\tau\\ &= \cot H(\mathrm{U}\tau' \tan P + \mathrm{U}\nu) = \mathrm{r}^{-1}R(\mathrm{U}\tau' \sin P + \mathrm{U}\nu \cos P)\\ &= r^4\nu\nu'\tau' = r^4\tau'\nu'\nu = \nu^{-1}\nu'\tau'^{-1} = \tau'^{-1}\nu'\nu^{-1}\\ &= \mathrm{r}^{-1}\nu(\rho - \sigma)(\kappa - \rho) = \mathrm{r}^{-1}(\kappa - \rho)(\rho - \sigma)\nu\\ &\qquad\qquad = \mathrm{r}^{-1}R\mathrm{U}(\nu(\rho - \sigma)(\kappa - \rho)) = \&\text{c.};\end{aligned}$$

if then we draw the *diameter of curvature* PM, and let fall a perpendicular KN from the centre K of the osculating circle on the new radius SM of the osculating sphere (as in the annexed Figure 82), *this perpendicular will touch*† *the locus of the centre* K, a result which agrees with the construction in 395, (6.); and we see, at the same time, that the *length* of the line KK$'$, or the *tensor* Tκ', may be expressed (comp. LXXIII.) as follows,

$$\text{XCII.} \ldots \overline{\mathrm{KK}}' = \mathrm{T}\kappa' = R\mathrm{T}\mathrm{r}^{-1} = r^2\mathrm{T}\nu' = \tan \angle\, \tau\tau''.$$

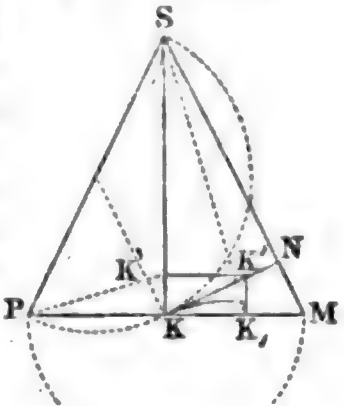

Fig. 82.

(38.) If we *project* the tangent KK$'$, into its two rectangular components, KK$_{,}$ and KK$^{`}$, on the diameter of curvature and the polar axis, we shall have by XCI. the expressions:

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* The cone of *this system* (36.), which is *touched* along the *binormal* by the *normal plane*, and which therefore *intersects* the *parabolic cylinder* LXXXIV. in a *new twisted cubic* (comp. (34.)), having *also* contact of the *third order* with the *curve*, is easily found to have, for its quaternion equation, the following:

$$2r^2(\mathrm{S}\tau'(\omega - \rho))^2 = 3\mathrm{r}\mathrm{S}\tau(\omega - \rho)\mathrm{S}\nu(\omega - \rho);$$

and with respect to *this cone* (comp. (35.)), the *polar of the rectifying plane* is the (*absolute*) *normal* (τ') *to the curve*.

† Geometrically, and by infinitesimals, if we conceive K$'$ to be an infinitely near point of the locus of K, and therefore in the normal plane at P, the angle PK$'$S (like PKS) will be right, and the point K$'$ will be *on the semicircle* PKS; but the *radius* of this semicircle drawn to K (comp. Fig. 82) is *parallel* to the line SM, to which line the *tangent* KK$'$ is therefore *perpendicular*, as above.

$$\text{XCIII.}\ldots\ \mathrm{KK}_{,} = rr'\tau' = r'\mathrm{U}\tau' = \frac{rr'}{\rho-\kappa} = \&\text{c.};$$

$$\text{XCIV.}\ldots\ \mathrm{KK}^{\backprime} = r^2\mathrm{r}^{-1}\nu = r\mathrm{r}^{-1}\mathrm{U}\nu = \frac{-rr'}{\sigma-\kappa} = \&\text{c.};$$

these two projections then, or the *vector-tangent* KK' itself, would *suffice* to determine r and r', or H and P, and thereby *all* the affections of the curve which depend on s^3, but *not* on s^4.

(39.) We have also the *similar triangles* (see again Fig. 82),

$$\text{XCV.}\ldots\ \Delta\,\mathrm{K_{,}K'K} \propto \mathrm{K^{\backprime}KK'} \propto \mathrm{KMS};$$

and the *vector equations*,

$$\text{XCVI.}\ldots\ \mathrm{KK}' : \mathrm{SM} = \mathrm{KK}_{,} : \mathrm{SK} = \mathrm{KK}^{\backprime} : \mathrm{KM} = \mathrm{KK}^{\backprime} : \mathrm{PK}$$
$$= \mathrm{r}^{-1}\tau = \textit{Vector of second curvature}\ (\text{IV.});$$

whence also result the *scalar expressions*,

$$\text{XCVII.}\ldots\ \tan \mathrm{KSK}_{,} = \tan \mathrm{KPK}^{\backprime} = \mathrm{r}^{-1} = \textit{Second}^{*}\ \textit{Curvature}\ (\text{III.}):$$

this last scalar being *positive* or *negative*, according as the *rotation* $\mathrm{KSK}_{,}$ (or $\mathrm{KPK}^{\backprime}$) *appears* to be positive or negative, when seen from *that side* of the *normal plane*, towards which the conceived *motion* (396, (1.)) along the given *curve*, or the *unit tangent* $+\tau$, is directed.†

(40.) Besides the *seven* expressions, III., XXVII., L., and XCVII., this important scalar r^{-1} admits of many others, of which the following, numbered for reference as 8, 9, &c., and deduced from formulæ and principles already laid down, are examples: and may serve as *exercises in transformation*, according to the rules of the present Calculus, while some of them may also be found useful, in future *geometrical applications*.

(41.) We have then (among others) the transformations:

$$\text{XCVIII.}\ldots\ \textit{Second Curvature} = \mathrm{r}^{-1}\,(= \textit{seven preceding expressions})$$
$$= p^{-1}r' = r^{-1}\cot H = \mathrm{T}\lambda\cos H = \mathrm{r}^{-1}r'\cot P \qquad (8,\ 9,\ 10,\ 11)$$
$$= r^2\mathrm{S}\nu'\tau' = -\mathrm{S}\nu'\tau'^{-1} = -r^2\mathrm{S}\tau\tau'\tau'' = \mathrm{S}\tau\tau'^{-1}\tau'' \qquad (12,\ 13,\ 14,\ 15)$$
$$= -r^2\mathrm{S}\nu\tau'' = \mathrm{S}\nu^{-1}\tau'' = -\mathrm{S}\nu\kappa' = \mathrm{S}\tau\kappa'\tau' \qquad (16,\ 17,\ 18,\ 19)$$
$$= \tau\kappa'(\sigma-\mu)^{-1} = \mathrm{S}\lambda\tau^{-1} = (\kappa-\rho)\mathrm{V}\lambda\nu = -\tau'^{-1}\mathrm{V}\lambda\nu \qquad (20,\ 21,\ 22,\ 23)$$
$$= r^2\tau'\mathrm{V}\lambda\nu = r^2\mathrm{S}\lambda\nu\tau' = \mathrm{S}\lambda\tau'\nu^{-1} = \mathrm{S}\lambda\tau'^{-1}\nu \qquad (24,\ 25,\ 26,\ 27$$
$$= r^2\mathrm{S}\nu'\lambda\tau = r^2\mathrm{S}\nu'\nu\tau = \mathrm{S}\tau\nu^{-1}\nu' = r^2\mathrm{S}\nu'\nu^{-1}\tau'' \qquad (28,\ 29,\ 30,\ 31)$$
$$= r^4\mathrm{S}\nu\nu'\tau'' = \tau''^{-1}\mathrm{V}\nu'\lambda = r^3r'^{-1}\mathrm{S}\nu'\lambda\tau' = r^3r'^{-1}\mathrm{S}\nu\lambda\tau'' \qquad (32,\ 33,\ 34,\ 35)$$
$$= \mathrm{S}\nu\lambda\tau''^{-1} = \mathrm{T}\tau''^{-2}\mathrm{S}\lambda\nu'\tau'' = \frac{-(r\nu)'}{r\tau'} = \frac{-r^2\nu'}{\sigma-\rho} \qquad (36,\ 37,\ 38,\ 39)$$

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* In illustration it may be observed, that if $\mathrm{d}s$ be treated as *infinitely small*, and if the line KK' be supposed to represent (not the *derivative* κ', but) the *differential* vector $\mathrm{d}\kappa = \kappa'\mathrm{d}s$, then the *projections* $\mathrm{KK}_{,}$ and $\mathrm{KK}^{\backprime}$ become $\mathrm{d}r$ and $r\mathrm{r}^{-1}\mathrm{d}s$ (comp. XCIII. and XCIV.); while $\mathrm{KPK}^{\backprime}$ (in Fig. 82) represents the *infinitesimal angle* $\mathrm{r}^{-1}\mathrm{d}s$, through which the *osculating plane* (comp. (1.)) *revolves*, round the tangent τ to the curve during the change $\mathrm{d}s$ of the *arc*.

† This direction of $+\tau$ is to be conceived (comp. Fig. 81) to be towards the *back* of Fig. 82, as drawn, if the scalars r' and r (and therefore also p) be *positive*.

$$= \frac{-r\nu'}{r\tau' + p\nu'} = \frac{r^2\tau'' + \tau}{\tau(\sigma-\rho)} = R^{-1}\tan\angle\,\mathrm{r}\tau\tau'' = R^{-1}\tan\angle\,\frac{\mathrm{V}\lambda\nu'}{\tau} \qquad (40,\ 41,\ 42,\ 43)$$

$$= \frac{rr'\nu}{\sigma-\kappa} = \frac{rr'\tau'}{(\sigma-\kappa)\tau} = \frac{r'}{\tau}\cdot\frac{\tau(\kappa-\rho)}{\sigma-\kappa} = \frac{rr'\tau}{(\sigma-\kappa)(\rho-\kappa)} \qquad (44,\ 45,\ 46,\ 47)$$

$$= \mathrm{S}\,\frac{rp\lambda}{(\sigma-\kappa)(\rho-\kappa)} = \mathrm{S}\,\frac{\rho + rp\lambda - \kappa}{(\sigma-\kappa)(\rho-\kappa)} = \mathrm{S}\,\frac{\mathrm{KL}}{\mathrm{KS}\,.\,\mathrm{KP}} \qquad (48,\ 49,\ 50)$$

$$= \mathrm{S}\,\frac{\mathrm{SL}}{\mathrm{PK}\,.\,\mathrm{KS}} = \frac{-(\mathrm{S}ar\nu)'}{r(\mathrm{S}a\tau)'} = \frac{-\mathrm{d}\cos\angle\,\frac{\nu}{a}}{r\mathrm{d}\cos\angle\,\frac{\tau}{a}} \qquad (51,\ 52,\ 53);$$

PKSL, in the forms 50 and 51, being points of the same *gauche quadrilateral* as in (18.); and a, in 52 and 53,* denoting *any constant vector:* while several other varieties of form may be deduced from the foregoing by very simple processes, such as the substitution of $\mathrm{U}\nu$ for $r\nu$, &c., which gives for instance (comp. XI'.), from the form 38, these others,

$$\text{XCVIII}'\ldots\ \mathrm{r}^{-1} = \frac{-(\mathrm{U}\nu)'}{r\tau'} = \frac{-(\mathrm{U}\nu)'}{\mathrm{U}\tau'} = \frac{-\mathrm{dU}\nu}{r\mathrm{d}\tau} \qquad (54,\ 55,\ 56).$$

We may also write, with the significations (10.) of Q_1 and Q_3, the following expression analogous to L.,

$$\text{XCVIII}''\ldots\ \mathrm{r}^{-1} = 6\mathrm{KP}\,.\,\lim.\,\frac{\mathrm{PQ}_3}{\mathrm{PQ}_1{}^3} \qquad (57),$$

which contains the law of the *inflexion* of the *plane curve,* into which the proposed curve of *double curvature* is *projected,* on its own *rectifying plane:* the *sign* of the *scalar,* to which this last expression ultimately reduces itself, being determined by the rules of quaternions.

(42.) And besides the various expressions for the positive scalar r^{-2}, which are immediately obtained by *squaring* the foregoing forms, the following are a few others:

$$\text{XCIX}\ldots\ \textit{Square of Second Curvature} = \mathrm{r}^{-2} = \mathrm{Tr}^{-2}$$

$$= \mathrm{T}\lambda^2 - r^{-2} = r^2\mathrm{S}\tau''\tau'\lambda - r^{-2} = r^2\mathrm{T}\nu'^2 - r^{-2}r'^2 \qquad (1,\ 2,\ 3)$$

$$= r^2\mathrm{S}\tau\nu'\tau'' - r^{-2}r'^2 = r^2\mathrm{T}\tau''^2 - r^{-2} - r^{-2}r'^2 = R^{-2}(r^4\mathrm{T}\tau''^2 - 1) \qquad (4,\ 5,\ 6)$$

$$= R^{-2}r^4\mathrm{T}\nu'^2 = R^{-2}\mathrm{T}\kappa'^2 = R^{-2}\tan^2\angle\,\tau\tau'' \qquad (7,\ 8,\ 9);$$

while the important vector τ'', besides its two original forms VI., admits of the following among other expressions (comp. XX. XXI.):

$$\text{C}\ldots\ \tau'' = \mathrm{D}_s{}^3\rho \ (= \text{the two expressions VI.})$$

$$= r^{-2}\mathrm{V}\lambda(\sigma-\rho) = \lambda\tau' - r^{-1}r'\tau' = \nu'\tau - r^{-2}\tau \qquad (3,\ 4,\ 5)$$

$$= \mathrm{rV}\nu'\lambda = r^{-2}\mathrm{r}^{-1}\tau(\sigma - \rho - \mathrm{r}) = r^{-3}p + r^{-2}\lambda(\sigma-\rho) \qquad (6,\ 7,\ 8)$$

$$= ((\rho-\kappa)^{-1})' = \tau'(\kappa' - \tau)\tau' = -r^{-2}\tau - \frac{r^{-1}r'}{\rho-\kappa} - \frac{r^{-1}r'}{\sigma-\kappa} \qquad (9,\ 10,\ 11).$$

(43.) As regards the general theory (396, (5.), &c.) of *emanant lines* (η) from curves, it might have been observed that if we write,

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* This last form 53 corresponds to and contains a theorem of M. Serret, alluded to in the second Note to page 563.

$$\text{CI.}\ldots\ \zeta = \mathrm{V}\frac{\tau}{\theta}, \quad \text{with} \quad \text{CII.}\ldots\ \theta = \mathrm{V}\frac{\eta'}{\eta}, \text{ as in 396, XXVII.,}$$

the equation 396, XXXII. takes the simplified form,

$$\text{CIII.}\ldots\ \mathrm{PH} = \omega_0 - \rho = \eta \mathrm{S}\eta^{-1}\zeta = \textit{projection of vector } \zeta \textit{ on emanant } \eta\,;$$

for example, when $\eta = \nu$, then $\theta = r^{-1}\tau$, and $\zeta = 0$, $\mathrm{PH} = 0$, or $\omega_0 = \rho$, as in (1.); and when $\eta = \tau$, then $\theta = \nu$, $\zeta = r^2\tau' \perp \eta$, so that the *projection* PH again vanishes, as in 396, (13.).

(44.) In an extensive class of applications, the *emanant lines* are *perpendicular to the given curve* ($\eta \perp \tau$); and since we have, by (43.),

$$\text{CIV.}\ldots\ \zeta = \frac{\mathrm{V}\tau\mathrm{V}\eta'\eta}{\eta^2\theta^2} = \eta^{-1}\theta^{-2}\mathrm{S}\tau\eta' = \frac{\eta^{-1}\mathrm{S}\eta\tau'}{\mathrm{T}\theta^2}, \quad \text{if} \quad \mathrm{S}\tau\eta = 0,$$

we may write, for this *case of normal emanation*, the formula,

$$\text{CV.}\ldots\ \mathrm{PH} = \zeta = \frac{\textit{projection of vector of curvature } (\tau') \textit{ on emanant line } (\eta)}{\textit{square of velocity } (\mathrm{T}\theta) \textit{ of rotation of that emanant}};$$

for example, when the emanant (η) coincides with the *absolute normal* (τ'), we have then $\theta = \lambda$, as in (3.), and the recent formula CV. becomes,

$$\text{CVI.}\ldots\ \mathrm{PH} = \omega_0 - \rho = \zeta = \tau'\mathrm{T}\lambda^{-2} = r^2\tau' \sin^2 H = (\kappa - \rho)\sin^2 H,$$

which agrees with the expression XXXVIII.

(45.) And in the corresponding case of *tangential emanant planes*, by making $\mathrm{S}\tau\eta = 0$ in the second equation 396, XXXVI., and passing to a second derived equation, we find for the *intercept* between the point P of the *curve*, and the point, say R, in which the *line of contact* of the *plane* with its own *envelope* touches the *cusp-edge* of that *developable surface*, the expression,

$$\text{CVII.}\ldots\ \mathrm{PR} = \frac{-\mathrm{V}\eta\eta'\mathrm{S}\eta\tau'}{\mathrm{S}\eta\eta'\eta''} = \frac{-\mathrm{S}\eta\tau'(\text{or} + \mathrm{S}\tau\eta')}{\textit{projection of } \eta'' \textit{ on } \theta};$$

which accordingly vanishes, as it ought to do, when $\eta = \nu$, that is, when the *emanant plane* $\mathrm{S}\eta(\omega - \rho) = 0$ coincides with the *osculating plane* XC.

(46.) Some additional light may be thrown on this whole theory, of the *affections of a curve in space* depending on the *third power of the arc*, and even on those affections which depend on *higher powers* of *s*, by that *conception* of an *auxiliary spherical curve*, which was employed in 379, (6.) and (7.), to supply *constructions* (or geometrical representations) for the *directions*, not only of the *tangent* (ρ') to the *given* curve, to which indeed the *unit-vector* (τ) of the *new* curve is *parallel*, but also of the *absolute normal*, the *binormal*, and the *osculating plane*; while the *same* auxiliary curve served also, in 389, (2.), to furnish a *measure* of the *curvature* of the original curve, which is in fact the *velocity** of motion in the *new* or *spherical* curve, if that in the *old* or *given* one be supposed to be *constant*, and be taken for *unity*.

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* Accordingly the *vector of velocity* τ', of this *conceived motion* in the *auxiliary curve*, is precisely what we have called (389, (4.), comp. 396, VI.) the *vector of curvature* of the proposed *curve in space*: and its *tensor* ($\mathrm{T}\tau'$) is equal to the *reciprocal* of the *radius* (r) of that curvature.

(47.) We might for instance have observed, that while the *normal plane* to the *curve in space* is represented (in direction) by the *tangent plane* to the *sphere*, the *rectifying plane* (as being perpendicular to the absolute normal) is represented similarly by the *normal* plane to the *spherical curve:* and it is not difficult to prove that the *rectifying line* has the direction of that *new radius* of the sphere, which is drawn to the *point* (say L) where the *normal arc* to the auxiliary curve *touches its own envelope.*

(48.) The *point* L thus determined is the *common spherical centre* (comp. 394, (5.)) *of curvature*, of the *auxiliary* curve *itself*, and of that *reciprocal*[*] *curve* on the same sphere, of which the *radii* have the directions (comp. 379, (7.)) of the *binormals* to the original curve; the *trigonometric tangent* of the *arcual radius of curvature* of the auxiliary curve is therefore ultimately equal to a *small arc* of that curve, *divided by the corresponding arc* of the reciprocal curve (or rather by the latter arc with its *direction reversed*, if the point L fall *between* the two curves upon the sphere); and therefore to the *first curvature* (r^{-1}) of the *given* curve, *divided by the second curvature* (r^{-1}): and thus we have not only a simple *geometrical interpretation* of the *quaternion equation* XI'., but also a *geometrical proof* (which may be said to require *no calculation*), of the important but known relation XVII., which connects the *ratio* ($\mathrm{r} : r$) *of the two curvatures*, with the *angle* (H) between the *tangent* (τ) and the *rectifying line* (λ), for *any curve in space.*

(49.) In whatever manner this known relation ($\tan H = \mathrm{r} : r$) has once been established, it is geometrically evident, that *if the ratio of the two curvatures be constant*, then, because the *curve crosses* the *generating lines* of its own *rectifying developable* (396) under a *constant angle* (H), that *developable surface* must be *cylindrical:* or in other words, the proposed *curve of double curvature* must, in the case supposed, be a *geodetic*[†] *on a cylinder* (comp. 380, (4.)). Accordingly the point L, in the two last sub-articles, becomes then a *fixed point* upon the *sphere*, and is the *common pole* of *two complementary small circles*, to which the *auxiliary spherical curve* (46.), and the *reciprocal curve* (48.), in the case here considered, reduce themselves; so that the *tangent* and the *binormal* to the *curve in space* make (in the same case) *constant angles*, with the *fixed radius* drawn to that point: and the *curve itself* is therefore (as before) a *geodetic line*, on *some* cylindrical surface.

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[*] The *reciprocity* here spoken of, between these *two spherical curves*, is of that known kind, in which each *point* of *one* is a *pole* of the *great-circle tangent*, at the *corresponding point* of the *other:* and accordingly, with our recent symbols, we have not only $\nu = \mathrm{V}\tau\tau'$, but also, $\mathrm{V}\nu\nu' = r^{-2}\mathrm{V}\nu'\nu^{-1} = r^{-2}\mathrm{r}^{-1}\tau \parallel \tau$.

[†] The writer has not happened to meet with the *geometrical proof* of this known theorem, which is attributed to M. Bertrand by M. Liouville, in page 558 of the already cited Additions to Monge; but the deduction of it as above, from the fundamental property (396) of the *rectifying line*, is sufficiently obvious, and appears to have suggested the method employed by M. de Saint-Venant, in the part (p. 26) of his Memoir *sur les lignes courbes non planes*, &c., before referred to, in which the result is enunciated. *Another*, and perhaps even a *simpler* method, *suggested by quaternions*, of *geometrically* establishing the same theorem, will be sketched in the present sub-article (49.); and in the following sub-article (50.), a proof by the *quaternion analysis* will be given, which seems to leave nothing to be desired on the side of simplicity of *calculation.*

(50.) By quaternions, when the *two curvatures* have thus a *constant ratio*, the equations XI'. and XVI. give,

$$\text{CVIII.} \ldots (r\lambda)' = (\mathrm{U}\nu + r\mathrm{r}^{-1}\tau)' = (r\mathrm{r}^{-1})'\tau = 0,$$

or

$$\text{CIX.} \ldots r\lambda = \textit{a constant vector};$$

the *tangent* (τ) makes therefore, in this case, a *constant angle* (H) with a *constant line* ($r\lambda$): and the *curve* is thus seen again, by this very simple *analysis*, to be a *geodetic on a cylinder*. And because it is easy to prove (comp. XXXI.), that we have in the same case the expression,

$$\text{CX.} \ldots r \sin^2 H = \textit{radius of curvature of base},$$

or of the *section* of the cylinder made by a plane perpendicular to the generating lines, this *other* known theorem results, with which we shall conclude the present series of sub-articles: *When both the curvatures are constant, the curve is a geodetic on a right circular cylinder* (or *cylinder of revolution*); or it is what has been called above, for simplicity and by eminence, a *helix*.*

398. When the *fourth power* (s^4) of the *arc* is taken into account, the expansion of the vector ρ_s involves *another term*, and takes the form (comp. 397, I.),

$$\text{I.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'' + \tfrac{1}{24}s^4\tau''',$$

in which

$$\text{II.} \ldots \tau''' = \mathrm{D}_s{}^4\rho, \quad \text{and} \quad \text{III.} \ldots \mathrm{S}\tau\tau''' = -3\mathrm{S}\tau'\tau'' = -3r^{-3}r';$$

so that the *new affections* of the curve, thus introduced, depend only on *two new scalars*, such as r' and r'', or r' and R', or H' and P', &c. We must be content to offer here a very few remarks on the theory of *such* affections, and on the manner in which it may be extended by the introduction of derivatives of *higher orders*.

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* In general, the expression XLIV. for the vector ω_s of the *osculating helix*, in which $\iota = -\mathrm{r}^{-1}\lambda^{-1} = \tau - \lambda^{-1}\tau'$, and $\rho - \omega_0 = \lambda^{-2}\tau'$, gives $\mathrm{T}\omega'_s = 1$; so that the *deviation* (8.) may be considered (comp. (13.)) to be measured from the extremity of an *arc of the helix*, which is *equal in length* to the arc s of the *curve*, and is set off from the same initial point P, with the same initial direction: while ω_0 does not *here* denote the value of ω_s answering to $s = 0$, but has a special signification assigned by the formula XXXVIII. It may also be noted that the *conception*, referred to in (46.), of an *auxiliary spherical curve*, corresponds to the ideal substitution of the *motion of a point* with a *varying velocity* upon a *sphere*, for a motion with an *uniform velocity* in *space*, in the investigation of the *general properties* of *curves of double curvature*: and that thus it is intimately connected (comp. 379, (9.)) with the general theory of *hodographs*.

(1.) The new vector τ''', on which everything here depends, is easily reduced to the following forms,* analogous to the expressions 397, VI. for τ'':

$$\text{IV.}\ldots\ \tau''' = \frac{r(r^{-3})'}{\tau} + \frac{(r^{-3}r')' - \tau''^2}{\tau'} - \frac{(r^{-2}\mathrm{r}^{-1})'}{\nu}$$
$$= 3r^{-3}r'\tau + (r(r^{-1})'' + \lambda^2)\tau' + (r^{-2}\mathrm{r}^{-1})'r^2\nu.$$

(2.) The first derivatives of the four vectors, ν', κ', λ, σ, taken in like manner with respect to the arc s of the curve, are the following:

$$\text{V.}\ldots\ \nu'' = (\mathrm{V}\tau\tau'')' = \mathrm{V}\tau\tau''' + r^{-2}\lambda$$
$$= r^{-2}\mathrm{r}^{-1}\tau + (r^{-2}\mathrm{r}^{-1})'\tau'^{-1} + (r(r^{-1})'' - \mathrm{r}^{-2})\nu;$$
$$\text{VI.}\ldots\ \kappa'' = -r^{-1}r'\tau + (rr'' - r^2\mathrm{r}^{-2})\tau' + (r^2\mathrm{r}^{-1})'\nu;$$
$$\text{VII.}\ldots\ \lambda' = (\mathrm{r}^{-1})'\tau + (r^{-1})'r\nu, \quad \text{or} \quad \text{VII}'\ldots\ (r\lambda)' = (r\mathrm{r}^{-1})'\tau \ (\text{comp. 397, CVIII.});$$
$$\text{VIII.}\ldots\ \sigma' = (\kappa + pr\nu)' = (p' + r\mathrm{r}^{-1})r\nu = RR'p^{-1}r\nu;$$

in which last the scalar derivatives p' and R' are determined, in terms of r'' and r', by the equations,

$$\text{IX.}\ldots\ p' = (r'\mathrm{r})' = r''\mathrm{r} + r'\mathrm{r}',$$

and $$\text{X.}\ldots\ R' = R^{-1}(pp' + rr') = p'\sin P + r'\cos P = (p' + \cot H)\sin P.$$

We have also the derivatives,

$$\text{XI.}\ldots\ H' = \frac{r\mathrm{r}' - r'\mathrm{r}}{r^2 + \mathrm{r}^2} = \frac{r^{-1}r' - \mathrm{r}^{-1}\mathrm{r}'}{r\mathrm{r}\lambda^2},$$
$$\text{XII.}\ldots\ P' = \frac{rp' - r'p}{r^2 + p^2} = \frac{(rr'' - r'^2)\mathrm{r} + rr'\mathrm{r}'}{R^2};$$

and the relations,

$$\text{XIII.}\ldots\ \mathrm{S}\tau\tau'\tau''' = \mathrm{S}\nu\tau''' = -(r^{-2}\mathrm{r}^{-1})';$$
$$\text{XIV.}\ldots\ \mathrm{S}\tau\tau''\tau''' = \mathrm{S}\nu'\tau''' = -r^{-3}\mathrm{r}^{-2}(p' - r\mathrm{r}\lambda^2);$$
$$\text{XV.}\ldots\ \mathrm{S}\tau'\tau''\tau''' = r^{-2}\mathrm{S}\lambda\tau''' = -r^{-5}(r\mathrm{r}^{-1})';$$

which may be proved in various ways, and by the two first (or the two last) of which, the derivatives r' and p', and therefore also H' and P', can be *separately* calculated, as *scalar functions* of the *four vectors* $\tau, \tau', \tau'', \tau'''$, or of some three of them, including the *new* vector τ'''.

(3.) We may also deduce, from either V. or VIII., the following *vector expressions*, of which the *geometrical signification* is evident from the recent theory (396, 397) of *emanant lines and planes*:

XVI. . . *Vector of Rotation of Radius* (R) *of Spherical Curvature*
= *Vector of Rotation of Tangent Plane to Osculating Sphere*

$$= (\text{say})\ \phi = \mathrm{V}\frac{\nu''}{\nu'} = \mathrm{V}\frac{\sigma' - \tau}{\sigma - \rho} = R^{-2}\tau(\nu^{-1}\sigma' + \sigma - \rho) \qquad (1, 2, 3)$$
$$= \frac{\tau}{R}\left(\frac{rR'}{p} + \frac{\sigma - \rho}{R}\right) = \frac{r\tau}{R^2}\left(p' + \frac{r}{\mathrm{r}} + r\tau' + p\nu\right) = R^{-2}r(r\lambda + p'\tau - p\tau') \qquad (4, 5, 6);$$

whence follows this *tensor value* for the *common angular velocity* of these two connected *rotations*, compared still with the velocity of *motion* along the *curve*,

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* In these new expressions, on the plan of the second Note to page 561, the scalars r', p', R', and the vector σ', are to be regarded as of the dimension *zero*; r'', H', P', and κ'' of the dimension -1; λ' of the dimension -2; and ν'' and τ''', as being each of the dimension -3.

XVII. . . *Velocity of Rotation of Radius* (R), *or of Tangent Plane to Sphere*,

$$= \mathrm{T}\phi = \mathrm{TV}\frac{\nu''}{\nu'} = R^{-1}\sqrt{(1 + R'^2 \cot^2 P)} = R^{-1}\sqrt{\{1 + (p' + \cot H)^2 \cos^2 P\}};$$

with the verifications, for the case of the *helix*, for which $p = 0$, $p' = 0$, $P = 0$, and $R = r$, that these expressions XVI. and XVII. become,

$$\text{XVI}'. \,.\,.\; \phi = \lambda, \quad \text{and} \quad \text{XVII}'. \,.\,.\; \mathrm{T}\phi = \mathrm{T}\lambda = r^{-1} \operatorname{cosec} H,$$

which agree with those found before, for the vector and velocity of rotation of the *radius* (r) of *absolute curvature.*

(4.) As another verification, we have $R' = 0$ for every *spherical curve*, and the general expressions take then the forms,

$$\text{XVI}''. \,.\,.\; \phi = \frac{-\tau}{\sigma - \rho}, \quad \text{and} \quad \text{XVII}''. \,.\,.\; \mathrm{T}\phi = R^{-1},$$

of which the interpretation is easy.

(5.) In general, the formula XVII. may also be thus written,

$$\text{XVIII}. \,.\,.\; R^2\phi^2 + 1 = -R'^2 \cot^2 P = R'^2 - p^{-2}R^2R'^2 = R'^2 + \sigma'^2 = \sigma'^2 \cos^2 P;$$

or thus, $\quad \text{XIX}. \,.\,.\; R\mathrm{T}\phi = \sqrt{(1 + \mathrm{T}\sigma'^2 \cos^2 P)} = \sqrt{(1 + \mathrm{T}\sigma'^2 - R'^2)};$

or finally, $\quad \text{XX}. \,.\,.\; R^2\mathrm{T}\phi = \sqrt{(R^2 - r^2\sigma'^2)} = \sqrt{(R^2 + r^2\mathrm{T}\sigma'^2)};$

so that the *small angle*, $s\mathrm{T}\phi$, between the *two near radii* of *spherical curvature*, R and R_s, is ultimately equal to the *square root of the sum of the squares* of the *two small angles*, in *two rectangular planes*, sR^{-1} and $rsR^{-2}\mathrm{T}\sigma'$, or PSP, and SPS$_s$, which are subtended, respectively, *at the centre* S *of the osculating sphere* by the *small arc* s of the *given curve*, and *at the given point* P by the *small corresponding arc* $s\mathrm{T}\sigma'$ of the *locus of centres* S of *spherical curvature*, or of the *cusp-edge* (395, (2.)) of the *polar developable;* exactly* as the *small angle* $s\mathrm{T}\lambda$, between *two near radii* (397, (5.)) of *absolute curvature*, r and r_s, is ultimately the square root of the sum of the squares of the *two other small angles*, sr^{-1} and sr^{-1}, or PKP, and KPK$_s$, which are likewise situated in two *rectangular planes*, and are subtended at the centre K of the *osculating circle* by the small arc s of the *curve*, and at the given point P by the corresponding arc $s\mathrm{T}\kappa'$ of the *locus* of the *centre* K (comp. 397, XXXIV., XCIV.).

(6.) The *point*, say V, in which the *radius* R of the osculating *sphere* at P approaches *most nearly* to the *near radius* R_s from P$_s$, is *ultimately* determined (comp. 397, CV. and X.) by the formula,

$$\text{XXI}. \,.\,.\; \mathrm{PV} = \zeta = \frac{\textit{Vector of Spherical Curvature}}{\textit{Square of Angular Velocity of Radius } (R)}$$

$$= (\rho - \sigma)^{-1}\mathrm{T}\phi^{-2} = \frac{\sigma - \rho}{1 + R'^2 \cot^2 P} = \frac{\sigma - \rho}{1 + p^{-2}r^2R'^2};$$

the *vector* of this point V (in its ultimate position) is therefore

$$\text{XXII}. \,.\,.\; \mathrm{OV} = \rho + \zeta = \frac{r^2R'^2\rho + p^2\sigma}{r^2R'^2 + p^2} = \frac{r^2R'^2\rho + r^2r'^2\sigma}{r^2R'^2 + r^2r'^2};$$

with the verification, that (by X., comp. XVII.) the scalar $p^{-1}rR'$ or $R' \cot P$ reduces itself to cot H, or to $r\mathrm{r}^{-1}$, for the case $p = 0$, $p' = 0$, $P = 0$ (comp. (3.)): and that thus the expression 397, XXXVIII., for the vector OH of the *point* of *nearest approach*, of a *radius* (r) *of absolute curvature* to a *consecutive** *radius* of the same kind, is reproduced.

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* It will soon be seen that these two results, and others connected with them, depend geometrically on one *common principle*, which extends to all systems of *normal emanants* (397, (44.)).

(7.) In general, if we introduce a *new auxiliary angle*, J, determined by the formula,

$$\text{XXIII.}\ldots \cot J = p^{-1}rR' = R' \cot P = (p' + \cot H) \cos P = R(\mathrm{r}^{-1} + P'),$$

the expression XXII. takes the simplified form (comp. again 397, XXXVIII.),

$$\text{XXIV.}\ldots \mathrm{OV} = \rho + \zeta = \rho \cos^2 J + \sigma \sin^2 J;$$

and the *segments*, into which the point V divides (internally) the radius R of the *sphere*, have the values (comp. 397, XXXIX.),

$$\text{XXV.}\ldots \overline{\mathrm{PV}} = R \sin^2 J, \quad \overline{\mathrm{VS}} = R \cos^2 J.$$

(8.) A *geometrical signification* may be assigned for this *new angle J*, which is *analogous* to the known signification of the angle H (397, XVII.). In fact, the *tangent plane* to the *osculating sphere* at P *touches* its own *developable envelope* along a *new right line*, of which the scalar equations are,

$$\text{XXVI.}\ldots \mathrm{S}(\sigma - \rho)(\omega - \rho) = 0, \quad \mathrm{S}(\sigma' - \tau)(\omega - \rho) = 0;$$

and because the *developable locus* of all such *lines* can be shown to be *circumscribed*, along the *given curve*, to the *locus of the osculating circle*, which is at the same time the *envelope of the osculating sphere*, we shall briefly call this *locus of the line* XXVI. the *Circumscribed Developable*. And the *inclination* of the *generatrix* of this *new developable surface*, to the *tangent* to the given *curve* at P, if suitably measured *in the tangent plane* to the *sphere*, is precisely the *angle* which has been above denoted by J.

(9.) To render this conception more completely clear, let us suppose that a *finite right line* PJ is set off from the given point P, *on the indefinite line* XXVI., so as to represent, by its *length* and *direction*, the *velocity* of the *rotation of the tangent plane* to the osculating sphere; and so to be, in the phraseology (396, (14.)) of the general theory of *emanants*, the *vector-axis* of that *rotation*. We shall then have the values,

$$\begin{aligned}\text{XXVII.}\ldots \mathrm{PJ} &= \phi\,(= \textit{the six expressions}\ \text{XVI.})\\ &= R^{-1}\tau(\cot J + \mathrm{U}(\sigma - \rho)) = R^{-1} \operatorname{cosec} J (\tau \cos J + \tau \mathrm{U}(\sigma - \rho) \sin J) \quad (7, 8);\end{aligned}$$

the angle J being determined by the formula XXIII., and a new expression, $\mathrm{T}\phi = R^{-1} \operatorname{cosec} J$, being thus obtained for the velocity XVII.

(10.) Hence the new angle J, if conceived to be included (like H) between the limits 0 and π, may be considered to be measured *from* τ *to* ϕ, or *from* the *unit-tangent* to the curve at P, *to the generating line* PJ of the *circumscribed developable* (8.), in the *direction* from τ to $\tau(\sigma - \rho)$: which *last* tangent to the osculating sphere makes generally, like the tangent ϕ or PJ itself, an *acute* angle with the positive binormal ν, as appears from the common *sign* of the scalar *coefficients* of that vector, in their developed expressions.

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* This usual expression, *consecutive*, is obviously borrowed here from the *language* of *infinitesimals*, but is supposed to be *interpreted*, like those used in other parts of the present series of Articles, by a reference to the conception of *limits*.

(11.) It may also be remarked, as an additional point of analogy, and as serving to verify some formulæ, that while the older angle H becomes *right*, when the given curve is *plane*, so the *new* angle $J = \frac{\pi}{2}$, for every *spherical curve*.

(12.) As *another geometrical illustration* of the properties of the angle J, and of some other results of recent sub-articles, which may serve to connect them, still more closely, with the general theory of *normal emanants* from curves (397, (44.)), let us conceive that AB, BC, CD are three successive right lines, perpendicular each to each; let us denote by a and b the angles BCA and CBD, and by c the inclination of the line AD to BC: and let us suppose that these two lines are intersected by their common perpendicular in the points G and H respectively.

(13.) Then, by completing the rectangle BCDE, and letting fall the perpendicular BF on the hypotenuse of the right-angled triangle ABE, we obtain the projections, AE and FB, of the two lines AD and GH, on the plane through B perpendicular to BC; and hence, by elementary reasonings, we can infer the relations:

$$\text{XXVIII.} \ldots \tan^2 c = \tan^2 \mathrm{ADE} = \tan^2 a + \tan^2 b;$$

and

$$\text{XXIX.} \ldots \frac{\mathrm{BH}}{\mathrm{BC}} = \frac{\mathrm{AG}}{\mathrm{AD}} = \frac{\mathrm{AF}}{\mathrm{AE}} = \frac{\mathrm{AB}^2}{\mathrm{AE}^2} = \sin^2 \mathrm{AEB},$$

or

$$\text{XXIX}'. \ldots \mathrm{BH} = \mathrm{BC} \sin^2 j, \quad \text{if} \quad \tan j = \tan a \cot b;$$

nothing here being supposed to be *small*. It may also be observed, that the *two rectilinear angles*, BCA and CBD, or a and b, represent respectively the inclinations of the *plane* ACD to the *plane* BCD, and of the plane ABD to the plane ABC.

(14.) Conceive next that PQ and $\mathrm{P_sQ_s}$ are two near normal emanants, touching the polar developable in the points Q and $\mathrm{Q_s}$, whereof Q is thus on the given polar axis KS, and $\mathrm{Q_s}$ is on the near polar axis $\mathrm{K_sQ_s}$; and let the second emanant be cut, in the points P′ and Q′, by planes through P and Q, perpendicular to the first emanant PQ. The line PP′ will then be very nearly tangential to the given curve at P; and the line QQ′ will be very nearly situated in the corresponding normal plane to that curve: so that these two new lines will be very nearly perpendicular to each other, and the gauche quadrilateral P′PQQ′ will ultimately have the properties of the recently considered quadrilateral ABCD.

(15.) This being perceived, if we denote by e the length of the emanant line PQ, the small angle a is very nearly $= e^{-1}s$; and if the small angle b be put under the form $b's$, then the new coefficient b' is ultimately equal (by XXIX′.) to $e^{-1} \cot j$: where j is an auxiliary angle, not generally small, and is such that we have ultimately $\mathrm{PH} = \mathrm{PQ} \,.\, \sin^2 j$, if H be the point in which the given normal emanant PQ approaches most closely to the consecutive emanant $\mathrm{P_sQ_s}$.

(16.) We have then the ultimate equation,

$$\begin{aligned} \text{XXX.} \ldots \cot j = eb' = \overline{\mathrm{PQ}} \times \lim. (s^{-1} \,.\, \mathrm{QPQ_s}) \\ &= \textit{length of emanant line } (\mathrm{PQ}) \\ &\times \textit{angular velocity of the tangential plane } (\mathrm{P'PQ}) \textit{ containing it}; \end{aligned}$$

this latter plane being here conceived as *turning*, for a moment, round the tangent to the given curve at P, and the velocity of *motion* along that curve being still taken for unity.

(17.) Accordingly, when we change e to r, b' to r^{-1}, and j to H, we recover in this way the fundamental value $\cot H = r\mathrm{r}^{-1}$ (397, XVII.), for the cotangent of the *older* angle H; and when, on the other hand, we treat the radius of *spherical* curvature as the normal emanant, supposing Q to coincide with S, and therefore changing e to R, and b' to $\mathrm{r}^{-1} + P'$, we recover the last of the expressions XXIII. for the cotangent of the *new* but analogous angle J, namely $\cot J = R\,(\mathrm{r}^{-1} + P')$, together with an *interpretation*, which may not have at first seemed obvious: although that expression *itself* was deducible, in the following among other ways, from equations previously established,

$$\text{XXXI.} \ldots R^{-1}\cot J - \mathrm{r}^{-1} = \frac{rR'}{pR} - \frac{r'}{p} = -\frac{R}{p}\left(\frac{r}{R}\right)' = -\frac{(\cos P)'}{\sin P} = P'.$$

(18.) As regards the *angular velocity*, say v, of the emanant *line* PQ, or the ultimate quotient of the *angle* between two such near lines, divided by the small *arc* s of the given curve, we see by XXVIII. (comp. (5.)) that this *small angle* vs is ultimately equal to the *square root of the sum of the squares* of the *two other small angles*, above denoted by a and b, and found to be equal, nearly, to $e^{-1}s$ and $e^{-1}s \cot j$ respectively: we may then establish the general formula,

$$\text{XXXII.} \ldots \textit{Angular Velocity of Normal Emanant} = v = e^{-1}\operatorname{cosec} j;$$

which reproduces the values, $r^{-1}\operatorname{cosec} H$, and $R^{-1}\operatorname{cosec} J$, already found for the angular velocities of the two radii, r and R.

(19.) And if we observe that the *projection of the vector of curvature*, KP^{-1}, *on the emanant* PQ, is easily proved to be $= \mathrm{QP}^{-1} = e^{-2}.\mathrm{PQ}$, we see by XXXII. that if this *projection* be *divided by the square of the angular velocity* (v) of the line PQ, the *quotient* is the line $\mathrm{PQ}\,.\,\sin^2 j$, or PH (15.): which reproduces the general result, 397, CV., for all *systems of normal emanants*, together with a geometrical *interpretation*.

(20.) As still *another geometrical illustration* of the properties of the new angle J, we may observe that in the construction (12.) and (13.) the corresponding auxiliary angle j was equal to AEB, or to ABF, and that the line BF (= HG) was *perpendicular* to *both* BC and AD, although not *intersecting* the latter. Substituting then, as in (14.), the quadrilateral P′PQQ′ for ABCD, and passing to the limit, we may say that if a new line PJ be a *common perpendicular*, at the given point P, to *two consecutive** *normal emanants*, PQ and P′Q′, the *general auxiliary angle* j is simply the *inclination* P′PJ, of that *common perpendicular* PJ, to the *tangent* PP′ to the *curve*.

(21.) And if, instead of *normally emanating lines* PQ, we consider a system of *tangential emanant planes* (as in 397, (45.)), to which those *lines* are *perpendicular*, we may then (comp. 396, (14.)) consider the recent line PJ as being a *generating line* of the *developable surface*, which is the *envelope* of all the *planes* of the system; the *auxiliary angle*,† j, is therefore *generally* by (20.) the *inclination* of this gene-

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* Compare the Note to page 581.

† In these *geometrical illustrations*, the angle j has been treated, for simplicity, as being both *positive* and *acute*; although the *general formulæ*, which involve the corresponding angles H and J, permit and require that we should occasionally attribute to them *obtuse* (but still positive) values: while those angles may also become *right*, in some particular cases (comp. (11.)).

ratrix to the *tangent:* a result which agrees with, and includes, the known and fundamental property (397, XVII.) of the angle H, in connexion with the *Rectifying Developable* (396); and also the *analogous* property of the newer angle J, connected (8.) with what it has been above proposed to call the *Circumscribed Developable.*

(22.) We shall soon return briefly on the theory of that *new developable surface* (8.), and of the *new locus* (of the osculating *circle*, or *envelope* of the osculating *sphere*) to which it has been said to be *circumscribed:* but may here observe, that if we write for abridgment (comp. VIII. and XXIII.),

$$\text{XXXIII.} \ldots n = \frac{\sigma'}{r\nu} = \frac{RR'}{p} = p' + \cot H = \cot J \sec P,$$

then what has been called the *coefficient of non-sphericity* (comp. 395, (14.) and (16.)) is easily seen to have by XIV. the values,

$$\text{XXXIV.} \ldots S - 1 = \frac{\mathrm{S}\tau^3\tau''\tau'''}{\mathrm{S}\tau\tau'^3\tau''} - 1 = -r^4\mathrm{r}\mathrm{S}\nu'\tau''' - 1 \qquad (1, 2)$$

$$= \frac{r}{\mathrm{r}}(p' - r\mathrm{r}\lambda^2) - 1 = \frac{r}{\mathrm{r}}\left(p' + \frac{r}{\mathrm{r}}\right) = nr\mathrm{r}^{-1} \qquad (3, 4, 5)$$

$$= \frac{\sigma'}{\mathrm{r}\nu} = \cot H \cot J \sec P = \frac{rRR'}{p\mathrm{r}} \qquad (6, 7, 8);$$

whence also the *deviation* of a near point P_s of the curve, from the osculating sphere at P, is ultimately (by 395, XXVII.).

$$\text{XXXV.} \ldots \overline{\mathrm{SP}_s} - \overline{\mathrm{SP}} = \frac{(S-1)s^4}{24r^2R} = \frac{ns^4}{24r\mathrm{r}R} = \frac{R's^4}{24r\mathrm{r}p};$$

and accordingly, the square of the vector $\rho_s - \sigma$ is given now (comp. I.) by the expression,

$$(\rho_s - \sigma)^2 = (\rho - \sigma)^2 - \frac{s^4}{12r^2}\{r^2\mathrm{S}(\sigma - \rho)\tau''' - 1\},$$

in which $\qquad r^2\mathrm{S}(\sigma - \rho)\tau''' = S = 1 + nr\mathrm{r}^{-1} = \&\text{c.}$, as above.

(23.) The same auxiliary scalar n enters into the following expressions for the *arc*, and for the *scalar radii* of the *first* and *second curvatures*, of the *locus of the centre* S of the osculating *sphere*, or of the *cusp-edge* of the *polar developable* (comp. 391, (6.), and 395, (2.)):

$$\text{XXXVI.} \ldots \pm\int n\,ds = \textit{Arc of that Cusp-Edge} \text{ (or of } \textit{locus of } \mathrm{S});$$

$$\text{XXXVI}'. \ldots r_1 = n\mathrm{r} = r + p'\mathrm{r} = \frac{RR'}{r'} = (\textit{Scalar}) \textit{ Radius of Curvature of same edge};$$

$$\text{XXXVI}''. \ldots \mathrm{r}_1 = nr = \sigma'\nu^{-1} = (\textit{Scalar}) \textit{ Radius of Second Curvature of same curve};$$

these two latter being here called *scalar* radii, because the *first* as well as the *second* (comp. 397, V.) is conceived to have an algebraic *sign*. In fact, if we denote by K_1 the *centre of the osculating circle* to the *cusp-edge* in question, its *vector* is (by the general formula 389, IV.),

$$\text{XXXVII.} \ldots \mathrm{OK}_1 = \kappa_1 = \sigma + \frac{\sigma'^3}{\mathrm{V}\sigma''\sigma'} = \sigma - nr\mathrm{r}\tau' = \rho - p'\mathrm{r}r\tau' + pr\nu = \sigma - r_1r\tau',$$

with the signification XXXVI'. of r_1; because by XXXIII. (comp. 397, XI'.),

$$\text{XXXVIII.} \ldots \sigma' = nr\nu, \quad \sigma'' = n'r\nu + n(r\nu)' = n'r\nu - nr\mathrm{r}^{-1}\tau',$$

and therefore

$$\text{XXXIX.} \ldots \sigma'^2 = -n^2, \quad \mathrm{V}\sigma'\sigma'' = n^3\mathrm{r}^{-1}\tau.$$

We may also observe that the relation $\sigma' \parallel \nu$ gives (by 397, IV.),

$$\text{XL.} \ldots \mathrm{V}\frac{\sigma''}{\sigma'} = \mathrm{V}\frac{\nu'}{\nu} = \mathrm{r}^{-1}\tau = \textit{Vector of Second Curvature of given curve};$$

and that we have the equation,

$$\text{XLI.} \ldots \frac{\mathrm{K}_1\mathrm{S}}{\mathrm{PK}} = \frac{\sigma - \kappa_1}{\kappa - \rho} = \frac{r_1}{r}, \quad \text{with} \quad r > 0, \quad \text{but} \quad r_1 > \text{ or } < 0,$$

according as the *cusp-edge* turns its *concavity* or its *convexity* towards the *given curve* at P.

(24.) The *radius* of (*first*) *curvature* of that cusp-edge, when regarded as a *positive* quantity, is therefore represented by the *tensor*,

$$\text{XLII.} \ldots \sqrt{r_1^2} = \pm r_1 = \mathrm{T}r_1 = R\mathrm{T}\frac{R'}{r'} = \pm \frac{R\mathrm{d}R}{\mathrm{d}r} \; (> 0);$$

and as regards the *scalar radius* XXXVI″. *of second curvature* of the same cusp-edge, its expression follows by XXXVIII. from the general formula 397, XXVII., which gives here,

$$\text{XLIII.} \ldots \mathrm{r}_1^{-1} = \mathrm{S}\frac{\sigma'''}{\mathrm{V}\sigma'\sigma''} = \frac{1}{nr}\,\mathrm{S}\frac{\nu''}{\mathrm{V}\nu\nu'} = n^{-1}r^{-1}, \text{ because } \text{XLIII}'. \ldots \mathrm{S}\frac{\nu''}{\mathrm{V}\nu\nu'} = 1;$$

the *two scalar derivatives*, n' and n'', which would have introduced the derived vectors τ^{IV} and τ^{V}, or $\mathrm{D}_s{}^5\rho$ and $\mathrm{D}_s{}^6\rho$, of the *fifth* and *sixth orders*, thus *disappearing from the expressions of the two curvatures* of the *locus of the centre* S of the *osculating sphere*, as was to be expected from geometrical* considerations.

(25.) For the *helix*, the formula XXXVII. gives $\kappa_1 = \rho$, or $\mathrm{K}_1 = \mathrm{P}$; we have then thus, as a verification, the known result, that the *given point* P of *this* curve is *itself* the *centre of curvature* K_1 of that *other helix* (comp. 389, (3.), and 395, (8.)), which is in this case the *common locus* of the *two coincident centres*, K and S. It is scarcely necessary to observe that for the helix we have also $J = H$.

(26.) In general, the *rectifying plane* of the *locus of* S is *parallel* to the rectifying plane of the *given curve*, because the *radii* of their *osculating circles* are parallel; the *rectifying lines* for these two curves are therefore not only *parallel* but *equal*; and accordingly we have here the formula,

$$\text{XLIV.} \ldots \lambda_1 = \mathrm{V}\frac{\tau_1''}{\tau_1'} = \mathrm{V}\frac{\tau''}{\tau'} = \lambda \text{ (by 397, XVI.)},$$

which will be found to agree with this other expression (comp. 397, XVII.),

$$\text{XLV.} \ldots \tan H_1 = \frac{\mathrm{r}_1}{\mathrm{T}r_1} = \frac{r}{\mathrm{r}}\,\mathrm{U}r_1 = \pm \cot H,$$

the upper or lower sign being taken, according as the *new curve* is *concave* (as in Figs. 81, 82), or is *convex* at S (comp. (23.)), towards the *old* (or given) curve at P: and the *new angle* H_1 being measured in the *new rectifying plane*, *from* the *new*

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[*] In fact, n represents here the velocity of motion of the point S along its own locus, while r^{-1} and r^{-1} represent respectively the velocities of rotation of the tangent and binormal to that curve: so that nr and $n\mathrm{r}$ must be, as above, the radii of its two curvatures.

tangent σ' or $n\tau\nu$, *to* the *new rectifying line* λ_1, and in the *direction* from that new tangent to the *new binormal* ν_1, or (comp. XL.) to a line from S which is equal to the *vector of second curvature* $\mathrm{r}^{-1}\tau$ of the *given curve*, multiplied by a *positive scalar*, namely by $\mathrm{T}n^{-1}$, or by the coefficient n^{-1} taken positively.

(27.) The *former rectifying line* λ *touches* the *cusp-edge* of the *rectifying developable* (396) of the given curve, in a *new point* R (comp. Fig. 81), of which by 397, (45.), and by XV., the vector *from the given point* is, generally,

$$\text{XLVI.}\ldots \mathrm{PR} = -\frac{\mathrm{V}\tau'^3\tau''}{\mathrm{S}\tau'\tau''\tau'''} = \frac{r^{-2}\lambda}{\mathrm{S}\lambda\tau'''} = -\frac{r\lambda}{(r\mathrm{r}^{-1})'} = \frac{\mathrm{U}\lambda\sin H}{H'};$$

with the verification that this expression becomes infinite (comp. 397, (49.), (50.)), when the *curve* is a *geodetic on a cylinder*.

(28.) In general, the vector OR of the *point of contact* R, which vector we shall here denote by υ, may be thus expressed,

$$\text{XLVII.}\ldots \upsilon = \mathrm{OR} = \rho + l\mathrm{U}\lambda, \quad \text{if} \quad \text{XLVIII.}\ldots l = \frac{\sin H}{H'} = \frac{-r\mathrm{T}\lambda}{(r\mathrm{r}^{-1})'};$$

and because $(r\lambda)' = (r\mathrm{r}^{-1})'\tau$, by VII'., its first derivative is,

$$\text{XLIX.}\ldots \upsilon' = r\lambda\left(\frac{\upsilon-\rho}{r\lambda}\right)' = \mathrm{U}\lambda\,\mathrm{cosec}\,H\,(l\sin H)' = \mathrm{U}\lambda\,(l' + \cos H);$$

in which however the new derived scalar l' involves H'', and so depends on τ^{iv}: while the *scalar coefficient* l itself represents the *portion* ($\pm\overline{\mathrm{PR}}$) of *the rectifying line*, intercepted *between* the *given curve*, and the *cusp-edge* (27.) of the rectifying developable, and considered as *positive* when the *direction* of this *intercept* PR coincides with that of the line $+\lambda$, but as negative in the contrary case.

(29.) For abridgment of discourse, the cusp-edge *last* considered, namely that of the *rectifying developable*, as being the locus of a point which we have denoted by the letter R, may be called simply "*the curve* (R);" while the *former* cusp-edge (23.), or that of the *polar* developable, may be called in like manner "*the curve* (S);" the *locus* of the *centre* K of (absolute) *curvature* may be called "*the curve* (K):" and the *given curve* itself (comp. again Figs. 81, 82) may be called, on the same plan, "*the curve* (P)."

(30.) The *arc* RR_s, of the *curve* (R), is (by XLIX., comp. XXXVI.),

$$\text{L.}\ldots \pm\int_0^s \mathrm{T}\upsilon'ds = l_s - l + \int_0^s \cos H ds;$$

this arc being treated as *positive*, when the direction of *motion* along it coincides with that of $+\lambda$.

(31.) The expression VII. for λ', combined with the former expression 397, XVI. for λ, gives easily by the general formula 389, IV.,

LI. . . *Vector of Centre of Curvature of the Curve* (R)

$$= \upsilon + \frac{\upsilon'}{\mathrm{V}\upsilon''\upsilon'^{-1}} = \upsilon + \frac{\upsilon'}{\mathrm{V}\lambda'\lambda^{-1}} = \upsilon + \frac{\upsilon'}{H'}\mathrm{U}\tau';$$

whence LII. . . *Radius of Curvature of Curve* (R) $= \mathrm{T}\dfrac{\upsilon'}{H'} = \mathrm{T}\dfrac{d\upsilon}{dH}$,

the scalar variable being here arbitrary.

(32.) We see, at the same time, that the *angular velocity of the rectifying line* λ, or of the *tangent* to this *curve* (R), is represented by $\pm H'$; or that the *small* *angle*[*] between *two such near lines*, λ and λ_s, is nearly equal to sH', or to $H_s - H$: while the *vector axis* ($\mathrm{V}\lambda'\lambda^{-1}$) *of rotation of the rectifying line*, set off from the point R, has $-H'\mathrm{U}\tau'$, or $-H'r\tau'$, for its expression.

(33.) As regards the *second curvature* of the same curve (R), we may observe that the expression (comp. VII. and LI.),

$$\text{LIII.}\ldots\ \lambda'' = (\mathrm{r}^{-1})''\tau + (r^{-1})''r\nu + r^{-1}(r\mathrm{r}^{-1})'\tau' = (\mathrm{r}^{-1})''\tau + (r^{-1})''r\nu + \mathrm{V}\lambda\lambda',$$

combined with the parallelism (XLIX.) of υ' to λ, gives, by the general formula 397, XXVII.,

LIV. . . *Radius of Second Curvature of Curve* (R)

$$= \left(\mathrm{S}\,\frac{\upsilon'''}{\mathrm{V}\upsilon'\upsilon''}\right)^{-1} = \frac{\upsilon'}{\lambda}\left(\mathrm{S}\,\frac{\lambda''}{\mathrm{V}\lambda\lambda'}\right)^{-1} = \frac{\upsilon'}{\lambda} = \frac{l' + \cos H}{\mathrm{T}\lambda};$$

with the verification, that while $l' + \cos H$ represents, by (30.), the velocity of *motion along this curve* (R), $\mathrm{T}\lambda$ represents, by 397, (3.), the velocity of *rotation of its osculating plane*, namely the *rectifying plane* of the *given curve* (P): and it is worth observing, that although *each* of these *two radii* of curvature, LII. and LIV., depends on τ^{IV} through l' (28.), yet *neither* of them depends on τ^{V} (comp. (24.)). As another verification, it can be shown that the *plane of the two lines* λ and τ' from P, namely the plane,

$$\text{LIV}'.\ldots\ \mathrm{S}\tau'\lambda(\omega - \rho) = 0,$$

which is the *normal plane to the rectifying developable* along the rectifying line, and contains the absolute normal to the given curve (P), *touches its own developable envelope along the line* RH, if H be the point determined by the formula 397, XXXVIII., or the *point of nearest approach* of a *radius of curvature* (r) of that *given* curve *to its consecutive* (comp. (6.); this line RH must therefore be the *rectifying line of the curve* (R): and accordingly (comp. 397, XVII.), the trigonometric tangent of its *inclination* to the *tangent* RP to this last curve has for expression (abstracting from sign),

$$\text{LIV}''.\ldots\ \tan \mathrm{PRH} = \overline{\mathrm{PH}} : \overline{\mathrm{PR}} = \pm\, l^{-1}r \sin^2 H = \pm\, rH' \sin H = \mathrm{T}\lambda^{-1}H'$$

$$= \frac{\textit{Radius}\ (\text{LIV.})\ \textit{of Second Curvature of Curve}\ (\mathrm{R})}{\textit{Radius}\ (\text{LII.})\ \textit{of First Curvature of same Curve}}.$$

(34.) Without even introducing τ^{IV}, we can assign as follows a *twisted cubic* (comp. 397, (34.)), which shall have *contact of the fourth order* with the given curve at P; or rather an *indefinite variety* of *such cubics*, or *gauche curves* of the *third degree*. Writing, for abridgment,

$$\text{LV.}\ldots\ x = -\mathrm{S}\tau(\omega - \rho), \quad y = -\mathrm{S}r\tau'(\omega - \rho), \quad z = -\mathrm{S}r\nu(\omega - \rho),$$

so that

$$\text{LVI.}\ldots\ \omega = \rho + x\tau + yr\tau' + zr\nu,$$

the scalar equation,

$$\text{LVII.}\ldots\ \left(\frac{2\mathrm{r}y}{r}\right)^2 = 6\left(\frac{\mathrm{r}}{r}\right)^3 xz + \left(\frac{\mathrm{r}^3}{r^2}\right)' yz + ez^2,$$

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[*] A result substantially equivalent to this is deduced, by an entirely different analysis, in the above cited Memoir of M. de Saint-Venant, and is illustrated by geometrical considerations: which also lead to expressions for the two curvatures (or, as he calls them, the *courbure* and *cambrure*), of the cusp-edge of the rectifying developable; and to a determination of the *rectifying line* of that *cusp-edge*.

in which e is an arbitrary but scalar constant, represents evidently, by its *form*, a *cone of the second order*, with its *vertex* at the given point P; and *this cone* can be proved to have *contact of the fourth order* with the *curve** at that point: or of the *third order* with the *cone of chords* from it (comp. 397, (31.), (32.)). In fact the coefficients will be found to have been so determined, that the difference of the two members of this equation LVII. contains s^6 as a factor, when we change ω to ρ_s, as given by the formula I., or when we substitute for xyz their approximate values for the curve, as functions of the arc s; namely, by the expressions IV. for τ''', and 397, VI. for τ'',

$$\text{LVIII.}\ldots\begin{cases} x_s = s - \dfrac{s^3}{6r^2} + \dfrac{r's^4}{8r^3}, \\ y_s = \dfrac{s^2}{2r} - \dfrac{r's^3}{6r^2} - \dfrac{s^4}{24}\left((r^{-2}r')' + r^{-3} + r^{-1}\mathrm{r}^{-2}\right), \\ z_s = \dfrac{s^3}{6r\mathrm{r}} + \dfrac{rs^4}{24}(r^{-2}\mathrm{r}^{-1})'; \end{cases}$$

where the terms set down are more than sufficient for the purpose of the proof. It may be added that the coefficient of $\frac{-s^4}{24}$ in y_s, which is the only one at all complex here, may be transformed as follows:

$$\text{LVIII}'.\ldots S r\tau'\tau''' = -(r^{-1})'' - r^{-1}\lambda^2 = r^{-3}S + p(r^{-2}\mathrm{r}^{-1})';$$

S being that scalar for which (or more immediately for its excess over unity) several expressions† have lately been assigned (22.), and which had occurred in an earlier investigation (395, (14.), &c.).

(35.) With the same significations LV. of the three scalars xyz, this other equation,

$$\text{LIX.}\ldots 18ry - (3x - r'y)^2 = (9 + r'^2 - 3rr'' - 3r^2\mathrm{r}^{-2})y^2,$$

or

$$\text{LIX}'.\ldots 2ry - (x - \tfrac{1}{3}r'y)^2 = (1 - \tfrac{1}{2}r^{\frac{4}{3}}(r^{\frac{2}{3}})'' - \tfrac{1}{3}r^2\mathrm{r}^{-2})y^2,$$

will be found to be satisfied when we substitute for x and y the values LVIII. of x_s and y_s, and neglect or suppress s^5; it therefore represents an *elliptic* (or *hyperbolic*) *cylinder*, which is *cut perpendicularly*, by the *osculating plane* to the given curve at P, in an *ellipse* (or *hyperbola*), having *contact* of the *fourth order* with the *projection* (comp. 397, (9.)), of that given *curve* upon that osculating *plane*: and the *cylinder itself* has *contact* of the *same* (*fourth*) *order* with the *curve in space*, at the

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* In the language of infinitesimals, the cone LVII. contains *five consecutive points* of the *curve*, or has *five-point contact* therewith: but it contains only *four consecutive sides* of the *cone of chords* from the given point, or has only *four-side* contact with *that cone*, except for one particular *value* of the *constant*, e, which we shall presently assign. It may be observed that xyz form *here* a (scalar) system of *three rectangular co-ordinates*, of the usual kind, with their *origin* at the point P of the curve, and with their *positive semiaxes* in the directions of the *tangent* τ, the *vector of curvature* τ', and the *binormal* ν.

† It might have been observed, in addition to the eight forms XXXIV., that we have also,

$$\text{XXXIV}'.\ldots S - 1 = R\mathrm{r}^{-1}\cot J = n\cot H \qquad (9,\ 10).$$

same given point P, so that we may call it (comp. 397, (31.)) the *Osculating Elliptic* (*or Hyperbolic*) *Cylinder, perpendicular to the osculating plane.*

(36.) As a verification, if we suppress the second member of either LIX. or LIX'., we obtain, under a new form, the equation of what has been already called the *Osculating Parabolic Cylinder* (397, LXXXIV.); and as another verification, the *coefficient of y^2* in that second member *vanishes*, as it ought to do, when the given curve is supposed to be a *parabola : that plane curve*, in fact, satisfying the *differential equation of the second order*,

$$\text{LX.}\ldots\ 3rr'' - r'^2 = 9, \quad \text{or} \quad \text{LX}'.\ldots\ r^{\frac{4}{3}}(r^{\frac{2}{3}})'' = 2,$$

or

$$\text{LX}''.\ldots\ r^{-\frac{4}{3}}\left(\left(\frac{\mathrm{d}r}{3\mathrm{d}s}\right)^2 + 1\right) = \text{const.} = p^{-\frac{2}{3}},$$

if r be still the *radius of curvature*, considered as a function of the *arc*, s, while p is here the *semiparameter*.

(37.) The *binormal* ν is, by the construction, a *generating line* of the cylinder LIX.; and although this line is *not generally* a *side* of the *cone* LVII., yet we can *make* it such, by assigning the particular value *zero* to the *arbitrary constant*, e, in its equation, or by suppressing the *term*, ez^2. And when this is done, the cone LVII. will *intersect* the *cylinder* LIX., not only in this *common side* ν (comp. 397, (33.)), but also in a certain *twisted cubic*, which will have *contact* of the *fourth order* with the *given curve* at P, as stated at the commencement of (34.).

(38.) But, as was also stated there, *indefinitely many* such *cubics* can be described, which shall have contact of the *same* (*fourth*) *order*, with the *same curve*, at the *same point*. For we may *assume any point* E of space, or *any vector* (comp. LVI.),

$$\text{LXI.}\ldots\ \text{OE} = \epsilon = \rho + a\tau + br\tau' + cr\nu,$$

in which a, b, c are *any three scalar constants;* and then the vector equation,

$$\text{LXII.}\ldots\ \omega = \rho_s + t(\epsilon - \rho),$$

in which t is a *new scalar variable*, will represent a *cylindric surface*, not *generally* of the *second order*, but passing *through the given curve*, and having the *line* PE for a *generatrix*. We can then *cut* (generally) this new cylinder by the *osculating plane* to the curve at P, and so obtain (generally) a *new* and *oblique projection* of the *curve* upon that *plane;* the x and y of which *new projected curve* will depend on the arc s of the original curve by the relations,

$$\text{LXIII.}\ldots\ x = x_s - ac^{-1}z_s, \quad y = y_s - bc^{-1}z_s;$$

with the approximate expressions LVIII. for $x_s y_s z_s$. And if we then determine *two new scalar constants*, B and C, by the condition that the substitution of these last expressions LXIII. for x and y shall satisfy this new equation,

$$\text{LXIV.}\ldots\ 2ry = x^2 + 2Bxy + Cy^2,$$

if only s^5 be neglected (comp. (35.)), or by *equating the coefficients of s^3 and s^4*, in the result of such substitution, then, on *restoring* the significations LV. of xyz, and writing for abridgment,

$$\text{LXV.}\ldots\ X = x - ac^{-1}z, \quad Y = y - bc^{-1}z,$$

the *equation of the second degree*,

$$\text{LXVI.} \ldots 2rY = X^2 + 2BXY + CY^2,$$

will represent generally an *oblique osculating elliptic* (or hyperbolic) *cylinder*, which has contact of the *fourth order* with the given curve at P, and contains the assumed line PE. If then we determine finally the constant e in LVII., by the result of the substitution of abc for xyz, or by the condition,

$$\text{LXVII.} \ldots \left(\frac{2\mathrm{r}b}{r}\right)^2 = 6\left(\frac{r}{\mathrm{r}}\right)^3 ac + \left(\frac{\mathrm{r}^3}{r^3}\right)' bc + ec^2,$$

the *cone* LVII., and the *cylinder* LXVI., will have that line PE for a *common side*; and will *intersect* each other, not only in *that line*, but *also* (as before) in a *twisted cubic*, although now a *new* one, which will have the *required* (*fourth*) *order of contact*, with the given curve at the given point.

(39.) If, after the substitution (38.) in LXIV., we equate the coefficients of the *three powers*, s^3, s^4, s^5, and then eliminate B and C, we are conducted to an *equation of condition*, which is found to be of the *form*,

$$\text{LXVIII.} \ldots \mathrm{a}b^3 + \mathrm{b}b^2c + \mathrm{c}bc^2 + \mathrm{e}c^3 = ac(b\mathrm{g} + c\mathrm{h});$$

in which the ratios of abc still serve to determine the direction of the generating line PE, while the coefficients a, b, c, e, g, h are *assignable functions* of r, r, r', r', r'', r'', and r''', depending on the vector τ^{IV}: and when this *condition* LXVIII. is satisfied, the *cylinder* LXVI. has *contact* of the *fifth order* with the given *curve* at P.

(40.) Again, if we *improve* the approximate expressions LVIII. for the three scalars x_s, y_s, z_s, by taking account of s^5, or by introducing the *new term* $\frac{s^5\tau^{\mathrm{IV}}}{120}$ (comp. I.) of ρ_s, and if we substitute the expressions so improved, instead of x, y, z, in the equation of the *cone* LVII. and then equate to zero (comp. (34.)) the coefficient of s^6 in the difference of the two members of that equation, we obtain a *definite expression* for the *constant*, e, which had been *arbitrary* before, but becomes now a *given function* of $r\mathrm{r}r'\mathrm{r}'r''$ and r'' (*not involving* r'''), namely the following

$$\text{LXIX.} \ldots e = \frac{\mathrm{r}^4}{5}\left(\frac{9}{r^4} - \frac{21}{r^2\mathrm{r}^2} + \frac{r'^2}{r^4} - \frac{3r''}{r^3} + \frac{3r'\mathrm{r}'}{r^3\mathrm{r}} - \frac{27\mathrm{r}'^2}{4r^2\mathrm{r}^2} + \frac{9\mathrm{r}''}{r^2\mathrm{r}}\right);$$

and when the constant e receives *this value*,* the *cone* has contact of the *fifth order* with the *curve* at the given point.

(41.) Finally, if we multiply the equation LXVII. by $b\mathrm{g} + c\mathrm{h}$, we can at once eliminate a by LXVIII., and so obtain a *cubic equation* in $b : c$, which has *at least one real root*, answering to a *real system of ratios* a, b, c, and therefore to a *real direction* of the *line* PE in (38.). It is therefore possible to assign *at least one real cylinder* of the second order (39.), which shall have *contact* of the *fifth order* with the curve at P, and shall at the same time have *one side* PE *common* with the *cone* of the second order (40.), which has contact of the *same* (*fifth*) *order* with the *curve* (or of the *fourth* order with the *cone of chords*): and consequently it is possible in this way to assign, as the *intersection* of this cylinder with this cone, at least *one real*

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* Compare the first Note to page 588.

twisted cubic, which has *contact of the fifth*[*] *order with the given curve of double curvature*, at the given point thereof. And *such* a cubic curve may be called, by eminence, an *Osculating*[†] *Twisted Cubic.*

(42.) Not intending to return, in these *Elements*, on the subject of such *cubic curves*, we may take this occasion to remark, that the very simple *vector equation*,[‡]

$$\text{LXX.}\ldots\ \mathrm{V}\alpha\rho = \rho\mathrm{V}\beta\rho,$$

represents a *curve of this kind*, if α and β be any two constant and non-parallel vectors. In fact, if we operate on this equation by the symbol $\mathrm{S}.\lambda$, in which λ is an *arbitrary* but constant vector, the *scalar equation* so obtained, namely,

$$\text{LXXI.}\ldots\ \mathrm{S}\lambda\alpha\rho = \mathrm{S}\lambda\rho\mathrm{S}\beta\rho - \rho^2\mathrm{S}\beta\lambda,$$

represents a *surface of the second order*, on which the *curve* is wholly contained; making then successively $\lambda = \alpha$ and $\lambda = \beta$, we get, in particular, the *two* equations,

$$\text{LXXII.}\ldots\ \mathrm{S}(\mathrm{V}\alpha\rho.\mathrm{V}\beta\rho) = 0, \quad \text{and} \quad \text{LXXIII.}\ldots\ (\mathrm{V}\beta\rho)^2 + \mathrm{S}\alpha\beta\rho = 0,$$

representing respectively a *cone* and *cylinder* of that order, with the vector β from the origin as a *common side*: and the *remaining part* of the *intersection* of these two surfaces, is precisely the *curve* LXX., which therefore is a *twisted cubic*, in the known sense already referred to.

(43.) *Other surfaces* of the same order, containing the same curve, would be obtained by assigning other values to λ; for example (comp. 397, (31.)), we should get generally an *hyperbolic paraboloid* from the form LXXI., by taking $\lambda \perp \beta$. But it may be more important here to observe, that *without* supposing *any acquaintance* with the theory of *curved surfaces*, the *vector equation* LXX. can be shown, *by*

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[*] Accordingly, it is known (see page 242 of Dr. Salmon's Treatise, already cited), that a *twisted cubic* can generally be described through any *six given points*; and also (page 248), that *three quadric cylinders* (or cylinders of the second order or degree) can be described, containing a *given cubic curve*, their *edges* being parallel to the *three* (real or imaginary) *asymptotes*.

[†] Compare the first Note to page 563.

[‡] This example was given in pages 679, &c., of the *Lectures*, with some connected transformations, the equation having been found as a certain *condition* for the *inscription of a gauche quadrilateral*, or other *even-sided polygon*, in a *given spheric surface* (comp. the sub-articles to 296): the $2n$ successive *sides* of the figure being obliged to pass through the same *even number* of *given points of space*. It was shown that the *curve* might be said to *intersect* the *unit-sphere* ($\rho^2 = -1$) in *two imaginary points at infinity*, and *also* in *two real* and *two imaginary points*, situated on *two real right lines*, which were *reciprocal polars* relatively to the sphere, and might be called *chords of solution*, with respect to the proposed *problem of inscription* of the polygon; and that *analogous results* existed for *even-sided polygons in ellipsoids*, and *other surfaces* of the *second order*: whereas the corresponding problem, of the inscription of an *odd-sided polygon* in such a *surface*, conducted only to the assignment of a *single chord of solution*, as happens in the known and analogous theory of *polygons in conics*, whether the number of sides be (in *that* theory) *even* or *odd*. But we cannot here pursue the subject, which has been treated at some length in the *Lectures*, and in the *Appendices* to them.

quaternions, to represent a *curve of the third degree*, in the sense that it is *cut*, by an *arbitrary plane*, in *three points* (real or imaginary). In fact, we may write the equation as follows,

$$\text{LXXIV.} \ldots \mathrm{V}q\rho = -a, \quad \text{if} \quad \text{LXXV.} \ldots q = g + \beta,$$

q being here a *quaternion*, of which the *vector part* β is *given*, but the *scalar part* g is *arbitrary;* and then, by *resolving* (comp. 347) this *linear equation* LXXIV., we may still further transform it as follows,

$$\text{LXXVI.} \ldots g(g^2 - \beta^2)\rho = \beta \mathrm{S}\beta a + g\mathrm{V}\beta a - g^2 a,$$

which conducts to a *cubic equation* in g, when combined with the equation,

$$\text{LXXVII.} \ldots \mathrm{S}\epsilon\rho = e,$$

of any proposed *secant plane.*

(44.) The vector equation LXX., however, is *not sufficiently general*, to represent an *arbitrary twisted cubic*, through an *assumed point* taken as *origin ;* for which purpose, *ten scalar constants* ought to be disposable, in order to allow of the curve being made to pass through *five*[*] *other arbitrary points :* whereas the equation referred to involves *only five* such *constants*, namely the *four* included in $\mathrm{U}a$ and $\mathrm{U}\beta$, and the *one* quotient of tensors $\mathrm{T}\beta : \mathrm{T}a$ (comp. 358).

(45.) It is easy, however, to accomplish the *generalization* thus required, with the help of that theory of *linear and vector functions* ($\phi\rho$) *of vectors*, which was assigned in the Sixth Section of the preceding Chapter (Arts. 347, &c.). We have only to write, instead of the equation LXX., this other but analogous *form* which *includes* it,

$$\text{LXXVIII.} \ldots \mathrm{V}a\rho + \mathrm{V}\rho\phi\rho = 0, \quad \text{or} \quad \text{LXXVIII}'. \ldots \phi\rho + c\rho = a,$$

and which gives, by principles and methods already explained (comp. 354, (1.)), the transformation,

$$\text{LXXIX.} \ldots \rho = (\phi + c)^{-1} a = \frac{\psi a + c\chi a + c^2 a}{m + m'c + m''c^2 + c^3};$$

a, ψa, and χa being here *fixed vectors*, and m, m', m'' being *fixed scalars*, but c being an *arbitrary* and *variable scalar*, which may receive *any value*, without the expression LXXIX. ceasing to satisfy the equation LXXVIII.

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[*] Compare the first Note to page 591. In general, when a *curve in space* is supposed to be represented (comp. 371, (5.)) by *two scalar equations*, each *new arbitrary point*, through which it is required to *pass*, introduces a necessity for *two new disposable constants*, of the *scalar* kind : and accordingly *each new order*, say the nth, of *contact* with such a *curve*, has been seen to introduce a *new vector*, $\mathrm{D}_s{}^n\rho$, or $\tau^{(n-1)}$, subject to a *condition* resulting from the general equation $\mathrm{TD}_s\rho = 1$, or $\tau^2 = -1$ (comp. 380, XXVI., and 396, III.), but involving virtually *two new scalar constants.* Thus, besides the *four* such constants, which enter through τ and τ' into the determination of the *directions* of the *rectangular system* of lines, *tangent*, *normal*, and *binormal* (comp. 379, (5.), or 396, (2.)), and of the *length* of the *radius* of (*first*) *curvature*, r, the *three successive derivatives*, r', r'', r''', of *that* radius, and the radius r of *second curvature*, with its *two* first derivatives, r' and r'', have been seen to enter, through the *three other vectors*, τ'', τ''', τ^{IV}, into the determination (41.) of the *osculating twisted cubic.*

(46.) The *curve* LXXVIII. is therefore *cut* (comp. (43.)) by the *plane* LXXVII. in *three points* (real or imaginary), answering to and determined by the *three roots* of the *cubic in* c, which is formed by substituting the expression LXXIX. for ρ in the equation of that secant plane; and consequently it is a curve of the *third degree*, the *three* (real or imaginary) *asymptotes* to which have directions corresponding to the *three values* of c, obtained by equating to zero the *denominator* of that expression LXXIX., or by making $M = 0$, in a notation formerly employed: so that they have the directions of the *three lines* β, which satisfy this *other* vector equation (comp. 354, I.),

$$\text{LXXX.} \ldots \mathrm{V}\beta\phi\beta = 0.$$

(47.) Accordingly, if β be *such* a line, and if γ be *any vector* in the *plane* of a and β, the curve LXXVIII. is a *part* of the *intersection* of the *two surfaces* of the *second order*,

$$\text{LXXXI.} \ldots \mathrm{S}a\rho\phi\rho = 0, \quad \text{and} \quad \text{LXXXII.} \ldots \mathrm{S}\gamma a\rho + \mathrm{S}\gamma\rho\phi\rho = 0,$$

whereof the first is a cone, and which have the *line* β from the origin for a *common side* (comp. (42.)): the *curve* is therefore found *anew* to be a *twisted cubic.*

(48.) And as regards the *number* of the *scalar constants*, which are to be conceived as entering into its *vector equation* LXXVIII., when we take for $\phi\rho$ the *form* $\mathrm{V}q_0\rho + \mathrm{V}\lambda\rho\mu$ assigned in 357, I., in which q_0 is an arbitrary but *constant quaternion*, such as $g + \gamma$, and λ, μ are *constant vectors*, the *term* $g\rho$ of $\phi\rho$ *disappears* under the symbol of operation $\mathrm{V}.\rho$, and the equation (45.) of the curve becomes,

$$\text{LXXXIII.} \ldots \mathrm{V}a\rho + \rho\mathrm{V}\gamma\rho + \mathrm{V}\rho\mathrm{V}\lambda\rho\mu = 0;$$

in which the *four versors*, $\mathrm{U}a$, $\mathrm{U}\gamma$, $\mathrm{U}\lambda$, $\mathrm{U}\mu$, introduce each *two* scalar constants, while the *two tensor quotients*, $\mathrm{T}\gamma : \mathrm{T}a$ and $\mathrm{T}\lambda\mu : \mathrm{T}a$, count as *two others*: so that the required number of *ten* such *constants* (44.) is exactly made up, the *curve* being still supposed to pass through an assumed *origin*, and therefore to have *one point given*. It is scarcely worth observing, that we can at once *remove* this last *restriction*, by merely *adding* a *new constant vector* to ρ, in the last equation, LXXXIII.

(49.) Although, for the determination of the *osculating twisted cubic* (41.), to a given *curve of double curvature*, it was necessary (comp. (40.)) to employ the vector τ^{IV} or $\mathrm{D}_s{}^5\rho$, or to take account of s^5 in the vector ρ_s, or in the connected scalars $x_s y_s z_s$ of (34.), and therefore to *improve* the expressions LVIII., by carrying in each of them (or at least in the two latter) the approximation *one step farther*, yet there are many *other problems* relating to *curves in space*, besides some that have been already considered, for which those *scalar expressions* LVIII. are *sufficiently approximate:* or for which the *vector expression* I. suffices.

(50.) Resuming, for instance, the questions considered in (22.) and (23.), we may throw some additional light on the *law of the deviation* of a *near point* P_s of the *curve*, from the *osculating sphere* at P, as follows. Eliminating n by XXXVI'. from XXXV., we find this *new expression*,

$$\text{LXXXIV.} \ldots \overline{\mathrm{SP}_s} - \overline{\mathrm{SP}} = \frac{r_1 s^4}{24r\mathrm{r}^2R};$$

the *direction* of this *deviation* from the *sphere* (R) depends therefore on the *sign* of the *scalar radius* r_1 (23.) of *curvature of the cusp-edge* (S) of the *polar developable:* and it is *outward* or *inward* (comp. 395, (14.)), according as that cusp-edge turns its *concavity* (comp. XLI.) or its *convexity*, at the *centre* S of the osculating sphere, towards the point P of the *given curve*, that is, *towards the point of osculation*.

(51.) Again, if we only take account of s^3, the deviation of P_s *from the osculating circle* at P has been seen to be a vector *tangential to the osculating sphere*, which may be thus expressed (comp. 397, IX., LII.),

$$\text{LXXXV.} \ldots \ \mathrm{C}_s\mathrm{P}_s = \frac{s^3}{6}\nu'\tau = \frac{s^3\tau(\sigma-\rho)}{6r^2\mathrm{r}},$$

if C_s be the point *on the circle*, which is distant from the given point P by an *arc of that circle* $= s$, with the *same initial direction* of motion, or of departure from P, represented by the *common unit tangent* τ; the *quantity* of *this* deviation is therefore expressed by the *scalar* $\frac{s^3R}{6r^2\mathrm{r}}$: that is, by the deviation $\frac{s^3}{6r\mathrm{r}}$ (comp. 397, (9.), (10.)) *from the osculating plane** at P, multiplied by the *secant* $(r^{-1}R)$ of the *inclination* (P) of the radius (R) of *spherical* curvature, to the radius (r) of *absolute curvature*, and *positive* when this *last* deviation has the direction of the *binormal* ν.

(52.) On the other hand (comp. (5.)) the *small angle*, which the *small arc* SS_s of the *cusp-edge* (S) of the *polar developable* subtends at the point P, is ultimately expressed by the *scalar*,

$$\text{LXXXVI.} \ldots \ \mathrm{SPS}_s = (\overline{\mathrm{PS}_s} - \overline{\mathrm{PS}}) \,.\, R^{-1}\cot P = \frac{rR's}{pR} = \frac{nrs}{R^2} \text{ (by XXXIII.)},$$

this *angle* being treated as *positive*, when the corresponding *rotation*† round $+\tau$ from

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* Besides the nine expressions in 397, (42.) for the *square* r^{-2} of the *second curvature*, the following may be remarked, as containing the *law of the regression* of the *projection of a curve* of double curvature *on its own normal plane*:

$$\mathrm{r}^{-2} = \frac{9}{2\mathrm{KP}} \,.\, \lim. \frac{\mathrm{PQ}_3^{\,2}}{\mathrm{PQ}_2^{\,3}}, \qquad 397,\ \text{XCIX.}, (10);$$

K being still the centre of the osculating circle, and Q_1, Q_2, Q_3 being still (as in 397, (10.)) the projections of a *near point* Q (or P_s), on the *tangent*, the absolute *normal* (or inward radius of curvature PK), and the *binormal* at P. In fact, the *principal terms* of the *three vector projections* corresponding, of the *small chord* PQ (or PP_s), are (comp. LVIII.):

$$\mathrm{PQ}_1 = s\tau; \quad \mathrm{PQ}_2 = (\tfrac{1}{2}s^2\tau' =)\,\frac{s^2}{2r}\,\mathrm{U}\tau'; \quad \mathrm{PQ}_3 = (\tfrac{1}{6}s^3\mathrm{r}^{-1}\nu =)\,\frac{s^3}{6r\mathrm{r}}\,\mathrm{U}\nu;$$

whence, ultimately,

$$\frac{9}{2} \,.\, \frac{\mathrm{PQ}_3^{\,2}}{\mathrm{PQ}_2^{\,3}} = -\mathrm{r}^{-2}r\mathrm{U}\tau' = \mathrm{r}^{-2} \,.\, \mathrm{KP}.$$

† Considered as a *rotation*, this small angle may be represented by the *small vector*, $rp^{-1}R'R^{-1}s\tau$; and if the *vector deviation* LXXXV. from the osculating *circle* be multiplied *by this*, the *quarter* of the *product* is (comp. XXXV.) the *vector deviation* from the osculating *sphere*, under the form,

$$\frac{s^4(\rho-\sigma)}{24R} \,.\, \frac{R'}{r\mathrm{r}p}.$$

PS to PS_s is positive: and if we multiply *this* scalar, by that which has just been assigned (51.), as an expression for the deviation $\mathrm{C}_s\mathrm{P}_s$ from the osculating *circle*, we get, by XXXV., the product,

$$\text{LXXXVII.}\ldots\ \frac{s^3R}{6r^2\mathrm{r}}\cdot\frac{rR's}{pR}=\frac{R's^4}{6r\mathrm{r}p}=4\,(\overline{\mathrm{SP}_s}-\overline{\mathrm{SP}}).$$

(53.) Combining then the recent results (50.), (51.), (52.), we arrive at the following *Theorem:*

The deviation of a near point P_s *of a curve in space, from the osculating sphere at the given point* P, *is ultimately equal to the quarter of the deviation of the same near point from the osculating circle at* P, *multiplied by the sine of the small angle which the arc* SS_s, *of the locus of centres of spherical curvature* (S), or of the *cusp-edge* of the *polar developable, subtends at the same point* P; and *this deviation* $(\overline{\mathrm{SP}_s}-\overline{\mathrm{SP}})$ *from the sphere has an outward or an inward direction, according as the same arc* SS_s *is concave or convex towards the same given point.*

(54.) The vector of the centre S_s, of the *near osculating sphere* at P_s, is (in the same order of approximation, comp. I.),

$$\text{LXXXVIII.}\ldots\ \mathrm{OS}_s=\sigma_s=\sigma+s\sigma'+\tfrac{1}{2}s^2\sigma''+\tfrac{1}{6}s^3\sigma'''+\tfrac{1}{24}s^4\sigma^{\mathrm{IV}};$$

and although $\sigma-\rho$ is already a function (by 397, IX., &c.) of τ, τ', τ'', so that σ' is (as in (2.) or (22.)) a function of τ', τ'', τ''', and σ'', σ''', σ^{IV} introduce respectively the new derived vectors τ^{IV}, τ^{V}, τ^{VI}, or $\mathrm{D}_s{}^5\rho$, $\mathrm{D}_s{}^6\rho$, $\mathrm{D}_s{}^7\rho$, which we are not at present employing (49.), yet we have seen, in (23.) and (24.), that some useful *combinations* of σ'' and σ''' can be expressed *without* τ^{IV}, τ^{V}: and the following is another remarkable example of the same species of *reduction*, involving not only σ'' and σ''' but also σ^{IV}, but still admitting, like the former, of a simple geometrical *interpretation.*

(55.) Remembering (comp. (22.), and 397, XV.) that

$$\text{LXXXIX.}\ldots\ (\sigma-\rho)^2+R^2=0,\quad\text{and}\quad\text{XC.}\ldots\ \mathrm{S}\tau'''(\sigma-\rho)=r^{-2}S=r^{-2}+nr^{-1}\mathrm{r}^{-1},$$

and reducing the successive derivatives of LXXXIX. with the help of the equations 397, XIX., and of their derivatives, we are conducted easily to the following system of equations, into which the derived vectors τ, τ', &c. do not expressly enter, but which involve σ', σ'', σ''', σ^{IV}, and R', R'', R''', R^{IV}:

$$\text{XCI.}\ldots\ \mathrm{S}\sigma'(\sigma-\rho)+RR'=0;\qquad\text{XCII.}\ldots\ \mathrm{S}\sigma'\sigma''(\sigma-\rho)=0;$$

$$\text{XCIII.}\ldots\ \mathrm{S}\sigma''(\sigma-\rho)+\sigma'^2+(RR')'=0;$$

$$\text{XCIV.}\ldots\ \mathrm{S}\sigma'''(\sigma-\rho)+3\mathrm{S}\sigma'\sigma''+(RR')''=0;$$

$$\text{XCV.}\ldots\ \mathrm{S}\sigma^{\mathrm{IV}}(\sigma-\rho)+4\mathrm{S}\sigma'\sigma'''+3\sigma''^2+(RR')'''=-\frac{RR'}{r\mathrm{r}p}=-\frac{n}{r\mathrm{r}};$$

auxiliary equations being,

$$\text{XCVI.}\ldots\ \mathrm{S}\sigma'\tau=0,\quad\mathrm{S}\sigma'\tau'=0,\quad\mathrm{S}\sigma''\tau=0,\qquad\text{comp. 395, X.}$$

and

$$\text{XCVII.}\ldots\ \mathrm{S}\sigma'''\tau=-\mathrm{S}\sigma''\tau'=\mathrm{S}\sigma'\tau''=\mathrm{S}\tau\tau''-\mathrm{S}(\sigma-\rho)\tau'''$$
$$=-r^{-2}(S-1)=-nr^{-1}\mathrm{r}^{-1}.$$

(56.) But, if R_s denote the radius of the near sphere, and if we still neglect s^5, we have,

$$\text{XCVIII.}\ldots\ \overline{\mathrm{P}_s\mathrm{S}_s}^2=-(\sigma_s-\rho_s)^2=R_s{}^2$$
$$=R^2+2sRR'+s^2(RR')'+\frac{s^3}{3}(RR')''+\frac{s^4}{12}(RR')''';$$

whence follows, by LXXXVIII., and by the recent equations, this very simple expression, from which (comp. (24.)) *everything depending on* τ^{IV}, τ^{V}, τ^{VI} *has disappeared*,

$$\text{XCIX.}\ldots\ (\sigma_s-\rho)^2+R_s^2=\frac{-RR's^4}{12r\mathrm{r}p}\,;$$

and which gives (within the same order of approximation, attending to XXXV.) the *geometrical relation*,

$$\text{C.}\ldots\ \overline{\mathrm{PS}_s}-\overline{\mathrm{P}_s\mathrm{S}_s}=\mathrm{T}(\sigma_s-\rho)-R_s=\frac{R's^4}{24r\mathrm{r}p}=\frac{ns^4}{24r\mathrm{r}R}=\overline{\mathrm{SP}_s}-\overline{\mathrm{SP}}\,;$$

or

$$\text{C}'.\ldots\ \overline{\mathrm{S}_s\mathrm{P}}-\overline{\mathrm{SP}_s}=\overline{\mathrm{S}_s\mathrm{P}_s}-\overline{\mathrm{SP}}=R_s-R.$$

(57.) This result might have been *foreseen*, from the following very simple consideration. When the *coefficient* $S-1$ *of non-sphericity* (395, (16.)), or of the deviation of a curve from a sphere, is *positive*, so that a *near point* P_s of the *curve* is *exterior* to (what we may call) the *given sphere*, which *osculates* to that curve at P, by an amount which is ultimately proportional to the *fourth power* of the *arc*, s, of the curve, then the *given point* P must be, for the same reason, *exterior to the near sphere*, which osculates at the point P_s; and the *two deviations*, $\overline{\mathrm{PS}_s}-\overline{\mathrm{P}_s\mathrm{S}_s}$, and $\overline{\mathrm{SP}_s}-\overline{\mathrm{SP}}$, which have been found by *calculation* to be *equal* (C.), if s^5 be neglected, must in fact bear to each other an *ultimate ratio of equality*, because the *two arcs*, $+s$ and $-s$, from P to P_s, and from P_s *back* to P, are *equally long*, although *oppositely directed*; or because $(+s)^4=(-s)^4$. And precisely the same reasoning applies, when the coefficient $S-1$ is *negative*, so that the *deviations*, equated in the formula C., are *both inwards*.

(58.) As regards the deviation (51.) of the near point P_s of the curve *from the osculating circle* at P, we may generalize and render more exact the expression LXXXV., by considering a point C_t of that circle, which is distant *by a circular arc* $=t$ from the given point P; and of which the vector is, *rigorously*, by 396, (18.),

$$\text{CI.}\ldots\ \mathrm{OC}_t=\omega_t=\rho+r\tau\sin\frac{t}{r}+r^2\tau'\operatorname{vers}\frac{t}{r}\,;$$

or if we only neglect t^5,

$$\text{CII.}\ldots\ \mathrm{OC}_t=\omega_t=\rho+\tau\left(t-\frac{t^3}{6r^2}\right)+r\tau'\left(\frac{t^2}{2r}-\frac{t^4}{24r^3}\right).$$

(59.) In this way we shall have (comp. (34.)) the *vector deviation*,

$$\text{CIII.}\ldots\ \mathrm{C}_t\mathrm{P}_s=\rho_s-\omega_t=X\tau+Yr\tau'+Zr\nu,$$

with the *scalar coefficients*,

$$\text{CIV.}\ldots\ X=x_s-r\sin\frac{t}{r},\quad Y=y_s-r\operatorname{vers}\frac{t}{r},\quad Z=z_s\,;$$

or, neglecting s^5 and t^5, and attending to the expressions LVIII. and LVIII'.,

$$\text{CV.}\ldots\ \begin{cases} X=s-t-\dfrac{s^3-t^3}{6r^2}+\dfrac{r's^4}{8r^3}\,; \\[2ex] Y=\dfrac{s^2-t^2}{2r}-\dfrac{p}{r}Z-\dfrac{s^4-t^4}{24r^3}-\dfrac{ns^4}{24r^2\mathrm{r}}\,; \\[2ex] Z=\dfrac{s^3}{6r\mathrm{r}}+\dfrac{rs^4}{24}(r^{-2}\mathrm{r}^{-1})'\,; \end{cases}$$

in which r, r', r, p, and n have the same significations as before.

(60.) Assuming then for the *circular arc* t the value,

$$\text{CVI.}\ldots t = s + \frac{r's^4}{8r^3},$$

which differs (as we see) by only a quantity of the *fourth order* from the arc s of the *curve*, we shall have, to the same order of approximation, the expressions,

$$\text{CVII.}\ldots X = 0, \quad Y = \frac{-p}{r} Z - \frac{ns^4}{24r^3\mathrm{r}}, \quad Z = z_s = \&\text{c., as before,}$$

the *deviation* at P_s from the *circle* being here measured in a direction *parallel to the normal plane* at P; and if s^4 be neglected (although the expressions enable us to take account of it), this deviation is also *parallel* (as before) to the *tangent* $\tau(\sigma - \rho)$ *to the osculating sphere* in that plane: while it is represented in quantity by $Rr^{-1}z_s$, which agrees with the result in (51.).

(61.) The expressions CVII. give also, *without neglecting* s^4,

$$\text{CVIII.}\ldots \frac{rY + pZ}{R} = -\frac{ns^4}{24r\mathrm{r}R} = \overline{\mathrm{SP}} - \overline{\mathrm{SP}_s};$$

such then is the *component* of the *deviation* from the osculating *circle*, which is *parallel to the normal* PS *to the sphere* at P; and we see that it only differs in *sign* (because it is *positive* when its *direction* is that of the *inward normal*, or *inward radius* PS), from the expression XXXV. (comp. C.), for the *outward deviation* $\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}}$ of the near point P_s, from the same osculating *sphere* at the given point P.

(62.) This *latter* component (61.) is *small*, even as *compared* with the *former* small component (60.); and the *small quotient*, of the latter divided by the former, is ultimately (by LXXXVI.),

$$\text{CIX.}\ldots \frac{rY + pZ}{rZ - pY} = \frac{-nrs}{4R^2} = -\tfrac{1}{4}\mathrm{SPS}_s;$$

where the *small angle* SPS_s is positive or negative, according to the rule stated in (52.), and may be replaced by its sine, or by its tangent.

(63.) Instead of cutting the given osculating circle, as in (60.), by a plane which is *parallel to the given normal plane* at P, we may propose to *cut that circle by the near normal plane* at P_s, or to satisfy this *new condition*,

$$\text{CX.}\ldots 0 = S\tau_s(\rho_s - \omega_t), \quad \text{or} \quad \text{CX}'.\ldots 0 = XS\tau\tau_s + YSr\tau'\tau_s + ZSr\nu\tau_s;$$

which is easily found to give by CV. the values (s and t being still supposed to be small, and s^5 being still neglected):

$$\text{CXI.}\ldots t = s - \frac{r's^4}{24r^3}, \quad \text{and} \quad \text{CXII.}\ldots X = \frac{r's^4}{6r^3}, \quad Y = \&\text{c.}, \ Z = \&\text{c., as in CVII.};$$

so that in passing to this *new near point* C_t of the *circle*, we only change X from *zero* to a small quantity of the *fourth order*, and make *no change* in the values of Y and Z.

(64.) The *new deviation* $\mathrm{C}_t\mathrm{P}_s$ from the *given circle* may be decomposed into *two partial deviations*, in the *near normal plane*, of which *one* has the direction of the *unit-tangent* $R_s^{-1}\tau_s(\sigma_s - \rho_s)$ to the *near sphere* at P_s, and the other has that of the *unit-normal* $R_s^{-1}(\sigma_s - \rho_s)$ to the same sphere at the same point (or the opposites of these two directions); and the *scalar coefficients* of these *two vector units*, if we attend only to *principal terms*, are easily found to be,

$$\text{CXIII.} \ldots \frac{rZ - pY}{R} = \frac{Rs^3}{6r^2\mathrm{r}}, \quad \text{and} \quad \text{CXIV.} \ldots \frac{rY + (p + ns)Z}{R} = \frac{ns^4}{8r\mathrm{r}R}.$$

(65.) We may then write :

CXV. . . *Deviation of near point* P_s *from given osculating circle, measured in the near normal plane to the curve at* P_s,

$$= \text{new } \mathrm{C}_t\mathrm{P}_s = \frac{Rs^3}{6r^2\mathrm{r}} \mathrm{U}\tau_s(\sigma_s - \rho_s) + \frac{ns^4}{8r\mathrm{r}R} \mathrm{U}(\sigma_s - \rho_s);$$

in which it may be observed, that the *second scalar coefficient* is equal to *three times* the *scalar deviation* $\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}}$ (XXXV. or C.), of the *near point* P_s of the *curve*, from the *given osculating sphere* (at P).

(66.) But we may also interpret the *new coefficient* last mentioned, as representing a *new deviation ;* namely, that of the *point* C_t of the *given circle, from the near osculating sphere* at P_s, considered as *positive* when that *new point* C_t is *exterior* to that near sphere ; or as denoting the *difference of distances*, $\overline{\mathrm{S}_s\mathrm{C}_t} - \overline{\mathrm{S}_s\mathrm{P}_s}$. We have therefore (comp. (56.)) this *new geometrical relation*, of an extremely simple kind :

$$\text{CXVI.} \ldots \overline{\mathrm{S}_s\mathrm{C}_t} - \overline{\mathrm{S}_s\mathrm{P}_s} = 3(\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}}) = 3(\overline{\mathrm{S}_s\mathrm{P}} - \overline{\mathrm{S}_s\mathrm{P}_s});$$

or

$$\text{CXVI}'. \ldots \overline{\mathrm{S}_s\mathrm{C}_t} = 3\overline{\mathrm{S}_s\mathrm{P}} - 2\overline{\mathrm{S}_s\mathrm{P}_s}.$$

(67.) Supposing, then, at first, that the *coefficient of non-sphericity* $S - 1$ is *positive* (comp. 395, (16.)), if we conceive a point to move *backwards, upon the curve*, from P_s to P, and then *forwards, upon the circle* which *osculates at* P, to the *new point* C_t (63.), we see that it will *first* attain (at P) a position *exterior to the sphere* which *osculates at* P_s, or will have an amount, determined in (56.), of *outward deviation*, with respect to that *near osculating sphere ;* and that it will *afterwards* attain (at the new point C_t) a *deviation of the same character* (namely *outwards*, if $S > 1$), from the *same near sphere*, but one of which the *amount* will be *threefold* the former : this last *relation* holding also when $S < 1$, or when *both* deviations are *inwards*.

(68.) It is easy also to infer from (65.), (comp. (57.)), that if we go *back from* P_s, *on the near circle* which *osculates at that near point*, through an *arc* (t) *of that circle*, which will only *differ* by a small quantity of the *fourth order* (comp. (60.)) from the *arc* (s) *of the curve*, so as to arrive at a point, which for the moment we shall simply denote by C, and in which (as well as in *another point of section*, not necessary here to be considered) the *near osculating circle* is *cut* by the *given normal plane* at P, the *vector deviation* of this *new point* C of the *new circle, from the given point* P of the *curve*, must be, nearly :

$$\text{CXVII.} \ldots \mathrm{PC} = \frac{Rs^3}{6r^2\mathrm{r}} \mathrm{U}\tau(\sigma - \rho) - \frac{ns^4}{8r\mathrm{r}R} \mathrm{U}(\sigma - \rho);$$

the coefficients being formed from those of the formula CXV., by first changing s to $-s$, and then changing the signs of the results : while the relation CXVI. or CXVI′. takes now the form,

$$\text{CXVIII.} \ldots \overline{\mathrm{SC}} - \overline{\mathrm{SP}} = 3(\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}}), \quad \text{or} \quad \text{CXVIII}'. \ldots \overline{\mathrm{SC}} = 3\overline{\mathrm{SP}_s} - 2\overline{\mathrm{SP}}.$$

(69.) Accordingly if, after going from P to P_s along the *curve*, we go forward or backward, through *any positive or negative arc*, t, of the *circle* which *osculates* at that point P_s, we shall arrive at a point which we may here denote by $\mathrm{C}_{s,t}$; and the *vector* (comp. again 396, (18.)) of this *near point* (*more general* than any of those hitherto considered) will be, *rigorously*,

$$\text{CXIX.} \ldots \omega_{s,t} = \mathrm{OC}_{s,t} = \rho_s + r_s\tau_s \sin\frac{t}{r_s} + r_s^2 r'_s \operatorname{vers}\frac{t}{r_s}.$$

And if we *develope* this new expression to the accuracy of the *fourth order* inclusive, we find that we satisfy the *new condition* (comp. (63.)),

$$\text{CXX.} \ldots \mathrm{S}\tau(\omega_{s,t} - \rho) = 0, \quad \text{when} \quad \text{CXXI.} \ldots t = -s - \frac{r's^4}{24r^3};$$

and that then the expression CXIX. agrees with CXVII., within the order of approximation here considered.

(70.) A *geometrical connexion* can be shown to exist, between the *two equivalents* which have been found above, one for the *quadruple* (LXXXVII., comp. (53.)), and the other for the *triple* (CXVIII.), of the *deviation* $\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}}$ of a *near point* P_s of the *curve*, from the *sphere* which *osculates* at the *given point* P: in such a manner that if *either* of those two expressions be regarded as *known*, the *other* can be *inferred* from it.

(71.) In fact if we draw, in the normal plane, perpendiculars PD and PE to the lines PS and PS_s, and determine points D and E upon them by drawing a parallel to PS through the point C of (68.), letting fall also a perpendicular CF on PS_s, the *two small lines* PD and DC will ultimately represent the *two terms* or components CXVII. of PC; and the *small angle* DPC will ultimately be equal to *three quarters* of the small angle SPS_s, and will correspond to the *same direction* of *rotation* round τ, because

$$\text{CXXII.} \ldots \frac{\mathrm{DC}}{\mathrm{PD}} = \tfrac{3}{4}.\frac{nrs\tau}{R^2} = \tfrac{3}{4}\mathrm{V}\frac{\sigma's}{\sigma - \rho},$$

or

$$\text{CXXIII.} \ldots \mathrm{DPC} = \tfrac{3}{4}\mathrm{SPS}_s = \tfrac{3}{4}\mathrm{DPE};$$

so that we shall have the *ultimate ratios* (comp. the annexed Fig. 83*):

$$\text{CXXIV.} \ldots \mathrm{DC} : \mathrm{DE} : \mathrm{CE}\ (\text{or } \mathrm{FP}) = 3 : 4 : 1.$$

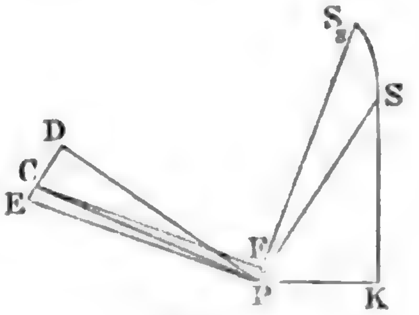

Fig. 83.

But the line CF is ultimately the *trace*, on the given normal plane, of the tangent plane at C to the near osculating sphere; the small line FP (or CE) represents therefore the *deviation* $\overline{\mathrm{S}_s\mathrm{P}} - \overline{\mathrm{S}_s\mathrm{P}_s}$ of the given point P from that near sphere, or the *equal* deviation (57.), $\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}}$; its ultimate *quadruple*, DE, represents the *product* mentioned in (52.); and the ultimate *triple*, DC, of the same small line CE, is a geometrical representation of that *other* deviation $\overline{\mathrm{SC}} - \overline{\mathrm{SP}}$, which has been more recently considered.

(72.) When the *two scalars*, s and t, are supposed capable of receiving *any values*, the *point* $\mathrm{C}_{s,t}$ in (69.) may be *any point of the Locus* (8.) *of the Osculating Circle* to the given curve of double curvature; and if we seek the direction of the *normal* to this superficial *locus*, at this point, on the plan of Art. 372, writing first the equation of the surface under the slightly simplified, but equally rigorous form,

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* In Figs. 81, 82, the little arc near S is to be conceived as *terminating there*, or as being a *preceding arc* of the curve which is the locus of S, if r', r, n, and therefore also p and r_1, be *positive* (comp. the second Note to page 574). In the new Figure 83, the triangle PDE is to be conceived as being in fact *much smaller* than PKS, though *magnified* to exhibit angular and other relations.

$$\text{CXXV.} \ldots \omega_{s,u} = \rho_s + r_s\tau_s \sin u + r_s^2\tau'_s \operatorname{vers} u,$$

with

$$\text{CXXVI.} \ldots u = r_s^{-1}t = \mathrm{P_sK_sC}_{s,t},$$

so that u is here a new scalar variable, representing the *angle subtended at the centre* K_s, of the osculating circle at P_s, *by the arc*, t, of that *circle*, we are led, after a few reductions, to the expression,

$$\text{CXXVII.} \ldots \mathrm{V}(\mathrm{D}_u\omega_{s,u} \,.\, \mathrm{D}_s\omega_{s,u}) = r_s\mathrm{r}_s^{-1}(\omega_{s,u} - \sigma_s) \operatorname{vers} u;$$

which proves, *by quaternions*, what was to be expected from *geometrical* considerations*, that the *locus of the osculating circle* is also (as stated in (8.) and (22.)) the *Envelope of the Osculating Sphere.*

(73.) The *normal* to this *locus*, at any proposed point $\mathrm{C}_{s,t}$ of any *one* osculating circle, is thus the *radius* of the *sphere* to which *that circle* belongs, or which has the *same point of osculation* P_s with the *given curve*, whether the *arc* (s) of that *curve*, and the arc (t) of the *circle*, be *small* or *large*. We must therefore consider the *tangent plane* to the *locus*, at the given point P of the *curve*, as *coinciding* with the tangent plane to the *osculating sphere* at that point; and in fact, while this latter plane ($\perp$ PS) *contains* the *tangent* τ to the *curve*, which is at the same time a *tangent* to the *locus*, it contains *also* the tangent $\tau(\sigma - \rho)$ to the *sphere*, which is by CXVII. *another tangent* to the locus, as being the tangent at P to the *section* of that surface, which is made by the *normal plane* to the curve.

(74.) But when we come to examine, with the help of the same equation CXVII., what is the *law of the deviation* DC (comp. Fig. 83) of that *normal section* of the *locus*, considered as a *new curve* (C), *from its own tangent* PD, we find that *this law* is ultimately expressed (comp. (71.)) by the formula,

$$\text{CXXVIII.} \ldots \frac{\mathrm{DC}^3}{\mathrm{PD}^4} = \frac{81}{32} \cdot \frac{n^3r^5\mathrm{r}(\sigma - \rho)}{R^8} = \text{const.};$$

hence $\overline{\mathrm{DC}}$ varies ultimately as the *power* of $\overline{\mathrm{PD}}$, which has the *fraction* $\frac{4}{3}$ for its exponent; the *limit* of $\overline{\mathrm{PD}}^2 : \overline{\mathrm{DC}}$ is therefore *null*, and the *curvature of the section* is *infinite at* P.

(75.) It follows that this point P is a *singular point* of the *curve* (C), in which the *locus* (8.) is *cut* (73.), by the normal plane to the given curve at that point; but it is *not a cusp* on that *section*, because the *tangential component* PD of the *vector chord* PC is ultimately proportional to an *odd power* (namely to the *cube*, by CXVII., comp. (71.)) of the *scalar variable*, s, and therefore has its *direction reversed*, when that *variable changes sign*: whereas the *normal component* DC of the same *chord* PC is proportional to an *even power* (namely the *fourth*, by the same equation CXVII.) of the same *arc*, s, of the *given curve*, and therefore retains its *direction unchanged*, when we pass from a near point P_s, on *one side* of the given point P, to a near point P_{-s} on the *other side* of it.

(76.) To illustrate this by a contrasted case, let G be the point in which the *tangent* to the given curve at P, is cut by the *normal plane* at P; or a point of the *section*, by that *plane*, of the *developable surface of tangents*. We shall then have the sufficiently approximate expressions,

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* In the language of infinitesimals, *two consecutive osculating spheres*, to any curve in space, *intersect* each other in an *osculating circle* to that curve.

$$\text{CXXIX.}\ldots \text{PG} = \rho_s - \rho - \left(s + \frac{s^3}{3r^2}\right)\tau_s = \frac{-s^2\tau'}{2} - \frac{s^3\nu}{3r} = -\text{PQ}_2 - 2\text{PQ}_3,$$

with the significations 397, (10.) of Q_2 and Q_3; hence the point P of the *curve* is (as is well known) a *cusp* of the *section* (G) of the developable *surface of tangents* (comp. 397, (15.)), because the *tangential component* ($-\text{PQ}_2$) of the *vector chord* (PG) has *here* a *fixed direction*, namely that of the *outward radius* (KP prolonged) of the *circle of curvature* at P: while it is *now* the *normal component* (-2PQ_3) which *changes direction*, when the arc s of the *curve* changes *sign*. At the same time we see* that the *equation* of this *last* section (G) may *ultimately* be thus expressed:

$$\text{CXXX.}\ldots \frac{(-2\text{PQ}_3)^2}{(-\text{PQ}_2)^3} = \frac{8\text{PK}}{9\text{r}^3} = \text{const.};$$

comparing which with the equation CXXVIII., we see that although, *in each case*, the *curvature of the section* is *infinite*, at the point P of the *curve*, yet the normal component (or co-ordinate) varies (ultimately) as the *power* $\frac{3}{2}$ of the *tangential* component, for the section (G) of the *Surface of Tangents:* whereas the former component varies by (74.) as the power $\frac{4}{3}$ of the latter, for the corresponding section (C) of the *Locus of the Osculating Circle.*

(77.) It follows also that the *curve* (P) *itself,* although it is *not a cusp-edge* of the last-mentioned *locus* (8.), while it *is* such on the *surface of tangents*, is yet a *Singular Line* upon *that locus likewise:* the nature and origin of which *line* will perhaps be seen more clearly, by reverting to the *view* (8.), (22.), (72.), according to which that *Locus of a Circle* is at the same time the *Envelope of a Sphere.*

(78.) In general, if we suppose that σ and R are *any two real functions*, of the vector and scalar kinds, *of any one real* and *scalar variable*, t, and that σ', R', and σ'', R'', &c. denote their successive *derivatives*, taken with respect to it, then σ may be conceived to be the *variable vector* of a *point* S of a *curve in space*, and R to be the *variable radius* of a *sphere*, which has its *centre* at that point S, but *alters* generally its *magnitude*, at the same time that it alters its *position*, by the *motion* of its centre *along the curve* (S).

(79.) Passing from *one* such sphere, with centre S and radius R, considered as *given*, and represented by the scalar equation,†

$$(\sigma - \rho)^2 + R^2 = 0, \qquad \text{LXXXIX.},$$

in which ρ is *now* conceived to be the vector of a *variable point* P *upon its surface*, to a *near sphere* of the *same system*, for which σ, S, and R are replaced by σ_t, S_t, and R_t, where t is supposed to be *small*, we easily infer (comp. 386, (4.)) that the equation,

$$S\sigma'(\sigma - \rho) + RR' = 0, \qquad \text{XCI.},$$

which is formed from LXXXIX. by once derivating σ and R with respect to t, but treating ρ as constant, represents the *real plane* (comp. 282, (12.)) of the (*real or imaginary*) *circle*, which is the *ultimate intersection* of the near sphere with the given one; the *radius* of this *circle*, which we shall call r, being found by the following formula,

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* Compare the first Note to page 594.

† This equation, and a few others which we shall require, occurred before in this series, but in a connexion so different, that it appears convenient to repeat them here.

$$\text{CXXXI.} \ldots r^2\sigma'^2 = R^2(R'^2 + \sigma'^2), \quad \text{or} \quad \text{CXXXI}'. \ldots r^2\text{T}\sigma'^2 = R^2(\text{T}\sigma'^2 - R'^2),$$

and being therefore *real* when

$$\text{CXXXII.} \ldots R'^2 + \sigma'^2 < 0, \quad \text{or} \quad \text{CXXXII}'. \ldots R'^2 < \text{T}\sigma'^2;$$

while the *centre*, say K, of the circle is *always real*, and its *vector* is,

$$\text{CXXXI}''. \ldots \text{OK} = \kappa = \sigma + RR'\sigma'^{-1};$$

and the *plane* XCI. of the same circle is *parallel* to the *normal plane* of the *curve* (S).

(80.) With the *condition* CXXXII., the *two scalar equations*, LXXXIX. and XCI., represent then *jointly* a *real circle*; and the *locus* of all such *circles* (comp. 386, (6.)) is easily proved to be also the *envelope* of all the *spheres*, of which one is represented by the equation LXXXIX. *alone*; *each such sphere touching this locus*, in the *whole extent* of the corresponding *circle* of the system.

(81.) The *plane* XCI., considered as *varying* with t, has a *developable surface* for its *envelope*; and the *real right line*, or *generatrix*, along which *one* touches the *other*, is represented (comp. again 386, (6.)) by the system of the *two* scalar equations, XCI. and

$$\text{S}\sigma''(\sigma - \rho) + \sigma'^2 + (RR')' = 0, \qquad \text{XCIII.};$$

where ρ is now the *variable vector of the line of contact*, although it has been *treated as constant* (comp. 386, (4.)), in the process by which we are here conceived to pass, by a *second derivation*, from LXXXIX. through XCI. to XCIII.

(82.) This real *right line* (81.) *meets* generally the *sphere*, and also the *circle* (as being in its plane), in *two* (real or imaginary) *points*, say P_1, P_2; and the *curvilinear locus* of all *such points* forms generally a species of *singular line*,* upon the *superficial locus* (or *envelope*) recently considered (80.); or rather it forms in general *two branches* (real or imaginary) of *such* a *line*: which *generally two-branched line* (or *curve*) is the (real or imaginary) *envelope* (comp. 386, (8.)), of all the *circles* of the system.

(83.) The equation,

$$\text{S}\sigma'\sigma''(\sigma - \rho) = 0, \qquad \text{XCII.},$$

which now represents (comp. 376, V.) the *osculating plane to the curve* (S), shows that this *plane through the centre* S of the *sphere* is *perpendicular to the right line* (81.), and consequently *contains* the perpendicular let fall *from that centre on that line:* the *foot* P of this last *perpendicular* is therefore found by combining the *three* linear and scalar equations, XCI., XCII., XCIII., and its *vector* is,

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* Called by Monge an *arête de rebroussement*, except in the *case* to which we shall next proceed, when its *two branches coincide*. The *envelope* (80.) of a *varying sphere* has been considered in two distinct Sections, § XXII. and § XXVI., of the *Application de l'Analyse à la Géométrie*; but the author of that great work does not appear to have perceived the *interpretation* which will soon be pointed out, of the *condition* of such *coincidence*. Meantime it may be mentioned, in passing, that quaternions are found to confirm the geometrical result, that when the *two branches* (P_1) (P_2) are *distinct*, then *each* is a *cusp-edge* of the *surface*; but that when they are *coincident*, the *singular line* (P) in which they merge has then a *different character*.

$$\text{CXXXIII.} \ldots \text{OP} = \rho = \sigma + \frac{g\sigma' + RR'\sigma''}{\mathrm{V}\sigma'\sigma''},$$

if

$$\text{CXXXIV.} \ldots g = -\sigma'^2 - R'^2 - RR'' = \mathrm{T}\sigma'^2 - (RR')'.$$

(84.) The *condition of contact* of the *right line* (81.) with the *sphere* (78.), or with the *circle* (79.), or the condition of contact between *two consecutive** *circles* of the system (80.), or finally the *condition of coincidence* of the *two branches* (82.) of that *singular line* upon the surface which is *touched by all those circles*, is at the same time the *condition of coexistence* of the *four scalar equations*, LXXXIX., XCI., XCII., XCIII.; it is therefore expressed by the equation (comp. CXXXIII.),

$$\text{CXXXV.} \ldots R^2(\mathrm{V}\sigma'\sigma'')^2 = (g\sigma' + RR'\sigma'')^2;$$

which may also be thus written,†

$$\text{CXXXVI.} \ldots (R\mathrm{S}\sigma'\sigma'' - R'g)^2 = (R'^2 + \sigma'^2)\,(R^2\sigma''^2 + g^2),$$

or thus,

$$\text{CXXXVII.} \ldots R^2(R'^2 + \sigma'^2)\,(\mathrm{V}\sigma'\sigma'')^2 = (g\sigma'^2 + RR'\mathrm{S}\sigma'\sigma'')^2;$$

the *scalar variable t* (78.), with respect to which the derivations are performed, remaining still entirely *arbitrary*, but the *point* P, which is determined by the formula CXXXIII., being *now* situated on both the *sphere* and the *circle:* and its *curvilinear locus*, which we may call the *curve* (P), being now the *singular line itself*, in its re-

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* Compare the Note to page 581.

† In page 372 of Liouville's Edition already cited, or in page 325 of the Fourth Edition (Paris, 1809), of the *Application de l'Analyse*, &c., it will be found that this condition is assigned by Monge, as that of the *evanescence* of a certain *radical*, under the form (an accidentally omitted exponent of π'' in the second part of the first member being here restored):

$$[a(\phi'\phi'' + \psi'\psi'' + \pi'\pi'') - h^2]^2 + h^2[a^2(\phi''^2 + \psi''^2 + \pi''^2) - h^4] = 0;$$

in which he writes, for abridgment,

$$h^2 = 1 - \phi'^2 - \psi'^2 - \pi'^2,$$

and ϕ, ψ, π are the three rectangular co-ordinates of the centre of a moving sphere, considered as functions of its radius a. Accordingly, if we change R to a, and σ to $i\phi + j\psi + k\pi$, supposing also that $R' = a' = 1$, and $R'' = a'' = 0$, whereby g is changed to $-h^2$, and $R'^2 + \sigma'^2$ to h^2, in the condition CXXXVI., that condition takes, by the rules of quaternions, the exact form of the equation cited in this Note: which, for the sake of reference, we shall call, for the present, the *Equation of Monge*, although it does not appear to have been either *interpreted* or *integrated* by that illustrious author. Indeed, if Monge had not hastened over this *case of coincident branches*, on which he seems to have designed to *return* in a subsequent Memoir (unhappily not written, or not published), he would scarcely have chosen such a symbol as h^2 (instead of $-h^2$), to denote a quantity which is *essentially negative*, whenever (as here) the *envelope* of the *sphere* is *real*.

duced and *one-branched* state. And the *last form* CXXXVII. shows, what was to be expected from geometry, that when this *condition of coincidence* is satisfied, the earlier *condition of reality* CXXXII. is satisfied also: together with this *other inequality*,

$$\text{CXXXVIII.} \ldots R^2\sigma''^2 + g^2 < 0,$$

which then results from the form CXXXVI.

(85.) The equations CXXXI., CXXXIV., and the general formula 389, IV., give the expressions,

$$\text{CXXXIX.} \ldots \frac{rr'}{RR'} = \frac{g\sigma'^2 + RR'\text{S}\sigma'\sigma''}{-\sigma'^4}; \qquad \text{CXL.} \ldots r_1^{-2} = \frac{(\text{V}\sigma'\sigma'')^2}{\sigma'^6};$$

where r is still the *radius* of the *circle of contact* of the *sphere* with its *envelope*, and r_1 is the *radius of curvature* of the *locus of the centre* s of the same variable sphere; whence it is easy to infer, that the *condition* CXXXV. may be reduced to the following very simple form (comp. XXXVI'. and XLII.):

$$\text{CXLI.} \ldots (r'r_1)^2 = (RR')^2; \quad \text{or} \quad \text{CXLI'.} \ldots r_1\text{d}r = \pm R\text{d}R;$$

the independent variable being still *arbitrary*.

(86.) If the *arc* of the *curve* (s) be taken as that variable t, the form CXXXVI. of the same condition is easily reduced to the following,

$$\text{CXLII.} \ldots R^2 = (RR')^2 + g^2r_1^2, \quad \text{with} \quad \text{CXLIII.} \ldots g = 1 - (RR')';$$

derivating then, and dividing by $2g$, we have this *new differential equation*, which is *of linear form with respect to* RR', whereas the *condition itself* may be considered as a differential equation of the *second degree*, as well as of the *second order*,[*]

$$\text{CXLIV.} \ldots RR' = r_1(gr_1)'; \quad \text{or} \quad \text{CXLV.} \ldots r_1^2u'' + r_1r_1'(u' - 1) + u = 0,$$

if $$\text{CXLVI.} \ldots u = RR' = R\text{D}_tR, \quad \text{and therefore} \quad \text{CXLVII.} \ldots u^2 = R^2 - r^2,$$

by CXXXI. or CXXXI'., because we have now,

$$\text{CXLVIII.} \ldots \sigma'^2 = -1, \quad \text{or} \quad \text{T}\sigma' = 1, \quad \text{or} \quad \text{d}t = \text{Td}\sigma:$$

so that the *new scalar variable*, RR', or u, with respect to which the *linear equation* CXLIV. or CXLV. is only of the *second order*, represents the perpendicular *height*[†] of the *centre* s of the *sphere*, above the *plane* of the *circle*, considered as a *function of the arc* (t) of the *curve* (s), and as *positive* when the *radius* R of the *sphere increases*, for *positive motion* along that curve, or for an *increasing value* of its *arc*.

(87.) If the curve (s) be *given*, or even if we only know the *law* according to which its *radius of curvature* (r_1) depends on its *arc* (t), the *coefficients* of the *linear equation* CXLV. are *known*; and if we succeed in *integrating* that equation, so as to find an expression for the *perpendicular* u as a function of that *arc* t, we shall then be able to express also, as functions of the *same* arc, the *radii* R and r of the *sphere* and *circle*, by the formulæ,

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[*] We shall soon assign the *complete integral* of the *differential equation in quaternions* (84.), and also that of the corresponding *Equation of Monge*, cited in the preceding Note.

[†] It will be found that this new scalar u, if we abstract from *sign*, corresponds precisely to the p of earlier sub-articles, although presenting itself in a *different connexion*: for the *sphere* (78.), and the *circle* (79.), under the *condition* (84.), will soon be shown to be the *osculating* sphere and circle to the recent *curve* (P), or to the *singular line* (84.) upon the *surface* at present considered, that is, on the *locus* or *envelope* (80.).

$$\text{CXLIX.} \ldots \pm r = gr_1 = r_1(1-u'), \quad \text{and} \quad \text{CL.} \ldots R^2 = 2\int u\,dt = u^2 + r_1^2(1-u')^2;$$

the *third scalar constant*, which the integral $2\int u\,dt$ would otherwise introduce into the expression for R^2, being in this manner *determined*, by means of the *other two*, which arise from the integration of the *equation* above mentioned.

(88.) For example, it may happen that the locus of the centre S of the sphere has a *constant curvature*, or that $r_1 = \text{const.}$; and then the *complete integral* of the linear equation CXLV. is at once seen to be of the form,

$$\text{CLI.} \ldots u = a\sin(r_1^{-1}t + b),$$

a and b being *two arbitrary* (but scalar) constants; after which we may write, by (87.),

$$\text{CLII.} \ldots \pm r = r_1 - a\cos(r_1^{-1}t + b); \quad \text{CLIII.} \ldots R^2 = r_1^2 - 2ar_1\cos(r_1^{-1}t + b) + a^2;$$

so that, *in this case*, *both* the *radii*, r and R, of *circle* and *sphere*, are *periodical functions* of the *arc* of the curve (S).

(89.) In general, if that *curve* (S) be *completely given*, so that the *vector* σ is a *known function* of a *scalar variable*, and if an expression have been *found* (or *given*) for the *scalar* R which satisfies any one of the forms of the *condition* (84.), we can then determine *also* the *vector* ρ, by the formula CXXXIII., as a function of the *same* variable; and so can assign the *point* P of the *singular line* (84.), which *corresponds* to any *given position* of the *centre* S of the *sphere*. For this purpose we have, when the arc of the curve (S) is taken, as in (86.), for the independent variable t, the formula,

$$\text{CLIV.} \ldots \rho = \sigma - u\sigma' - (1-u')\sigma''^{-1} = \kappa_1 - u\sigma' - r_1^2u'\sigma'',$$

if κ_1 be the *vector* of the *centre*, say K$_1$, of the *osculating circle* at S *to that given curve*, so that (comp. 389, XI.) it has the value,

$$\text{CLV.} \ldots \text{OK}_1 = \kappa_1 = \sigma - \sigma''^{-1} = \sigma + r_1^2\sigma'', \quad \text{with} \quad \text{CLV}'. \ldots \sigma''^2 + r_1^{-2} = 0.$$

If then we denote by v the *distance* of the point P from *this* centre K$_1$, and attend to the *linear equation* CXLV., we see that

$$\text{CLVI.} \ldots v = \overline{\text{K}_1\text{P}} = \text{T}(\rho - \kappa_1) = \sqrt{(u^2 + r_1^2u'^2)},$$

and $\quad \text{CLVI}'. \ldots vv' = r_1r_1'u_1', \quad \text{with} \quad \text{T}\sigma' = 1;$

or more generally, $\quad \text{CLVII.} \ldots vv's_1' = r_1r'_1u',$

if $\quad \text{CLVII}'. \ldots u = RR's_1'^{-1}, \quad \text{and} \quad \text{CLVII}''. \ldots s_1 = \int \text{T}d\sigma,$

while $\quad \text{CLVI}''. \ldots v^2 = u^2 + r_1^2u'^2s_1'^{-2};$

so that s_1 denotes the *arc* of the *curve* (S), when the independent variable t is again left arbitrary. This *distance*, v, is therefore *constant* ($= a$) in the *case* (88.), namely when the *radius of curvature* r_1 of that curve is *itself* a constant quantity.

(90.) When $s_1' = \text{T}\sigma' = 1$, as in CXLVIII., the part $\sigma - u\sigma'$ of the first expression CLIV. for ρ becomes $= \kappa$, by CXXXI''. and CXLVI.; attending then to CLV., we have the *scalar quotient*,

$$\text{CLVIII.} \ldots \frac{\kappa - \rho}{\sigma - \kappa_1} = 1 - u';$$

whence generally,

$$\text{CLVIII}'.\ldots\ \frac{\kappa-\rho}{\sigma-\kappa_1}=1-\frac{1}{s_1'}\left(\frac{RR'}{s_1'}\right)'=1-\left(\frac{\mathrm{d}}{\mathrm{d}s_1}\right)^2\left(\frac{R^2}{2}\right),$$

the independent variable t being again arbitrary. Accordingly, if we combine the general expression CXXXIII. for ρ, with the expression CXXXI''. for κ, and with the following for κ_1 (comp. 389, IV.),

$$\text{CLIX.}\ldots\ \kappa_1=\sigma+\frac{\sigma'^3}{\mathrm{V}\sigma''\sigma'},\ \text{for an } \textit{arbitrary} \text{ scalar variable,}$$

we easily deduce this new form of the scalar quotient,

$$\text{CLIX}'.\ldots\ \frac{\kappa-\rho}{\sigma-\kappa_1}=1+((RR')'-RR'\mathrm{S}\sigma'^{-1}\sigma'')\,\sigma'^{-2};$$

which agrees with CLVIII'., because $-\sigma'^2=s_1'^2$, and $\mathrm{S}\,\dfrac{\sigma''}{\sigma'}=\dfrac{s_1''}{s'_1}$.

(91.) It has then been fully shown, how to *determine the vector* ρ as a *function of the scalar* t, when σ and R are *two known functions* of that variable, which satisfy any one of the forms of the *condition* (84.). It must then be possible to determine also the *derived vectors*, ρ', ρ'', &c., as functions of the same variable: and accordingly this can be done, by *derivating* any *three* of the *four* scalar equations, LXXXIX. XCI. XCII. XCIII., of which that condition (84.) expresses the *coexistence*. Now if we derivate a first time the two first of these, and then reduce by the second and fourth, we get the equations,

$$\text{CLX.}\ldots\ \mathrm{S}\rho'(\sigma-\rho)=0,\quad \mathrm{S}\rho'\sigma'=0,\quad \text{whence}\quad \text{CLX}'.\ldots\ \rho'\parallel\mathrm{V}\sigma'(\sigma-\rho);$$

and although this last formula only determines the *direction* of the *tangent* to the *singular line* at P, namely that of the *common tangent* at that point to *two consecutive circles* (84.), yet it enables us to infer, by the remaining equation XCII., that

$$\text{CLXI.}\ldots\ \rho'\perp\sigma'',\quad \rho'\parallel\mathrm{V}\sigma'\sigma'',\quad \text{and}\quad \text{CLXI}'.\ldots\ \mathrm{S}\rho'\sigma''=0;$$

reducing by which the derivative of XCIII., we find,

$$\mathrm{S}\sigma'''(\sigma-\rho)+3\mathrm{S}\sigma'\sigma''+(RR')''=0, \qquad \text{XCIV.},$$

the scalar variable being still arbitrary. And *conversely*, the *system*[*] of the *four equations* LXXXIX. XCI. XCIII. XCIV. gives the *three* equations CLX. CLXI'., and so conducts to the equation XCII., and thence to the *condition* (84.); *unless we suppose that* ρ *is a constant vector* α, or that the *variable sphere* passes *through a fixed point* A, a case which we do not here consider, because in it the singular *line* (P) would reduce itself to that one *point*.

(92.) Derivating the two equations CLX., and reducing with the help of CLXI'., we find these new equations,

$$\text{CLXII.}\ldots\ \mathrm{S}\rho''(\sigma-\rho)-\rho'^2=0,\quad \mathrm{S}\rho''\sigma'=0;$$

whence

$$\text{CLXIII.}\ldots\ \mathrm{S}\rho'''(\sigma-\rho)-3\mathrm{S}\rho'\rho''=0.$$

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[*] In the language of infinitesimals, this system of equations expresses that *four consecutive spheres intersect*, in one *common point* P. When that point happens to be a *fixed* one, the *condition* (84.) requires that we should have the relation $\mathrm{S}\sigma'\sigma''(\sigma-\alpha)=0$; or geometrically, that the *curve* (s) should be *in a plane through the fixed point*, which is then a *singular point* of the envelope.

We are led then, by *elimination of the derivatives of* σ, to the system of the three equations 395, VII.; and we conclude, that *the point* S *is the centre*, and the *radius* R *is the radius, of the osculating sphere** *to the singular line* (P): whence it is easy to infer also, that the *plane of contact* (79.) of the *sphere* with its *envelope* is the *osculating plane*, and that the *circle of contact* (80.) is the *osculating circle* (comp. (72.)), to the same *curve* (P), at the *point* where *two consecutive circles touch one another* (84.).

(93.) *In general*, and even *without* the condition (84.), the *tangent* to a *branch* (82.) of the *curvilinear envelope* of the *circles* of the system, at any point P_1 of that branch, has the direction represented by the vector $V\sigma'(\sigma-\rho_1)$, of the tangent to the circle at that point; but when that *condition* is satisfied, so that the *two branches* of the singular line *coincide*, the *point* P of that line is *in the osculating plane* (83.) *to the curve* (S): and *then* the equation XCII. shows that the *tangent* ρ', or $V\sigma'(\sigma-\rho)$, to the *line*, is perpendicular to σ'', or parallel to $V\sigma'\sigma''$ (comp. CLXI.), and therefore that *the singular line crosses that plane at right angles*.

(94.) It follows that, with the *condition* (84.), the singular line (P) is an *orthogonal trajectory* to the *system of osculating planes* to the curve (S); and whereas, when this last curve is given, there ought to be *one* such trajectory for *every point* of a *given* osculating plane, this circumstance is analytically represented, in our recent calculations, by the *biordinal form* of the *differential equation* CXLV., of which the *complete integral* must be conceived (87.) to involve generally, as in the case (88.), *two arbitrary constants*.

(95.) It follows also that, with the same condition of *coincidence of branches*, the *singular line* (P) must have the *curve* (S) for the *cusp-edge* of its *polar developable*; or that the *sphere*, with S for centre, and with R for radius, must be the *osculating sphere* to the *curve* (P), as otherwise found by calculation in (92.): while the *circle* (80.) must be, as before, the *osculating circle* to that curve.

(96.) Accordingly, *all equations*, and *inequalities*, which have been stated in the recent sub-articles (79.), &c., respecting the envelope of a moving sphere with variable radius, under that *condition* (84.), and without any special *selection* of the independent variable, admit of being *verified*, by means of the *earlier* formulæ for the osculating circle and sphere to a curve (P) treated as a *given* one, when the *arc* (s) of *that* curve is taken as such a variable.

(97.) For example, we had lately the *two inequalities*, $R'^2+\sigma'^2<0$, CXXXII., and $R^2\sigma''^2+g^2<0$, CXXXVIII. And accordingly the earlier sub-articles (22.), (23.) give, for those two combinations, the *essentially negative values*,

$$\text{CLXIV.}\ldots\ R'^2+\sigma'^2=-p^{-2}r^2R'^2; \qquad \text{CLXV.}\ldots\ R^2\sigma''^2+g^2=-((nr)')^2;$$

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* In the language of infinitesimals (comp. the preceding Note), if every *four consecutive spheres* of a system *intersect* in *one point of a curve*, then *each sphere* passes *through four consecutive points* of that curve. Simple as this geometrical reasoning is, the writer is not aware that it has been anticipated; and indeed he is at present led to suppose that this whole theory, of the *Locus of the Osculating Circle*, as the *Envelope of the Osculating Sphere*, is new. Monge had however considered, but rejected (page 374 of Liouville's Edition), the case of a *system of circles* having each a *simple contact* with a *curve in space*.

in obtaining which last, the following transformations have been employed:

$$\text{CLXVI.} \ldots \sigma''^2 = -n'^2 - n^2\mathrm{r}^{-2}; \qquad \text{CLXVII.} \ldots g = -n'p + nr\tau^{-1}.$$

(98.) As regards the verification of the *equations*, it may be sufficient to give one example; and we shall take for it the *last general form* CLVII. of the differential *equation of condition* (84.). For this purpose we may now write, by (22.) and (23.),

$$\text{CLXVIII.} \ldots s_1' = \pm n, \quad u = \pm p, \quad u' = \pm p', \quad r_1 u_1' s_1'^{-1} = p' r_1 n^{-1} = p'\mathrm{r};$$

and have only to observe that

$$\text{CLXIX.} \ldots \tfrac{1}{2}(p^2 + p'^2\mathrm{r}^2)' = p'\mathrm{r}(r + p'\mathrm{r})', \quad \text{because} \quad p = r'\mathrm{r}.$$

(99.) If we denote by c_1, c_2, c_3 the first members of the equations XCI., XCIII., XCIV., then besides the equation LXXXIX., which may be regarded as a mere *definition* of the radius R, we have $c_1 = 0$ for the *whole* of the superficial *locus* or *envelope* (80.); but we have not *also* $c_2 = 0$, except for a point on one or other of the *two* (generally distinct) *branches* of the *singular line* (82.) upon that locus. And if, at any *other* and *ordinary point*, we *cut* the surface by a *plane* perpendicular to the *circle* at that point, we find, by a process of the same kind as some which have been already employed, expressions for the *tangential* and *normal components* of the vector chord, whereof the *principal terms* involve the scalar c_2 as a *factor*, while the latter varies (ultimately) as the *square* of the former, so that the *curvature of the section* is *finite* and known, but *tends* to become *infinite* when c_2 tends to *zero*.

(100.) If the *condition of coincidence* (84.) be *not satisfied*, so that the two branches of the singular line (82.) remain *distinct*, and that thus $c_2 = 0$, but *not* $c_3 = 0$ (comp. (91.)), for any ordinary point on *one* of those two branches, then if we *cut* the surface at that point by a *plane* perpendicular to the *branch*, or to the *circle* which *touches* it there, we find an ultimate expression for the vector chord which involves the scalar c_3 as a factor, and of which the *normal* component varies as the *sesquiplicate power* of the *tangential* one: so that we have here the case of a *semicubical cusp*, and *each branch* of the *singular line* is a *cusp-edge** of the surface, exactly in the *same known sense* (comp. (76.)) as that in which a *curve* of double curvature is *generally such*, on the *developable locus* of its *tangents*.

(101.) But when the condition (84.) *is* satisfied, so that the two branches *coincide*, and that thus (comp. again (91.)) we have at once the *three* equations,

$$\text{CLXX.} \ldots c_1 = 0, \quad c_2 = 0, \quad c_3 = 0,$$

then the *terms*, which were lately the *principal* ones (100.), *disappear*: and a *new expression* arises, for the vector chord of a section of the surface, made by a plane perpendicular to the singular line, which (when we take $t = s$, as in (96.)) is found to admit of being identified with the formula CXVII., and of course conducts to precisely the same system of consequences; the *tangential* component *now* varying ultimately as the *cube*, and the *normal* component as the *fourth power* of a small variable, so that the *cuspidal property* of the point P of the *section* no longer exists: although the *curvature* at that point is still *infinite*, as in (74.): and the *Singular Line*, reduced now to a *single branch*, to which all the *circles* of the system *osculate*,

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* Compare the Note to page 602.

(92.), (95.), is *not a cusp-edge* of the *Surface*, as had been otherwise found before (77.), but a line of a *different character*,* which may thus be regarded, with reference to a *more general Envelope* (80.), as the result of a *Fusion* (84.) of *Two Cusp-Edges*.

(102.) The *condition* of such *fusion* (or coincidence) has been seen (84.) to be expressible by the *differential equation* of the *second order*, and *second degree*,

$$(RS\sigma'\sigma'' - R'g)^2 = (R'^2 + \sigma'^2)(R^2\sigma''^2 + g^2), \qquad \text{CXXXVI.}$$

with

$$g = -\sigma'^2 - (RR')', \qquad \text{CXXXIV.}$$

and with the independent variable arbitrary. And we are now prepared to assign the *complete general integral*† of this differential equation; namely the system of the *two* following equations (comp. 395, (7.) and (14.)), of the *vector* and *scalar* kinds,

$$\text{CLXXI.} \ldots \sigma = \rho + \frac{3V\rho'\rho''S\rho'\rho'' + V\rho'''\rho'^3}{S\rho'\rho''\rho'''}, \quad \text{and} \quad \text{CLXXII.} \ldots R = T(\sigma - \rho);$$

in which ρ is an *arbitrary vector function* of *any scalar variable*, t, and which express, when geometrically interpreted, that σ is the *variable vector* of the *centre* S, and that R is the *variable radius*, of the *osculating sphere* to an *arbitrary curve* (P), of which the variable vector of a point P is ρ.

(103.) In fact, if we *met* the cited *equation of condition* CXXXVI., g representing therein the expression CXXXIV., without any previous knowledge of its meaning or origin, we might first, by the *rules of quaternions*, and as a mere affair of *calculation*, *transform* it to the equation CXXXV.; which would evidently allow the assumption of the formula CXXXIII., ρ being treated as an *auxiliary vector*, which satisfies (in virtue of the supposed *condition*) the system of the *four scalar equations*, LXXXIX., XCI., XCII., XCIII.; whence derivating and combining, as in (91.) and (92.), we are led to a *new system*‡ of *four* scalar equations, whereof *one* is again the equation LXXXIX., and may be written under the form CLXXII.; while the *three others* are those formerly numbered as 395, VII., and conduct (except in a particular *case* which we shall presently consider) to the *vector expression* CLXXI., which conversely is *sufficient* to represent them, all *derivatives* of σ and of R being thus *eliminated*.

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* Compare the Note to page 602. Monge (in page 372 of Liouville's Edition) has the remark, that (when a certain radical vanishes) "les deux branches de la courbe touchée par toutes les caractéristiques se confondent en une seule: et cette courbe, sans cesser d'être une ligne singulière de la surface, n'est plus une arête de rebroussement, elle est une ligne de striction." The propriety of this last *name*, "line of striction," appears to the present writer questionable: although he has confirmed, as above, by calculations with quaternions, the result that, in the case referred to, the *singular line* is *not a cusp-edge*. Monge does not seem to have perceived that, in the same *case of fusion*, the curved *line* in question is not merely *touched*, but *osculated*, by all the *circles* of the system.

† Compare the first Note to page 604. We say here, *general integral*, because a *less* general one, although involving *one arbitrary function* (of the scalar kind), will soon be pointed out.

‡ The *Equation of Monge* (comp. the second Note to page 603) may be considered as the *condition of coexistence* of the *four* following equations, in which ϕ, ψ, π are supposed to be functions of a, and to be differentiated or derivated as such ·

(104.) The *case* just now alluded to, in which the *general integral* (102.) is replaced by a *less general form*, is the case (91.) when the *variable sphere* passes through a *fixed point* A, to which *point*, in that case, the *singular line* reduces itself. And the *integral equations*,* which then replace CLXXI. and CLXXII., may be thus written:

CLXXIII... $\sigma = \alpha + t\beta + u\gamma$, with $u = F(t)$, and CLXXIV... $R = \mathrm{T}(t\beta + u\gamma)$;

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(1). . . $(x-\phi)^2 + (y-\psi)^2 + (z-\pi)^2 = a^2$;
(2). . . $(x-\phi)\phi' + (y-\psi)\psi' + (z-\pi)\pi' + a = 0$;
(3). . . $(x-\phi)\phi'' + (y-\psi)\psi'' + (z-\pi)\pi'' + 1 - \phi'^2 - \psi'^2 - \pi'^2 = 0$;
(4). . . $(x-\phi)(\psi'\pi'' - \pi'\psi'') + (y-\psi)(\pi'\phi'' - \phi'\pi'') + (z-\pi)(\phi'\psi'' - \psi'\phi'') = 0$;

whereof the first *three* have been employed by Monge himself, but the *fourth* does not seem to have been perceived by him, the condition of *evanescence of a radical* having been used in its stead. And by a *translation* of quaternion results, above deduced, into the usual language of analysis, it is found that the *complete* and *general integral*, of the *non-linear differential equation* of the *second order*, which is obtained by the elimination of x, y, z between these four, is expressed by a *new system* of four equations, the equation (1) being *one* of them; and the *three others*, in which x, y, z are now treated as *arbitrary functions* of a, and are derivated as such, being the following:

(5). . . $(x-\phi)x' + (y-\psi)y' + (z-\pi)z' = 0$;
(6). . . $(x-\phi)x'' + (y-\psi)y'' + (z-\pi)z'' + x'^2 + y'^2 + z'^2 = 0$;
(7). . . $(x-\phi)x''' + (y-\psi)y''' + (z-\pi)z''' + 3(x'x'' + y'y'' + z'z'') = 0$.

By treating a as a function of some *other* independent variable, t, the terms $+ a$ and $+ 1$, in (2) and (3), come to be replaced by $+ aa'$ and $+ aa'' + a'^2$; and the slightly *more general form*, which Monge's Equation thus assumes, has *still* its complete *general integral* assigned by the system (1) (5) (6) (7), if x, y, z (as well as a) be now regarded as arbitrary functions of the *new* variable t, in the place of which it is permitted (for instance) to take x, and so to write $x' = 1$, $x'' = 0$: only *two arbitrary functions* thus entering, in the last analysis, into the *general solution*, as was to be expected from the form of the equation.

* The *particular integral* corresponding, of the *Equation of Monge*, is expressed by the following system:

$$\phi = a + et + lu, \quad \psi = b + ft + mu, \quad \pi = c + gt + nu,$$
$$(et + lu)^2 + (ft + mu)^2 + (gt + nu)^2 = a^2;$$

abcdefglmn being *nine arbitrary constants*, while t and u are *two functions* of a, whereof *one* is *arbitrary*, but the *other* is algebraically *deduced* from it, by means of the fourth equation. The writer is not aware that either of these integrals has been assigned before.

the *second scalar coefficient*, u, being here an *arbitrary function* of the *first* scalar coefficient, or of the independent variable t, and α, β, γ being *three arbitrary* but *constant vectors:* so that the *curve* (S) is now obliged to lie *in some one plane*[*] through the *fixed point* A, but remains in other respects *arbitrary*. Accordingly it will be found that this *last integral system*, although *less general* than the former system (102.), and not properly *included* in it, *satisfies* the differential equation CXXXVI.; whereof the two members acquire, by the substitutions indicated, this *common value*,

$$\text{CLXXV.} \ldots (R\text{S}\sigma'\sigma'' - R'g)^2 = \&\text{c.} = R^{-2}t^2(tu' - u)^2 u''^2 (\text{V}\beta\gamma)^4.$$

(105.) Other problems might be proposed and resolved, with the help of formulæ[†] already given, respecting the properties or affections of curves in space which depend on the *fourth power* (s^4) of the *arc*, or on the *fourth derivative* $\text{D}_s{}^4\rho$ or τ''' of the *vector* ρ_s; but it is time to conclude this series of sub-articles, which has extended to a much greater length than was designed, by observing that, in virtue of the *vector form* 396, XI. for the equation of a *circle of curvature*, the *Locus* (8.) *of the Osculating Circle* may be concisely but sufficiently represented by the *Vector Equation*,

$$\text{CLXXVI.} \ldots \text{V}\frac{2\tau_s}{\omega - \rho_s} + \nu_s = 0,$$

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[*] Compare the Note to page 606.

[†] We might for example employ the formula VI. for κ'', in conjunction with one of the expressions 397, XCI. for κ', to determine, by the general formula 389, IV., the *vector* (say ξ) of the *centre of curvature* of the *curve* (K), and therefore also the *radius of curvature* of that curve, which is the *locus of the centres of curvature* of the *given curve* (P), supposed to be *in general* one of *double* curvature. After a few reductions, with the help of XII., we should thus find the equations,

$$\text{CLXXVII.} \ldots \text{V}\frac{\kappa''}{\kappa'} = \frac{-r'\tau}{r\kappa'} + (\mathrm{r}^{-1} - P')\tau,$$

$$\text{CLXXVIII.} \ldots \xi = \kappa + \frac{\kappa'}{\text{V}\frac{\kappa''}{\kappa'}} = \kappa + \frac{\sigma - 2\kappa + \rho}{1 - \frac{\mathrm{r}\text{d}P}{\text{d}s} + \frac{p\text{d}s}{r\text{d}\kappa}},$$

in which last the denominator is a quaternion, and the scalar variable is arbitrary: whence also,

CLXXIX. . . *Radius of curvature of curve* (K),

or of *locus of centres of osculating circles to a given curve* (P) *in space*,

$$= \text{T}(\xi - \kappa) = R\left\{\left(1 - \frac{\mathrm{r}\text{d}P}{\text{d}s}\right)^2 + \left(\frac{p\mathrm{r}}{Rr}\right)^2\right\}^{-\frac{1}{2}}$$

$$= \pm\frac{R\text{d}r}{p\text{d}s}\left\{\left(\frac{1}{\mathrm{r}} - \frac{\text{d}P}{\text{d}s}\right)^2 + \left(\frac{p}{Rr}\right)^2\right\}^{-\frac{1}{2}};$$

with the verification, that for the case of a *plane curve* (P), for which therefore $\frac{R}{p} = 1$, and $\frac{1}{\mathrm{r}} = 0 = \frac{\text{d}P}{\text{d}s}$, we have thus the elementary expression,

$$\text{CLXXX.} \ldots \textit{Radius of Curvature of Plane Evolute} = \pm\frac{r\text{d}r}{\text{d}s},$$

r being still the radius of curvature, and s the *arc*, of the *given curve*.

which *apparently* involves only *one scalar variable*, *s*, namely, the *arc of the curve* (P), the *other* scalar variable, such as *t*, which corresponds (69.) to the *arc of the circle*, *disappearing* under the *sign* V: and that the *surface*, which was called in (8.) the *Circumscribed Developable*, is now seen to be in fact *circumscribed* to that *Locus*, or *Envelope*, in a certain *singular* (or eminent) *sense*, as *touching it along its Singular Line*.

399. When we take account of the *fifth power* (s^5) of the *arc*, the expression for ρ_s receives a *new term*, and becomes (comp. 398, I.),

$$\text{I.}\ldots\ \rho_s=\rho+s\tau+\tfrac{1}{2}s^2\tau'+\tfrac{1}{6}s^3\tau''+\tfrac{1}{24}s^4\tau'''+\tfrac{1}{120}s^5\tau^{\text{IV}};$$

and although some of the consequences of such an expression have been already considered, especially as regards the general determination of what has been above called the *Osculating Twisted Cubic* to a curve of double curvature, or the *gauche curve of the third degree* which has *contact of the fifth order* with a given *curve in space*, yet, without repeating any calculations already made, some additional light may be thrown on the subject as follows.

(1.) As regards the successive deduction of the derived vectors in the formula I., it may be remarked that if we write (comp. 398, LVI., LXI.),

$$\text{II.}\ldots\ \mathrm{D}^{n+1}_s\rho=\tau^{(n)}=a_n\tau+b_nr\tau'+c_nr\nu,$$

we shall have, generally,

$$\text{III.}\ldots\ a_{n+1}=a'_n-r^{-1}b_n,\quad b_{n+1}=b'_n+r^{-1}a_n-\mathrm{r}^{-1}c_n,\quad c_{n+1}=c'_n+\mathrm{r}^{-1}b_n,$$

with the initial values,

$$\text{IV.}\ldots\ a_0=1,\quad b_0=0,\quad c_0=0,\quad\text{or}\quad\text{IV}'.\ldots\ a_1=0,\quad b_1=r^{-1},\quad c_1=0;$$

$$\text{whence}\quad\text{V.}\ldots\begin{cases}a_2=-r^{-2},\quad b_2=(r^{-1})',\quad c_2=r^{-1}\mathrm{r}^{-1},\\ a_3=3r^{-3}r',\quad b_3=(r^{-1})''-r^{-3}-r^{-1}\mathrm{r}^{-2},\quad c_3=r(r^{-2}\mathrm{r}^{-1})',\end{cases}$$

as in the expressions 397, VI. for τ'', and 398, IV. for τ'''; the corresponding coefficients of τ^{IV} being in like manner found to be,

$$\text{VI.}\ldots\begin{cases}a_4=-2(r^{-2})''+((r^{-1})')^2+r^{-2}(r^{-2}+\mathrm{r}^{-2});\\ b_4=(r^{-1})'''-2(r^{-3})'-3(r^{-1}\mathrm{r}^{-1})'\mathrm{r}^{-1};\\ c_4=r^{-1}(\mathrm{r}^{-1})''+3((r^{-1})'\mathrm{r}^{-1})'-r^{-1}\mathrm{r}^{-1}(r^{-2}+\mathrm{r}^{-2});\end{cases}$$

and being sufficient for the investigation of all affections or properties of a curve in space, which depend only on the *fifth power* of the arc *s*.

(2.) For the *helix* the two curvatures are *constant*, so that all the derivatives of the two radii r and r vanish; the expressions become therefore greatly simplified, and a *law* is easily perceived, allowing us to *sum* the infinite *series* for ρ_s, and so to obtain the following *rigorous expressions* for the *co-ordinates** x_s, y_s, z_s of this particular curve, instead of those which were developed *generally* in 398, LVIII., but only as far as s^4 inclusive:

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* We have here, and in this whole investigation, an instance of the facility with which *quaternions* can be combined with *co-ordinates*, whenever the *geometrical na-*

$$\text{VII.} \ldots x_s = l^3(\mathrm{r}^{-2}t + r^{-2}\sin t); \quad y_s = l^2 r^{-1}\operatorname{vers} t; \quad z_s = l^3 r^{-1}\mathrm{r}^{-1}(t - \sin t);$$

where l and t are an auxiliary constant and variable, namely,

$$\text{VIII.} \ldots l = (r^{-2} + \mathrm{r}^{-2})^{-\frac{1}{2}} = r\sin H, \quad t = l^{-1}s,$$

l being thus what was denoted in earlier formulæ by $\mathrm{T}\lambda^{-1}$, and t being the angle between two axial planes; while the origin is still placed at the point P of the curve, and the tangent, normal, and binormal are still made the axes of xyz.

(3.) The *cone* of the second order, 398, (40.), which has *generally* a *contact of the fifth order* with a proposed *curve in space*, at a point P taken for vertex, has in this case of the *helix* the equation (comp. 398, LVII.* and LXIX.),

$$\text{IX.} \ldots y^2 = \frac{3\,\mathrm{r}}{2\,r}\left\{x + \left(\frac{3}{10}\frac{\mathrm{r}}{r} - \frac{7}{10}\frac{r}{\mathrm{r}}\right)z\right\}z.$$

Accordingly it can be shown, by elementary methods, that if we write, for a moment,

$$\text{X.} \ldots f(t) = 3\,(t - \sin t)\;(3t + 7\sin t) - 20\operatorname{vers}^2 t,$$

we have the *eight* evanescent values,

$$\text{XI.} \ldots f0 = f'0 = f''0 = f'''0 = f^{\text{iv}}0 = f^{\text{v}}0 = f^{\text{vi}}0 = f^{\text{vii}}0 = 0;$$

whence it is easy to infer that this *cone* IX. has (in the present example, although not generally) a contact as high as the *sixth order*† with the *curve*, of which the co-ordinates have here the expressions VII.; and consequently that the cone in question must wholly *contain* the *osculating twisted cubic* to that curve.

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ture of a question may render it convenient so to combine them, by offering to our notice any obvious planes of reference. If it be thought useful to pass to a system connected more immediately with the *right cylinder* than with the *helix*, we may write,

$$\text{VII}'. \ldots \begin{cases} \mathrm{x}_s = l(r^{-1}x_s - \mathrm{r}^{-1}z_s) = l^2 r^{-1}\sin t, \\ \mathrm{y}_s = l^2 r^{-1} - y_s \qquad\quad = l^2 r^{-1}\cos t, \\ \mathrm{z}_s = l(\mathrm{r}^{-1}x_s + r^{-1}z_s) = l^2\mathrm{r}^{-1}t, \end{cases}$$

where $l^2 r^{-1} = r\sin^2 H$ is the radius of the cylinder, with converse formulæ easily assigned.

* In the corresponding equation 398, LXVII., the coefficient of $6ac$ ought to have been printed as $\left(\frac{\mathrm{r}}{r}\right)^3$, like the coefficient of $6xz$ in the equation LVII.

† Or in modern language, *seven-point contact*, in the sense that the cone passes, *in this case*, through *seven consecutive points* of the curve. It may be remarked that the *gauche curve* of the *fourth degree*, or the *quartic curve*, in which this *cone cuts* the *cylinder of revolution* whereon the helix is traced (cutting also in it a certain *other* cylinder of the second order), and which has the point P for a *double point*, *crosses the helix* by *one* of its *two branches* at that point, while it has *seven-point contact* with the same helix by its *other branch*: and that thus the fact of calculation, expressed by the formula XI., is geometrically accounted for.

(4.) In general, to find a *second locus* for such a *cubic curve*, the method of recent sub-articles (398, (38.) &c.) leads us to form the equation (398, LXVI.) of a *cylinder* of the *second order*, or briefly of a *quadric* cylinder*, which like the *quadric cone* (3.) shall have *contact* of the *fifth order* with the proposed *curve* in space, at the given point P; the *ratios* of *abc*, which determine the *direction* of a generating line PE, being obliged for this purpose to satisfy a certain *equation of condition* (398, LXVIII.), of which the *form* indicates that the *locus* of this *line* PE is *generally* a certain *cubic cone*, having the *tangent* (say PT) to the *curve* for a *nodal side:* along which side it is *touched*, not only (like the quadric cone) by the *osculating plane* ($z=0$) to that given curve, but *also* by a *second plane*, whereof the equation ($\mathrm{g}y+\mathrm{h}z=0$, or after reductions $y-\frac{1}{2}\mathrm{r}'z=0$) shows that the *second branch* of the cubic cone *crosses the first branch*, or the quadric cone, or the osculating plane to the curve, at an *angle* of which the trigonometric *cotangent* is equal to *half the differential of the radius* (r) *of second curvature, divided by the differential of the arc* (s); so that this *second tangent plane* to the *cone* coincides with the *rectifying plane* to the *curve*, when the second curvature happens to be *constant*. The *tangent* PT therefore *counts as three* of the *six common sides* of the *two cones* with P for vertex: and the *three other* common sides, for the assigning of which it has been shown (in 398, (41.)) how to form a *cubic equation* in $b:c$, are the *parallels* from that point P to the three real or imaginary *asymptotes*† of the *twisted cubic*, and are *generating lines* PE of *three quadric cylinders*, whereof *one* at least is necessarily *real*, and *contains*, as a *second locus*, that sought *osculating gauche curve* of the *third degree*.

(5.) In applying this *general method* to the case of the *helix*, it is found that the cubic cone breaks up, in this example, into a system of a *new quadric cone*, which *touches* the *former* quadric cone IX. along the *tangent* PT to the curve (the *two other* common sides of *these* two cones being *imaginary*), and of a *plane* ($y=0$), namely the *rectifying plane* (comp. (4.)) of the helix, or the *tangent plane* to the *cylinder of revolution* on which that given curve is traced: and that this last plane cuts the *first* quadric cone in *two real right lines*, the *tangent* being again *one* of them, and the *other* having the *sought direction* of a *real asymptote* to the sought *osculating twisted cubic*. Without entering here into details of calculation, the resulting equation of the *real*‡ *quadric cylinder*, on which that sought gauche curve is situated, may be at once stated to be (with the present system of co-ordinates),

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* So called by Dr. Salmon, in his Treatise already cited. Compare the first Note to page 591 of these *Elements*.

† Compare again the Note last referred to.

‡ As regards the *two imaginary quadric cylinders*, their equations can be formed by the same general method, employing as generating lines the *two imaginary common sides* (5.), of the *cone* IX., and of that *other quadric cone* above referred to, which is *here a separable part* of the *general cubic locus*, and has for equation,

$$\mathrm{IX}'.\ \ldots\ \frac{20}{9}y^2 = 5\frac{\mathrm{r}}{r}xz + \left(8\frac{\mathrm{r}^2}{r^2} - 2\right)z^2.$$

It seems sufficient here to remark, that by taking the sum and difference of the equations of those two imaginary cylinders, *two new real quadric surfaces* are obtained.

$$\text{XII.} \ldots 2ry = \left\{x + \left(\frac{3}{10}\frac{\mathrm{r}}{r} - \frac{7}{10}\frac{r}{\mathrm{r}}\right)z\right\}^2 + \frac{3}{5}\left(1 + \frac{r^2}{\mathrm{r}^2}\right)y^2;$$

in such a manner that *if we set aside the right line*,

$$\text{XIII.} \ldots y = 0, \quad x + \left(\frac{3}{10}\frac{\mathrm{r}}{r} - \frac{7}{10}\frac{r}{\mathrm{r}}\right)z = 0,$$

which is a *common side* of the *cone* IX. and of the *cylinder* XII., the *curve*, which is the *remaining part* of their *complete* intersection, is the *twisted cubic sought*. As an elementary verification of the fact, that *this gauche curve* of intersection IX. XII. has *contact of the fifth order* with the *helix* at the point P, it may be observed that if we change the co-ordinates xyz in XII. to the expressions VII., and write for abridgment,

$$\text{XIV.} \ldots F(t) = (3t + 7t \sin t)^2 - 200 \operatorname{vers} t + 60 \operatorname{vers}^2 t,$$

we have then (comp. X. XI.) the *six* evanescent values,

$$\text{XV.} \ldots F0 = F'0 = F''0 = F'''0 = F^{\mathrm{iv}}0 = F^{\mathrm{v}}0 = 0.$$

(6.) As another *verification*, which is at the same time a sufficient *proof*, of the *à posteriori* kind, that the *gauche curve* IX. XII. has in fact *contact* of the *fifth order* with the *helix*, it can be shown that while the co-ordinates y_s and z_s of the *latter* may (by VII., writing simply x for x_s, and neglecting x^7) be thus developed,

$$\text{XVI.} \ldots \begin{cases} y_s = \dfrac{x^2}{2r} + \dfrac{x^4}{24r}\left(\dfrac{3}{r^2} - \dfrac{1}{\mathrm{r}^2}\right) + \dfrac{x^6}{720r}\left(\dfrac{45}{r^4} - \dfrac{24}{r^2\mathrm{r}^2} + \dfrac{1}{r^4}\right), \\ z_s = \dfrac{x^3}{6r\mathrm{r}} + \dfrac{x^5}{120r\mathrm{r}}\left(\dfrac{9}{r^2} - \dfrac{1}{\mathrm{r}^2}\right), \end{cases}$$

the corresponding co-ordinates y and z of the *former*, that is, of the *curvilinear part* of the *intersection* of the cone IX. with the cylinder XII., have (in the same order of approximation) developments which may be thus abridged,

$$\text{XVII.} \ldots y = y_s - \frac{(r^{-2} + \mathrm{r}^{-2})^2 x^6}{800r}, \quad z = z_s.$$

(7.) The *deviation* of the helix from the gauche curve IX. XII. is therefore of the *sixth order* (with respect to x, or s), and it has an *inward direction*, or in other words, the *osculating twisted cubic* deviates *outwardly* from the *helix*, with respect to the right cylinder; the *ultimate* (or initial) *amount* of this deviation, or the *law* according to which it *tends* to vary, being represented by the formula,

$$\text{XVII}'. \ldots y_s - y = \frac{(r^{-2} + \mathrm{r}^{-2})^2 s^6}{800r} = \frac{t^4 y_s}{400};$$

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which *also contain* the *osculating twisted cubic*, and *intersect* each other in that gauche curve: namely *two hyperbolic paraboloids*, which have a *common side at infinity*, and of which the equations can be *otherwise deduced* (by way of verification), *without imaginaries*, through easy algebraical combinations of the two real equations IX. and XII.

where t denotes as in (2.) the *angle*, which a *plane* drawn through a near point P_s, and through the axis of the *right cylinder*,*

$$\text{XVIII.} \ldots 2ry = \left(x - \frac{r}{\mathrm{r}} z \right)^2 + \left(1 + \frac{r^2}{\mathrm{r}^2} \right) y^2,$$

whereon the helix is traced, makes with the plane drawn through the same *axis of revolution*, or through the right line,

$$\text{XIX.} \ldots x = \frac{r}{\mathrm{r}} z, \quad y = r^{-1}(r^{-2} + \mathrm{r}^{-2})^{-1} = l^2 r^{-1},$$

and through the given point P: while y_s is still the (inward) distance of the same near point P_s, from the tangent plane to the same cylinder at the same given point P.

(8.) If we cut the cone IX., and the cylinder XII., by any plane,

$$\text{XX.} \ldots 2ry = w \left\{ x + \left(\frac{3}{10} \frac{\mathrm{r}}{r} - \frac{7}{10} \frac{r}{\mathrm{r}} \right) z \right\},$$

drawn through their common side XIII., we obtain two other sides, one for each of these two quadric surfaces; and these two new right lines, in this plane XX., intersect each other in a a new point,† of which the co-ordinates xyz are given, as functions of the new variable w, by the *three fractional expressions*,‡

$$\text{XXI.} \ldots x = \frac{w + \left(\frac{7}{r^2} - \frac{3}{\mathrm{r}^2} \right) \frac{w^3}{60}}{1 + \frac{3}{20} \frac{w^2}{l^2}}; \quad 2ry = \frac{w^2}{1 + \frac{3}{20} \frac{w^2}{l^2}}; \quad 6r\mathrm{r}z = \frac{w^3}{1 + \frac{3}{20} \frac{w^2}{l^2}};$$

while the *twisted cubic*, which *osculates* (as above) to the helix at P, is the *locus* of all the *points of intersection* thus determined. Accordingly, if we develop xyz by XXI., in ascending powers of w, neglecting w^7 (or x^7), we are conducted, by elimination of w, to expressions for y and z in terms of x, which agree with those found in (6.), and thereby establish in a new way the existence of the required contact of the *fifth order*, between the two curves of double curvature.

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* With the co-ordinates VII'. of a recent Note (to page 612), the equation of this cylinder would be,

$$\text{XVIII}'. \ldots \mathbf{x}^2 + \mathbf{y}^2 = l^4 r^{-2}.$$

† The plane XX., as containing the line XIII., is parallel to an asymptote, and therefore meets the cubic at infinity; it also passes through the given point P: and therefore it can only cut the twisted cubic in *one other point*, of which the position is expressed by the equations XXI.

‡ *Quaternions* suggest such fractional expressions, through the formula 398, LXXIX. for the *vector* $(\phi + c)^{-1} \alpha$; but it is proper to state that expressions of *fractional form*, for the co-ordinates of a *curve in space* of the *third order* (or degree) were given by Möbius, who appears to have been the first to discover the existence of *such* gauche curves, and who published several of their principal properties in his *Barycentric Calculus* (der barycentrische Calcul, Leipzig, 1827). Compare the Notes to pages 23 and 35, and Note B at the end of these Elements.

(9.) The *real asymptote* to the cubic curve is found by supposing the auxiliary variable w to tend to infinity in the expressions XXI.; it is therefore the right line (comp. XX.),

$$\text{XXII.} \ldots y = \frac{10}{3}\frac{l^2}{r}, \quad x + \left(\frac{3}{10}\frac{\mathrm{r}}{r} - \frac{7}{10}\frac{r}{\mathrm{r}}\right)z = 0,$$

namely the *second side* in which the *elliptic cylinder* XII. is cut by a normal plane through the side XIII.; and by comparing the value of its y with the equation XIX., we see that the *least distance between the real asymptote to the osculating twisted cubic, and the axis of revolution of the cylinder* on which the *helix* is traced, is equal to *seven-thirds of the radius of that right cylinder.*

(10.) As regards the *two imaginary asymptotes*, they correspond to the two imaginary values of w, which cause the *common denominator* of the expressions XXI. to vanish; but it may be sufficient here to observe, that because those expressions give, generally,

$$\text{XXIII.} \ldots x + \left(\frac{6}{5}\frac{\mathrm{r}}{r} + \frac{1}{5}\frac{r}{\mathrm{r}}\right)z = w,$$

the two imaginary lines in question are to be considered as being contained in *two imaginary planes*, which are both *parallel to the real plane** *through* P,

$$\text{XXIV.} \ldots x + \left(\frac{6}{5}\frac{\mathrm{r}}{r} + \frac{1}{5}\frac{r}{\mathrm{r}}\right)z = 0;$$

namely to a certain *common normal plane* to the *two real cylinders* XII. and XVIII., or to the *elliptic* and *right cylinders* already mentioned.

(11.) *In general*, instead of seeking to determine, as above, a *cylinder* of the second order, which shall have *contact* of the *fifth order* with *any given curve* of double curvature, at a given point P, we may propose to find a *second cone* of the same (second) order, which shall have *such* contact with that curve at that point, its *vertex* being at *some other point* of space (abc). Writing (comp. 398, LXVI.) the equation of such a *cone* under the form,

$$\text{XXV.} \ldots 2r(cy - bz)(c - z) = (cx - az)^2 + 2B(cx - az)(cy - bz) + C(cy - bz)^2;$$

substituting for xyz the co-ordinates $x_s y_s z_s$ of the curve, under the forms (comp. 398, LVIII.),

$$\text{XXVI.} \ldots \begin{cases} x_s = s - \dfrac{s^3}{6r^2} + \dfrac{a_3 s^4}{24} + \dfrac{a_4 s^5}{120}, \\ y_s = \dfrac{s^2}{2r} - \dfrac{r' s^3}{6r^2} + \dfrac{b_3 s^4}{24} + \dfrac{b_4 s^5}{120}, \\ z_s = \dfrac{s^3}{6r\mathrm{r}} + \dfrac{c_3 s^4}{24} + \dfrac{c_4 s^5}{120}, \end{cases}$$

in which the coefficients $a_3 b_3 c_3$ and $a_4 b_4 c_4$ have the values assigned in (1.); developing according to powers of s, neglecting s^6, and comparing coefficients of s^3, s^4, s^5; we find first the expressions,

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* The *right line at infinity*, in this plane XXIV., is the *common side* of the *two hyperbolic paraboloids* mentioned in the third Note to page 614, as each containing the whole twisted cubic.

$$\text{XXVII.}\ldots B=\frac{-1}{3}\left(r'+\frac{b}{c}\frac{r}{\mathrm{r}}\right),\quad C=-\frac{4}{9}\left(r'+\frac{b}{c}\frac{r}{\mathrm{r}}\right)^2+\frac{4}{3}\left(1+\frac{a}{c}\frac{r}{\mathrm{r}}\right)+\frac{r^3}{3}\left(b_3-\frac{b}{c}c_3\right),$$

which are the *same* for *cone* as for cylinder: and then are led to the *new equation of condition*,

$$\text{XXVIII.}\ldots \frac{r}{5}\left(b_4-\frac{b}{c}c_4\right)=a_3-\frac{a}{c}c_3+\frac{2}{cr\mathrm{r}}+B\left(b_3-\frac{b}{c}c_3-\frac{2}{r^3}-\frac{2a}{cr^2\mathrm{r}}\right)$$
$$-2C\left(\frac{r'}{r^3}+\frac{b}{cr^2\mathrm{r}}\right),$$

which *differs* from the corresponding equation for the determination of a *cylinder* having the *same* (fifth) *order of contact* with the curve, but only by the *one term* $\frac{2}{cr\mathrm{r}}$ in the second member, which term *vanishes* when the co-ordinate c of the vertex is *infinite*.

(12.) Eliminating B and C, and substituting for $a_3b_3c_3$ and $a_4b_4c_4$ their values V. and VI., we find that the condition XXVIII. may be thus expressed (comp. 398, LXVIII.):

$$\text{XXIX.}\ldots ac\left(b-\frac{\mathrm{r}'}{2}c\right)-\mathrm{r}c^2=\mathrm{a}b^3+\mathrm{b}b^2c+\mathrm{c}bc^2+\mathrm{e}c^3;$$

in which we have written, for abridgment,

$$\text{XXX.}\ldots\begin{cases}\mathrm{a}=\dfrac{4}{9}\dfrac{r}{\mathrm{r}};\qquad \mathrm{b}=\dfrac{r'}{3}-\dfrac{r}{\mathrm{r}}\dfrac{\mathrm{r}'}{2};\\[1ex] \mathrm{c}=\dfrac{1}{30}(6r''\mathrm{r}-3r\mathrm{r}''-2r^{-1}r'^2\mathrm{r}-6r'\mathrm{r}'+6r\mathrm{r}^{-1}\mathrm{r}'^2-18r^{-1}\mathrm{r}+12r\mathrm{r}^{-1});\\[1ex] \mathrm{e}=\dfrac{1}{90}(9r'''\mathrm{r}^2-9r^{-1}r'r''\mathrm{r}^2+4r^{-2}r'^3\mathrm{r}^2+36r^{-2}r'\mathrm{r}^2+18r'-27r\mathrm{r}^{-1}\mathrm{r}').\end{cases}$$

The *locus* of the *vertex* of the sought *quadric cone* XXV. is therefore that *cubic surface*, or surface of the *third order*, which is represented by the equation XXIX. in *abc*; this surface, then, is a *second locus* (comp. (4.)) for the *osculating twisted cubic*, whatever the *given curve* in space may be: a *first locus* for that *cubic curve* being still the *quadric cone* (comp. (3.)), of which the equation in *abc* is (by 398, LXVII.* and LXIX.),

$$\text{XXXI.}\ldots 4\left(\frac{\mathrm{r}}{r}\right)b^2=6\left(\frac{\mathrm{r}}{r}\right)_3 ac+\left(\frac{\mathrm{r}^3}{r^3}\right)'bc$$
$$+\frac{\mathrm{r}^4}{5}\left(\frac{9}{r^4}-\frac{21}{r^2\mathrm{r}^2}+\frac{r'^2}{r^4}-\frac{3r''}{r^3}+\frac{3r'\mathrm{r}'}{r^3\mathrm{r}}-\frac{27\mathrm{r}'^2}{4r^2\mathrm{r}^2}+\frac{9\mathrm{r}''}{r^2\mathrm{r}}\right)c^2,$$

and which has contact of the *fifth order* with the *curve*, while its *vertex* is *at the given point* P of osculation.

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* After making the correction indicated in a former Note (to page 613), so as to bring the cited equation into agreement with the earlier formula 398, LVII. The quadric cone XXXI. may be said to have *five-side contact* with the *cone of chords* of the given curve (compare the first Note to page 588).

(13.) Instead of thus introducing, as *data*, the *derivatives* of the two *radii of curvature*, r and r, taken with respect to the *arc*, s, it may be more convenient in many applications to treat the two co-ordinates y and z of the curve as functions of the third co-ordinate x, assumed as the independent variable: and so to write (comp. (6.)) these new developments,

$$\text{XXXII.}\ldots\ y_x = \frac{x^2}{2r} + \frac{y'''x^3}{6} + \frac{y^{\mathrm{IV}}x^4}{24} + \frac{y^{\mathrm{V}}x^5}{120}, \quad z_x = \frac{x^3}{6r\mathrm{r}} + \frac{z^{\mathrm{IV}}x^4}{24} + \frac{z^{\mathrm{V}}x^5}{120};$$

and then the equation of the quadric cone XXXI. will be found to become (in xyz),

$$\text{XXXIII.}\ldots\ y^2 = \frac{3}{2}\frac{r}{\mathrm{r}}xz + 2gyz + hz^2,$$

with the coefficients,

$$\text{XXXIV.}\ldots\ g = r\mathrm{r}\left(y''' - \frac{3}{8}\mathrm{r}z^{\mathrm{IV}}\right), \quad h = \frac{3}{2}r\mathrm{r}^2\left(y^{\mathrm{IV}} - \frac{3}{10}\mathrm{r}z^{\mathrm{V}}\right)$$
$$- r^2\mathrm{r}^2\left(y'''^2 + \frac{3}{4}\mathrm{r}z^{\mathrm{IV}}y''' - \frac{9}{16}\mathrm{r}^2z^{\mathrm{IV}2}\right);$$

while the cubic surface XXIX. will also come to be represented by an equation of the *same form* as before, namely (in xyz) by the following,

$$\text{XXXV.}\ldots\ xz\,(y + \mathrm{h}z) - \mathrm{r}z^2 = \mathrm{a}y^3 + \mathrm{b}y^2z + \mathrm{c}yz^2 + \mathrm{e}z^3,$$

in which the coefficients are,

$$\text{XXXVI.}\ldots\left\{\begin{array}{l} \mathrm{a} = \dfrac{4}{9}\dfrac{r}{\mathrm{r}}\ \text{(as before)}; \quad \mathrm{b} = -\dfrac{4}{3}r^2y''' + \dfrac{r^2\mathrm{r}}{2}z^{\mathrm{IV}}; \quad \mathrm{h} = -r\mathrm{r}y''' + \frac{1}{3}r\mathrm{r}^2z^{\mathrm{IV}}; \\ \mathrm{c} = \dfrac{4}{9}r^3\mathrm{r}y'''^2 - \frac{1}{3}r^3\mathrm{r}^2y'''z^{\mathrm{IV}} - \frac{1}{3}r^2\mathrm{r}y^{\mathrm{IV}} + \frac{1}{10}r^2\mathrm{r}^3z^{\mathrm{V}}; \\ \mathrm{e} = -\dfrac{4}{9}r^4\mathrm{r}^2y'''^3 + \frac{1}{3}r^3\mathrm{r}^2y'''y^{\mathrm{IV}} - \frac{1}{10}r^2\mathrm{r}^3y^{\mathrm{V}}. \end{array}\right.$$

(14.) Whichever set of expressions for the *coefficients* we may adopt, some general consequences may be drawn from the mere *forms* of the *equations*, XXXI. and XXIX., or XXXIII. and XXXV., of the *quadric cone* and *cubic surface*, considered as *two loci* (12.) of the *osculating twisted cubic* to a given curve of double curvature. Thus, if we eliminate ac (comp. 398, (41.)) from XXIX. by XXXI., or xz by XXXIII. from XXXV., we get an equation between b, c, or between y, z, which rises *no higher* than the *third degree*, and is of the *form*,

$$\text{XXXVII.}\ldots\ 2\mathrm{r}z^2 = \mathrm{a}y^3 + \mathrm{b}_{,}y^2z + \mathrm{c}_{,}yz^2 + \mathrm{e}_{,}z^3,$$

with the same value of a as before; such then is the equation of the *projection of the twisted cubic, on the normal plane* to the curve; and we see that, as was to be expected, the *plane cubic* thus obtained has a *cusp* at the given point P, which (when we neglect s^7 or x^7) *coincides* with the *corresponding cusp** of the projection of the given curve of double curvature *itself*, on the same normal plane.

(15.) The equation XXXVII. may also be considered as representing a *cubic cylinder*, which is a *third locus* of the twisted cubic; and on which the *tangent* PT to the curve is a *cusp-edge*, in such a manner that an arbitrary *plane through this line*, suppose the plane

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* Compare the first formula of the first Note to page 594.

$$\text{XXXVIII.} \ldots 3rz = vy,$$

where v is any assumed constant, *cuts* the cylinder *in that line twice*, and a *third* time in a *real* and parallel *right line*, which intersects the *quadric cone* in a point at infinity (because the tangent PT is a *side* of that cone), and in *another real point*, which is *on the twisted cubic*, and may be made to be *any point* of that sought curve, by a suitable value of v: in fact, the plane XXXVIII. *touches both curves* at P, and therefore *intersects* the *cubic* curve in *one* other real point. And thus may *fractional expressions* (comp. (8.)) for the co-ordinates of the *osculating cubic* be found *generally*, which we shall not here delay to write down.

(16.) Without introducing the *cubic cylinder* XXXVII., it is easy to see that *any plane*, such as XXXVIII., which is *tangential* to the given curve at P, *cuts* the *cubic surface* XXXV. in a section which may be said to consist of the *tangent twice taken*, and of a certain *other right line*, which varies with the direction of this secant plane, so that the *locus* XXXV. or XXIX. is a *Ruled Cubic Surface*, with the given tangent PT for a *singular** *line*, which is *intersected* by *all the other right lines* on that surface, determined as above: and if we *set aside this line*, the *remaining part* of the *complete intersection* of that *cubic surface* with the *quadric cone* XXXIII. or XXXI. is the *twisted cubic sought*. We may then consider ourselves to have *completely* and *generally determined* the *Osculating Twisted Cubic* to a curve of double curvature, *without requiring* (as in 398, (41.)), the *solution* of any *cubic* or other *equation*.†

(17.) As illustrations and verifications, it may be added that the general *ruled cubic surface*, and *cubic cylinder*, lately considered, take for the case of the *helix* (2.), the particular forms,‡

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* If the *cubic surface* be cut by a plane *perpendicular* to the tangent PT, at any point T *distinct* from the point P itself, the *section* is a *plane cubic*, which has T for a *double point*; and this point counts for *three* of the *six common points*, or points of *intersection*, of the plane cubic just mentioned with the *plane conic* in which the *quadric cone* is cut by the same secant plane, because *one branch*, or *one tangent*, of the plane cubic at T *touches* the plane conic at that point, in the osculating plane to the given curve at P, while the *other branch*, or the *other tangent*, *cuts* that plane conic there.

† It may be remarked that, by equating the second member of XXXVII. to zero, and changing y, z to b, c, we obtain generally the *cubic equation*, referred to in 398, (41.); and that by suppressing the term $- rc^2$ in XXIX., or the term $- rz^2$ in XXXV., we pass, in like manner *generally*, from the *cubic surface* of recent sub-articles, to the earlier *cubic cone* (4.).

‡ By suppressing the term $- rz^2$, dividing by $\frac{ry}{5r}$, and transposing, we pass for the case of the *helix* from the equation XXXIX. of the *cubic locus*, to the equation IX'. in the last Note to page 614; namely to the equation of that *quadric cone* which forms (in this example) a *separable part* of the general *cubic cone*, the *other part* being here the *tangent plane* ($y = 0$) to the *right cylinder*.

$$\text{XXXIX.}\ldots\ xyz - \mathrm{r}z^2 = \frac{4}{9}\frac{r}{\mathrm{r}}y^3 + \left(\frac{2}{5}\frac{r}{\mathrm{r}} - \frac{3}{5}\frac{\mathrm{r}}{r}\right)yz^2,$$

and

$$\text{XL.}\ldots\ \mathrm{r}z^2 = \frac{2}{9}\frac{r}{\mathrm{r}}y^3 + \frac{3}{10}\left(\frac{r}{\mathrm{r}} + \frac{\mathrm{r}}{r}\right)yz^2;$$

and that accordingly these two last equations are satisfied, independently of w, when the *fractional expressions* XXI. are substituted for xyz.

400. The general theory* of *evolutes of curves in space* may be briefly treated by quaternions, as follows: a *second curve* (in space, or in one plane) being *defined* to bear to a *first curve* the relation of *evolute* to *involute*, when the *first cuts the tangents to the second at right angles.*

(1.) Let ρ and σ be corresponding vectors, OP and OS, of involute and evolute, and let ρ', σ', ρ'', σ'' denote their first and second derivatives, taken with respect to a scalar variable t, on which they are both conceived to depend. Then the two fundamental equations, which express the relation between the two curves, as above defined, are the following:

$$\text{I.}\ldots\ S(\sigma-\rho)\rho' = 0; \qquad \text{II.}\ldots\ V(\sigma-\rho)\sigma' = 0;$$

which express, respectively, that the point S is *in the normal plane* to the *involute* at P, and that the latter point is *on the tangent* to the *evolute* at S: so that the *locus of* P (the involute) is a *rectangular trajectory* to all such *tangents to the locus of* S (the evolute).

(2.) Eliminating $\sigma-\rho$ between the two preceding equations, and taking their derivatives, we find,

$$\text{III.}\ldots\ S\rho'\sigma' = 0, \quad \text{IV.}\ldots\ S(\sigma-\rho)\rho'' - \rho'^2 = 0, \quad \text{V.}\ldots\ V(\sigma-\rho)\sigma'' - V\rho'\sigma' = 0;$$

whence also,

$$\text{VI.}\ldots\ S\rho'\sigma'\sigma'' = 0.$$

(3.) Interpreting these results, we see *first*, by IV. combined with I. (comp. 391, (5.)), that the *point* S of the evolute is *on the polar axis* of the involute at P, and therefore that the *evolute itself* is *some curve on the polar developable* of the *involute;* and *second*, by VI. (comp. 380, I.), that this curve is a *geodetic line* on that *polar surface*, because the *osculating plane to the evolute* at S contains the *tangent to the involute* at P, and therefore also the (parallel) *normal to the locus of evolutes.*

(4.) The *locus of centres of curvature* (395, (6.)) of a *curve in space* is *not* generally an *evolute* of that curve, because the *tangents*† KK′ to that *locus* do not generally *intersect* the *curve* at all; but a *given plane involute* has always the locus just

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* Invented by Monge.

† It might have been remarked, in connexion with a recent series of sub-articles (397), that this tangent KK′ or κ' is inclined to the rectifying line λ, at an angle of which the cosine is,

$$-SU\kappa'\lambda = \pm R^{-1}T\lambda^{-1} = \pm \sin H \cos P;$$

upper or lower signs being taken, according as the second curvature r^{-1} is positive or negative, because $S\kappa'\lambda = -\mathrm{r}^{-1}$.

mentioned for *one* of its evolutes; and has, besides, indefinitely many *others*,[*] which are all *geodetics on the cylinder* which rests perpendicularly on that one *plane evolute* as its *base*.

(5.) An easy combination of the foregoing equations gives,

$$\text{VII.} \ldots (\mathrm{T}(\sigma-\rho))' = -\mathrm{S}(\mathrm{U}(\sigma-\rho).(\sigma'-\rho')) = \mp \mathrm{S}\sigma'\mathrm{U}\sigma' = \pm \mathrm{T}\sigma',$$

or with differentials, $\text{VIII.} \ldots \mathrm{dT}(\sigma-\rho) = \pm \mathrm{Td}\sigma$;

whence by an immediate integration (comp. 380, XXII. and 397, LIV.),

$$\text{IX.} \ldots \Delta\mathrm{T}(\sigma-\rho) = \pm \int \mathrm{Td}\sigma = \pm \textit{ arc of the evolute}:$$

this arc then, *between two points* such as S and S$_1$ of the latter curve, is equal to the *difference between the lengths* of the *two lines*, PS and P$_1$S$_1$, intercepted *between the two curves* themselves.

(6.) Another quaternion combination of the same equations gives, after a few steps of reduction, the differential formula (comp. 335, VI.),

$$\text{X.} \ldots \mathrm{d}\cos \mathrm{OPS} = -\mathrm{dSU}\,\frac{\sigma-\rho}{\rho} = \frac{\mathrm{dT}\rho}{\mathrm{T}(\sigma-\rho)}.\mathrm{S}\frac{\sigma}{\rho};$$

if then the *involute* be a curve *on a given sphere*, with its centre at the origin O, so that the *evolute* is a *geodetic on a concentric cone*, this differential X. *vanishes*, and we have the integrated equation,

$$\text{XI.} \ldots \cos \mathrm{OPS} = \text{const.}, \quad \text{or simply,} \quad \text{XI}'. \ldots \mathrm{OPS} = \text{const.};$$

the *tangents* PS to the evolute being thus *inclined* (in the case here considered) at a *constant angle*,[†] to the *radii* OP of the *sphere*.

(7.) In general, if we denote by R the *interval* $\overline{\mathrm{PS}}$ between two *corresponding points* of involute and evolute, we shall have the equation,

$$\text{XII.} \ldots (\sigma-\rho)^2 + R^2 = 0, \quad \text{or} \quad \text{XII}'. \ldots \mathrm{T}(\sigma-\rho) = R;$$

and the formula VII. may be replaced by the following,

$$\text{XIII.} \ldots R'^2 + \sigma'^2 = 0, \quad \text{or} \quad \text{XIII}'. \ldots \mathrm{D}_t R = \pm \mathrm{TD}_t\sigma,$$

in which the independent variable t is still left *arbitrary*.

(8.) But if we take for that variable the *arc* S$_0$S$_t$ of the *evolute*, measured from some fixed point of that curve, we may then write,

$$\text{XIV.} \ldots t = \int \mathrm{Td}\sigma, \qquad \text{XV.} \ldots \mathrm{d}R_t = \pm \mathrm{d}t, \qquad \text{XVI.} \ldots \mathrm{D}_t R_t = \pm 1;$$

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[*] Compare the first Note to page 534; from the formulæ of which page it *now* appears, that if the *involute* be an *ellipse*, with $\beta = \mathrm{OB}$ and $\gamma = \mathrm{OC}$ for its major and minor semiaxes, and therefore with the scalar equations,

$$(\mathrm{S}\beta^{-1}\rho)^2 + (\mathrm{S}\gamma^{-1}\rho)^2 = 1, \quad \mathrm{S}\beta\gamma\rho = 0,$$

the *evolutes* are geodetics on the *cylinder* of which the corresponding equation is,

$$(\mathrm{S}\beta\sigma)^{\frac{2}{3}} + (\mathrm{S}\gamma\sigma)^{\frac{2}{3}} = (\beta^2 - \gamma^2)^{\frac{2}{3}}.$$

[†] This property of the *evolutes of a spherical curve* was deduced by Professor De Morgan, in a Paper *On the Connexion of Involute and Evolute in Space* (Cambridge and Dublin Mathematical Journal for November, 1851); in which also a *definition* of involute and evolute was proposed, substantially the same as that above adopted.

whence

$$\text{XVII.}\ldots\ \mathrm{D}_t(R_t \mp t) = 0, \quad \text{and} \quad \text{XVIII.}\ldots\ R_t \mp t = \text{const.} = R_0,$$

the integral IX. being thus under a new form reproduced.

(9.) In this *last mode* of obtaining the result,

$$\text{XIX.}\ldots\ \Delta\,\overline{\mathrm{PS}} = R_t - R_0 = \pm t = \pm\, \textit{arc}\ \widehat{\mathrm{S}_0\mathrm{S}_t}\ \textit{of evolute},$$

no use is made of *infinitesimals*,[*] or even of *small differentials*. We only infer, as in XVIII. (comp. 380, (9.)), that the quantity $R_t \mp t$ is *constant*,[†] because its *derivative is null:* it having been previously proved (380, (8.)), as a consequence of our *definition of differentials* (320, 324) that if s be the arc and ρ the vector of *any curve*, then the equation $\mathrm{d}s = \mathrm{Td}\rho$ (380, XXII.) is *rigorously satisfied*, whatever the *independent variable* t may be, and whether the two connected and *simultaneous differentials* be *small* or *large*.

(10.) But when we employ the *notation of integrals*, and introduce, as above, the symbol $\int \mathrm{T}\mathrm{d}s$, we are then led to *interpret* that *symbol* as denoting the *limit of a sum* (comp. 345, (12.)); or to write, generally,

$$\text{XX.}\ldots\ \int \mathrm{Td}\rho = \lim.\ \Sigma\mathrm{T}\Delta\rho, \quad \text{if} \quad \lim.\ \Delta\rho = 0,$$

with analogous formulæ for other cases of *integration in quaternions*. Geometrically, the equation,

$$\text{XXI.}\ldots\ \int \mathrm{Td}\rho = \Delta s, \quad \text{or} \quad \text{XXI}'.\ldots\ \int \mathrm{Td}\sigma = \Delta\, t,$$

if s and t denote *arcs* of curves of which ρ and σ are *vectors*, comes thus to be interpreted as an expression of the well-known principle, that the *perimeter of any curve* (or of any *part* thereof) is the *limit of the perimeter* of an *inscribed polygon* (or of the corresponding *portion* of that polygon), when the *number* of the *sides* is indefinitely *increased*, and when their *lengths* are *diminished* indefinitely.

(11.) The equations I. and XII. give,

$$\text{XXII.}\ldots\ \mathrm{S}\sigma'(\sigma - \rho) + RR' = 0,$$

the independent variable t being again arbitrary; but these equations XII. and XXII. coincide with the formulæ 398, LXXXIX. and XCI.; we may then, by 398, (79.) and (80.), consider the *locus of the point* P as the *envelope of a variable sphere*, namely of the sphere which has S for centre and R for radius, and is represented by the recent equation XII., if $\rho = \mathrm{OP}$ be the vector of a *variable point* thereon.

(12.) But whereas *such* an envelope has been seen to be *generally* a *surface*, which is *real* or *imaginary* (398. (79.)) according as $R'^2 + \sigma'^2 <$ or > 0, we have *here* by XIII. the *intermediate* or *limiting case* (comp. 398, CXXXI.), for which the *circles* of the system become *points*, and the surface itself degenerates into a *curve*, which is here the *involute* (P) above considered. The *involutes of a given curve* (S) are therefore *included*, as a *limit*, in that general *system of envelopes* which was considered in the lately cited subarticles, and in others immediately following.

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[*] In general, it may have been observed that we have hitherto *abstained*, at least in the *text* of this whole Chapter of *Applications*, from making any use of *infinitesimals*, although they have been often referred to in these *Notes*, and employed therein to assist the *geometrical investigation* or *enunciation* of results. But as regards the *mechanism* of *calculation*, it is at least as easy to use *infinitesimals in quaternions* as in any other system: as will perhaps be shown by a few examples, farther on.

[†] Compare the Note to page 516.

(13.) The *equation of condition*, 398, CXXXVI., is in this case satisfied by XIII., both members vanishing; but we cannot now put it under the form 398, CXLI., because in the passage to that form, in 398, (85.), there was tacitly effected a *division by* r^2, which is not now allowed, the radius r of the circle on the envelope being in the present case equal to zero. For a similar reason, we cannot now *divide* by g, as was done in 398, (86.); and because, in virtue of II., the *two* equations 398, CLX. reduce themselves to *one*, they no longer conduct to the formulæ 398, CLX'. CLXI. CLXI'. CLXIII. XCIV.; nor to the second equation 398, CLXII.

(14.) The general geometrical relations of the curves (P) and (S), which were investigated in the sub-articles to 398 for the case when the *condition*[*] above referred to is satisfied, are therefore only very *partially* applicable to a system of *involute* and *evolute* in space: at least if we still consider the *former* curve (the involute) as being a *rectangular trajectory* to the *tangents* to the *latter* (the evolute), instead of being, like the curve (P) previously considered, a rectangular trajectory (398, (94.)) to the *osculating planes*[†] of the curve (S).

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[*] If, without thinking of *evolutes*, we merely suppose that the *condition* 398, CXXXVI. is satisfied, as lately in (13.), by our having the relation $R'^2 + \sigma'^2 = 0$, it will be found (comp. the symbolical expression 274, XX. for $0^{\frac{1}{2}}$, and the imaginary solution in 353, (18.) of the system $\mathrm{S}\gamma\rho = 0,\ \rho^2 = 0$), that the *envelope* of the *sphere* $(\sigma - \rho)^2 + R^2 = 0$, or the *locus* of the (null) *circles* in which such spheres are (conceived to be) *cut* by the (tangent) *planes*, $\mathrm{S}\sigma'(\sigma - \rho) + RR' = 0$, may be said to be *generally* the system of all those *imaginary points*, of which the vectors (or the *bivectors*, comp. 214, (6.)) are assigned by the formula,

$$\rho = \sigma - RR'^{-1}\sigma' + (\mathrm{U}\sigma' + \sqrt{-1})\,\mathrm{V}\sigma'\mu\,;$$

where μ is an *arbitrary vector*, and $\sqrt{-1}$ is the *old imaginary* of algebra. By making $\mu = 0$ we reduce this expression for ρ to the *real vector form*,

$$\rho = \sigma - RR'^{-1}\sigma' = \sigma + RR'\sigma'^{-1},$$

= the κ of 398, CXXXI."; and thus the *curve* (P), which is here the *locus of the centres of the null circles of contact*, and coincides with the *involute* in the present series of sub-articles, may *still* be called a *Singular Line* upon the *Envelope of the Sphere* (with *One Variable Parameter*), as being in the present case the *only real part* of that *elsewhere imaginary surface*.

[†] The curve to the *osculating planes* of which another curve is thus an *orthogonal trajectory*, and which is therefore (398, (95.)) the *cusp-edge* of the *polar developable* of the latter curve, was called by Lancret its *evolute by the plane* (developpée par le plan); whereas the curve (S) of the *present series* (400) of sub-articles, to whose *tangents* the corresponding curve (P) is an orthogonal trajectory, has been called by way of distinction the *evolute by the thread* (developpée par le fil) of this last curve. It would be improper to delay here on subjects so well known to geometers: but the student may be invited to read again, in connexion with them, the sub-articles (88.) and (89.) to Art. 398.

(15.) If the *arc* of the *evolute* be again taken for the independent variable t, and if the positive direction of motion along that arc be always *towards* the *involute*, we may write,

$$\text{XXIII.}\ldots\ \rho = \dot{\sigma} + R\sigma', \quad R' = -1, \quad \sigma'^2 = -1, \text{ \&c.};$$

whence

$$\text{XXIV.}\ldots\ \rho' = R\sigma'', \quad \rho'' = R\sigma''' - \sigma'', \quad \mathrm{V}\rho''\rho' = R^2\mathrm{V}\sigma'''\sigma'';$$

if then $\kappa = \text{OK}$ be the vector of the centre K of the circle which osculates to the involute at P, the general formula 389, IV. gives, after a few reductions,* the expression (comp. 397, XVI. XXXIV., and XCVIII. (15)),

$$\text{XXV.}\ldots\ \kappa = \rho + \frac{\rho'^3}{\mathrm{V}\rho''\rho'} = \sigma + R\left(\sigma' + \frac{\sigma''^3}{\mathrm{V}\sigma'''\sigma''}\right)$$
$$= \sigma + \frac{R\mathrm{S}\sigma'\sigma'''\sigma''}{\mathrm{V}\sigma'''\sigma''} = \sigma - \frac{R\mathrm{S}\sigma'\sigma''^{-1}\sigma'''}{\mathrm{V}\sigma'''\sigma''^{-1}}$$
$$= \sigma - R\mathrm{r}_1^{-1}\lambda_1^{-1} = \sigma + \mathrm{U}\lambda_1 . R \cos H_1,$$

if r_1, H_1, and λ_1 be what r, H, and λ in 397 become, when we pass from the curve (P) to the curve (S), with the present relations between those two curves; this *centre of curvature* K is therefore the *foot of the perpendicular* let fall *from the point* P *of the involute, on the rectifying line* λ_1 *of the evolute*: as indeed is evident from geometrical considerations, because by (3.) this *rectifying line* of the curve (S) is the *polar axis* of the curve (P).

(16.) If we conceive (comp. 389, (2.)) an *auxiliary spherical curve* to be described, of which the variable unit-vector shall be,

$$\text{XXVI.}\ldots\ \text{OT} = \tau = \sigma' = \mathrm{U}(\rho - \sigma) = R^{-1}(\rho - \sigma),$$

and suppose that υ is the vector OU of the centre of curvature of this *new curve*, at the point T which corresponds to the point S of the evolute, we shall then have by XXV. the expression,

$$\text{XXVII.}\ldots\ \text{TU} = \upsilon - \tau = \frac{\tau'^3}{\mathrm{V}\tau''\tau'} = \frac{\sigma''^3}{\mathrm{V}\sigma'''\sigma''} = \frac{\kappa - \rho}{R} = \text{PK} : \overline{\text{PS}};$$

we have therefore this *theorem*, that the *inward radius of curvature of the hodograph of the evolute* (conceived to be an *orbit* described, as in 379, (9.), with a *constant velocity* taken for *unity*) is equal to the *inward radius of curvature of the involute, divided by the interval R* between the two curves (P) and (S): and that these *two radii of curvature*, TU and PK, have one *common direction*, at least if the direction of *motion on the evolute* be supposed, as in (15.), to be *towards the involute*.

(17.) The following is perhaps a simpler enunciation of the theorem† just stated:—*If* P, P_1, P_2, .. *and* S, S_1, S_2, .. *be corresponding points of involute and evo-*

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* Especially by observing that $\mathrm{V}\sigma'\mathrm{V}\sigma'''\sigma'' = -\sigma''^3$, because $\mathrm{S}\sigma'\sigma'' = 0$, and $\mathrm{S}\sigma'\sigma''' = -\sigma''^2$.

† Some additional light may be thrown on this theorem, by comparing it with the *construction* in 397, (48.); and by observing that the equations 397, XVI. XXXIV. give generally, in the notations of the Article referred to, for the vector of the *centre of curvature* of the *hodograph of any curve*, the transformations,

$$\tau + \frac{\tau'}{\mathrm{V}\tau''\tau'^{-1}} = \tau + \frac{\tau'}{\lambda} = -\mathrm{r}^{-1}\lambda^{-1} = \mathrm{U}\lambda . \cos H.$$

lute, and if we draw lines $\mathrm{ST}_1 \parallel \mathrm{S}_1\mathrm{P}_1$, $\mathrm{ST}_2 \parallel \mathrm{S}_2\mathrm{P}_2, \ldots$ *with a common length* $= \overline{\mathrm{SP}}$, *the spherical curve* $\mathrm{PT}_1\mathrm{T}_2 \ldots$ *will then have contact of the second order with the curve* $\mathrm{PP}_1\mathrm{P}_2 \ldots$, that is with the involute at P.

401. The fundamental formula 389, IV., for the vector of the centre of the osculating circle to a curve in space, namely the formula,

$$\text{I.} \ldots \kappa = \rho + \frac{\rho'^3}{V\rho''\rho'}, \quad \text{or} \quad \text{II.} \ldots \kappa = \rho + \frac{d\rho^3}{Vd^2\rho d\rho},$$

which has been so extensively employed throughout the present Section, has hitherto been established and used in connexion with *derivatives* and *differentials* of vectors, rather than with *differences*, great or small. We may however establish, in another way, an essentially *equivalent* formula, into which *differences* enter by their *limits* (or rather by their *limiting relations*), namely, the following,

$$\text{III.} \ldots \kappa = \rho + \lim. \frac{\Delta\rho^3}{V\Delta^2\rho\Delta\rho}, \quad \text{if} \quad \lim. \Delta\rho = 0, \quad \text{and} \quad \lim. \frac{\Delta^2\rho}{\Delta\rho} = 0,$$

the denominator $V\Delta^2\rho\Delta\rho$ being understood to signify the same thing as $V(\Delta^2\rho . \Delta\rho)$; and *then* may, if we think fit, *interpret* the *differential expression* II. as if $d\rho$ and $d^2\rho$ in it denoted *infinitesimals*,* of the *first and second orders:* with similar interpretations in other but analogous investigations.

(1.) If in the second expression 316, L., for the perpendicular from O on the line AB, we change α and β to their reciprocals (comp. Figs. 58, 64) and then take the reciprocal of the result, we obtain this new expression,

$$\text{IV.} \ldots \mathrm{OD} = \delta = \frac{\alpha^{-1} - \beta^{-1}}{V\beta^{-1}\alpha^{-1}} = \frac{\alpha(\beta - \alpha)\beta}{V\beta\alpha} = \frac{\mathrm{OA}.\mathrm{AB}.\mathrm{OB}}{V(\mathrm{OB}.\mathrm{OA})},$$

in the denominator of which, OB may be replaced by AB, or by AO + AB, for the *diameter* OD of the *circle* OAB; so that if C be the *centre* of this circle, its vector $\gamma = \mathrm{OC} = \frac{1}{2}\mathrm{OD} = \frac{1}{2}\delta =$ &c. Supposing then that P, Q, R are *any three points* of *any given curve in space*, while O is as usual an *arbitrary origin*, and writing

$$\text{V.} \ldots \mathrm{OP} = \rho, \quad \mathrm{OQ} = \rho + \Delta\rho, \quad \mathrm{OR} = \rho + 2\Delta\rho + \Delta^2\rho,$$

and therefore

$$\text{VI.} \ldots \mathrm{PQ} = \Delta\rho, \quad \mathrm{QR} = \Delta\rho + \Delta^2\rho, \quad \tfrac{1}{2}\mathrm{PR} = \Delta\rho + \tfrac{1}{2}\Delta^2\rho,$$

the *centre* C of the circle PQR has the following *rigorous expression* for its vector:

$$\text{VII.} \ldots \mathrm{OC} = \gamma = \rho + \frac{\Delta\rho(\Delta\rho + \Delta^2\rho)(\Delta\rho + \frac{1}{2}\Delta^2\rho)}{V(\Delta^2\rho . \Delta\rho)};$$

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* Compare 345, (17.), and the first Note to page 623.

whence passing to the *limit*, we obtain successively the expressions III. and II. for the vector κ of the *centre of curvature* to the curve PQR at P; the two other points, Q and R, being *both* supposed to *approach indefinitely* to the given point P, according to *any law* (comp. 392, (6.)), which allows the two successive vector chords, PQ and QR, to bear to each other an *ultimate ratio of equality.*

(2.) Instead of thus *first* forming a *rigorous* expression, such as VII., involving the *differences* $\Delta\rho$ and $\Delta^2\rho$; then *simplifying* the formula so found, by the rejection of *terms*, which become *indefinitely small*, with respect to the terms retained; and finally changing differences to *differentials* (comp. 344, (2.)), namely $\Delta\rho$ to $\mathrm{d}\rho$, and $\Delta^2\rho$ to $\mathrm{d}^2\rho$, in the *homogeneous* expression which results, and of which the *limit* is to be taken: we may *abridge* the *calculation*, by *at once writing* the *differential symbols*, in place of *differences*, and *at once suppressing* any terms, of which we *foresee* that they must *disappear from the final result.* Thus, in the recent example, when we have perceived, by quaternions, that if K be the centre of the circle PQR, the equation

$$\text{VIII.} \ldots \mathrm{PK} = \frac{\mathrm{PQ}\,.\,\mathrm{QR}\,.\,\frac{1}{2}(\mathrm{PQ}+\mathrm{QR})}{\mathrm{V}\{(\mathrm{QR}-\mathrm{PQ})\,\mathrm{PQ}\}}$$

is *rigorous*, we may *at once* change *each* of the *three factors* of the *numerator* to $\mathrm{d}\rho$, while the factor $\mathrm{QR}-\mathrm{PQ}$ in the *denominator* is to be changed to $\mathrm{d}^2\rho$; and thus the *differential expression* II., for the *inward vector-radius of curvature* $\kappa-\rho$, is at once obtained.

(3.) It is scarcely necessary to observe, that this expression for that *radius*, as a *vector*, agrees with and *includes* the known expressions for the same radius of curvature of a curve in space, considered as a (positive) *scalar*, which has been denoted in the present Section by the italic letter r (because the more usual symbol ρ would have *here* caused confusion). Thus, while the formula II. gives *immediately* (because $\mathrm{Td}\rho = \mathrm{d}s$) the equation,

$$\text{IX.} \ldots r^{-1}\mathrm{d}s^3 = \mathrm{TVd}\rho\mathrm{d}^2\rho,$$

it gives also (because $\mathrm{d}\rho^2 = -\mathrm{d}s^2$, and $\mathrm{Sd}\rho\mathrm{d}^2\rho = -\mathrm{d}s\mathrm{d}^2s$) the transformed equation,

$$\text{X.} \ldots r^{-1}\mathrm{d}s^2 = \sqrt{}(\mathrm{Td}^2\rho^2 - \mathrm{d}^2s^2);$$

and it conducts (by 389, VI.) to this still simpler formula (comp. the equation $r^{-1} = \mathrm{T}\tau'$, 396, IX.),

$$\text{XI.} \ldots r^{-1}\mathrm{d}s = \mathrm{TdUd}\rho.$$

(4.) Accordingly, if we employ the *standard trinomial form* (295, I.) for a *vector*,

$$\text{XII.} \ldots \rho = ix + jy + kz,$$

which gives, by the laws of the symbols ijk (182, 183),

$$\text{XIII.} \ldots \left\{\begin{array}{ll} \mathrm{d}\rho = i\mathrm{d}x + j\mathrm{d}y + k\mathrm{d}z, & \mathrm{d}s = \mathrm{Td}\rho = \sqrt{}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \\ \mathrm{d}^2\rho = i\mathrm{d}^2x + j\mathrm{d}^2y + k\mathrm{d}^2z, & \mathrm{Td}^2\rho = \sqrt{}(\mathrm{d}^2x^2 + \mathrm{d}^2y^2 + \mathrm{d}^2z^2), \\ \mathrm{Vd}\rho\mathrm{d}^2\rho = i(\mathrm{d}y\mathrm{d}^2z - \mathrm{d}z\mathrm{d}^2y) & +\, j(\mathrm{d}z\mathrm{d}^2x - \mathrm{d}x\mathrm{d}^2z) + k(\mathrm{d}x\mathrm{d}^2y - \mathrm{d}y\mathrm{d}^2x), \\ \mathrm{Ud}\rho = i\dfrac{\mathrm{d}x}{\mathrm{d}s} + j\dfrac{\mathrm{d}y}{\mathrm{d}s} + k\dfrac{\mathrm{d}z}{\mathrm{d}s}, & \mathrm{dUd}\rho = i\mathrm{d}\dfrac{\mathrm{d}x}{\mathrm{d}s} + \ldots, \end{array}\right.$$

the recent equations IX. X. XI. take these *known forms*:

$$\text{IX}'. \ldots r^{-1}\mathrm{d}s^3 = \sqrt{}((\mathrm{d}y\mathrm{d}^2z - \mathrm{d}z\mathrm{d}^2y)^2 + \ldots);$$

$$\text{X}'. \ldots r^{-1}\mathrm{d}s^2 = \sqrt{}(\mathrm{d}^2x^2 + \mathrm{d}^2y^2 + \mathrm{d}^2z^2 - \mathrm{d}^2s^2);$$

$$\text{XI}'. \ldots r^{-1}\mathrm{d}s = \sqrt{\left(\left(\mathrm{d}\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 + \left(\mathrm{d}\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2\right)}.$$

(5.) The formula IV., which lately served us to determine a diameter of a circle through three given points, may be more symmetrically written as follows. *If* AD *be a diameter of the circle* ABC, *then*

$$\text{XIV.}\ldots \quad \mathrm{AD}.\mathrm{V}(\mathrm{AB}.\mathrm{BC}) = \mathrm{AB}.\mathrm{BC}.\mathrm{CA}\,;$$

an equation* in which $\mathrm{V}(\mathrm{AB}.\mathrm{BC})$ may be changed to $\mathrm{V}(\mathrm{AB}.\mathrm{AC})$, &c., and in which it may be remarked that each member is an expression (comp. 296, V.) for a *vector* AT, which *touches at* A *the segment* ABC: while its *length* is at once a representation of the *product of the lengths of the sides of the triangle* ABC, and also of the *double area of that triangle* (comp. 281, XIII.), *multiplied by the diameter of the circumscribed circle.*

(6.) In general, if PQRS be *any four concircular points*, they satisfy (by 260, IX., comp. 296, (3.)) the *condition of concircularity*,

$$\text{XV.}\ldots \quad \mathrm{V}\left(\frac{\mathrm{PS}}{\mathrm{SQ}}.\frac{\mathrm{QR}}{\mathrm{RP}}\right) = 0,$$

which may be thus transformed:†

$$\text{XVI.}\ldots \quad \mathrm{V}\left(\frac{\mathrm{PQ}}{\mathrm{PS}} + \frac{\mathrm{QP}+\mathrm{QR}}{\mathrm{PR}}\right) = \mathrm{V}\left(\frac{1}{\mathrm{PS}}.\mathrm{PQ}.\frac{\mathrm{QP}+\mathrm{QR}}{\mathrm{PR}}\right).$$

Writing then (comp. VI., and the remarks in (2.)),

$$\text{XVII.}\ldots \quad \mathrm{PS} = \omega - \rho, \quad \mathrm{PQ} = \mathrm{d}\rho, \quad \mathrm{PR} = 2\mathrm{d}\rho + \mathrm{d}^2\rho, \quad \mathrm{QP} + \mathrm{QR} = \mathrm{d}^2\rho,$$

the second member is seen to be, *on the present plan*, an *infinitesimal* of the *second order*, which is therefore to be *suppressed*, because the first member is only of the *first order*; and thus we obtain at once the following *vector equation of the osculating circle* to the curve PQR at P,

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* A student might find it useful practice to verify, that if we write in like manner,

$$\text{XIV}'.\ldots \quad \mathrm{BE}.\mathrm{V}(\mathrm{BC}.\mathrm{CA}) = \mathrm{BC}.\mathrm{CA}.\mathrm{AB},$$

so that BE is a *second diameter*, then $\mathrm{AB} = \mathrm{ED}$, or ABDE is a *parallelogram*. He may employ the principles, that $\alpha\beta\gamma = \gamma\beta\alpha$, if $\mathrm{S}\alpha\beta\gamma = 0$, and that $\beta\gamma - \gamma\beta = 2\mathrm{V}\beta\gamma$; in virtue of which, after subtracting XIV′. from XIV., and dividing by $\mathrm{V}(\mathrm{BC}.\mathrm{CA})$, or by its equal $\mathrm{V}(\mathrm{AB}.\mathrm{BC})$, the equation $\mathrm{AD} - \mathrm{BE} = 2\mathrm{AB}$ is obtained, and proves the relation mentioned. It is easy also to prove that

$$\text{XIV}''.\ldots \quad \mathrm{BD}.\mathrm{V}(\mathrm{BC}.\mathrm{CA}) = \mathrm{AB}.\mathrm{S}(\mathrm{BC}.\mathrm{CA}),$$

and therefore that ABDE is a *rectangle.*

† Without having recourse to this *transformation* XVI., we might treat the *condition* XV. by *infinitesimals*, as follows:

$$\text{XVII}'.\ldots \quad \begin{cases} \dfrac{\mathrm{PS}}{\mathrm{QS}} = 1 + \dfrac{\mathrm{PQ}}{\mathrm{QS}} = 1 + \dfrac{\mathrm{d}\rho}{\omega - \rho - \mathrm{d}\rho} = 1 + \dfrac{\mathrm{d}\rho}{\omega - \rho}\,; \\[2ex] \dfrac{2\mathrm{QR}}{\mathrm{PR}} = 1 + \dfrac{\mathrm{QP}+\mathrm{QR}}{\mathrm{PR}} = 1 + \dfrac{\mathrm{d}^2\rho}{2\mathrm{d}\rho + \mathrm{d}^2\rho} = 1 + \dfrac{\mathrm{d}^2\rho}{2\mathrm{d}\rho} \end{cases}$$

equating then to zero the vector part of the product of these two expressions, and suppressing the infinitesimal of the second order, the equation XVIII. of the osculating circle is obtained anew.

$$\text{XVIII.}\ldots \mathrm{V}\left(\frac{\mathrm{d}\rho}{\omega-\rho}+\frac{\mathrm{d}^2\rho}{2\mathrm{d}\rho}\right)=0\,;$$

which agrees with the equation 392, VI., although deduced in a quite different manner, and conducts anew to the expression II. for $\kappa - \rho$, under the form,

$$\text{XIX.}\ldots \frac{\mathrm{d}\rho}{\kappa-\rho}+\mathrm{V}\frac{\mathrm{d}^2\rho}{\mathrm{d}\rho}, \text{ as in 392, VIII.}$$

(7.) Again, if $\mathrm{OD} = \delta$ be the *diameter* from the origin, of any *sphere* through that point O, which passes also through any *three other given points* A, B, C, with $\mathrm{OA} = \alpha$, &c., we have by 296, XXVI. the formula,

$$\text{XX.}\ldots \delta\mathrm{S}\alpha\beta\gamma = \mathrm{V}\alpha(\beta-\alpha)(\gamma-\beta)\gamma\,;$$

writing then (comp. XVII.),

$$\text{XXI.}\ldots \alpha=\mathrm{d}\rho,\quad \beta-\alpha=\mathrm{d}\rho+\mathrm{d}^2\rho,\quad \gamma-\beta=\mathrm{d}\rho+2\mathrm{d}^2\rho+\mathrm{d}^3\rho,$$

and

$$\text{XXII.}\ldots \delta=2\mathrm{PS}=2(\sigma-\rho),$$

where σ is (as in 395, &c.) the vector OS (from an *arbitrary origin* O) of the *centre* S of the *osculating sphere* to a curve of double curvature at P, we have by *infinitesimals*, suppressing terms which are of the *seventh* and higher *orders*, because the first member is only of the *sixth* order, and reducing* by the rules of quaternions,

$$\begin{aligned}\text{XXIII.}\ldots(\sigma-\rho)\mathrm{S}\mathrm{d}\rho\mathrm{d}^2\rho\mathrm{d}^3\rho &= \tfrac{1}{2}\mathrm{V}\mathrm{d}\rho(\mathrm{d}\rho+\mathrm{d}^2\rho)(\mathrm{d}\rho+2\mathrm{d}^2\rho+\mathrm{d}^3\rho)(3\mathrm{d}\rho+3\mathrm{d}^2\rho+\mathrm{d}^3\rho)\\ &= 3\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho\mathrm{S}\mathrm{d}\rho\mathrm{d}^2\rho+\mathrm{d}\rho^2\mathrm{V}\mathrm{d}^3\rho\mathrm{d}\rho\,;\end{aligned}$$

which agrees precisely with the formula 395, XIII., although obtained by a process so different.

(8.) Finally as regards the *osculating plane*, and the *second curvature*, of a curve in space, *infinitesimals* give *at once* for that *plane* the equation,

$$\text{XXIV.}\ldots \mathrm{S}(\omega-\rho)\mathrm{d}\rho\mathrm{d}^2\rho=0, \text{ agreeing with 376, V.;}$$

and if *three consecutive elements* of the curve be represented (comp. XXI.) by the differential expressions,

$$\text{XXV.}\ldots \mathrm{PQ}=\mathrm{d}\rho,\quad \mathrm{QR}=\mathrm{d}\rho+\mathrm{d}^2\rho,\quad \mathrm{RS}=\mathrm{d}\rho+2\mathrm{d}^2\rho+\mathrm{d}^3\rho,$$

the second curvature r^{-1}, defined as in 396, is easily seen to be connected as follows with the *angle* of a certain *auxiliary quaternion* q, which *differs infinitely little from unity*:

$$\text{XXVI.}\ldots r^{-1}\mathrm{d}s=\angle q, \quad\text{if}\quad \text{XXVII.}\ldots q=\frac{\mathrm{V}(\mathrm{QR}\ \ \mathrm{RS})}{\mathrm{V}(\mathrm{PQ}\,.\,\mathrm{QR})}=1+\frac{\mathrm{V}\mathrm{d}\rho\mathrm{d}^3\rho}{\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho}\,;$$

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* Of the eighteen terms which would follow the sign of operation $\tfrac{1}{2}\mathrm{V}$, if the second member of XXIII. were fully developed, *one* is of the *fourth order*, but is a *scalar*; *three* are of the *fifth* order, but have a *scalar sum*; *nine* are of orders *higher* than the sixth; and *two* terms of the *sixth* order are *scalars*, so that there remain only *three terms* of that order to be considered. In this manner it is found that the second member in question reduces itself to the sum of the *two vector parts*,

$$\tfrac{3}{2}\mathrm{V}.(\mathrm{d}\rho\mathrm{d}^2\rho)^2=3\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho\,.\,\mathrm{S}\mathrm{d}\rho\mathrm{d}^2\rho,$$

and

$$\tfrac{1}{2}\mathrm{d}\rho^2\mathrm{V}(\mathrm{d}\rho\mathrm{d}^3\rho+3\mathrm{d}^3\rho\mathrm{d}\rho)=\mathrm{d}\rho^2\mathrm{V}\mathrm{d}^3\rho\mathrm{d}\rho\,;$$

and thus the third member of XXIII. is obtained.

we have then the expression,

$$\text{XXVIII.}\ .\quad \textit{Second Curvature} = \mathrm{r}^{-1} = \frac{\mathrm{V}q}{\mathrm{d}\rho} = \mathrm{S}\,\frac{\mathrm{d}^3\rho}{\mathrm{Vd}\rho\mathrm{d}^2\rho},$$

which agrees with the formula 397, XXVII., and has been illustrated, in the sub-articles to 397 and 398, by numerous geometrical applications.

(9.) On the whole, then, it appears that although the *logic of derived vectors*, and of *differentials of vectors* considered as *finite lines*, *proportional* to such *derivatives*, is perhaps *a little clearer* than that of *infinitesimals*, because it shows *more evidently* (especially when combined with *Taylor's Series adapted to Quaternions*, 342, 375) that *nothing is neglected*, yet it is perfectly *possible* to *combine** quaternions, *in practice*, with methods founded on the *more usual notion* of *Differentials*, as *infinitely small differences*: and that when this combination is judiciously made, *abridgments of calculation* arise, *without any ultimate error*.

SECTION 7.—*On Surfaces of the Second Order; and on Curvatures of Surfaces.*

402. As early as in the First Book of these *Elements*, some specimens were given of the treatment or expression of *Surfaces of the Second Order* by *Vectors*; or by *Anharmonic Equations* which were *derived* from the theory of *vectors*, without any introduction, at that stage, of *Quaternions* properly so called. Thus it was shown, in the sub-articles to 98, that a very simple *anharmonic equation* ($xz = yw$) might represent either a *ruled paraboloid*, or a *ruled hyperboloid*, according as a certain *condition* ($ac = bd$) was or was not satisfied, by the *constants* of the surface. Again, in the sub-articles to 99, *two* examples were given, of *vector expressions* for *cones* of the *second order* (and *one* such expression for a *cone* of the *third* order, with a *conjugate ray* (99, (5.)); while an expression of the same sort, namely,

$$\text{I.}\ .\ .\ \rho = x\alpha + y\beta + z\gamma, \quad \text{with} \quad x^2 + y^2 + z^2 = 1,$$

was assigned (99, (2.)) as representing *generally* an *ellipsoid*,† with α, β, γ, or OA, OB, OC, for three *conjugate semidiameters*. And finally,

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* Compare the first Note to page 623. It will however be of course necessary, in any *future applications* of quaternions, to specify in *which* of these *two senses*, as a *finite differential*, or as an *infinitesimal*, such a *symbol* as $\mathrm{d}\rho$ is employed.

† In like manner the expression,

$$\text{II.}\ .\ .\ \rho = x\alpha + y\beta + z\gamma, \quad \text{with} \quad x^2 + y^2 - z^2 = 1, \ \text{or} = -1,$$

represents a *general hyperboloïd*, of *one sheet*, or of *two*, with $\alpha\beta\gamma$ for conjugate semidiameters: while, with the *scalar equation* $x^2 + y^2 - z^2 = 0$, the *same vector expression* represents their *common asymptotic cone* (not generally of *revolution*).

in the sub-articles (11.) and (12.) to Art. 100, an instance was furnished of the determination of a *tangential plane to a cone*, by means of *partial derived vectors*.

403. In the Second Book, a much greater *range of expression* was attained, in consequence of the introduction of the *peculiar symbols*, or *characteristics of operation*, which belong to the present Calculus; but still with that *limitation* which was caused, by the *conception* and *notation* of a *Quaternion* being confined, in that Book, to *Quotients of Vectors* (112, 116, comp. 307, (5.)), without yet admitting *Products* or *Powers* of *Directed Lines in Space*: although *versors*, *tensors*, and even *norms*[*] of such *vectors* were already introduced (156, 185, 273).

(1.) The *Sphere*,[†] for instance, which has its *centre* at the *origin*, and has the *vector* OA, or α, with a *length* $\mathrm{T}\alpha = a$, for one of its *radii*, admitted of being represented, not only (comp. 402, I.) by the *vector expression*,

$$\text{I.} \ldots \rho = x\alpha + y\beta + z\gamma, \quad x^2 + y^2 + z^2 = 1,$$

with

$$\text{I}'. \ldots \mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma = a, \quad \text{and} \quad \text{I}''. \ldots \mathrm{S}\frac{\beta}{\alpha} = \mathrm{S}\frac{\gamma}{\alpha} = \mathrm{S}\frac{\gamma}{\beta} = 0,$$

but also by any one of the following *equations*, in which it is permitted to change α to $-\alpha$:

$$\text{II.} \ldots \frac{\alpha}{\rho} = \mathrm{K}\frac{\rho}{\alpha}; \qquad \text{III.} \ldots \frac{\rho}{\alpha}\mathrm{K}\frac{\rho}{\alpha} = 1; \quad \text{IV.} \ldots \mathrm{N}\frac{\rho}{\alpha} = 1; \quad 145, (8.), (12.)$$

$$\text{V.} \ldots \mathrm{T}\rho = a; \qquad \text{VI.} \ldots \mathrm{T}\rho = \mathrm{T}\alpha; \quad \text{VII.} \ldots \mathrm{T}\frac{\rho}{\alpha} = 1; \quad 186, (2.), \; 187, (1.)$$

$$\text{VIII.} \ldots \mathrm{S}\frac{\rho - \alpha}{\rho + \alpha} = 0; \qquad \text{IX.} \ldots \mathrm{N}\frac{\rho}{\iota} = \mathrm{N}\frac{\alpha}{\iota}; \quad \text{X.} \ldots \mathrm{N}\rho = \mathrm{N}\alpha; \quad 200, (11.), \; 215, (10.), \; 273, (1.)$$

$$\text{XI.} \ldots \left(\mathrm{S}\frac{\rho}{\alpha}\right)^2 - \left(\mathrm{V}\frac{\rho}{\alpha}\right)^2 = 1; \quad \text{XII.} \ldots \mathrm{NS}\frac{\rho}{\alpha} + \mathrm{NV}\frac{\rho}{\alpha} = 1; \quad 204, (6.), \text{XXV., XXVI.}$$

$$\text{XIII.} \ldots \mathrm{N}\left(\mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\alpha}\right) = 1; \quad \text{XIV.} \ldots \mathrm{T}\left(\mathrm{S}\frac{\rho}{\alpha} + \mathrm{V}\frac{\rho}{\alpha}\right) = 1; \quad 204, (9.)$$

or by the *system of equations*,

$$\text{XV.} \ldots \mathrm{S}\frac{\rho}{\alpha} = x, \quad \left(\mathrm{V}\frac{\rho}{\alpha}\right)^2 = x^2 - 1 \; (\leqq 0), \qquad 204, (4.)$$

representing a *system of circles*, with the *spheric surface* for their *locus*.

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[*] The *notation* $\mathrm{N}\alpha$, for $(\mathrm{T}\alpha)^2$, although not formally introduced before Art. 273, had been used by anticipation in 200, (3.), page 188.

[†] That is to say, the *spheric surface* through A, with O for centre. Compare the Note to page 197.

(2.) *Other forms* of equation, for the same spheric surface, may on the same principles be assigned; for example we may write,

$$\text{XVI.} \ldots \frac{\rho}{\alpha} = \mathrm{K}\frac{\alpha}{\rho}; \qquad \text{XVII.} \ldots \mathrm{N}\frac{\alpha}{\rho} = 1; \qquad \text{XVIII.} \ldots \mathrm{T}\frac{\alpha}{\rho} = 1;$$

$$\text{XIX.} \ldots \angle\frac{\rho - \alpha}{\rho + \alpha} = \frac{\pi}{2}; \qquad \text{XX.} \ldots \mathrm{S}\frac{2\alpha}{\rho + \alpha} = 1; \qquad \text{XXI.} \ldots \mathrm{S}\frac{2\rho}{\rho + \alpha} = 1;$$

or (comp. 186, (5.), and 200, (3.)),

$$\text{XXII.} \ldots \mathrm{T}(\rho - c\alpha) = \mathrm{T}(c\rho - \alpha), \quad c^2 \gtrless 1;$$

under which *last form*, the *sphere* may be considered to be generated by the *revolution* of the *circle*, which has been already spoken of as the *Apollonian** *Locus*.

(3.) And from *any one* to *any other*, of all these various *forms*, it is possible, and easy to *pass*, by general *Rules of Transformation*,† which were established in the Second Book: while *each* of them is capable of receiving, on the principles of the same Book, a *Geometrical Interpretation*.

(4.) But we could not, on the principles of the Second Book *alone*, advance to such subsequent *equations* of the same *sphere*, as

$$\text{XXIII.} \ldots \rho^2 = \alpha^2, \quad \text{or} \quad \text{XXIV.} \ldots \rho^2 + \alpha^2 = 0, \qquad 282, \text{ VII. XIII.}$$

whereof the latter includes (282, (9.)) the important equation $\rho^2 + 1 = 0$, or $\rho^2 = -1$, of what we have called the *Unit-Sphere* (128); nor to such an *exponential expression* for the *variable vector* ρ of the *same spheric surface*, as

$$\text{XXV.} \ldots \rho = \alpha k^t j^s k j^{-s} k^{-t}, \qquad 308, \text{ XVIII.}$$

in which j and k belong to the fundamental system ijk of *three rectangular unit-lines* (295), connected by the fundamental Formula A of Art. 183, namely,

$$i^2 = j^2 = k^2 = ijk = -1, \qquad \text{(A)}$$

while s and t are *two arbitrary* and *scalar variables*, with simple *geometrical*‡ *significations*: because we were not *then* prepared to introduce any *symbol*, such as ρ^2, or k^t, which should represent a *square* or other *power* of a *vector*.§ And similar remarks apply to the representation, by quaternions, of *other surfaces of the second order*.

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* Compare the first Note to page 128.

† This *richness of transformation*, of *quaternion expressions* or equations, has been noticed, by some friendly critics, as a *characteristic* of the present *Calculus*. In the preceding parts of this work, the reader may compare pages 128, 140, 183, 573, 574, 575; in the two last of which, the *variety* of the expressions for the *second curvature* (r^{-1}) of a *curve in space* may be considered worthy of remark. On the other hand, it may be thought remarkable that, in this Calculus, a *single expression*, such as that given by the first formula (389, IV.) of page 532, adapts itself with *equal ease* to the determination of the vector (κ) of the *centre* of the *osculating circle*, to a *plane curve*, and to a curve of *double curvature*, as has been sufficiently exemplified in the foregoing Section.

‡ Compare the second Note to page 365.

§ It is true that the formula A was established in the course of the Second Book (page 160); but it is to be remembered that the symbols ijk were *there* treated as denoting a system of *three right versors*, in *three mutually rectangular planes* (181):

404. A brief review, or *recapitulation*, of some of the chief expressions connected with the *Ellipsoid*, for example, which have been already established in these *Elements*, with *references* to a few others, may not be useless here.

(1.) Besides the *vector expression* $\rho = x\alpha + y\beta + z\gamma$, with the *scalar relation* $x^2 + y^2 + z^2 = 1$, and with *arbitrary vector values* of the constants α, β, γ, which was lately cited (402) from the First Book, or the equations 403, I., *without* the *conditions* 403, I'., II'. which are peculiar to the *sphere*, there were given in the Second Book (204, (13.), (14.)) equations which differed from those lately numbered as 403, XI. XII. XIII. XIV. XV., only by the substitution of $\mathrm{V}\frac{\rho}{\beta}$ for $\mathrm{V}\frac{\rho}{\alpha}$; for instance, there was the equation,

$$\mathrm{I.} \ldots \left(\mathrm{S}\frac{\rho}{\alpha}\right)^2 - \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = 1, \qquad 204,\ (14.)$$

analogous to 403, XI., and representing generally* an *ellipsoid*, regarded as the *locus* of a certain system of *ellipses*, which were thus substituted for the *circles*† (403, XV.) of the *sphere*, by a species of geometrical *deformation*, which led to the establishment of certain *homologies* (developed in the sub-articles to 274).

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although it has *since* been found possible and useful, in this Third Book, to *identify* those right *versors* with their own *indices* or *axes* (295), and so to treat them as a system of *three rectangular lines*, as above.

* In the *case of parallelism* of the two vector constants ($\beta \parallel \alpha$), the equation I. represents generally a *Spheroid of revolution*, with its *axis* in the direction of α; while in the contrary *case of perpendicularity* ($\beta \perp \alpha$), the same equation I. represents an *elliptic Cylinder*, with its *generating lines* in the direction of β. Compare 204, (10.), (11.), and the Note to page 224.

† The equation I. might also have been thus written, on the principles of the Second Book,

$$\mathrm{I'.} \ldots \left(\mathrm{S}\frac{\rho}{\alpha} + \mathrm{S}\frac{\rho}{\beta}\right)\left(\mathrm{S}\frac{\rho}{\alpha} - \mathrm{S}\frac{\rho}{\beta}\right) + \left(\mathrm{T}\frac{\rho}{\beta}\right)^2 = 1;$$

whence it would have followed at once (comp. 216, (7.)), that the *ellipsoid* I. is *cut in two circles*, with a common radius $= \mathrm{T}\beta$, by the *two diametral planes*,

$$\mathrm{I''.} \ldots \mathrm{S}\frac{\rho}{\alpha} + \mathrm{S}\frac{\rho}{\beta} = 0, \quad \mathrm{S}\frac{\rho}{\alpha} - \mathrm{S}\frac{\rho}{\beta} = 0.$$

In fact, this equation I'. is what was called in 359 a *cyclic form*, while I. itself is what was there called a *focal form*, of the equation of the surface; the lines $\alpha^{-1} \pm \beta^{-1}$ being, by the Third Book, the *two* (real) *cyclic normals*, while β is *one* of the two (real) *focal lines* of the (imaginary) *asymptotic cone*. Compare the Note to page 474.

(2.) Employing still *only quotients of vectors*, but introducing *two other pairs* of *vector-constants*, γ, δ and ι, κ, instead of the pair α, β in the equation I., which were however connected with that pair and with each other by certain assigned *relations*, that equation was transformed successively to

$$\text{II.} \ldots \mathrm{T}\left(\frac{\rho}{\gamma} + \mathrm{K}\frac{\rho}{\delta}\right) = 1, \qquad \text{216, X.}$$

and to a form which may be written thus (comp. 217, (5.)),

$$\text{III.} \ldots \mathrm{T}\left(\iota + \mathrm{K}\frac{\kappa}{\rho}\cdot\rho\right)\mathrm{T}\rho = \mathrm{T}\iota^2 - \mathrm{T}\kappa^2; \qquad \text{217, XVI.}$$

and this last *form* was *interpreted*, so as to lead to a *Rule of Construction** (217, (6.), (7.)), which was illustrated by a *Diagram* (Fig. 53), and from which many *geometrical properties* of that surface were deduced (218, 219) in a very simple manner, and were confirmed by calculation with quaternions: the equation and construction being also modified afterwards, by the introduction (220) of a *new pair* of vector-constants, ι' and κ', which were shown to admit of being substituted for ι and κ, in the recent form III.

(3.) And although the *Equation of Conjugation*,

$$\text{IV.} \ldots \mathrm{S}\frac{\lambda}{\alpha}\,\mathrm{S}\frac{\mu}{\alpha} - \mathrm{S}\left(\mathrm{V}\frac{\lambda}{\beta}\cdot\mathrm{V}\frac{\mu}{\beta}\right) = 1, \qquad \text{316, LXIII.}$$

which connects the vectors λ, μ of any two points L, M, whereof *one* is on the *polar plane* of the other, with respect to the ellipsoid I., was not *assigned* till near the end of the First Chapter of the present Book, yet it was there *deduced* by principles and processes of the Second Book *alone*: which thus were *adequate*, although not in the most practically convenient way, to the treatment of questions respecting *tangent planes* and *normals* to an *ellipsoid*, and similarly for *other surfaces*† of the same second order.

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* This *Construction of the Ellipsoid*, by means of a *Generating Triangle* and a *Diacentric Sphere* (page 227), is believed to have been new, when it was deduced by the writer in 1846, and was in that year stated to the Royal Irish Academy (see its *Proceedings*, vol. iii. pp. 288, 289), as a result of the *Method of Quaternions*, which had been previously communicated by him to that Academy (in the year 1843).

† The following are a few other references, on this subject, to the Second Book. Expressions for a *Right Cone* (or for a single *sheet* of such a cone) have been given in pages 119, 179, 220, 221. In page 179 the equation $\mathrm{S}\frac{\rho}{\alpha}\ \mathrm{S}\frac{\beta}{\rho} = 1$, has been assigned, with a transformation in page 180, to represent generally a *Cyclic Cone*, or a cone of the *second order*, with its vertex at the origin; and to exhibit its *cyclic planes*, and *subcontrary sections* (pp. 181, 182). *Right Cylinders* have occurred in pages 193, 196, 197, 198, 199, 218. A case of an *Elliptic Cylinder* has been already mentioned (the case when $\beta \perp \alpha$ in I.); and a transformation of the equation III. of the *Ellipsoid*, by means of *reciprocals* and *norms* of *vectors*, was assigned in page 298. And several expressions (comp. 403), for a *Sphere* of which the ori-

(4.) But in this Third Book we have been able to write the equation III. under the simpler form,*

$$\text{V.} \ldots \mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad 282, \text{XXIX.}$$

which has again admitted of numerous transformations; for instance, of all those which are obtained by equating $(\kappa^2 - \iota^2)^2$ to any one of the expressions 336, (5.), for the *square* of this last *tensor* in V., or for the *norm* of the *quaternion* $\iota\rho + \rho\kappa$; *cyclic forms*† of equation thus arising, which are easily converted into focal forms (359); while a *rectangular transformation* (373, XXX.) has subsequently been assigned, whereby the *lengths* (abc), and also the *directions*, of the *three semiaxes* of the surface, are expressed in terms of the *two vector-constants*, ι, κ: the results thus obtained by *calculation* being found to agree with those previously deduced, from the geometrical *construction* (2.) in the Second Book.

(5.) The equation V. has also been *differentiated* (336), and a *normal vector* $\nu = \phi\rho$ has thus been deduced, such that, for the ellipsoid in question,

$$\text{VI.} \ldots S\nu d\rho = 0, \quad \text{and} \quad \text{VII.} \ldots S\nu\rho = 1;$$

a process which has since been *extended* (361), and appears to furnish one of the best *general methods* of treating *surfaces*‡ *of the second order* by quaternions: especially when combined with that theory of *linear and vector functions* ($\phi\rho$) *of vectors*, which was developed in the Sixth Section§ of the Second Chapter of the present Book.

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gin was *not* the *centre*, occurred in pages 164, 179, 189, and perhaps elsewhere, without any employment of *products of vectors*.

[*] Mentioned by anticipation in the Note to page 233.

[†] Compare the second Note to page 633. The vectors ι and κ are here the *cyclic normals*, and $\iota - \kappa$ is *one* of the *focal lines*; the *other* being the line $\iota' - \kappa'$ of page 232.

[‡] The following are a few additional references to preceding parts of this Third Book, which has extended to a much greater length than was designed (page 302). In the First Chapter, the reader may consult pages 305, 306, 307, for some other forms of equation of the ellipsoid and the sphere. In the Second Chapter, pages 416, 417 contain some useful practice, above alluded to, in the differentiation and transformation of the equation $r^2 = \mathrm{T}(\iota\rho + \rho\kappa)$. As regards the Sixth Section of that Chapter, which we are about to *use* (405), as one supposed to be familiar to the reader, it may be sufficient here to mention Arts. 357-362, and the Notes (or some of them) to pages 464, 466, 468, 474, 481, 484. In this Third Chapter, the sub-articles (7.)-(21.) to 373 (pages 504, &c.) might be re-perused; and perhaps the investigations respecting *cones* and *sphero-conics*, in 394 and its sub-articles (pages 541, &c.), including remarks on an *hyperbolic cylinder*, and its *asymptotic planes* (in page 547). Finally, in a few longer and later series of sub-articles, to Arts. 397, &c., a certain degree of *familiarity* with some of the chief properties of surfaces of the second order has been assumed; as in pages 571, 588, 591, and generally in the recent investigations respecting the *osculating twisted cubic* (pages 591, 620, &c.), to a *helix*, or *other curve in space*.

[§] It appears that this *Section* may be conveniently *referred to*, as III. ii. 6; and similarly in other cases.

405. Dismissing then, at least for the present, the *special* consideration of the *ellipsoid*, but still confining ourselves, for the moment, to *Central Surfaces* of the *Second Order*, and using freely the principles of this Third Book, but especially those of the Section (III. ii. 6) last referred to, we may denote any such *central* and *non-conical surface* by the scalar equation (comp. 361),

$$\text{I.} \ldots f\rho = \mathrm{S}\rho\phi\rho = 1;$$

the *asymptotic cone* (real or imaginary) being represented by the connected equation,

$$\text{II.} \ldots f\rho = \mathrm{S}\rho\phi\rho = 0;$$

and the *equation of conjugation*, between the vectors ρ, ρ' of any *two points* P, P′, which are *conjugate* relatively to this surface I. (comp. 362, and 404, (3.), see also 373, (20.)), being,

$$\text{III.} \ldots f(\rho, \rho') = f(\rho', \rho) = \mathrm{S}\rho\phi\rho' = \mathrm{S}\rho'\phi\rho = 1;$$

while the *differential equation* of the surface is of the form (361),

$$\text{IV.} \ldots 0 = \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho, \quad \text{with} \quad \text{V.} \ldots \nu = \phi\rho;$$

this *vector-function* $\phi\rho$, which represents the *normal* ν to the surface, being at once *linear* and *self-conjugate* (361, (3.)); and the *surface itself* being the *locus* of all the *points* P which are *conjugate to themselves*, so that its *equation* I. may be thus written,

$$\text{I}'. \ldots f(\rho, \rho) = 1, \quad \text{because} \quad f(\rho, \rho) = f\rho, \quad \text{by 362, IV.}$$

(1.) Such being the *form* of $\phi\rho$, it has been seen that there are always *three real* and *rectangular unit-lines*, α_1, α_2, α_3, and *three real scalars*, c_1, c_2, c_3, such as to satisfy (comp. 357, III.) the *three vector equations*,

$$\text{VI.} \ldots \phi\alpha_1 = -c_1\alpha_1, \quad \phi\alpha_2 = -c_2\alpha_2, \quad \phi\alpha_3 = -c_3\alpha_3;$$

whence also these *three scalar equations* are satisfied,

$$\text{VII.}\ f\alpha_1 = c_1, \quad f\alpha_2 = c_2, \quad f\alpha_3 = c_3;$$

and therefore (comp. 362, VII.),

$$\text{VIII.} \ldots f(c_1^{-\frac{1}{2}}\alpha_1) = f(c_2^{-\frac{1}{2}}\alpha_2) = f(c_3^{-\frac{1}{2}}\alpha_3) = 1.$$

(2.) It follows then that the *three* (real or imaginary) *rectangular lines*,

$$\text{IX.} \ldots \beta_1 = c_1^{-\frac{1}{2}}\alpha_1, \quad \beta_2 = c_2^{-\frac{1}{2}}\alpha_2, \quad \beta_3 = c_3^{-\frac{1}{2}}\alpha_3,$$

are the *three* (real or imaginary) *vector semiaxes* of the surface I.; and that the *three* (positive or negative) *scalars*, c_1, c_2, c_3, namely the *three roots* of the *scalar* and *cubic equation** $M = 0$ (comp. 357, (1.)), are the (always real) *inverse squares* of the three (real or imaginary) *scalar semiaxes*, of the same central surface of the second order.

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* It is unnecessary *here* to write $M_0 = 0$, as in page 462, &c., because the function ϕ is here supposed to be *self-conjugate;* its *constants* being also *real*.

(3.) For the *reality* of that *surface* I., it is necessary and sufficient that *one at least* of the three scalars c_1, c_2, c_3 should be *positive;* if *all* be such, the surface is an *ellipsoid;* if *two*, but not the third, it is a *single-sheeted hyperboloid;* and if only *one*, it is a *double-sheeted hyperboloid:* those scalars being here supposed to be each *finite*, and different from *zero*.

(4.) We have already seen (357, (2.)) how to obtain the *rectangular transformation*,

$$\text{X.} \ldots f\rho = c_1(S\alpha_1\rho)^2 + c_2(S\alpha_2\rho)^2 + c_3(S\alpha_3\rho)^2,$$

which may now, by IX., be thus written,

$$\text{XI.} \ldots f\rho = (S\beta_1^{-1}\rho)^2 + (S\beta_2^{-1}\rho)^2 + (S\beta_3^{-1}\rho)^2;$$

but it is to be remembered that, by (2.) and (3.), *one* or even *two* of these three *vectors* $\beta_1\beta_2\beta_3$ may become *imaginary*, without the *surface* ceasing to be *real*.

(5.) We had also the *cyclic transformation* (357, II. II.'),

$$\text{XII.} \ldots f\rho = g\rho^2 + S\lambda\rho\mu\rho = \rho^2(g - S\lambda\mu) + 2S\lambda\rho S\mu\rho,$$

in which the scalar g and the vector λ, μ are *real*, and the latter have the directions of the two (real) cyclic normals;* in fact it is obvious on inspection, that the surface is *cut in circles*, by *planes perpendicular* to these *two last lines*.

(6.) It has been proved that the *four real scalars*, $c_1c_2c_3g$, and the *five real vectors*, $\alpha_1\alpha_2\alpha_3\lambda\mu$, are connected by the relations† (357, XX. and XXI.),

$$\text{XIII.} \ldots c_1 = -g - T\lambda\mu, \qquad c_2 = -g + S\lambda\mu, \quad c_3 = -g + T\lambda\mu;$$
$$\text{XIV.} \ldots \alpha_1 = U(\lambda T\mu - \mu T\lambda), \quad \alpha_2 = UV\lambda\mu, \qquad \alpha_3 = U(\lambda T\mu + \mu T\lambda);$$

at least if the *three roots* $c_1c_2c_3$ of the cubic $M = 0$ be arranged in *algebraically ascending order* (357, IX.), so that $c_1 < c_2 < c_3$.

(7.) It may happen (comp. (3.)), that *one* of these three roots *vanishes;* and in that case (comp. (2.)), *one* of the three *semiaxes* becomes *infinite*, and the *surface* I. becomes a *cylinder*.

(8.) Thus, in particular, if $c_1 = 0$, or $g = -T\lambda\mu$, so that the *two other* roots are both *positive*, the equation takes (by XII., comp. 357, XXII.) a form which may be thus written,

$$\text{XV.} \ldots (S\lambda\mu\rho)^2 + (S\lambda\rho T\mu + S\mu\rho T\lambda)^2 = T\lambda\mu - S\lambda\mu > 0;$$

and it represents an *elliptic cylinder*.

(9.) Again, if $c_2 = 0$, or $g = S\lambda\mu$, the equation becomes,

$$\text{XVI.} \ldots 2S\lambda\rho S\mu\rho = 1,$$

and represents an *hyperbolic cylinder;* the root c_1 being in this case *negative*, while the remaining root c_3 is *positive*.

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* Compare the Note to page 468; see also the proof by quaternions, in 373, (16.), &c., of the known theorem, that any two *subcontrary* circular sections are *homospherical*, with the *equation* (373, XLIV.) of their *common sphere*, which is found to have its *centre* in the *diametral plane* of the *two cyclic normals* λ, μ.

† These relations and a few others mentioned are so useful that, although they occurred in an earlier part of the work, it seems convenient to restate them here.

(10.) But if we suppose that $c_3 = 0$, or $g = T\lambda\mu$, so that c_1 and c_2 are *both negative*, the equation may (by 357, XXIII.) be reduced to the form,

$$\text{XVII.} \ldots (S\lambda\mu\rho)^2 + (S\lambda\rho T\mu - S\mu\rho T\lambda)^2 = -T\lambda\mu - S\lambda\mu < 0;$$

it represents therefore, in *this* case, *nothing real*, although it may be *said* to be, in the same case, the equation of an *imaginary** *elliptic cylinder*.

(11.) It is scarcely worth while to remark, that we have here supposed *each* of the *two vectors* λ and μ to be not only *real* but *actual* (Art. 1); for if *either* of them were to *vanish*, the equation of the surface would take by XII. the form,

$$\text{XVIII.} \ldots \rho^2 = g^{-1}, \quad \text{or} \quad \text{XVIII}'. \ldots T\rho = (-g)^{-\frac{1}{2}},$$

and would represent a *real* or *imaginary sphere*, according as the scalar constant g was *negative* or *positive*: λ and μ have also *distinct directions*, except in the case of *surfaces of revolution*.

(12.) In general, it results from the relations (6.), that the *plane* of the *two* (real) *cyclic normals*, λ, μ, is *perpendicular* to the (real) *direction* of that (real or imaginary) *semiaxis*, of which, when considered as a *scalar* (2.), the *inverse square* c_2 is *algebraically intermediate* between the inverse squares c_1, c_3 of the *other two*; or that the *two diametral and cyclic planes* ($S\lambda\rho = 0$, $S\mu\rho = 0$) *intersect* in that *real line* ($V\lambda\mu$) which has the direction of the *real unit-vector* α_2 (1.), corresponding to the *mean root* c_2 of the cubic equation $M = 0$: all which agrees with known results, respecting the *circular sections* of the (real) *ellipsoid*, and of the two *hyperboloids*.

406. Some additional light may be thrown on the theory of the *central surface* 405, I., by the consideration of its *asymptotic cone* 405, II.; of which *cone*, by 405, XII., the equation may be thus written,

$$\text{I.} \ldots f\rho = g\rho^2 + S\lambda\rho\mu\rho = \rho^2(g - S\lambda\mu) + 2S\lambda\rho S\mu\rho = 0;$$

and which is *real* or *imaginary*, according as we have the inequality,

$$\text{II.} \ldots g^2 < \lambda^2\mu^2, \quad \text{or} \quad \text{III.} \ldots g^2 > \lambda^2\mu^2;$$

that is, by 405, (6.), according as the *product* c_1c_3 of the *extreme roots* of the *cubic* $M = 0$ is *negative* or *positive*; or finally, according as the *surface* $f\rho = 1$ is a (real) *hyperboloid*, or an *ellipsoid* (real or imaginary†).

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* In the Section (III. ii. 6) above referred to, many symbolical results have been established, respecting *imaginary cyclic normals*, or *focal lines*, &c., on which it is unnecessary to return. But it may be remarked that as, when the *scalar function* $f\rho$ admits of *changing sign*, for a *change of direction* of the *real vector* ρ, so as to be *positive* for *some* such directions, and *negative* for *others*, although $f(-\rho) = f(+\rho)$, the two equations, $f\rho = +1$, $f\rho = -1$, represent then *two real* and *conjugate hyperboloids*, of *different species*: so, when the function $f\rho$ is either *essentially positive*, or else *essentially negative*, for *real* values of ρ, the equations $f\rho = 1$ and $f\rho = -1$ may *then* be said to represent *two conjugate ellipsoids*, *one real*, and the *other imaginary*.

† Compare the Note immediately preceding; also the second Note to page 474.

(1.) As regards the asserted *reality* of the cone I., when the *condition* II. is satisfied, it may suffice to observe that if we cut the cone by the *plane*,

$$\text{IV.}\ldots\ \mathrm{S}\lambda(\rho-\mu)=-g,$$

the *section* is a *circle* of the *real* and *diacentric sphere*,

$$\text{V.}\ldots\ \rho^2=2\mathrm{S}\mu\rho,\quad\text{or}\quad \text{V}'.\ldots\ (\rho-\mu)^2=\mu^2;$$

and a *real circle*, because it is on the *real cylinder of revolution*,

$$\text{VI.}\ldots\ \mathrm{TV}(\rho-\mu)\mathrm{U}\lambda=(\mathrm{T}\mu^2-g^2\mathrm{T}\lambda^{-2})^{\frac{1}{2}},$$

so that its *radius* is equal to this last *real radical*.

(2.) For example, the cone

$$\text{VII.}\ldots\ \mathrm{S}\frac{\rho}{\alpha}\,\mathrm{S}\frac{\beta}{\rho}=1,\quad\text{or}\quad \text{VII}'.\ldots\ 2(\mathrm{S}\alpha\rho\mathrm{S}\beta\rho-\alpha^2\rho^2)=0,$$

which under the form VII. occurred as early as 196, (8.), and for which $\lambda=\alpha$, $\mu=\beta$, $g=\mathrm{S}\alpha\beta-2\alpha^2$, and therefore $\mathrm{T}\lambda\mu+g>0$, the condition II. reduces itself to $\mathrm{T}\lambda\mu-g>0$; or after division by $2\mathrm{T}\alpha^2$, &c., to the form (comp. 199, XII.),

$$\text{VIII.}\ldots\ \tfrac{1}{2}(\mathrm{T}+\mathrm{S})\frac{\beta}{\alpha}>1,\quad\text{or}\quad \text{VIII}'.\ldots\ \mathrm{S}\sqrt{\frac{\beta}{\alpha}}>1;$$

and accordingly, when either of these two last inequalities exists, it will be found that the *sphere* $\mathrm{S}\frac{\beta}{\rho}=1$ is *cut* by the *plane* $\mathrm{S}\frac{\rho}{\alpha}=1$ in a *real circle*, the base of a *real cone* VII.

(3.) As an example of the *variety of processes* by which problems in this Calculus may be treated, we might propose to determine, by the general formula 389, IV., the vector κ of the *centre* of the *osculating circle* to the *curve* IV. V., considered merely as an *intersection* of two surfaces. The first derivatives of the equations would allow us to assume $\rho'=\mathrm{V}\lambda(\rho-\mu)$, and therefore $\rho''=\lambda\rho'$; whence, by the formula, we have

$$\text{IX.}\ldots\ \kappa=\rho+\frac{\rho'^3}{\mathrm{V}\rho''\rho'}=\rho+\frac{\rho'}{\lambda}=\frac{\mathrm{S}\rho\lambda+\mathrm{V}\mu\lambda}{\lambda}=\mu-g\lambda^{-1};$$

the *section* is therefore a *circle*, because its *centre of curvature* is *constant*; and its *radius* is,

$$\text{X.}\ldots\ r=\mathrm{T}(\rho-\kappa)=\mathrm{T}(\rho-\mu+g\lambda^{-1})=(\mathrm{T}\mu^2-g^2\mathrm{T}\lambda^{-2})^{\frac{1}{2}},$$

= the radius of the cylinder VI.

(4.) When the *opposite inequality* III. exists, the *radius* X., the *cylinder* VI., the *circle* IV. V., and the *cone* I., become all four *imaginary*; the *plane* IV. being then wholly *external* to the *sphere* V., as happens, for instance, with the plane and sphere in (2.), when the condition VIII. or VIII'. is *reversed*.

(5.) In the *intermediate case*, when

$$\text{XI.}\ldots\ g^2=\lambda^2\mu^2,\quad\text{or}\quad \text{XI}'.\ldots\ g=\mp\mathrm{T}\lambda\mu,$$

the radius r in X. *vanishes*; the right cylinder VI. reduces itself to its *axis*; and the circle IV. V. becomes a *point*, in which the sphere is *touched* by the plane. In this case, then, the cone I. is reduced to a *single* (*real**) *right line*, which has

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* It may however be said, that in this case the *cone* consists of a *pair of imaginary planes*, which *intersect* in a *real right line*.

(compare the equations of the *elliptic cylinders*, 405, XV. XVII.) the direction of $\lambda T\mu - \mu T\lambda$, if $g = -T\lambda\mu$, but the perpendicular direction of $\lambda T\mu + \mu T\lambda$, if $g = +T\lambda\mu$.

(6.) In general (comp. 405, X.), the equation of the cone I. admits of the *rectangular transformation*,

$$\text{XII.} \ldots f\rho = c_1(Sa_1\rho)^2 + c_2(Sa_2\rho)^2 + c_3(Sa_3\rho)^2 = 0\,;$$

and the two *sub-cases* last considered (5.) correspond respectively (by 405, (6.)) to the *evanescence* of the *roots* c_1, c_3 of the cubic $M = 0$, with the resulting *directions* a_1, a_3 of the only *real side* of the cone. An analogous but intermediate case (comp. 405, (9.)) is that when $c_2 = 0$, or $g = S\lambda\mu$; in which case, the *cone* I. reduces itself to the *pair of (real) planes*,

$$\text{XIII.} \ldots S\lambda\rho \,.\, S\mu\rho = 0,$$

namely to the *asymptotic planes* of the *hyperbolic cylinder* 405, XVI., or to those which are usually the *two cyclic* planes* of the cone.

(7.) The *case* (comp. 394, (29.)),

$$\text{XIV.} \ldots g = -S\lambda\mu, \quad \text{or} \quad \text{XIV}'. \ldots c_1 - c_2 + c_3 = 0,$$

for which the equation I. of the cone becomes,

$$\text{XV.} \ldots 0 = f\rho = 2(S\lambda\rho S\mu\rho - \rho^2 S\lambda\mu) = 2S(V\lambda\rho \,.\, V\mu\rho),$$

may deserve a moment's attention. In this case, the *two planes*, of $\lambda\rho$ and $\mu\rho$, which connect the *two cyclic normals* λ and μ with an *arbitrary side* ρ of the cone, are always *rectangular* to each other; and these two *normals* to the cyclic planes are at the same time *sides* of the cone, which thus is *cut in circles*, by planes *perpendicular to those two sides*. And because the equation of the cone may (in the same case) be thus written,

$$\text{XVI.} \ldots TV(\lambda + \mu)\rho = TV(\lambda - \mu)\rho,$$

while the *lengths* of λ and μ may *vary*, if their *product* $T\lambda\mu$ be left unchanged, so that $\lambda + \mu$ and $\lambda - \mu$ may represent *any two lines* from the vertex, in the *plane of the two cyclic normals*, and *harmonically conjugate with respect to them*, it follows that, *for this cone* XV., the *sines of the inclinations of an arbitrary side* ρ, *to these two new lines*, have a *constant ratio* to each other.

(8.) *In general*, the second form I. of $f\rho$ shows (comp. 394, (23.)), that the *constant product of the sines of the inclinations*, of a *side* ρ of the *cone* to the two *cyclic planes*, has for expression,

$$\text{XVII.} \ldots \cos\angle\frac{\rho}{\lambda} \,.\, \cos\angle\frac{\rho}{\mu} = \tfrac{1}{2}\left(\frac{g}{T\lambda\mu} + \cos\angle\frac{\mu}{\lambda}\right);$$

while the first form I. of the same function $f\rho$ reproduces the *condition of reality* II., by showing that $g : T\lambda\mu$ is (for a *real cone*) the *cosine* of a *real angle*, namely, that of the quaternion product $\lambda\rho\mu\rho$, since it gives the relation,

$$\text{XVIII.} \ldots \frac{g}{T\lambda\mu} = SU\lambda\rho\mu\rho = \cos\angle\,\lambda\rho\mu\rho = \cos\angle\frac{\rho\mu^{-1}\rho}{\lambda}.$$

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* The cones and surfaces which have a common centre, and common values of the vectors λ and μ, but different values of the scalar g, may thus be said, in a known phraseology, to be *biconcyclic*.

(9.) We may also observe that in the *case of reality* II., with exclusion of the sub-case (6.), if α_3 have the direction of the *internal axis* of the cone, so that

$$\text{XIX.} \ldots c_1 < 0, \quad c_2 < 0, \quad c_3 > 0, \quad \text{or} \quad \text{XIX}'. \ldots g > \mathrm{S}\lambda\mu, \quad g < \mathrm{T}\lambda\mu,$$

the *two sides* (of one sheet) in the plane of $\lambda\mu$ have the directions,

$$\text{XX.} \ldots \rho_1 = c_3^{-\frac{1}{2}}\alpha_3 + (-c_1)^{-\frac{1}{2}}\alpha_1, \quad \rho_2 = c_3^{-\frac{1}{2}}\alpha_3 - (-c_1)^{-\frac{1}{2}}\alpha_1;$$

if then their mutual *inclination*, or the *angle of the cone* in the *plane of the cyclic normals*, be denoted by 2b, we have the values,

$$\text{XXI.} \ldots \tan^2 \mathrm{b} = \frac{c_3}{-c_1}, \quad \text{XXI}'. \ldots \cos 2\mathrm{b} = \frac{-c_1 - c_3}{-c_1 + c_3} = \frac{g}{\mathrm{T}\lambda\mu};$$

the *angle of the quaternion* $\lambda\rho\mu\rho$ is therefore (by XVIII.), equal to this angle 2b, namely to the *arcual minor axis* of the *sphero-conic*, in which the cone is cut by the concentric unit-sphere.

(10.) The same condition of reality II. may be obtained in a quite different way, as that of the reality of the *reciprocal cone*, which is the *locus of the normal vector*,

$$\text{XXII.} \ldots \nu = \phi\rho = g\rho + \mathrm{V}\lambda\rho\mu.$$

Inverting this *linear function* ϕ, by the method of the Section III. ii. 6, we find first the expression (comp. 354, (12.), and 361, (6.)[*]),

$$\text{XXIII.} \ldots m\rho = \psi\nu = \mu^2\lambda\mathrm{S}\lambda\nu + \lambda^2\mu\mathrm{S}\mu\nu - g(\lambda\mathrm{S}\mu\nu + \mu\mathrm{S}\lambda\nu) + (g^2 - \lambda^2\mu^2)\nu,$$

in which $\text{XXIV.} \ldots m = (g - \mathrm{S}\lambda\mu)(g^2 - \lambda^2\mu^2) = -c_1c_2c_3;$

and next the *reciprocal equation* (comp. 361, XXVII.),

$$\text{XXV.} \ldots 0 = \mathrm{S}\nu\psi\nu = \mu^2(\mathrm{S}\lambda\nu)^2 + \lambda^2(\mathrm{S}\mu\nu)^2 - 2g\mathrm{S}\lambda\nu\mathrm{S}\mu\nu + (g^2 - \lambda^2\mu^2)\nu^2,$$

which may be put under the form,

$$\text{XXVI.} \ldots \cos\left(\angle\frac{\nu}{\lambda} + \angle\frac{\nu}{\mu}\right) = \frac{-g}{\mathrm{T}\lambda\mu};$$

the *quotient* $g : \mathrm{T}\lambda\mu$ thus presenting itself anew as a *cosine*, namely as that of the *supplement of the sum of the inclinations of the normal* ν (to the cone I.), *to the two cyclic normals* λ, μ (of *that* cone); or as the cosine[†] of $\pi - \mathrm{A} - \mathrm{B}$, if A and B denote (comp. Fig. 80) the *two spherical angles*, which the *tangent arc* to the *sphero-conic* (9.) makes with the two *cyclic arcs:* so that by comparison of XXI′. and XXVI. we have the relation,

$$\text{XXVII.} \ldots \mathrm{A} + \mathrm{B} = \angle\frac{\nu}{\lambda} + \angle\frac{\nu}{\mu} = \pi - 2\mathrm{b}.$$

(11.) Comparing the expression XXI′. for cos 2b, with the last expression

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[*] In the expression 361, XXVI. for $\psi\nu$, the second term ought to have been printed as $-\mathrm{V}\lambda\mu\mathrm{S}\lambda\nu\mu$; or else the sign should have been changed.

[†] This relation was mentioned by anticipation in 394, (3.); and the relation in XXVII. may easily be verified, by conceiving the point of contact P in Fig. 80 (page 543) to tend towards a minor summit of the conic, or the tangent arc APB to tend to pass through the two points C, C′, in which the cyclic arcs intersect.

XVIII. for $g : \mathrm{T}\lambda\mu$, we derive the following *construction* for a *sphero-conic*, which may easily be verified by geometry:*

Having assumed two points (L, M) *on a sphere, and having described a small circle round one of them* (say L), *bisect the arcs* (MQ) *which are drawn to its circumference from the other point; the locus of the bisecting points* (P) *will be a sphero-conic, with the two fixed points for its two cyclic poles* (or for the poles of its cyclic arcs), *and with an arcual minor axis* (2b) *equal to the arcual radius of the small circle.*

(12.) As regards the *arcual major axis* (say 2a) of the same sphero-conic, it is (with the conditions XIX.) the angle between the two sides (comp. XX.),

$$\text{XXVIII.} \ldots \rho_3 = c_3^{-\frac{1}{2}}\alpha_3 + (-c_2)^{-\frac{1}{2}}\alpha_2, \quad \rho_4 = c_3^{-\frac{1}{2}}\alpha_3 - (-c_2)^{-\frac{1}{2}}\alpha_2;$$

whence (comp. XXI.),

$$\text{XXIX.} \ldots \tan^2 \mathrm{a} = \frac{c_3}{-c_2}, \quad \text{or} \quad \text{XXIX}'. \ldots \cos 2\mathrm{a} = \frac{-c_2 - c_3}{-c_2 + c_3} = (\text{say})\ e,$$

and therefore, a few easy reductions being made,

$$\text{XXX.} \ldots \frac{\sin \mathrm{b}}{\sin \mathrm{a}} = \sqrt{\left\{\tfrac{1}{2}\left(1 + \mathrm{SU}\frac{\mu}{\lambda}\right)\right\}} = \cos \tfrac{1}{2} \angle \frac{\mu}{\lambda};$$

from which we can at once infer, that if a *focus* of the conic be determined, by drawing from a minor summit to the major axis an arc equal to the major semiaxis a, *the minor axis subtends at this focus* (or at the other) *a spherical angle equal to the angle between the two cyclic arcs.*

(13.) For the *two real unifocal transformations* of the equation of the *cone*, or the *forms*,

$$\text{XXXI.} \ldots a(\mathrm{V}\alpha\rho)^2 + b(\mathrm{S}\beta\rho)^2 = 0, \quad \text{and} \quad \text{XXXI}'. \ldots a(\mathrm{V}\alpha'\rho)^2 + b(\mathrm{S}\beta'\rho)^2 = 0,$$

with one *common* set of real values of the scalar *coefficients*, a and b, but with *two real focal unit lines* α, α', and *two real directive normals* β, β' corresponding, it may be sufficient here to refer to the sub-articles to 358; except that it should be noticed, that if the cone be *real*, and if the line α_3 have the direction of its *internal axis*, so that the inequalities XIX. are satisfied, and therefore also (by 405, (6.)),

$$\text{XXXII.} \ldots c_3^{-1} > 0 > c_1^{-1} > c_2^{-1},$$

instead of the inequalities 358, III., or 359, XXXVII., we are now to change, in the earlier formulæ referred to, the symbols $c_1c_2c_3\alpha_1\alpha_2\alpha_3$ to $c_3c_1c_2\alpha_3\alpha_1\alpha_2$, so that we have now the values,

$$\text{XXXIII.} \ldots a = -c_1, \quad b = c_3 - c_1 + c_2, \quad \text{if} \quad \mathrm{T}\beta = \mathrm{T}\beta' = 1.$$

(14.) And as regards the *interpretation* of the *unifocal form* XXXI., with these last values, it is evidently contained in this other equation,

$$\text{XXXIV.} \ldots \sin \angle \frac{\rho}{\alpha} \,.\, \sec \angle \frac{\rho}{\beta} = \frac{\mathrm{TV}\alpha\rho}{-\mathrm{S}\beta\rho} = \left(\frac{c_3 - c_1 + c_2}{-c_1}\right)^{\frac{1}{2}} = \text{const.};$$

the *sines of the inclinations* of an arbitrary *side* (ρ) of the *cone*, to a *focal line* (α), and to the corresponding *director plane* ($\perp \beta$), thus bearing to each other (as is well known) a *constant ratio*, which remains unchanged when we pass to the *other* (real) *focal line* (α'), and at the same time to the *other* (real) *director plane* ($\perp \beta'$): and the *focal plane* of these *two lines* (α, α') being *perpendicular* to *that one* of the *three axes*, which corresponds to the *root* (here c_1, by XXXII.) of the *cubic*, of which the *reciprocal* is algebraically *intermediate* between the reciprocals of the other two.

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* In fact, the bisecting radii OP are parallel to the supplementary chords M'Q, if MM' be a diameter of the sphere; and the locus of all such chords is a cyclic cone, resting on the small circle as its base.

(15.) It is, however, more *symmetric* to employ the *bifocal transformation* (comp. 360, VI.*),

$$\text{XXXV.} \ldots 0 = (S\alpha\rho)^2 - 2eS\alpha\rho S\alpha'\rho + (S\alpha'\rho)^2 + (1 - e^2)\rho^2;$$

in which the scalar constant e has the value (comp. XXIX'.),

$$\text{XXXVI.} \ldots e = \cos 2\text{a};$$

and α, α' are the *two*† real and *focal unit lines*, recently considered (13.).

(16.) The equation XXXV., for the case of a *real cone*, may be thus written (comp. XXVI. XXXVI.),

$$\text{XXXVII.} \ldots \angle \frac{\rho}{\alpha} + \angle \frac{\rho}{\alpha'} = \cos^{-1} e = 2\text{a};$$

the *sum*‡ of the *inclinations* of the *side* ρ to the *two* focal lines α, α' being thus *constant*, and equal (as is well known) to the *major axis* of the *spherical conic*: and although, when $e > 1$, the *cone* becomes *imaginary*, yet it is then asymptotic to a *real ellipsoid*, as we shall shortly see.

407. The *bifocal form* (406, XXXV.) of the equation of a *cone* may suggest the corresponding *form*,

$$\text{I.} \ldots C = Cf\rho = (S\alpha\rho)^2 - 2eS\alpha\rho S\alpha'\rho + (S\alpha'\rho)^2 + (1 - e^2)\rho^2,$$

in which α and α' are given and generally non-parallel unit-lines, while e and C are scalar constants, as capable of representing *generally* (comp. 360, (2.), (3.)) a *central* but *non-conical surface* ($f\rho = 1$) of the *second order*. And we shall find that if, in passing from *one* such surface to *another*, we suppose α and α' to remain *unchanged*, but e and C to *vary together*, so as to be always connected by the relation,

$$\text{II.} \ldots C = (e^2 - 1)(e + S\alpha\alpha')l^2,$$

in which l is some real, positive, and *given scalar*, then all the surfaces I. so deduced, or in other words the surfaces represented by the common equation,

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* It is to be remembered that, in the formula here cited, the symbols α, α' did *not* denote unit-vectors.

† When these two vectors α, α' remain *constant*, but the scalar e *changes*, there arises a system of *biconfocal cones*: or, by their intersections with a concentric sphere, a system of *biconfocal sphero-conics*. Compare the Note to page 640.

‡ Or the *difference*, according to the choice between two opposite directions, for *one* of the two focal lines. The *angular transformation* XXXVII. may be accomplished, by *resolving* the equation XXXV. as a *quadratic* in e, and then interpreting the result.

$$\text{III.}\ldots\ l^2 = l^2 f\rho = \frac{(\mathrm{S}\alpha\rho)^2 - 2e\mathrm{S}\alpha\rho\mathrm{S}\alpha'\rho + (\mathrm{S}\alpha'\rho)^2 + (1-e^2)\rho^2}{(e^2-1)\,(e+\mathrm{S}\alpha\alpha')},$$

with e for the *only variable parameter*, compose a *Confocal System.*

(1.) The *scalar form* III. of $f\rho$ gives the connected *vector form*,

$$\text{IV.}\ldots\ l^2\nu = l^2\phi\rho = \frac{\alpha\mathrm{S}(\alpha - e\alpha')\rho + \alpha'\mathrm{S}(\alpha' - e\alpha)\rho + (1-e^2)\rho}{(e^2-1)\,(e+\mathrm{S}\alpha\alpha')},$$

which may also be thus written, with the value II. of C,

$$\text{V.}\ldots\ C\nu = C\phi\rho = (\alpha - e\alpha')\mathrm{S}\alpha\rho + (\alpha' - e\alpha)\mathrm{S}\alpha'\rho + (1-e^2)\rho,$$

so that the function ϕ is *self-conjugate*, as it ought to be.

(2.) And because we have thus,

$$\text{VI.}\ldots\ (e^2-1)l^2\phi\alpha = \alpha' - e\alpha, \quad (e^2-1)l^2\phi\alpha' = \alpha - e\alpha',$$

if we write, for abridgment,

$$\text{VII.}\ldots\ a^2 = (e+1)l^2, \quad b^2 = (e+\mathrm{S}\alpha\alpha')l^2, \quad c^2 = (e-1)l^2,$$

we shall have the values,

$$\text{VIII.}\ldots\ \begin{cases}\phi(\alpha+\alpha') = -a^{-2}(\alpha+\alpha'),\\ \phi\mathrm{V}\alpha\alpha' \quad = -b^{-2}\mathrm{V}\alpha\alpha',\\ \phi(\alpha-\alpha') = -c^{-2}(\alpha-\alpha');\end{cases}$$

comparing which with 405, (1.), (2.), we see that the three (real or imaginary) lines,

$$\text{IX.}\ldots\ a\mathrm{U}(\alpha+\alpha'), \quad b\mathrm{UV}\alpha\alpha', \quad c\mathrm{U}(\alpha-\alpha'),$$

of any one of which the direction may be reversed, are the *three vector semiaxes* of the *surface* $f\rho = 1$; and therefore, by VII., that the *system* III. is one of *confocals*, as asserted.

(3.) The *rectangular transformations*, scalar and vector, are now (comp. 405, X., and 357, V. VIII.):

$$\text{X.}\ldots\ l^2 = l^2 f\rho = \frac{(\mathrm{S}\rho\mathrm{U}(\alpha+\alpha'))^2}{e+1} + \frac{(\mathrm{S}\rho\mathrm{UV}\alpha\alpha')^2}{e+\mathrm{S}\alpha\alpha'} + \frac{(\mathrm{S}\rho\mathrm{U}(\alpha-\alpha'))^2}{e-1};$$

$$\text{XI.}\ldots\ l^2\nu = l^2\phi\rho = \frac{\mathrm{U}(\alpha+\alpha').\mathrm{S}\rho\mathrm{U}(\alpha+\alpha')}{e+1} + \frac{\mathrm{UV}\alpha\alpha'.\mathrm{S}\rho\mathrm{UV}\alpha\alpha'}{e+\mathrm{S}\alpha\alpha'} + \frac{\mathrm{U}(\alpha-\alpha').\mathrm{S}\rho\mathrm{U}(\alpha-\alpha')}{e-1};$$

which can both be established, by the rules of the present Calculus, in several other ways, and from the first of which it follows that (as is well known) *through any proposed point* P of space there can in general be drawn *three confocal surfaces*, of a *given system* III.; *one* an *ellipsoid*, for which $e > 1$, and therefore $a^2 > b^2 > c^2 > 0$; *another* a *single-sheeted hyperboloid*, for which $e < 1$, $e > -\mathrm{S}\alpha\alpha'$, $a^2 > b^2 > 0 > c^2$; and the *third* a *double-sheeted hyperboloid*, for which $e < -\mathrm{S}\alpha\alpha'$, $e > -1$, $a^2 > 0 > b^2 > c^2$.

(4.) From the *other* rectangular transformation XI. it follows, that if we denote by $\nu_1 = \phi_1\rho$ what the normal vector $\nu = \phi\rho$ becomes, when ρ remains the same, but e is changed to a *second root* e_1 of the equation III. or X. of the *surface*, considered as a *cubic* in e, then

$$\text{XII.} \ldots \frac{\nu_1 - \nu}{e_1 - e} = l^2\phi\nu_1 = l^2\phi_1\nu = l^2\phi_1\phi\rho = l^2\phi\phi_1\rho\,;$$

but

$$\text{XIII.} \ldots \mathrm{S}\rho\nu_1 = \mathrm{S}\rho\nu = f_1\rho = f\rho = 1,$$

$f_1\rho$ being formed from $f\rho$, by the substitution of e_1 for e; therefore,

$$\text{XIV.} \ldots 0 = \mathrm{S}\rho\phi\nu_1 = \mathrm{S}\nu_1\phi\rho = \mathrm{S}\nu_1\nu,$$

and the known theorem results, that *confocal surfaces cut each other orthogonally.*[*]

(5.) It follows, from V. and VI., that the *inverse function* $\phi^{-1}\rho$ can be expressed as follows:

$$\text{XV.} \ldots \phi^{-1}\rho = l^2(a\mathrm{S}a'\rho + a'\mathrm{S}a\rho) - b^2\rho\,;$$

or that ρ may be deduced from ν by the formula,

$$\text{XVI.} \ldots \rho = \phi^{-1}\nu = l^2(a\mathrm{S}a'\nu + a'\mathrm{S}a\nu) - b^2\nu,$$

which can easily be otherwise established. Hence (comp. 361, (4.)), the equation of the surface *reciprocal* to the surface I. or III., or of that *new surface* which has ν (instead of ρ) for its *variable vector*, is

$$\text{XVII.} \ldots 1 = F\nu = \mathrm{S}\nu\phi^{-1}\nu = 2l^2\mathrm{S}a\nu\mathrm{S}a'\nu - b^2\nu^2\,;$$

the *fixed focal lines* a, a' of the *confocal system* III., or of the corresponding system of the *asymptotic cones*, becoming thus (in agreement with known results) the *fixed cyclic normals* (or *cyclic lines*, comp. 361, (6.)) of the *reciprocal system* XVII.

(6.) In thus deducing the equation XVII. from III., *no use* has been made of the *rectangular transformations* X. XI., of the functions $f\rho$ and $\phi\rho$. *Without* the transformations last referred to, we could therefore have inferred, by a slight modification of the form XVII., that the *reciprocal surface* ($F\nu = 1$) with ν for its variable vector, which has the *same rectangular system* of *directions* for its three semiaxes as the *original surface* ($f\rho = 1$), but with *inverse squares* (the roots of *its* cubic) equal to the *direct* squares of the *original* semiaxes, has for equation (comp. 405, XII.),

$$\text{XVIII.} \ldots 1 = F\nu = l^2(\mathrm{S}a\nu a'\nu - e\nu^2) = \mathrm{S}\lambda\nu\mu\nu + g\nu^2,$$

if

$$\text{XIX.} \ldots \lambda = la, \quad \mu = la', \quad g = -el^2 = -e\mathrm{T}\lambda\mu\,;$$

the values VII. of a^2, b^2, c^2 being thus deduced *anew*, but by a process quite different from that employed in (2.), under the forms (comp. 405, XIII.),

$$\text{XX.} \ldots a^2 = c_3 = -g + \mathrm{T}\lambda\mu\,; \quad b^2 = c_2 = -g + \mathrm{S}\lambda\mu\,; \quad c^2 = c_1 = -g - \mathrm{T}\lambda\mu\,;$$

while the *directions* IX. of the corresponding semiaxes may be deduced as those of α_3, α_2, α_1, from the formulæ 405, XIV.

(7.) If the symbol $\omega(\nu)$, or simply $\omega\nu$, be used to denote a *new* linear and self-conjugate vector function of ν, defined by the equation,

$$\text{XXI.} \ldots \omega\nu = \rho\mathrm{S}\rho\nu - l^2(a\mathrm{S}a'\nu + a'\mathrm{S}a\nu),$$

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[*] We shall soon see that the *same formula* XII., by expressing that ν, ν_1, and $\phi\nu_1$ or $\phi_1\nu$ are *complanar*, contains this *other part* of the known theorem referred to, that the *intersection* is a *line of curvature*, on *each* of the two *confocals*.

with ρ *here* treated as a vector constant, then (because $S\rho\nu = 1$) the equation XVI. may be thus written (comp. 354, &c.),

$$\text{XXII.} \ldots (\omega + b^2)\nu = 0;$$

the *three rectangular directions*, of the *three normals* ν, ν_1, ν_2 to the *three confocals* through P, are therefore those which satisfy (comp. again 354) the *vector quadratic* equation,

$$\text{XXIII.} \ldots V\nu\omega\nu = 0;$$

and they are the *directions* of the *axes* of this *new surface* of the second order (comp. 357, &c.),

$$\text{XXIV.} \ldots S\nu\omega\nu = (S\rho\nu)^2 - 2l^2 Sa\nu Sa'\nu = 1,$$

in which ρ is still treated as a *constant* vector, but ν as a *variable* one.

(8.) The *inverse squares* of the *scalar semiaxes* of this *new* surface ($S\nu\omega\nu = 1$), are the *direct* squares b^2, b_1^2, b_2^2 of what may be called the *mean semiaxes* of the *three confocals;* these *latter* squares must therefore be the *roots* of this *new cubic*,

$$\text{XXV.} \ldots 0 = m + m'b^2 + m''(b^2)^2 + (b^2)^3,$$

in which the coefficients m, m', m'', deduced here from the *new function* ω, as they were deduced from ϕ in the Section III. ii. 6, have the values,

$$\text{XXVI.} \ldots \begin{cases} m = l^4(Saa'\rho)^2; \\ m' = l^4(Vaa')^2 + 2l^2 S(Va\rho . Va'\rho); \\ m'' = \rho^2 - 2l^2 Saa'. \end{cases}$$

Accordingly, if we observe that (because $Ta = Ta' = 1$) we have among others the transformation,

$$\text{XXVII.} \ldots (Saa'\rho)^2 = \rho^2(Vaa')^2 - (Sa\rho)^2 - 2Saa'Sa\rho Sa'\rho - (Sa'\rho)^2,$$

we can express this last cubic equation XXV., with these values XXVI. of its coefficients, under the form,

$$\text{XXVIII.} \ldots 0 = (b^2 + \rho^2)\{(b^2 - l^2 Saa')^2 - l^4\} \\ + 2l^2(b^2 - l^2 Saa')Sa\rho Sa'\rho - l^4((Sa\rho)^2 + (Sa'\rho)^2);$$

which, when we change b^2 by VII. to its value $l^2(e + Saa')$, and divide by l^4, becomes the *cubic* in e, or the equation III. under the form,

$$\text{XXIX.} \ldots 0 = (e^2 - 1)\{l^2(e + Saa') + \rho^2\} + 2eSa\rho Sa'\rho - (Sa\rho)^2 - (Sa'\rho)^2.$$

(9.) As an additional test of the *consistency* of this whole theory and method, the *directions* of the *three axes* of the *new surface* XXIV., or those of the *three normals* (7.) to the confocals, or the *three vector roots* (354) of the equation XXIII., ought to admit of being assigned by three expressions of the forms,

$$\text{XXX.} \ldots \begin{cases} n\nu = \psi\sigma + b^2\chi\sigma + b^4\sigma, \\ n_1\nu_1 = \psi\sigma_1 + b_1^2\chi\sigma_1 + b_1^4\sigma_1, \\ n_2\nu_2 = \psi\sigma_2 + b_2^2\chi\sigma_2 + b_2^4\sigma_2; \end{cases}$$

in which b^2, b_1^2, b_2^2 are the *three scalar roots* of the cubic XXV. or XXVIII., while σ, σ_1, σ_2 are *three arbitrary vectors;* n, n_1, n_2 are *three scalar coefficients*, which can be determined by the conditions $S\rho\nu = S\rho\nu_1 = S\rho\nu_2 = 1$ (comp. XIII.); and ψ, χ are two new *auxiliary linear* and *vector functions*, to be deduced here from the function ω, in the same manner as they were deduced from ϕ in the Section lately referred to.

(10.) Accordingly, by the method of that Section, taking for convenience the given[*] vector ρ (instead of the arbitrary vectors σ, σ_1, σ_2) as the subject of the operations ψ and χ, we find the expressions,

$$\text{XXXI.} \ldots \psi\rho = l^4 \mathrm{V}\alpha\alpha'\mathrm{S}\alpha\alpha'\rho, \quad \chi\rho = l^2(\alpha\mathrm{S}\alpha'\rho + \alpha'\mathrm{S}\alpha\rho - 2\rho\mathrm{S}\alpha\alpha');$$

whence, after a few reductions, with elimination of n by the relation $\mathrm{S}\rho\nu = 1$, and by the cubic in b^2, the first equation XXX. becomes:

$$\text{XXXII.} \ldots 0 = (b^2\nu + \rho)\{(b^2 - l^2\mathrm{S}\alpha\alpha')^2 - l^4\} \\ + l^2(b^2 - l^2\mathrm{S}\alpha\alpha')(\alpha\mathrm{S}\alpha'\rho + \alpha'\mathrm{S}\alpha\rho) - l^4(\alpha\mathrm{S}\alpha\rho + \alpha'\mathrm{S}\alpha'\rho);$$

which is in fact a *form* of the relation between ν and ρ, for any *one* of the confocals, as appears (for instance) by again changing b^2 to $l^2(e + \mathrm{S}\alpha\alpha')$, and comparing with the equation IV.

(11.) Another and a more interesting *auxiliary surface*, of which the axes have still the directions of the normals ν, is found by *inverting* the new linear function ω, or by forming from XXII. the *inverse equation*,

$$\text{XXXIII.} \ldots (\omega^{-1} + b^{-2})\nu = 0;$$

in which,

$$\text{XXXIV.} \ldots \omega^{-1}\nu . (\mathrm{S}\alpha\alpha'\rho)^2 = \mathrm{V}\alpha\alpha'\mathrm{S}\alpha\alpha'\nu + l^{-2}(\mathrm{V}\alpha\rho\mathrm{S}\alpha'\rho\nu + \mathrm{V}\alpha'\rho\mathrm{S}\alpha\rho\nu);$$

and from which it follows that the *normals* ν to the *confocals* through P have the directions of the *axes* of this *new cone*,

$$\text{XXXV.} \ldots \mathrm{S}\nu\omega^{-1}\nu = 0, \quad \text{or} \quad \text{XXXVI.} \ldots 0 = l^2(\mathrm{S}\alpha\alpha'\nu)^2 + 2\mathrm{S}\alpha\rho\nu\mathrm{S}\alpha'\rho\nu,$$

with ρ treated as a constant, as before.

(12.) The *vertex* of this auxiliary cone being placed at the given point P, of intersection of the three confocals, we may inquire *in what curve* is the *cone cut*, by the *plane* of the given *focal lines*, α, α', drawn through the *common centre* O of all the surfaces III. Denoting by $\sigma = t\alpha + t'\alpha'$ the vector of a point S of this sought *section*, and writing

$$\text{XXXVII.} \ldots \nu = \sigma - \rho = t\alpha + t'\alpha' - \rho,$$

the equation XXXVI. gives the relation,

$$\text{XXXVIII.} \ldots tt' = \frac{l^2}{2} = \frac{a^2 - c^2}{4} = \text{const.};$$

the section is therefore an *hyperbola*, which is *independent of the point* P, and has the focal lines of the *system* for its *asymptotes*. And because its *vector equation* may be thus written (comp. 371, II.),

$$\text{XXXIX.} \ldots \sigma = t\alpha + \tfrac{1}{2}l^2t^{-1}\alpha',$$

or what may be called its *quaternion equation* as follows (comp. 371, I.),

$$\text{XL.} \ldots 2\mathrm{V}\alpha\sigma . \mathrm{V}\sigma\alpha' = l^2(\mathrm{V}\alpha\alpha')^2,$$

it satisfies the *two scalar equations*,

$$\text{XLI.} \ldots m = 0, \quad m' = 0,$$

with the significations XXVI. of m and m'; it is therefore that important curve, which is known by the name of the *Focal Hyperbola*:[†] namely the *limit* to which the *section* of the *confocal surface* by the *plane* of its *extreme*[*] *axes* tends, when the *mean axis* ($2b$) tends to *vanish*. We are then led thus to the known theorem, that *if, with any assumed point* P *for vertex, and with the focal hyperbola*[†] *for base, a cone be constructed, the axes of this focal cone have the directions of the normals to the confocals through* P.

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[*] The general expressions for $\psi\sigma$ and $\chi\sigma$ include terms, which vanish when $\sigma = \rho$.

[†] Compare the Notes to pages 231, 505.

(13.) As regards the *Focal Ellipse*, its two scalar equations may be deduced from the rectangular form X., by equating to zero both the numerator and the denominator of its last term; they are therefore,

$$\text{XLII.} \ldots \mathrm{S}(\alpha-\alpha')\rho=0, \quad 2l^2=(\mathrm{S}\rho\mathrm{U}(\alpha+\alpha'))^2+\left(\frac{\mathrm{S}\rho\mathrm{UV}\alpha\alpha'}{\mathrm{S}\sqrt{\alpha\alpha'}}\right)^2;$$

the curve being thus given as a *perpendicular section* of an *elliptic cylinder*, with $l\sqrt{2}$ and $l\sqrt{(1+\mathrm{S}\alpha\alpha')}$, or $(a^2-c^2)^{\frac{1}{2}}$ and $(b^2-c^2)^{\frac{1}{2}}$, for the semiaxes of its base, or of the ellipse itself.

(14.) The same curve may also be represented by the equations,

$$\text{XLIII.} \ldots \mathrm{S}\alpha\rho=\mathrm{S}\alpha'\rho, \quad \mathrm{TV}\alpha\rho=(b^2-c^2)^{\frac{1}{2}},$$

or

$$\text{XLIII}'. \ldots \mathrm{S}\alpha'\rho=\mathrm{S}\alpha\rho, \quad \mathrm{TV}\alpha'\rho=(b^2-c^2)^{\frac{1}{2}};$$

which express that it is the *common intersection* of its own *plane* ($\perp \alpha-\alpha'$) with *two right cylinders*,[‡] which have the *two focal lines* α, α' of the system for their *axes of revolution*, and have *equal radii*, denoted each by the radical last written.

(15.) In general, the *unifocal form* (comp. 406, (13.)) of the equation III., namely,

$$\text{XLIV.} \ldots 0=(1-e^2)\left((\mathrm{V}\alpha\rho)^2+b^2\right)+(\mathrm{S}(\alpha'-e\alpha)\rho)^2,$$

in which α and α' may be interchanged, shows that the *two equal right cylinders*,

$$\text{XLV.} \ldots (\mathrm{V}\alpha\rho)^2+b^2=0, \quad \text{XLV}'. \ldots (\mathrm{V}\alpha'\rho)^2+b^2=0,$$

or

$$\text{XLVI.} \ldots \mathrm{TV}\alpha\rho=b, \quad \text{XLVI}'. \ldots \mathrm{TV}\alpha'\rho=b,$$

which are *real* if their common *radius* b be such, that is, if the confocal (e) be either an *ellipsoid* (supposed to be *real*), or else a *single-sheeted hyperboloid*, and which have the *focal lines* α, α' of the system for their *axes* of revolution, *envelope*[§] *that confocal surface*; the *planes* of the two *ellipses of contact* (which again are *real curves*, if b be real) being given by the equations,

$$\text{XLVII.} \ldots \mathrm{S}(\alpha'-e\alpha)\rho=0, \qquad \text{XLVII}'. \ldots \mathrm{S}(\alpha-e\alpha')\rho=0;$$

so that they pass *through the centre* O of the surface (or of the system), and are the (real) *director planes* (comp. 406, (14.)) of the *asymptotic cone* (real or imaginary), to the particular confocal (e).

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[*] Namely, those two of which the *squares* algebraically include between them that of the third; this latter being, for the same reason, considered here as the *mean*.

[†] We shall soon see that quaternions give, with equal ease, a more general known theorem, in which this is included as a limit.

[‡] The reader may consult page 513 of the *Lectures*, for the case of this theorem which answers to a given *ellipsoid*. The *focal ellipse* may also be represented generally by the expression (comp. page 382 of these *Elements*),

$$\rho=(a^2-c^2)^{\frac{1}{2}}\,\mathrm{V}.\alpha^t\mathrm{U}(\alpha+\alpha');$$

or by the same expression, with α and α' interchanged.

[§] Compare pages 199, 228, 233, 299.

(16.) Whether the mean *semiaxis* (b) be real or imaginary, the *surface* III. (supposed to be itself *real*) is always, by the form XLIV. of its equation, the *locus* of a system of *real ellipses* (comp. 404, (1.)), in planes *parallel* to the *director plane* XLVII., which have their *centres on the focal line* a, and are orthogonally *projected into circles* on a plane *perpendicular to that line.*

(17.) The *same surface* is also the locus of a *second system* of such ellipses, related similarly to the second focal line a', and to the second director plane XLVII'.; and it appears that *these two systems* of *elliptic sections* of a surface of the second order, which from some points of view are nearly as interesting as the *circular sections*, may conveniently be called its *Centro-Focal Ellipses.*

(18.) For example, when the *first quaternion form* (204, (14.), or 404, I.) of the equation of the *ellipsoid* is employed, *one* system of such ellipses coincides with the system (204, (13.)) of which, in the *first generation*[*] of the surface, the ellipsoid

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[*] Besides that *first generation* (I) of the Ellipsoid, which was a *double* one, in the sense that a *second system* (17.) of *generating ellipses* might be employed, and which served to connect the surface with a *concentric sphere*, by certain relations of *homology* (274); and the *second double generation* or *construction* (II), by means of either of *two diacentric spheres* (217, (4.), (6.), (7.), and 220, (3.)), which was illustrated by Fig. 53 (page 226): several *other* generations of the same important surface were deduced from quaternions in the *Lectures*, to which it is only possible here to *refer*. A reader, then, who happens to have a copy of that earlier work, may consult page 499 for a *generation* (III) of a system of *two reciprocal ellipsoids*, with a *common mean axis* ($2b$), by means of a *moving sphere*, of which the *radius* ($=b$) is *given*, but of which the *centre* has the *original* ellipsoid for its *locus*; while the *corresponding point* on the *reciprocal surface*, and also the *normals* at the two points, are easily deduced from the construction. In page 502, he will find another and perhaps a *simpler generation* (IV), of the *same pair of reciprocal ellipsoids*, by means of *quadrilaterals inscribed in a fixed sphere* (the *common mean sphere*, comp. 216, (10.)); the *directions* of the *four sides* of such a quadrilateral being *given*, and *one pair* of *opposite sides intersecting* in a point of *one surface*, while the *other pair* have for their intersection the *corresponding point* of the *other* (or reciprocal) ellipsoid. In the page last cited, and in the following page, there is given a *new double generation* (V) of any *one* ellipsoid; its *circular sections* (of either system) being constructed as *intersections of two equal spheres* (or spheric surfaces), of which the *line of centres* retains a *fixed direction*, while the *spheres slide* within *two equal and right cylinders*, whose *axes* intersect each other (in the *centre* of the generated surface), and of which the *common radius* is the *mean semiaxis* (b). Finally, in page 699 of the same volume, there will be found a *new generation* (VI) of the *original ellipsoid* (abc), *analogous* to the generation (IV) by the *fixed* (*mean*) *sphere*, but with *new directions* of the *sides* of the *quadrilaterals*, which are also (in this *last* generation) *inscribed* in the *circles* of a certain *mean ellipsoid* (or *prolate spheroid*) of *revolution*, which has the *mean axis* ($2b$) for *its major axis*, and has *two medial foci* on that axis, whose *common distance* from the *centre* is represented by the expression,

$$\frac{\sqrt{(a^2-b^2)}\sqrt{(b^2-c^2)}}{\sqrt{(a^2-b^2+c^2)}};$$

was treated as the *locus*; and an *analogous generation* of the *two hyperboloids*, by geometrical *deformation* of two corresponding surfaces of *revolution*, with certain resulting *homologies* (comp. sub-arts. to 274), through substitution of (*centro-focal*) *ellipses* for *circles*, conducts to *equations* of those hyperboloids of the same *unifocal form*; namely, if α and β have significations analogous to those in the cited equation of the ellipsoid (so that β and *not* α is *here* a *focal line*),

$$\text{XLVIII.} \ldots \left(\mathrm{S}\frac{\rho}{\alpha}\right)^2 + \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = \mp 1;$$

the upper or the lower *sign* being taken, according as the surface consists of *one sheet* or of *two*.

(19.) It may also be remarked that as, by changing β to α in the corresponding equation of the *ellipsoid*, we could return (comp. 404, (1.)) to a form (403, XI.) of the equation of the *sphere*, so the same change in XLVIII. conducts to equations of the *equilateral hyperboloids of revolution*, of *one* sheet and of *two*, under the very simple forms* (comp. 210. XI.),

$$\text{XLIX.} \ldots \mathrm{S}\left(\frac{\rho}{\alpha}\right)^2 = -1, \quad \text{and} \quad \text{L.} \ldots \mathrm{S}\left(\frac{\rho}{\alpha}\right)^2 = +1;$$

in which it seems unnecessary to insert *points* after the signs S, and of which the geometrical *interpretations* become obvious when then they are written thus (comp. 199, V.),

$$\text{LI.} \ldots \mathrm{T}\frac{\rho}{\alpha} = \sqrt{\sec 2\left(\frac{\pi}{2} - \angle\frac{\rho}{\alpha}\right)}, \quad \text{LII.} \ldots \mathrm{T}\frac{\rho}{\alpha} = \sqrt{\sec 2\angle\frac{\rho}{\alpha}};$$

where $\mathrm{T}\frac{\rho}{\alpha} = \overline{\mathrm{OP}} : \overline{\mathrm{OA}}$, while $\angle\frac{\rho}{\alpha}$ is the inclination AOP of the *semidiameter* OP to the *axis* of revolution OA, and $\frac{\pi}{2} - \angle\frac{\rho}{\alpha}$ is the inclination of the same semidiameter to a *plane* perpendicular to that axis.

(20.) The *real cyclic forms* of the equation of the surface III. might be deduced from the *unifocal form* XLIV., by the general method of the subarticles to 359; but since we have ready the *rectangular form* X., it is simpler to obtain them from that form, with the help of the identity,

$$\text{LIII.} \ldots -\rho^2 = (\mathrm{S}\rho\mathrm{U}(\alpha + \alpha'))^2 + (\mathrm{S}\rho\mathrm{U}\mathrm{V}\alpha\alpha')^2 + (\mathrm{S}\rho\mathrm{U}(\alpha - \alpha'))^2,$$

by eliminating the *first* of these *three terms* for the case of a *single-sheeted hyperbo-*

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the *common tangent planes*, to this *mean* (or *medial*) ellipsoid, and to the *given* (or *generated*) ellipsoid (abc), which are *parallel* to their *common axis* ($2b$), being parallel also to the *two umbilicar diameters* of the latter surface.

* The *same forms*, but with σ for ρ, and β for α, may be deduced from XLVIII. on the plan of 274, (2.), (4.), by assuming an auxiliary vector σ such that $\mathrm{S}\frac{\sigma}{\beta} = \pm\mathrm{S}\frac{\rho}{\alpha}$, and $\mathrm{V}\frac{\sigma}{\beta} = \mathrm{V}\frac{\rho}{\beta}$; the *homologies*, above alluded to, between the *general hyperboloid* of either species, and the *equilateral hyperboloid of revolution* of the same species, admitting also thus of being easily exhibited.

loid (for which $b^{-2} > a^{-2} > 0 > c^{-2}$); the *second* for an *ellipsoid* ($c^{-2} > b^{-2} > a^{-2} > 0$); and the *third* for a *double-sheeted hyperboloid* ($a^{-2} > 0 > c^{-2} > b^{-2}$).

(21.) Whatever the *species* of the surface III. may be, we can always derive from the unifocal form XLIV. of its equation what may be called an *Exponential Transformation;* namely the vector expression,

$$\text{LIV.} \ldots \rho = x\alpha + y\mathrm{V}\alpha^t\beta, \quad \text{with} \quad \text{LV.} \ldots x^2 f\alpha + y^2 f\mathrm{UV}\alpha\alpha' = 1;$$

the *scalar exponent*, t, remaining *arbitrary*, but the *two scalar coefficients*, x and y, being *connected* by this last equation of the second degree: provided that the *new constant vector* β be derived from α, α', and e, by the formula,

$$\text{LVI.} \ldots \beta = \frac{(\alpha' - e\alpha)\,\mathrm{UV}\alpha\alpha'}{e + \mathrm{S}\alpha\alpha'},$$

which gives after a few reductions (comp. the expression 315, III. for α^t, when $\mathrm{T}\alpha = 1$),

$$\text{LVII.} \ldots \mathrm{V}\alpha\beta = \mathrm{UV}\alpha\alpha',\ \mathrm{S}(\alpha' - e\alpha)\beta = 0,\ \mathrm{S}\alpha\alpha'\beta = 0;$$

$$\text{LVIII.} \ldots \mathrm{V}\alpha^t\beta = \beta\mathrm{S}.\alpha^t + \mathrm{UV}\alpha\alpha'.\mathrm{S}.\alpha^{t-1}; \quad \text{LIX.} \ldots \mathrm{V}.\alpha\mathrm{V}\alpha^t\beta = \alpha^t\mathrm{UV}\alpha\alpha' = \mathrm{T}^{-1}1;$$

$$\text{LX.} \ldots \mathrm{S}(\alpha' - e\alpha)\rho = x(e + \mathrm{S}\alpha\alpha'),\ \mathrm{V}\alpha\rho = y\alpha^t\mathrm{UV}\alpha\alpha';$$

while $$\text{LXI.} \ldots f\alpha = a^{-2}b^2c^{-2}, \quad \text{and} \quad \text{LXII.} \ldots f\beta = f\mathrm{UV}\alpha\alpha' = b^{-2}.$$

(22.) If we treat the *exponent*, t, as the *only variable* in the expression LIV. for ρ, then (comp. 314, (2.)) that *exponential expression* represents what we have called (17.) a *centro-focal ellipse;* the *distance of its centre* (or of its *plane*) from the centre of the *surface*, measured *along the focal line* α, being represented by the coefficient x; and the *radius* of the *right cylinder*, of which the ellipse is a *section*, or the radius of the *circle* (16.) into which that ellipse is *projected*, on a plane $\perp \alpha$, being represented by the *other* coefficient, y: while $\frac{1}{2}t\pi$ is the *excentric anomaly*.

(23.) If, on the contrary, we treat the *exponent* t *as given*, but the *coefficients* x and y *as varying together*, so as to satisfy the equation LV. of the second degree, the expression LIV. then represents a *different section* of the *surface* III., made by a *plane through the line* α, which makes with the *focal plane* (of α, α') an angle $= \dfrac{t\pi}{2}$; this latter section (like the former) being *always real*, if the surface *itself* be such: but being an *ellipse* for an *ellipsoid*, and an *hyperbola* for *either hyperboloid*, because

$$\text{LXIII.} \ldots f\alpha.f\mathrm{UV}\alpha\alpha' = a^{-2}c^{-2} \text{ by LXI. and LXII.}$$

(24.) And it is scarcely necessary to remark, that by interchanging α and α' we obtain a *Second Exponential Transformation*, connected with the *second system* (17.) of *centro-focal ellipses*, as the *first* exponential transformation LIV. is connected with the *first system* (16.).

(25.) The *asymptotic cone* $f\rho = 0$ has likewise its *two systems* of *centro-focal ellipses*, and its equation admits in like manner of *two exponential transformations*, of the form LIV.; the only difference being, that the equation LV. is replaced by the following,

$$\text{LXIV.} \ldots x^2 f\alpha + y^2 f\mathrm{UV}\alpha\alpha' = 0,$$

in which, for a *real cone*, the coefficients of x^2 and y^2 have *opposite signs* by (23.).

(26.) Finally, as regards the *confocal relation* of the surfaces III., which may represent *any confocal system* of surfaces of the second order, it may be perceived from (4.) that an *essential character* of such a relation is expressed by the equation,

$$\text{LXV.} \ldots \mathrm{V}\nu_{,}\phi\nu_{,} = \mathrm{V}\nu\phi_{,}\nu\,;$$

which may perhaps be called, on that account, the *Equation of Confocals.*

(27.) It is understood that the *two confocal surfaces* here considered, are represented by the two scalar equations,

$$\text{LXVI.} \ldots \mathrm{S}\rho\phi\rho = 1, \quad \mathrm{S}\rho\phi_{,}\rho = 1, \quad \text{or} \quad \text{LXVI}'. \ldots f\rho = 1, \quad f_{,}\rho = 1\,;$$

and that the *two linear and vector functions*, ν and $\nu_{,}$, of an *arbitrary vector* ρ, which represent *normals* to the two *concentric* and *similar* and *similarly posited surfaces*,

$$\text{LXVII.} \ldots f\rho = \text{const.}, \quad f_{,}\rho = \text{const.},$$

passing through any proposed point P, are expressed as follows,

$$\text{LXVIII.} \ldots \nu = \phi\rho, \quad \nu_{,} = \phi_{,}\rho.$$

(28.) It is understood also, that the two surfaces LXVI. or LXVI′. are not only *concentric*, as their equations show, but also *coaxal*, so far as the *directions* of their *axes* are concerned: or that the *two vector quadratics* (comp. 354),

$$\text{LXIX.} \ldots \mathrm{V}\rho\phi\rho = 0, \quad \text{and} \quad \text{LXX.} \ldots \mathrm{V}\rho\phi_{,}\rho = 0,$$

are satisfied by one *common system* of *three rectangular unit lines*. And with these understandings, it will be found that the equation LXV., which has been called above the *Equation of Confocals*, is not only *necessary* but *sufficient*, for the establishment of the relation required.

(29.) It is worth while however to observe, before closing the present series of subarticles, that the equations XII., and those formed from them by introducing e_2 and ν_2, give the following among other relations:

$$\text{LXXI.} \ldots f\mathrm{U}\nu_1 = (b^2 - b_1^2)^{-1} = -f_1\mathrm{U}\nu\,; \quad f_1\mathrm{U}\nu_2 = (b_1^2 - b_2^2)^{-1} = -f_2\mathrm{U}\nu_1\,; \text{ \&c.};$$

and $$\text{LXXII.} \ldots f(\nu_1, \nu_2) = f_1(\nu_2, \nu) = f_2(\nu, \nu_1) = 0\,;$$

and therefore,

$$\text{LXXIII.} \ldots f_1\{(b_2^2 - b_1^2)^{\frac{1}{2}}\mathrm{U}\nu_2 \pm (b_1^2 - b^2)^{\frac{1}{2}}\mathrm{U}\nu\} = 0\,;$$

whence it is easy to see that the *two vectors* under the functional sign f_1 in this last expression have the directions of the *generating lines* of the *single-sheeted hyperboloid* (e_1) through P, if we suppose that $b_2^2 > b_1^2 > 0 > b^2$, so that the confocal (e_2) is *here* an *ellipsoid*, and (e) a *double-sheeted hyperboloid.*

(30.) But if σ be taken to denote the variable vector of the *auxiliary surface* XXIV., the equation of that surface may by (7.) and (8.) be brought to the following rectangular form, with the meaning XXI. of ω,

$$\text{LXXIV.} \ldots 1 = \mathrm{S}\sigma\omega\sigma = (\mathrm{S}\rho\sigma)^2 - 2l^2\mathrm{S}\alpha\sigma\mathrm{S}\alpha'\sigma = b^2(\mathrm{S}\sigma\mathrm{U}\nu)^2 \\ + b_1^2(\mathrm{S}\sigma\mathrm{U}\nu_1)^2 + b_2^2(\mathrm{S}\sigma\mathrm{U}\nu_2)^2\,;$$

hence, with the inequalities (29.), its *cyclic normals*, or those of its *asymptotic cone* $\mathrm{S}\sigma\omega\sigma = 0$, or the *focal lines* of the *reciprocal cone* $\mathrm{S}\sigma\omega^{-1}\sigma = 0$, that is of the cone XXXVI., or finally the *focal lines of the focal** *cone* (12.), which *rests on the focal hyperbola*, have the directions of the lines LXXIII.; *those focal lines* are therefore

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* A more general known theorem, including this, will soon be proved by quaternions.

(by what has just been seen) *the generating lines of the hyperboloid* (e_1), which passes through the given point P.

(31.) And for an *arbitrary* σ we have the transformation,

$$\text{LXXV.} \ldots l^{-2}(\mathrm{S}\rho\sigma)^2 - \mathrm{S}\alpha\sigma\alpha'\sigma = e(\mathrm{S}\sigma\mathrm{U}\nu)^2 + e_1(\mathrm{S}\sigma\mathrm{U}\nu_1)^2 + e_2(\mathrm{S}\sigma\mathrm{U}\nu_2)^2.$$

408. The general *equation*[*] *of conjugation*,

$$\text{I.} \ldots (f\rho, \rho') = 1, \qquad 405, \text{III.}$$

connecting the vectors ρ, ρ' of any two *points* P, P′ which are conjugate with respect to the *central* but *non-conical surface* $f\rho = 1$, may be called for that reason the *Equation of Conjugate Points;* while the analogous equation,

$$\text{II.} \ldots f(\rho, \rho') = 0,$$

which replaces the former for the case of the *asymptotic cone* $f\rho = 0$, may be called by contrast the *Equation of Conjugate Directions:* in fact, it is satisfied by any two *conjugate semidiameters*, as may be at once inferred from the *differential equation* $f(\rho, \mathrm{d}\rho) = 0$ of the surface $f\rho = \text{const.}$ (comp. 362). Each of these two formulæ admits of numerous applications, among which we shall here consider the deduction, and some of the transformations, of the *Equation of a Circumscribed Cone*,

$$\text{III.} \ldots (f(\rho, \rho') - 1)^2 = (f\rho - 1)(f\rho' - 1);$$

which may also be considered as the *Condition of Contact*, of the *right line* PP′ with the *surface* $f\rho = 1$.

(1.) In this last view, the equation III. may be at once deduced, as the condition of *equal roots* in the scalar and *quadratic* equation (comp. 216, (2.), and 316, (30.)),

$$\text{IV.} \ldots 0 = f(x\rho + x'\rho') - (x + x')^2,$$

or

$$\text{V.} \ldots 0 = x^2(f\rho - 1) + 2xx'(f(\rho, \rho') - 1) + x'^2(f\rho' - 1);$$

which gives in general the *two vectors of intersection*, as the two values of the expression $\dfrac{x\rho + x'\rho'}{x + x'}$.

(2.) If we treat the *point* P′ as *given*, and denote the *two secants* drawn from it in any *given direction* τ by $t_1^{-1}\tau$ and $t_2^{-1}\tau$, then t_1 and t_2 are the roots of this other quadratic, $f(\rho' + t^{-1}\tau) = 1$, or

$$\text{VI.} \ldots 0 = f(t\rho' + \tau) - t^2 = t^2(f\rho' - 1) + 2tf(\rho', \tau) + f\tau;$$

denoting then by $t_0^{-1}\tau$ the *harmonic mean* of these two secants, so that $2t_0 = t_1 + t_2$, and writing $\rho = \rho' + t_0^{-1}\tau$, we have

$$\text{VII.} \ldots t_0(1 - f\rho') = f(\rho', \tau), \quad f(\rho, \rho') = 1;$$

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[*] For the notation used, Art. 362 may be again referred to.

we are then led in this way to the formula I., as the ***Equation of the Polar Plane*** of the point P′, if that plane be here supposed to be ***defined*** by its well-known ***harmonic property*** (comp. 215, (16.), and 316, (31.), (32.)).

(3.) At the same time we obtain this ***other form*** of the ***condition of contact*** III., as that of equal roots in VI.,

$$\text{VIII.} \ldots f(\rho', \tau)^2 = f\tau . (f\rho' - 1),$$

the first member being an abridgment of $(f(\rho', \tau))^2$: and because this last equation VIII. is ***homogeneous with respect to*** τ, it represents a cone, namely the *Cone of Tangents* (τ) to the given surface $f\rho = 1$, from the given point P′. Accordingly it is easy to prove that the equation III. may be thus written,

$$\text{IX.} \ldots f(\rho', \rho - \rho')^2 = f(\rho - \rho').(f\rho' - 1),$$

under which last form it is seen to be homogeneous with respect to $\rho - \rho'$.

(4.) Without expressly introducing τ, the transformation IX. shows that the equation III. represents ***some cone***, with the given point P′ for its vertex; and because the ***intersection*** of this ***cone*** with the given ***surface*** is expressed by the ***square*** of the equation I. of the ***polar plane*** of that point, the cone must be (as above stated) ***circumscribed*** to the surface $f\rho = 1$, touching it ***along the curve*** (real or imaginary) in which that surface is ***cut*** by that ***plane*** I.

(5.) Another important transformation, or set of transformations, of the equation III. may be obtained as follows. In general, for ***any two vectors*** ρ and ρ', if the scalar constant m, the vector function ψ, and the scalar function F, be derived from the linear and vector function ϕ, which is here ***self-conjugate*** (405), by the method of the Section III. ii. 6, we have successively,

$$\begin{aligned}\text{X.} \ldots f(\rho, \rho')^2 - f\rho . f\rho' &= S\rho\phi\rho' . S\rho'\phi\rho - S\rho\phi\rho . S\rho'\phi\rho' = S(V\rho\rho' . V\phi\rho\phi\rho') \\ &= S . \rho\rho'\psi V\rho\rho' = mS . \rho\rho'\phi^{-1}V\rho\rho' = mFV\rho\rho';\end{aligned}$$

and thus the equation III. of the ***circumscribed cone*** becomes,

$$\text{XI.} \ldots mFV\rho\rho' + f(\rho - \rho') = 0, \quad \text{or} \quad \text{XII.} \ldots mFV\tau\rho' + f\tau = 0,$$

if $\tau = \rho - \rho'$ be a ***tangent*** from P′. Or because $\phi\psi = m$, and $m = -c_1c_2c_3 = -a^{-2}b^{-2}c^{-2}$, by 406, XXIV., we may write (with $\tau = \rho - \rho'$) either

$$\text{XIII.} \ldots 0 = S\tau\psi^{-1}\tau + S\upsilon\phi^{-1}\upsilon, \quad \text{if} \quad \upsilon = V\tau\rho' = V\rho\rho',$$

or

$$\text{XIV.} \ldots FV\rho\rho' = a^2b^2c^2 f(\rho - \rho'),$$

as the ***condition of contact*** of the ***line*** PP′ with the ***surface*** $f\rho = 1$.

(6.) A ***geometrical interpretation***, of this last ***form*** XIV. of that ***condition***, can easily be assigned as follows. Supposing at first for simplicity that the surface is an ellipsoid, let P be the point of contact, so that $f\rho = 1$, $f(\rho, \tau) = 0$; and let the tangent PP′ be taken equal to the parallel semidiameter OT, so that $f\tau = f(\rho - \rho') = 1$. Then, with the signification XIII. of υ, the equation XIV. becomes,

$$\text{XV.} \ldots \sqrt{F\upsilon} = T\upsilon . \sqrt{FU\upsilon} = abc;$$

in which the factor $T\upsilon$ represents the area of the parallelogram under the conjugate semidiameters OP, OT of the given surface $f\rho = 1$; while the other factor $\sqrt{FU\upsilon}$ represents the reciprocal of the semidiameter of the reciprocal surface $F\nu = 1$, which is perpendicular to their plane POT; or the perpendicular distance between that plane, and a parallel plane which touches the given ellipsoid: so that their product $\sqrt{F\upsilon}$ is equal, by elementary principles, to the product of the three semiaxes, as stated in the formula XV. And the result may easily be extended by squaring, to other central surfaces.

(7.) It may be remarked in passing, that if ρ, σ, τ be ***any three conjugate semidiameters*** of any central surface $f\rho = 1$, so that

$$\text{XVI.} \ldots f\rho = f\sigma = f\tau = 1, \quad \text{and} \quad \text{XVII.} \ldots f(\rho, \sigma) = f(\sigma, \tau) = f(\tau, \rho) = 0,$$

and if $x\rho + y\sigma + z\tau$ be ***any other*** semidiameter of the same surface, we have then the scalar equation,

$$\text{XVIII.} \ldots f(x\rho + y\sigma + z\tau) = x^2 + y^2 + z^2 = 1;$$

a relation between the coefficients, x, y, z, which has been already noticed for the ***ellipsoid*** in 99, (2.), and in 402, I., and is indeed deducible for ***that*** surface, from principles of ***real scalars*** and ***real vectors*** alone: but in extending which to the ***hyperboloids***, one at least of those three ***coefficients*** becomes ***imaginary***, as well as one at least of the three ***vectors*** ρ, σ, τ.

(8.) Under the same conditions XVI. XVII., we have also,

$$\text{XIX.} \ldots \text{V}\rho\sigma = \pm abc\phi\tau = \pm(-m)^{-\frac{1}{2}}\phi\tau;$$

$$\text{XX.} \ldots \tau = \pm(-m)^{\frac{1}{2}}\phi^{-1}\text{V}\rho\sigma = \mp(-m)^{-\frac{1}{2}}\text{V}\phi\rho\phi\sigma;$$

$$\text{XXI.} \ldots \text{S}\rho\sigma\tau = \pm abc = \pm(-m)^{-\frac{1}{2}};$$

together with this very simple relation,

$$\text{XXII.} \ldots \text{S}\rho\sigma\tau . \text{S}\phi\rho\phi\sigma\phi\tau = -1.$$

(9.) Under the same conditions, if $x\rho + y\sigma + z\tau$ and $x'\rho + y'\sigma + z'\tau$ have only ***conjugate directions***, that is, if they have the ***directions*** of any two ***conjugate semidiameters***, the six scalar coefficients must satisfy (comp. II.) the equation,

$$\text{XXIII.} \ldots xx' + yy' + zz' = 0.$$

(10.) The equation VIII., with ρ for ρ', may be written under the form,

$$\text{XXIV.} \ldots 0 = \text{S}\sigma\tau = \text{S}\tau\omega\tau, \quad \text{if XXV.} \ldots \sigma = \omega\tau = \phi\rho\text{S}\rho\phi\tau + \phi\tau(1 - f\rho),$$

= a ***new*** linear and vector function, which represents a ***normal to the cone of tangents*** from P, to the surface $f\rho = 1$. ***Inverting*** this last function, we find

$$\text{XXVI.} \ldots \tau = \omega^{-1}\sigma = \frac{\phi^{-1}\sigma - \rho\text{S}\rho\sigma}{1 - f\rho};$$

the equation in σ of the ***reciprocal cone***, or of the ***cone of normals*** to the ***circumscribed cone*** from P, is therefore,

$$\text{XXVII.} \ldots \text{S}\sigma\omega^{-1}\sigma = 0, \quad \text{or} \quad \text{XXVIII.} \ldots F\sigma = (\text{S}\rho\sigma)^2, \quad \text{or finally}$$

$$\text{XXVIII}'. \ldots F(\sigma : \text{S}\rho\sigma) = 1;$$

a remarkably ***simple form***, which admits also of a simple ***interpretation***. In fact, the line $\sigma : \text{S}\rho\sigma$ is the ***reciprocal of the perpendicular***, from the centre O, ***on a tangent plane to the cone***, which is also a tangent plane ***to the surface***; it is therefore ***one of the values of the vector*** ν (comp. (6.), and 878, (21.)), and consequently it is a ***semidiameter of the reciprocal surface*** $F\nu = 1$.

(11.) As an application of the equation XXVIII., let the surface be the ***confocal*** (e), represented by the equation 407, III. or X., of which the ***reciprocal*** is represented by 407, XVII. or XVIII. Substituting for $F\sigma$ its value thus deduced, the equation of the ***reciprocal cone*** (10.), with σ for a ***side***, becomes,*

$$\text{XXIX.} \ldots 2l^2\text{S}\alpha\sigma\text{S}\alpha'\sigma - (\text{S}\rho\sigma)^2 = b^2\sigma^2, \quad \text{or} \quad \text{XXIX}'. \ldots \text{S}\alpha\sigma\alpha'\sigma - l^{-2}(\text{S}\rho\sigma)^2 = e\sigma^2;$$

if then the ***vertex*** P be ***fixed***, but the ***confocal vary***, by a change of e, or of b^2 which varies with it, the *cone* XXIX. will *also vary*, but will belong to a *biconcyclic system*; whence it follows that the (*direct* or) *circumscribed cones from a given point* are all *biconfocal*: and also, by 407, (30.), that their *common focal lines* are the *generating lines* of the *confocal hyperboloid** of one sheet, which passes through their *common vertex*.

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* It may be observed that, when $b = 0$, this equation XXIX. represents the ***asymptotic cone*** to the ***auxiliary surface*** 407, XXIV.; and at the same time the ***reciprocal*** of that ***focal cone***, 407, XXXVI., which rests on the ***focal hyperbola***.

(12.) Changing e to $e_{,}$ in XXIX'., and using the transformation 407, LXXV., with the identity (comp. 407, LIII.),

$$-\sigma^2 = (\mathrm{S}\sigma\mathrm{U}\nu)^2 + (\mathrm{S}\sigma\mathrm{U}\nu_1) + (\mathrm{S}\sigma\mathrm{U}\nu_2)^2,$$

we find that if σ be a *normal* to the *cone of tangents* from P to $(e_{,})$, it satisfies the equation,

$$\text{XXX.} \ldots 0 = (e - e_{,})(\mathrm{S}\sigma\mathrm{U}\nu)^2 + (e_1 - e_{,})(\mathrm{S}\sigma\mathrm{U}\nu_1)^2 + (e_2 - e_{,})(\mathrm{S}\sigma\mathrm{U}\nu_2)^2;$$

and therefore that if τ be a *tangent* from the same point P, to the same confocal $(e_{,})$, it satisfies this other condition,

$$\text{XXXI.} \ldots 0 = (e - e_{,})^{-1}(\mathrm{S}\tau\mathrm{U}\nu)^2 + (e_1 - e_{,})^{-1}(\mathrm{S}\tau\mathrm{U}\nu_1)^2 + (e_2 - e_{,})^{-1}(\mathrm{S}\tau\mathrm{U}\nu_2)^2,$$

which thus is a form of the equation of the *circumscribed cone* to $(e_{,})$, with its *vertex* at a *given point* P: the *confocal character* (11.) of all *such* cones being hereby exhibited anew.

(13.) It follows also from XXXI., that the *axes* of every *cone* thus *circumscribed* have the directions of the *normals* ν, ν_1, ν_2 to the *three confocals through* P; and this known theorem† may be otherwise deduced, from the *Equation of Confocals* (407, LXV.), by our general method, as follows. That equation gives

$$\nu_{,} - \nu \parallel \phi_{,}\nu \text{ (because } \phi\nu_{,} = \phi_{,}\nu\text{), and therefore,}$$

$$\text{XXXII.} \ldots (\nu_{,} - \nu)\mathrm{S}\nu\nu_{,} = \phi_{,}\nu(f_{,}\rho - 1), \quad \mathrm{V}\nu\nu_{,}\mathrm{S}\nu\nu_{,} + \mathrm{V}\nu\phi_{,}\nu(1 - f_{,}\rho) = 0;$$

changing then V to S, and ν to τ, we see that ν, ν_1, ν_2, as being the *roots* (354) of this last *vector quadratic* XXXII., have the *directions* of the axes of the cone, with τ for side,

$$\text{XXXIII.} \ldots f_{,}(\rho, \tau)^2 + f_{,}\tau \cdot (1 - f_{,}\rho) = 0;$$

that is, by VIII., the directions of the *axes of the cone of tangents*, from P to $(e_{,})$.

(14.) As an application of the formula XIV., with the abridged symbols τ and v of (5.) for $\rho - \rho'$ and $\mathrm{V}\rho\rho'$, the *condition of contact* of the *line* PP' with the *confocal* (e) becomes, by the expressions 407, III., XVIII., and VII. for the functions f, F, and the squares a^2, b^2, c^2, the following *quadratic* in e:

$$\text{XXXIV.} \ldots (\mathrm{S}a\tau)^2 - 2e\mathrm{S}a\tau\mathrm{S}a'\tau + (\mathrm{S}a'\tau)^2 + (1 - e^2)\tau^2 = l^{-2}(\mathrm{S}ava'v - ev^2);$$

there are therefore in general (as is known) *two confocals*, say (e) and $(e_{,})$, of a *given system*, which touch *a given right line*; and their *parameters*,‡ e and $e_{,}$, are the *two roots* of the last equation: for instance, their *sum* is given by the formula,

$$\text{XXXV.} \ldots (e + e_{,})\tau^2 = l^{-2}v^2 - 2\mathrm{S}a\tau\mathrm{S}a'\tau.$$

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* This theorem (which includes that of 407, (30.)) is cited from Jacobi, and is proved, in page 143 of Dr. Salmon's Treatise, referred to in several former Notes.

† Compare the second Note to page 648.

‡ This name of *parameter* is here given, as in 407, to the arbitrary constant $e = \dfrac{a^2 + c^2}{a^2 - c^2}$, of which the value distinguishes *one* confocal (e) of a system from another.

(15.) Conceive then that ρ is a *given semidiameter* of a *given confocal* (e), and that $\mathrm{d}\rho$ is a *tangent*, *given in direction*, at its extremity; the equation XXXIV. will then of course be satisfied,[*] if we change τ to $\mathrm{d}\rho$, and ν to $\mathrm{V}\rho\mathrm{d}\rho$, retaining the *given value* of e; but it will *also* be satisfied, for the *same* ρ and $\mathrm{d}\rho$ (or for the same τ and ν), when we change e to this *new parameter*,

$$\text{XXXVI.}\ldots e_{,} = -e + 2\mathrm{S}\alpha\mathrm{U}\mathrm{d}\rho.\mathrm{S}\alpha'\mathrm{U}\mathrm{d}\rho - l^{-2}(\mathrm{V}\rho\mathrm{U}\mathrm{d}\rho)^2;$$

that is to say, the *new confocal* $(e_{,})$, with a parameter determined by this last formula, will *touch the given tangent* to the *given* confocal (e).

(16.) If we *at once* make $l^2 = 0$ in the equation 407, III. of a *Confocal System of Central Surfaces*, leaving the parameter e *finite*, we fall back on the system 406, XXXV. of *Biconfocal Cones;* but if we conceive that l^2 only *tends* to *zero*, and that e at the same time tends to *positive infinity*, in such a manner that their *product* tends to a *finite limit*, r^2, or that

$$\text{XXXVII.}\ldots \lim.l = 0, \quad \lim.e = \infty, \quad \lim.el^2 = r^2,$$

then the equation of the surface (e) tends to this *limiting form*,

$$\text{XXXVIII.}\ldots \rho^2 + r^2 = 0, \quad \text{or} \quad \text{XXXVIII}'.\ldots \mathrm{T}\rho = r;$$

a *system of biconfocal cones* is therefore to be *combined* with a system of *concentric spheres*, in order to make up a *complete confocal system.*

(17.) Accordingly, any *given right line* PP′ is in general *touched* by only *one cone* of the system just referred to, namely by *that* particular *cone* (e), for which (comp. XXXIV.) we have the value,

$$\text{XXXIX.}\ldots e = \mathrm{S}\alpha\nu\alpha'\nu^{-1}, \quad \text{or} \quad \text{XXXIX}'.\ldots e + \mathrm{S}\alpha\alpha' = 2\mathrm{S}\alpha\nu\mathrm{S}\alpha'\nu^{-1},$$

with $\nu = \mathrm{V}\rho\rho'$, as before, so that ν is *perpendicular to the given plane* OPP′, which contains the *vertex* and the *line;* in fact, the *reciprocals* of the *biconfocal cones* 406, XXXV., when α, α' are treated as *given unit lines*, but e as a *variable parameter*, compose the *biconcyclic*[†] *system* (comp. 407, XVIII.),

$$\text{XL.}\ldots \mathrm{S}\alpha\nu\alpha'\nu = e\nu^2.$$

But, besides the *tangent cone* thus found, there is a *tangent sphere* with the same centre O; of which, by passing to the limits XXXVII., the radius r may be found from the same formula XXXIV. to be,

$$\text{XLI.}\ldots r = \mathrm{T}\frac{\nu}{\tau} = \mathrm{T}\frac{\mathrm{V}\rho\rho'}{\rho - \rho'};$$

and such is in fact an expression (comp. 316, L.) for the length of the perpendicular from the origin on the given line PP′.

(18.) In general, the equation XXXIV. is a form of the *equation of the cone*, with ρ for its variable vector, which has a *given vertex* P′, and is *circumscribed* to a *given confocal* (e). Accordingly, by making $e = -\mathrm{S}\alpha\alpha'$ in that formula, we are

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[*] In fact it follows easily from the transformations (5.), that

$$f\rho.f\mathrm{d}\rho - a^{-2}b^{-2}c^{-2}F\mathrm{V}\rho\mathrm{d}\rho = f(\rho, \mathrm{d}\rho)^2.$$

[†] The *bifocal form* of the equation of this *reciprocal system* of cones XL. was given in 406, XXV., but with *other constants* (λ, μ, g), connected with the *cyclic form* (406, I.) of the equation of the *given system.*

led (after a few reductions, comp. 407, XXVII.) to an equation which may be thus written,

$$\text{XLII.} \ldots \; 0 = l^2(\mathrm{S}\alpha\alpha'\tau)^2 + 2\mathrm{S}\alpha\rho'\tau\mathrm{S}\alpha'\rho'\tau,$$

with the variable side $\tau = \rho - \rho'$, as before; and which differs only by the substitution of ρ' and τ for ρ and ν, from the equation 407, XXXVI. for that *focal cone*, which rests on the *focal hyperbola*. The *other* (real) *focal cone* which has the same arbitrary vertex P′, but rests on the *focal ellipse*, has for equation,

$$\text{XLIII.} \ldots \; l^2(\mathrm{S}(\alpha - \alpha')\tau)^2 = \mathrm{S}\alpha\nu\alpha'\nu - \nu^2,$$

as is found by changing e to 1 in the same formula XXXIV.

(19.) It is however simpler, or at least it gives more symmetric results, to change $e_{,}$ in XXXI. to $-\mathrm{S}\alpha\alpha'$ for the focal hyperbola, and to $+1$ for the focal ellipse, in order to obtain the *two real focal cones* with P for vertex, which rest on those two curves; while that *third* and wholly *imaginary focal cone*, which has the same vertex, but rests on the known *imaginary focal curve*, in the plane of b and c, is found by changing $e_{,}$ to -1. This imaginary focal cone, and the two real ones which rest as above on the hyperbola and ellipse respectively, may thus be represented by the three equations,

$$\text{XLIV.} \ldots \; 0 = a^{-2}(\mathrm{S}\tau\mathrm{U}\nu)^2 + a_1^{-2}(\mathrm{S}\tau\mathrm{U}\nu_1)^2 + a_2^{-2}(\mathrm{S}\tau\mathrm{U}\nu_2)^2;$$
$$\text{XLV.} \ldots \; 0 = b^{-2}(\mathrm{S}\tau\mathrm{U}\nu)^2 + b_1^{-2}(\mathrm{S}\tau\mathrm{U}\nu_1)^2 + b_2^{-2}(\mathrm{S}\tau\mathrm{U}\nu_2)^2;$$
$$\text{XLVI.} \ldots \; 0 = c^{-2}(\mathrm{S}\tau\mathrm{U}\nu)^2 + c_1^{-2}(\mathrm{S}\tau\mathrm{U}\nu_1)^2 + c_2^{-2}(\mathrm{S}\tau\mathrm{U}\nu_2)^2;$$

τ being in each case a side of the cone, and ν, ν_1, ν_2 having the same significations as before.

(20.) On the other hand, if we place the *vertex of* a circumscribed cone at a point P of a *focal curve*, real or imaginary, the *enveloped surface* being the *confocal* $(e_{,})$, we find first, by XXX., for the *reciprocal* cones, or *cones of normals* σ, with the same order of succession as in (19.), the three equations,

$$\text{XLVII.} \ldots \; a^2(\mathrm{SU}\nu\sigma)^2 = a_{,}^2;$$
$$\text{XLVIII.} \ldots \; b^2(\mathrm{SU}\nu\sigma)^2 = b_{,}^2;$$
$$\text{XLIX.} \ldots \; c^2(\mathrm{SU}\nu\sigma)^2 = c_{,}^2;$$

and next, for the *circumscribed cones* themselves, or cones of *tangents* τ, the connected equations:

$$\text{L.} \ldots \; a^2(\mathrm{VU}\nu\tau)^2 + a_{,}^2 = 0;$$
$$\text{LI.} \ldots \; b^2(\mathrm{VU}\nu\tau)^2 + b_{,}^2 = 0;$$
$$\text{LII.} \ldots \; c^2(\mathrm{VU}\nu\tau)^2 + c_{,}^2 = 0;$$

all which have the *forms* of equations of *cones of revolution*, but on the geometrical *meanings* of the three last of which it may be worth while to say a few words.

(21.) The cone L. has an *imaginary vertex*, and is always *itself* imaginary; but the *two other cones*, LI. and LII., have *each* a *real vertex* P, with $b^2 > 0$ for the first, and $c^2 < 0$ for the second; b being the mean semiaxis of the *ellipsoid*, which passes through a given point of the *focal hyperbola*, and c^2 being the negative and algebraically least square of a scalar semiaxis of the *double-sheeted hyperboloid*, which passes through a given point of the *focal ellipse*: while, *in each case*, ν has the direction of the *normal* to the *surface*, which is also the *tangent* to the *curve* at that point, and is at the same time the *axis* of revolution of the *cone*.

(22.) The *semiangles* of the two last cones, LI. and LII., have for their respective *sines* the two quotients,

$$\text{LIII.}\ldots b_{,} : b, \quad \text{and} \quad \text{LIV.}\ldots (-c_{,}^{2})^{\frac{1}{2}} : (-c^{2})^{\frac{1}{2}};$$

each of those *two cones* is therefore *real*, if circumscribed to a *single-sheeted hyperboloid*, because, for *such* an enveloped surface $(e_{,})$, $b_{,}$ is *real*, and *less* than the b of *any confocal ellipsoid*, while $c_{,}$ is imaginary, and its *square* is algebraically *greater* (or nearer to zero) than the square of the imaginary semiaxis c of *every double-sheeted hyperboloid*, of the same confocal system; but the *cone* LI. is *imaginary*, if the *enveloped surface* $(e_{,})$ be either an hyperboloid of *two* sheets ($b_{,}$ imaginary), or an *exterior ellipsoid* $(b_{,} > b)$; and the *other* cone LII. is imaginary, if the surface $(e_{,})$ be either *any ellipsoid* ($c_{,}$ real), or else an *exterior* and *double*-sheeted hyperboloid $(a_{,}^{2} < a^{2}, c_{,}^{2} < c^{2}, -c_{,}^{2} > -c^{2})$. Accordingly it is known that the focal *hyperbola*, which is the *locus of the vertex* of the cone LI., lies entirely *inside every double*-sheeted hyperboloid of the system; while the focal *ellipse*, which is in like manner the *locus* of the *vertex* of the cone LII., is *interior* to *every ellipsoid*: and *real tangents* to a *single*-sheeted hyperboloid can be drawn, from *every real point* of space.

(23.) The *twelve points* (whereof only *four* at most can be *real*), in which a *surface* (e) or (abc) is cut by the *three focal curves*, are called the *Umbilics* of that surface; the vectors, say ω, $\omega_{,}$, $\omega_{,,}$, of *three* such umbilics, in the respective planes of ca, ab, bc, are:

$$\text{LV.}\ldots \omega = \frac{a}{2}(\alpha + \alpha') + \frac{c}{2}(\alpha - \alpha');$$

$$\text{LVI.}\ldots \omega_{,} = \frac{a(\alpha + \alpha')}{1 - \text{S}\alpha\alpha'} + \frac{\sqrt{-1}\,b\text{V}\alpha\alpha'}{1 - \text{S}\alpha\alpha'};$$

$$\text{LVII.}\ldots \omega_{,,} = \frac{c(\alpha - \alpha')}{1 + \text{S}\alpha\alpha'} - \frac{\sqrt{-1}\,b\text{V}\alpha\alpha'}{1 + \text{S}\alpha\alpha'};$$

and the others can be formed from these, by changing the signs of the terms, or of some of them. The four *real* umbilics of an *ellipsoid* are given by the formula LV., and those of a *double-sheeted hyperboloid* by LVI., with the changes of sign just mentioned.

(24.) In transforming expressions of this sort, it is useful to observe that the expressions for the squares of the semiaxes,

$$a^{2} = l^{2}(e + 1), \quad b^{2} = l^{2}(e + \text{S}\alpha\alpha'), \quad c^{2} = l^{2}(e - 1), \qquad 407,\ \text{VII.}$$

combined with $\text{T}\alpha = \text{T}\alpha' = 1$, give not only $a^{2} - c^{2} = 2l^{2}$, but also,

$$\text{LVIII.}\ldots \text{T}\frac{\alpha + \alpha'}{2} = \sqrt{\frac{1 - \text{S}\alpha\alpha'}{2}} = \cos\tfrac{1}{2}\angle\frac{\alpha'}{\alpha} = \left(\frac{a^{2} - b^{2}}{a^{2} - c^{2}}\right)^{\frac{1}{2}};$$

$$\text{LIX.}\ldots \text{T}\frac{\alpha - \alpha'}{2} = \sqrt{\frac{1 + \text{S}\alpha\alpha'}{2}} = \sin\tfrac{1}{2}\angle\frac{\alpha'}{\alpha} = \left(\frac{b^{2} - c^{2}}{a^{2} - c^{2}}\right)^{\frac{1}{2}};$$

and $$\text{LX.}\ldots \text{TV}\alpha\alpha' = \sqrt{(1 - (\text{S}\alpha\alpha')^{2})} = \sin\angle\frac{\alpha'}{\alpha} = l^{-2}(a^{2} - b^{2})^{\frac{1}{2}}(b^{2} - c^{2})^{\frac{1}{2}},$$

with the verification, that because

$$\text{LXI.}\ldots (\alpha - \alpha')(\alpha + \alpha') = 2\text{V}\alpha\alpha',$$

therefore $$\text{LXI}'.\ldots \text{T}(\alpha - \alpha').\text{T}(\alpha + \alpha') = 2\text{TV}\alpha\alpha'.$$

We have also the relations,

$$\text{LXII.}\ldots \text{T}(\alpha + \alpha')^{-2} + \text{T}(\alpha - \alpha)^{-2} = (\text{TV}\alpha\alpha')^{-2};$$

$$\text{LXIII.}\ldots \text{T}(\alpha + \alpha')^{-2} - \text{T}(\alpha - \alpha')^{-2} = \text{S}\alpha\alpha'.(\text{TV}\alpha\alpha')^{-2};$$

with others easily deduced.

(25.) The expression LV. conducts to the following among other consequences, which all admit of elementary verifications,* and may be illustrated by the annexed Fig. 84. Let U, U′ be the two real points in which an ellipsoid (abc) is cut by one branch of the focal hyperbola, with H for summit, and with F for its interior focus; the adjacent major summit of the surface being E, and R, R′ being (as in the Figure) the adjacent points of intersection of the same surface with the focal lines α, α', that is, with the asymptotes to the hyperbola. Let also V, T be the points in which the same asymptotes α, α' meet the tangent to the hyperbola at U, or the normal to the ellipsoid at that real umbilic, of which we may suppose that the vector OU is the ω of the formula LV.; and let S be the foot of the perpendicular on this normal to the surface, or tangent TV to the curve, let fall from the centre O. Then, besides the obvious values,

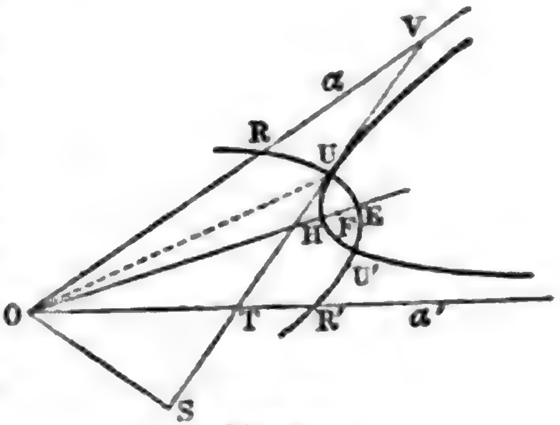

Fig. 84.

$$\text{LXIV.} \ldots \overline{\text{OE}} = a, \quad \overline{\text{OF}} = (a^2 - c^2)^{\frac{1}{2}}, \quad \overline{\text{OH}} = (a^2 - b^2)^{\frac{1}{2}},$$

and the obvious relations, that the intercept TV is bisected at U, and that the point F is at once a summit of the focal ellipse, and a focus of that other ellipse in which the surface is cut by the plane (ac) of the figure, we shall have these vector expressions (comp. 371, (3.), and 407, VIII. LXI.):

$$\text{LXV.} \ldots \text{OV} = (a + c)\,\alpha, \quad \text{OT} = (a - c)\,\alpha', \quad \text{TV} = a(\alpha - \alpha') + c(\alpha + \alpha');$$

$$\text{LXVI.} \ldots \text{SU}^{-1} = \phi\omega = -\frac{a^{-1}}{2}(\alpha + \alpha') - \frac{c^{-1}}{2}(\alpha - \alpha'), \quad \text{SU} = -ac : \text{TU};$$

$$\text{LXVII.} \ldots \text{OR} = \frac{\alpha}{\sqrt{f\alpha}} = ab^{-1}c\alpha, \quad \text{OR}' = \frac{\alpha'}{\sqrt{f\alpha'}} = ab^{-1}c\alpha';$$

whence follow by (24.) these other values,

$$\text{LXVIII.} \ldots \overline{\text{OV}} = a + c, \quad \overline{\text{OT}} = a - c, \quad \overline{\text{TV}} = 2b;$$

$$\text{LXIX.} \ldots \overline{\text{TU}} = \overline{\text{UV}} = b, \quad \overline{\text{SU}} = \overline{\text{OR}} = \overline{\text{OR}}' = ab^{-1}c;$$

$$\text{LXX.} \ldots \overline{\text{OU}} = \text{T}\omega = (a^2 - b^2 + c^2)^{\frac{1}{2}};$$

$$\text{LXXI.} \ldots \overline{\text{OS}} = (a^2 - b^2 + c^2 - a^2b^{-2}c^2)^{\frac{1}{2}} = b^{-1}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}}.$$

(26.) It follows that the lengths of the *sides* OV, OT, TV of the *umbilicar triangle* TOV are equal to the *sum* and *difference* ($a \pm c$) of the *extreme semiaxes*, and to the *mean axis* ($2b$) of the *ellipsoid;* while the *area* of that triangle $= \overline{\text{OS}} . \overline{\text{TU}} = (a^2 - b^2)^{\frac{1}{2}} (b^2 - c^2)^{\frac{1}{2}} =$ the *rectangle* under the *two semiaxes* of the *hyperbola*, if *both* be treated as *real*. The length $(\text{T}\phi\omega)^{-1}$, or $\overline{\text{SU}}$, of the *perpendicular* from the *centre* O, on the *tangent plane* at an umbilic U, is $ab^{-1}c$; and the *sphere* concentric with the ellipsoid, which *touches the four umbilicar tangent planes*, passes *through the points* R, R′ of *intersection* of that ellipsoid with the *focal lines* α, α', that is, as before, with the

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* Some such verifications were given in the *Lectures*, pages 691, 692, in connexion with Fig. 102 of that former volume, which answered in several respects to the present Fig. 84.

asymptotes to the *hyperbola*; or, by (21.)(22.), with the *axes* of the *two circumscribed right cylinders.** And finally the length, say u, of the *umbilicar semidiameter* OU, is given by the formula,

$$\text{LXXII.} \ldots u^2 = a^2 - b^2 + c^2;$$

all which agrees (25.) with known results.

(27.) An *umbilic* of a surface of the second order may be otherwise *defined* (comp. (23.)), as a real or imaginary point at which the *tangent plane is parallel* to a *cyclic plane*; and accordingly it is easy to prove (comp. 407, (20.)) that the *umbilicar normal* $\phi\omega$ in LXVI. has the direction of a *cyclic normal*. To employ this known property in verification of the recent expressions (25.), (26.), for the lengths of OU and SU, it is only necessary to observe that the *common radius* of the diametral and *circular sections* of the ellipsoid is the *mean semiaxis* b (comp. 216, (7.) (9.), &c.); and that, by a slight extension of the analysis in (7.), (8.), (9.), it can be shown that if ρ, σ, τ and ρ', σ', τ' be *any two systems* of *three conjugate semidiameters of any central surface*, $f\rho = 1$, then

$$\text{LXXIII.} \ldots \rho'^2 + \sigma'^2 + \tau'^2 = \rho^2 + \sigma^2 + \tau^2, \quad \text{and} \quad \text{LXXIV.} \ldots (S\rho'\sigma'\tau')^2 = (S\rho\sigma\tau)^2.$$

(28.) A less elementary verification of the value LXXII. of u^2, but one which is useful for other purposes, may be obtained from either the cubic in b^2, or that in e, assigned in 407, (8.). For if b_0^2, b_1^2, b_2^2 be the roots of the former cubic, and e_0, e_1, e_2 the roots of the latter, inspection of those equations shows at once that we have *generally*,

$$\text{LXXV.} \ldots -\rho^2 = b_0^2 + b_1^2 + b_2^2 - 2l^2 S\alpha\alpha' = l^2(e_0 + e_1 + e_2 + S\alpha\alpha');$$

or

$$\text{LXXVI.} \ldots \overline{\text{OP}}^2 = T\rho^2 = a_0^2 + b_1^2 + c_2^2 = b_0^2 + c_1^2 + a_2^2 = \&\text{c.},$$

where the semiaxes a_0, b_1, c_2 belong to the three confocals through *any* proposed point P. Making then,

$$\text{LXXVII.} \ldots a_0^2 = a^2, \quad b_1^2 = 0, \quad c_2^2 = c^2 - b^2,$$

we recover the expression assigned above, for the square of the length u of an *umbilicar semidiameter* of an *ellipsoid*.

(29.) For *any* central surface, the principle (27.) shows that if λ, μ be, as in 405, (5.), &c., the *two real cyclic normals*, and if g be the *real scalar* associated with them as before, then the vectors of the *four real umbilics* (if such exist) must admit of being thus expressed:

$$\text{LXXVIII.} \ldots \pm \phi^{-1}\lambda : \sqrt{F\lambda} = \pm abc\,(gU\lambda + \mu T\lambda);$$
$$\text{LXXIX.} \ldots \pm \phi^{-1}\mu : \sqrt{F\mu} = \pm abc\,(gU\mu + \lambda T\mu);$$

and thus we see anew, that an *hyperboloid* with *one sheet* has (as is well known) *no*

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* Compare 218, (5.), and 220, (4.); in which the points B, B' (comp. also Fig. 53, page 226) may now be conceived to coincide with the points R, R' of the new Figure 84. It is obvious that the theory of *circumscribed cylinders* is included in that of circumscribed *cones*; so that the cylinder circumscribed to the confocal (e), with its generating lines parallel to a given (real or imaginary) semidiameter γ of that surface ($f\gamma = 1$), may be represented (comp. III. XIV.) by the equation,

$$\text{III}' \ldots f(\rho, \gamma)^2 = f\rho - 1; \quad \text{or} \quad \text{XIV}' \ldots FV\gamma\rho = a^2b^2c^2;$$

with interpretations easily deduced, from principles already established.

real umbilic, because for *that* surface the product *abc* of the semiaxes is imaginary; or because it has *no real tangent plane parallel* to either of its two real planes of *circular section*.

(30.) Of whatever *species* the surface may be, the *three umbilicar vectors* (23.), of which only *one* at most can be *real*, with the particular *signs* there given, but which have the *forms* of lines *in the three principal planes*, must be conceived, in virtue of their *expressions* LV. LVI. LVII., to *terminate on an imaginary right line*, of which the vector equation is,

$$\text{LXXX.} \ldots \rho = \frac{-\alpha(e'+1)}{\alpha+\alpha'} - \sqrt{-1}\,\frac{b(e'+S\alpha\alpha')}{V\alpha\alpha'} + \frac{c(e'-1)}{\alpha-\alpha'};$$

e' being a scalar variable, which receives the three values, $-S\alpha\alpha'$, $+1$, and -1, when ρ comes to coincide with ω, $\omega_{,}$, and $\omega_{,,}$ respectively. And *such* an imaginary *right line*, which is easily proved to *satisfy*, for *all values* of the variable e', both the *rectangular* and the *bifocal forms* of the *equation of the surface* (e), or to be (in an imaginary sense) *wholly contained* upon that surface, may be called an *Umbilicar Generatrix*.

(31.) There are in general *eight such generatrices* of *any central surface* of the second order, whereof *each* connects *three umbilics*, in the *three principal planes*, *two* passing through *each* of the *twelve* umbilicar points (28.); and because e'^2 disappears from the *square* of the expression LXXX. for ρ, which square reduces itself to the following,

$$\text{LXXXI.} \ldots \rho^2 = -l^2(2e'+e+S\alpha\alpha') = -b^2 - 2l^2e',$$

they may be said to be the *eight generating lines* through the *four imaginary points*, in which the *surface meets* the *circle at infinity*.

(32.) In general, from the cubics in e and in b^2, or from either of them, it may be without difficulty inferred (comp. (28.)), that the *eight intersections* (real or imaginary) of *any three confocals* (e_0) (e_1) (e_2) have their vectors ρ represented by the formula:

$$\text{LXXXII.} \ldots \rho = \frac{\pm a_0a_1a_2}{l^2(\alpha+\alpha')} \pm \frac{\sqrt{-1}\,b_0b_1b_2}{l^2V\alpha\alpha'} \pm \frac{c_0c_1c_2}{l^2(\alpha-\alpha')};$$

comparing which with the vector expression LXXX., we see that the three confocals, through the point determined by that former expression, for any given value of e', are (e), (e'), and (e') *again*; and therefore that *two* of the *three* confocal surfaces through *any point* of an *umbilicar generatrix* (30.) *coincide*: a result which gives in a new way (comp. LXXV.) the expression LXXXI. for ρ^2.

(33.) The *locus* of all *such* generatrices, for *all* the confocals (e) of the system, is a certain *ruled surface*, of which the doubly variable *vector* may be thus expressed, as a function of the *two* scalar variables, e and e':

$$\text{LXXXIII.} \ldots \rho_{e,e'} = \frac{\pm l(e+1)^{\frac{1}{2}}(e'+1)}{\alpha+\alpha'} \pm \frac{\sqrt{-1}\,l(e+S\alpha\alpha')^{\frac{1}{2}}(e'+S\alpha\alpha')}{V\alpha\alpha'} \pm \frac{l(e-1)^{\frac{1}{2}}(e'-1)}{\alpha-\alpha'};$$

and because we have thus, for *any one set* of *signs*, the *differential relation*,

$$\text{LXXXIV.} \ldots D_e^2\rho_{e,e'} = \tfrac{1}{2}D_{e,e'}\rho_{e,e'},$$

it follows that *this ruled locus* is a *Developable Surface:* its *edge of regression* being that wholly *imaginary curve*, of which the *vector* is $\rho_{e,e}$, and which is therefore by (32.) the *locus* of all the *imaginary points*, through each of which pass *three coincident confocals*.

(34.) The *only real part* of this *imaginary developable* consists of the *two real focal curves*, which are *double lines* upon it, as are also the *imaginary focal*, and the *circle at infinity* (31.); and the *scalar equation* of the same *imaginary surface*, obtained by elimination of the two arbitrary scalars e and e', is found to be of the *eighth degree*, namely the following:

$$\text{LXXXV.}\ldots\left\{\begin{array}{l} 0=\Sigma m^2x^8+2\Sigma m(m-n)x^6y^2+\Sigma(p^2-6mn)x^4y^4 \\ +2\Sigma(3m^2-np)x^4y^2z^2+2\Sigma m^2(n-p)x^6+2\Sigma m(mp-3n^2)x^4y^2 \\ +2(m-n)(n-p)(p-m)x^2y^2z^2+\Sigma m^2(m^2-6np)x^4 \\ +2\Sigma mn(mn-3p^2)x^2y^2+2\Sigma m^2np(p-n)x^2+m^2n^2p^2; \end{array}\right.$$

in which we have written, for abridgment,

$$\text{LXXXVI.}\ldots x=-S\rho U(a+a'),\quad y=-S\rho UVaa',\quad z=-S\rho U(a-a'),$$

and

$$\text{LXXXVII.}\ldots m=b^2-c^2,\quad n=c^2-a^2,\quad p=a^2-b^2,$$

so that

$$\text{LXXXVIII.}\ldots m+n+p=0;$$

while each sign Σ indicates a sum of three or of six terms, obtained by cyclical or binary* interchanges.

(35.) From the manner in which the equation of this *imaginary surface* (33.) or (34.) has been deduced, we easily see by (32.) that it has the double property: I.st of being (comp. (20.)) the *locus* of the *vertices* of all the (real or imaginary) *right cones*, which can be *circumscribed* to the *confocals* of the system; and II.nd of being at the same time the *common envelope* of all those confocals: which *envelope* accordingly is known to be a *developable*† *surface*.

(36.) The *eight imaginary lines* (31.) will come to be mentioned again, in connexion with the *lines of curvature* of a surface of the second order; and before closing the present series of subarticles, it may be remarked that the equation in (15.), for the determination of the *second confocal* ($e_{,}$) which *touches a given tangent*, $d\rho$ or PP′, to a *given surface* (e) of the same system, will soon appear under a new form, in connexion with that theory of *geodetic lines*, on surfaces of the second order, to which we next proceed.

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* When xyz and abc are *cyclically* changed to yzx and bca, then mnp are similarly changed to npm; but when, for instance, retaining x and a unchanged, we make only *binary interchanges* of y, z, and of b, c, we then change m, n, and p, to $-m$, $-p$, and $-n$ respectively.

† This theorem is given, for instance, in page 157 of the several times already cited Treatise by Dr. Salmon, who also mentions the *double lines* &c. upon the surface; but the present writer does not yet know whether the theory above given, of the *eight umbilicar generatrices*, has been anticipated: the *locus* (33.) of which *imaginary right lines* (30.) is *here* represented by the *vector equation* LXXXIII., from which the *scalar equation* LXXXV. has been above deduced (34.), and ought to be found to agree (notation excepted) with the known co-ordinate equation of the *developable envelope* (35.) of a *confocal system*.

409. A general theory of *geodetic lines*, as treated by quaternions, was given in the Fifth Section (III. iii. 5) of the present Chapter; and was illustrated by applications to several different *families of surfaces*. We can only here spare room for applying the same theory to the deduction, in a new way, of a few known but *principal properties* of geodetics on *central surfaces of the second order*; the differential equation employed being one of those formerly used, namely (comp. 380, IV.),

$$\text{I.} \ldots \mathrm{V}\nu \mathrm{d}^2\rho = 0, \quad \text{if} \quad \text{II.} \ldots \mathrm{Td}\rho = \text{const.};$$

that is, if the *arc* of the geodetic be made the independent variable.

(1.) In general, for *any surface*, of which ν is a normal vector, so that the *first* differential equation of the surface is $\mathrm{S}\nu\mathrm{d}\rho = 0$, the *second differential* equation $\mathrm{dS}\nu\mathrm{d}\rho = 0$ gives, by I., for a *geodetic* on that surface, the expression,

$$\text{III.} \ldots \mathrm{d}^2\rho = -\nu^{-1}\mathrm{Sd}\nu\mathrm{d}\rho.$$

(2.) Again, the surface $f\rho = \text{const.}$ being still quite *general*, if we write (comp. 363, X'., 373, III., &c.),

$$\text{IV.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho = 2\mathrm{S}\phi\rho\mathrm{d}\rho, \quad \text{we shall have} \quad \text{V.} \ldots \mathrm{d}f\mathrm{d}\rho = 2\mathrm{S}(\phi\mathrm{d}\rho \,.\, \mathrm{d}^2\rho);$$

and therefore, by III., for a *geodetic*,

$$\text{VI.} \ldots \frac{\mathrm{d}f\mathrm{d}\rho}{\mathrm{Sd}\rho\mathrm{d}\phi\rho} + 2\mathrm{S}\frac{\phi\mathrm{d}\rho}{\phi\rho} = 0.$$

(3.) For a *central surface* of the *second order*, $\phi\rho$ is a *linear function*, and we may write (comp. 361, IV.),

$$\text{VII.} \ldots \phi\mathrm{d}\rho = \mathrm{d}\phi\rho = \mathrm{d}\nu, \quad \mathrm{Sd}\rho\mathrm{d}\phi\rho = \mathrm{Sd}\rho\phi\mathrm{d}\rho = f\mathrm{d}\rho;$$

the general differential equation VI. becomes therefore here,

$$\text{VIII.} \ldots \frac{\mathrm{d}f\mathrm{d}\rho}{f\mathrm{d}\rho} + 2\mathrm{S}\frac{\mathrm{d}\nu}{\nu} = 0;$$

and gives, by a first integration, with the condition II.,

$$\text{IX.} \ldots \nu^2 f\mathrm{d}\rho = h\mathrm{d}\rho^2, \quad \text{or} \quad \text{IX'.} \ldots \mathrm{T}\nu^2 f\mathrm{Ud}\rho = h = \text{const.};$$

or $$\text{X.} \ldots P^{-2}D^{-2} = h, \quad \text{or} \quad \text{X'.} \ldots P.D = h^{-\frac{1}{2}} = \text{const.};$$

where $P = \mathrm{T}\nu^{-1}$ = *perpendicular from centre on tangent plane*,

and $D = (f\mathrm{Ud}\rho)^{-\frac{1}{2}}$ = *semidiameter parallel to tangent;*

these two last quantities being treated as scalars, whereof the latter may be real or imaginary,* together with the last scalar constant $h^{-\frac{1}{2}}$.

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* For the case of the *ellipsoid*, for which the product $P.D$ is necessarily *real*, the foregoing deduction, by quaternions, of Joachimstal's celebrated first integral, $P.D$ = const., was given (in substance) in page 580 of the *Lectures*.

(4.) The following is a quite different way of accomplishing a first integration, which conducts to another known result of not less interest, although rather of a *graphic* than of a *metric* kind. Operating on the equation 407, XVI. by $\mathrm{S.d}\rho$, and remembering that $\mathrm{S}\rho\nu = 1$, and $\mathrm{S}\nu\mathrm{d}\rho = 0$, we obtain the differential equation,

$$\text{XI.} \ldots \mathrm{S}\rho\nu\mathrm{S}\rho\mathrm{d}\rho = l^2(\mathrm{S}\alpha'\nu\mathrm{S}\alpha\mathrm{d}\rho + \mathrm{S}\alpha\nu\mathrm{S}\alpha'\mathrm{d}\rho);$$

that is, by I. and II.,

$$\text{XII.} \ldots \mathrm{S}\rho\mathrm{d}\rho.\mathrm{S}\rho\mathrm{d}^2\rho - \rho^2\mathrm{S}\mathrm{d}\rho\mathrm{d}^2\rho = l^2\mathrm{d}(\mathrm{S}\alpha\mathrm{d}\rho.\mathrm{S}\alpha'\mathrm{d}\rho),$$

in which the first member, like the second, is an exact differential, because

$$\text{XIII.} \ldots \mathrm{S}(\mathrm{V}\rho\mathrm{d}\rho.\mathrm{V}\rho\mathrm{d}^2\rho) = \tfrac{1}{2}\mathrm{d}(\mathrm{V}\rho\mathrm{d}\rho)^2;$$

hence, for the geodetic,

$$\text{XIV.} \ldots l^{-2}(\mathrm{V}\rho\mathrm{d}\rho)^2 - 2\mathrm{S}\alpha\mathrm{d}\rho\mathrm{S}\alpha'\mathrm{d}\rho = h'\mathrm{d}\rho^2,$$

or

$$\text{XV.} \ldots 2\mathrm{S}\alpha\mathrm{Ud}\rho.\mathrm{S}\alpha'\mathrm{Ud}\rho - l^{-2}(\mathrm{V}\rho\mathrm{Ud}\rho)^2 = h',$$

h' being a new scalar constant.

(5.) Comparing this last equation with the formula 408, XXXVI., we find that the new constant h' is the *sum*, $e + e_,$, of what have been above called the *parameters*,* of the *given surface* (e) on which the geodetic is traced, and of the *confocal* $(e_,)$ which *touches* a given *tangent* to that curve: whence follows the known† theorem, that *the tangents to a geodetic, on any central surface of the second order, all touch one common confocal.*‡

(6.) The new constant $e_,(=h'-e)$ may, by 407, LXXV. and 408, LXXV. (with e for e_0), be thus transformed:

$$\text{XVI.} \ldots e_, = e_1(\mathrm{TVU}\nu_1\mathrm{d}\rho)^2 + e_2(\mathrm{TVU}\nu_2\mathrm{d}\rho)^2$$
$$= e_1(\mathrm{SU}\nu_2\mathrm{d}\rho)^2 + e_2(\mathrm{SU}\nu_1\mathrm{d}\rho)^2 = \text{const.};$$

where e_1, e_2 are the parameters of the two confocals through the point P of the geodetic on (e), and ν_1, ν_2 are as before the normals at that point, to those two surfaces (e_1), (e_2).

(7.) In fact, the two equations last cited give the *general* transformation,

$$\text{XVII.} \ldots l^{-2}(\mathrm{V}\rho\sigma)^2 - 2\mathrm{S}\alpha\sigma\mathrm{S}\alpha'\sigma$$
$$= e(\mathrm{V}\sigma\mathrm{U}\nu)^2 + e_1(\mathrm{V}\sigma\mathrm{U}\nu_1)^2 + e_2(\mathrm{V}\sigma\mathrm{U}\nu_2)^2;$$

σ being an *arbitrary* vector, which may for instance be replaced by $\mathrm{d}\rho$. Equating then this last expression to $(e + e_,)\sigma^2$, or to $e(\mathrm{V}\sigma\mathrm{U}\nu)^2 - e_,\mathrm{T}\sigma^2$, since $\mathrm{S}\nu\sigma = 0$, we obtain the first and therefore also the second transformation XVI., because the three normals $\nu\nu_1\nu_2$ compose a rectangular system (comp. 407, (4.), &c.).

(8.) It is, however, simpler to deduce the second expression XVI. from the equation 408, XXXI. of the cone of tangents from P to $(e_,)$, by changing τ to $\mathrm{Ud}\rho$; and then if we write

$$\text{XVIII.} \ldots v_1 = \angle\frac{\mathrm{d}\rho}{\nu_1},$$

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[*] Compare the last Note to page 656.

[†] Discovered by M. Chasles.

[‡] This touched confocal becomes a *sphere*, when the given confocal is a *cone*. Compare 380, (5.), and 408, (16.), (17.); also the Note to page 517.

so that v_1 denotes the angle at which the geodetic crosses the normal ν_1 to (e_1), considered as a tangent to the given surface (e), the first integral XVI. takes the form,*

$$\text{XIX.} \ldots e_{,} = e_1 \sin^2 v_1 + e_2 \cos^2 v_1,$$

or

$$\text{XX.} \ldots a_{,}^2 = a_1^2 \sin^2 v_1 + a_2^2 \cos^2 v_1, \ \&\text{c.};$$

in which the constant $a_{,}$ is the primary semiaxis of the touched confocal (5.).

(9.) Without supposing that $\mathrm{Td}\rho$ is constant, we may investigate as follows the differential of the real scalar h in IX. or X., or of the product $P^{-2}.D^{-2}$, for *any curve* on a central surface of the second order. Leaving *at first* the *surface arbitrary*, as in (1.) and (2.), and resolving $\mathrm{d}^2\rho$ in the three rectangular directions of ν, $\mathrm{d}\rho$, and $\nu\mathrm{d}\rho$, we get the *general* expression,

$$\text{XXI.} \ldots \mathrm{d}^2\rho = -\nu^{-1}\mathrm{Sd}\nu\mathrm{d}\rho + \mathrm{d}\rho^{-1}\mathrm{Sd}\rho\mathrm{d}^2\rho + (\nu\mathrm{d}\rho)^{-1}\mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho;$$

of which, under the conditions I. and II., the two last terms vanish, as in III. Without assuming those conditions, if we *now* introduce the relations VII. which belong to a central surface of the second order, we have by V. and IX. the expression,†

$$\text{XXII.} \ldots \tfrac{1}{2}\mathrm{d}h\,.\,\mathrm{d}\rho^2 = \nu^2\mathrm{Sd}\nu\mathrm{d}^2\rho + \mathrm{S}\nu\mathrm{d}\nu\mathrm{Sd}\nu\mathrm{d}\rho - h\mathrm{Sd}\rho\mathrm{d}^2\rho = \mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho^{-1}.\mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho,$$

or

$$\text{XXIII.} \ldots \mathrm{d}h = \mathrm{d}\,.\,\nu^2\mathrm{Sd}\nu\mathrm{d}\rho^{-1} = \mathrm{d}\,.\,P^{-2}D^{-2} = 2\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho^{-1}\mathrm{S}\nu\mathrm{d}\rho^{-1}\mathrm{d}^2\rho;$$

or finally,

$$\text{XXIV.} \ldots \mathrm{d}h\,.\,\mathrm{d}\rho^4 = 2\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho\,.\,\mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho,$$

the scalar variable with respect to which the differentiations are performed being here entirely arbitrary.

(10.) For a *geodetic line* on *any surface*, referred thus to *any scalar variable*, we have by 380, II. the differential equation,

$$\text{XXV.} \ldots \mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho = 0;$$

and therefore by XXIV., for *such* a line on a central surface of the *second order*, we have *again*, as in (3.),

$$\text{XXVI.} \ldots \mathrm{d}h = 0, \quad \text{or} \quad \text{XXVI}'. \ldots h = \text{const.},$$

with $h = P^{-2}D^{-2}$ as in X.

(11.) But we now see, by XXIV., that for such a surface the condition XXVI. is satisfied, not only by this *differential equation* of the *second order* XXV. but also by this *other* differential equation,

$$\text{XXVII.} \ldots \mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho = 0;$$

the product P^2D^2 (or PD itself) is therefore *constant*, *not only* as in (3.) for *every*

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* Under this form XX., the integral is easily seen to coincide with that of M. Liouville,

$$\mu^2\cos^2 i + \nu^2\sin^2 i = \mu'^2 = \text{const.},$$

cited in page 290 of Dr. Salmon's Treatise.

† In deducing this expression, it is to be remembered that

$$\mathrm{dSd}\nu\mathrm{d}\rho = \mathrm{d}f\mathrm{d}\rho = 2\mathrm{Sd}\nu\mathrm{d}^2\rho;$$

in fact, the linear and self-conjugate form of $\nu = \phi\rho$ gives,

$$\mathrm{Sd}\rho\mathrm{d}^2\nu = f(\mathrm{d}\rho,\, \mathrm{d}^2\rho) = \mathrm{Sd}\nu\mathrm{d}^2\rho.$$

geodetic on the surface, but *also* for *every curve* of *another set*,[*] represented by this *last* equation XXVII., which is only of the *first order*, and the geometrical meaning of which we next propose to consider.

410. In general, if ν and $\nu + \Delta\nu$ have the directions of the normals to *any surface*, at the extremities of the vectors ρ and $\rho + \Delta\rho$, the *condition of intersection* (or parallelism) of these two normals is, rigorously,

$$\text{I.} \ldots \mathrm{S}\nu\Delta\nu\Delta\rho = 0;$$

the *differential equation*[†] of what are called the *Lines of Curvature*, on an *arbitrary surface*, is therefore (comp. 409, XXVII.),

$$\text{II.} \ldots \mathrm{S}\nu d\nu d\rho = 0;$$

from which we are now to deduce a few *general* consequences, together with some that are peculiar to surfaces of the *second order*.

(1.) The differential equation of the *surface* being, as usual,

$$\text{III.} \ldots \mathrm{S}\nu d\rho = 0,$$

the normal vector ν is generally *some* function of ρ, although *not* generally *linear*, because the surface is as yet *arbitrary*: its differential $d\nu$ is therefore generally some function of ρ and $d\rho$, which is linear relatively to the *latter*. And if, attending only to the dependence of $d\nu$ on $d\rho$, we write

$$\text{IV.} \ldots d\nu = \phi d\rho,$$

it results from what has been already proved (363), that *this linear* and *vector function* ϕ is at the same time *self-conjugate*.

(2.) Denoting then by τ *a tangent*[‡] PT to a line of curvature, drawn at the given extremity P of ρ, we see that the vector τ must satisfy the *two* following scalar equations, in which ν is supposed to be given,

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[*] Namely, the *lines of curvature*, as is known, and as will presently be proved by quaternions.

[†] In this equation II., $d\rho$ and $d\nu$ are *two simultaneous differentials*, which may (according to the theory of the present Chapter, and of the one preceding it) be at pleasure regarded, either as *two finite right lines*, whereof $d\rho$ is (rigorously) *tangential* to the surface, and to the line of curvature; or else as *two infinitely small vectors*, $d\rho$ being, on this latter plan, an *infinitesimal chord* $\Delta\rho$. (Compare pages 99, 392, 497, 626, and the first Notes to pages 623, 630.) The treatment of the *equations* is the *same*, in these *two views*, whereof one may appear clearer to some readers, and the other view to others.

[‡] This symbol τ is used here partly for abridgment, and partly that the reader may not be *obliged* to interpret $d\rho$ as denoting a *finite tangent*, although the principles of this work *allow* him so to interpret it.

$$\mathrm{V.} \ldots S\nu\tau = 0, \quad \text{and} \quad \mathrm{VI.} \ldots S\nu\tau\phi\tau = 0;$$

this tangent τ admits therefore (355) of *two real and rectangular directions*, but *not* in general of *more*: *opposite* directions being *not here* counted as *distinct*. Hence, as is indeed well known, *through each point of any surface there pass generally two lines of curvature*: and *these two curves intersect each other at right angles*.

(3.) A *construction* for the two rectangular directions of τ can easily be assigned as follows. Assuming, as we may, that the *length* of the tangent τ varies with its *direction*, according to the law,

$$\mathrm{VII.} \ldots S\tau\phi\tau = 1,$$

which gives

$$\mathrm{VIII.} \ldots S(\phi\tau . d\tau) = 0, \quad \text{or briefly} \quad \mathrm{VIII'.} \ldots S\phi\tau d\tau = 0,$$

by the properties above mentioned of ϕ; and remembering that ν is treated as a constant in V., so that we may write,

$$\mathrm{IX.} \ldots S\nu d\tau = 0, \quad \text{and therefore (by VI.),} \quad \mathrm{X.} \ldots S\tau d\tau = 0;$$

we see that, under the conditions of the question, the above mentioned *length* $T\tau$, of this tangential vector τ, is a *maximum* or *minimum*: and therefore that the *two directions* sought are those of the *two axes* of the *plane conic* V. VII., which has its *centre at the given point* P of the surface, and is *in the tangent plane* at that point.

(4.) This plane conic V. VII. may be called the *Index Curve*, for the given surface at the given point P; in fact it is easily proved to coincide, if we abstract from mere dimensions, with the known *indicatrix* (la courbe indicatrice) of Dupin,* who first pointed out the coincidence (3.) of the directions of its *axes*, with those of the lines of curvature; and also established a more general relation of *conjugation* between *two tangents* to a surface at one point, which exists when they have the directions of any two *conjugate semidiameters* of that curve: so that the lines of curvature are distinguished by this *characteristic property*, that the *tangent to each* is *perpendicular to its conjugate*.

(5.) In our notations, this relation of conjugation between two tangents τ, τ', which satisfy as such the equations,

$$\mathrm{V.} \ldots S\nu\tau = 0, \quad \text{and} \quad \mathrm{V'.} \ldots S\nu\tau' = 0,$$

is expressed by the formula,

$$\mathrm{XI.} \ldots S\tau\phi\tau' = 0, \quad \text{or} \quad \mathrm{XI'.} \ldots S\tau'\phi\tau = 0;$$

we have therefore the parallelisms,†

$$\mathrm{XII.} \ldots \tau \parallel V\nu\phi\tau', \quad \mathrm{XII'.} \ldots \tau' \parallel V\nu\phi\tau;$$

so that the equation VI. may be written under the very simple form,

$$\mathrm{XIII.} \ldots S\tau\tau' = 0,$$

which gives at once the *rectangularity* lately mentioned.

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* *Développements de Géométrie* (Paris, 1813), pages 48, 145, &c.

† The *conjugate character* of these two parallelisms, or the relation,

$$V.\nu\phi V\nu\phi\tau \parallel \tau, \quad \text{if} \quad S\nu\tau = 0,$$

may easily be deduced from the *self-conjugate* property of ϕ, with the help of the formula 348, VII., in page 440.

(6.) The parallelism XII′. may be otherwise expressed by saying (comp. (4.)) that

$$\text{XIV.} \ldots d\rho \quad \text{and} \quad \mathrm{V}\nu d\nu$$

have the directions of *conjugate tangents;* or that the two vectors,

$$\text{XV.} \ldots \Delta\rho \quad \text{and} \quad \mathrm{V}\nu\Delta\nu,$$

have *ultimately* such directions, when $\mathrm{T}\Delta\rho$ diminishes indefinitely. But whatever may be this *length* of the *chord* $\Delta\rho$, the vector $\mathrm{V}\nu\Delta\nu$ has the *direction* of the *line of intersection* of the *two tangent planes* to the surface, drawn at its two extremities: another theorem of Dupin* is therefore reproduced, namely, that *if a developable be circumscribed to any surface, along any proposed curve thereon, the generating lines of this developable are everywhere conjugate, as tangents to the surface, to the corresponding tangents to the curve,* with the recent definition (4.) of such conjugation.

(7.) The following is a very simple mode of proving by quaternions, that *if* a tangent τ satisfies the equation VI., then the *rectangular tangent,*

$$\text{XVI.} \ldots \tau' = \nu\tau,$$

satisfies the same equation. For this purpose we have only to observe, that the *self-conjugate* property of ϕ gives, by VI. and XVI.,

$$\text{XVII.} \ldots 0 = \mathrm{S}\tau'\phi\tau = \mathrm{S}\tau\phi\tau' = \nu^{-2}\mathrm{S}\nu\tau'\phi\tau'.$$

(8.) Another way of exhibiting, by quaternions, the mutual rectangularity of the lines of curvature, is by employing (comp. 357, I.) the self-conjugate *form,*

$$\text{XVIII.} \ldots \phi\tau = g\tau + \mathrm{V}\lambda\tau\mu;$$

in which the vectors λ, μ, and the scalar g, depend only on the surface and the point, and are independent of the direction of the tangent. The equation VI. then becomes by V.,

$$\text{XIX.} \ldots 0 = \mathrm{S}\nu\tau\lambda\tau\mu = \mathrm{S}\nu\tau\lambda\mathrm{S}\mu\tau + \mathrm{S}\nu\tau\mu\mathrm{S}\lambda\tau;$$

assuming then the expression,

$$\text{XX.} \ldots \tau = x\mathrm{V}\nu\lambda + y\mathrm{V}\nu\mu,$$

we easily find that

$$\text{XXI.} \ldots y^2(\mathrm{V}\nu\mu)^2 = x^2(\mathrm{V}\nu\lambda)^2, \quad \text{or} \quad \text{XXI}'. \ldots y\mathrm{TV}\nu\mu = \pm x\mathrm{TV}\nu\lambda;$$

the *two directions* of τ are therefore those of the two lines,

$$\text{XXII.} \ldots \mathrm{UV}\nu\lambda \pm \mathrm{UV}\nu\mu,$$

which are evidently perpendicular† to each other.

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* Dupin proved *first* (*Dév. de Géométrie*, pp. 43, 44, &c.), that two such tangents as are described in the text have a relation of *reciprocity* to each other, on which account he *called* them "*tangentes conjuguées:*" and *afterwards* he gave a sort of *image*, or *construction*, of this relation and of others connected with it, by means of the *curve* which he named "*l'indicatrice*" (in his already cited page 48, &c.).

† *This mode*, however, of determining *generally* the directions of the lines of curvature, gives only an illusory result, when the normal ν has the direction of either λ or μ, which happens at an *umbilic* of the surface. Compare 408, (27.), (29.), and the first Note to page 466.

(9.) An *interpretation*, of some interest, may be given to this last expression XXII., by the introduction of a certain *auxiliary surface* of the *second order*, which may be called the *Index Surface*, because the *index curve* (4.) is the *diametral section* of this *new* surface, made by the *tangent plane* to the *given* one. With the recent signification of ϕ, this *index surface* is represented by the equation VII., if τ be *now* supposed (comp. (2.)) to represent a line PT drawn *in any direction* from the given point P, and therefore *not now obliged* to satisfy the *condition* V. of *tangency*. Or if, for greater clearness, we denote by $\rho + \rho'$ the vector from the origin O to a point of the index surface, the equation to be satisfied is, by the form XVIII. of ϕ (comp. 357, II.),

$$\text{XXIII.} \ldots 1 = S\rho'\phi\rho' = g\rho'^2 + S\lambda\rho'\mu\rho';$$

the *centre* of this auxiliary surface being thus at P, and its two (real) *cyclic normals* being the lines λ and μ: so that $V\nu\lambda$ and $V\nu\mu$ have the directions of the *traces of its two cyclic planes, on that diametral plane* ($S\nu\rho' = 0$) which *touches the given surface*. We have therefore, by XXII., this general *theorem*, that *the bisectors of the angle formed by these two traces are the tangents to the two lines of curvature, whatever the form of the given surface may be.*

(10.) Supposing *now* that the *given surface* is *itself* one of the *second order*, and that its *centre* is at the *origin* O, so that it may be represented (comp. 405, XII.) by the equation,

$$\text{XXIV.} \ldots 1 = S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho,$$

with *constant values* of λ, μ, and g, which will reproduce *with those values* the *form* XVIII. of ϕ, we see that the *index surface* (9.) becomes in this case simply that *given* one, with its *centre transported* from O to P; and therefore with a *tangent plane at the origin*, which is *parallel to the given tangent plane*. And thus the *traces* (9.), of the *cyclic planes* on the *diametral plane* of the *index surface*, become *here* the *tangents* to the *circular sections* of the *given surface*. We recover then, as a *case* of the general theorem in (9.), this *known* but *less general theorem*: that *the angles formed by the two circular sections, at any point of a surface of the second order, are bisected by the lines of curvature*, which pass through the same point.

(11.) And because the *tangents* to these latter lines *coincide generally*, by (3.) (4.) (9.), with the *axes* of the *diametral section* of the *index surface*, made by the *tangent plane* to the *given surface*, they are *parallel*, in the *case* (10.), as indeed is well known, *to the axes* of the *parallel section* of a given surface of the *second order*.*

(12.) And if we now look back to the *Equation of Confocals* in 407, (26.), and to the earlier formulæ of 407, (4.), we shall see that because the vector ν_1, in the last cited sub-article, represents a *tangent* to the given surface $S\rho\phi\rho = 1$, *complanar** with the *normal* ν and the *derived vector* $\phi\nu_1$, so that it satisfies (comp. 407, XII. XIV., and the recent formulæ V. VI.) the two scalar equations,

$$\text{XXV.} \ldots S\nu\nu_1 = 0, \quad \text{and} \quad \text{XXVI.} \ldots S\nu\nu_1\phi\nu_1 = 0,$$

which are likewise satisfied (comp. (7.)) when we change ν_1 to the *rectangular tan-*

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* Compare the Note to page 645.

gent ν_2, it follows that ***these two vectors***, ν_1 and ν_2, which are the ***normals to the two confocals*** to (e) ***through*** P, are also the ***tangents to the two lines of curvature*** on that ***given surface*** of the second order at that point: whence follows this other theorem[*] of Dupin, that ***the curve of orthogonal intersection*** (407, (4.)), ***of two confocal surfaces, is a line of curvature on each.***

(13.) And by combining this known theorem, with what was lately shown respecting the ***umbilicar generatrices*** (in 408, (30.), (32.), comp. also (35.), (36.)), we may see that while, on the one hand, the ***lines of curvature*** on a central surface of the ***second order*** have ***no real envelope***, yet on the other hand, ***in an imaginary sense***, they have for their ***common envelope***[†] the ***system*** of the ***eight imaginary right lines*** (408, (31.)), which ***connect the twelve*** (real or imaginary) ***umbilics*** of the surface, ***three by three***, and are at once ***generating lines*** of the ***surface itself***, and also of the known ***developable envelope*** of the ***confocal system.***

(14.) It may be added, as another curious property of these ***eight imaginary right lines***, that ***each*** is, in an imaginary sense, ***itself a line of curvature*** upon the surface: or rather, each represents ***two coincident lines*** of that kind. In fact, if we denote the variable vector 408, LXXX. of such a generatrix by the expression,

$$\text{XXVII.} \ldots \rho = e'\sigma + \sigma',$$

in which e' is a ***variable scalar***, but σ, σ' are two ***given*** or ***constant*** but ***imaginary vectors***, such that

$$\text{XXVIII.} \ldots \sigma^2 = 0, \quad \mathrm{S}\sigma\sigma' = -l^2, \quad \sigma'^2 = -b^2,$$

and $$\text{XXIX.} \ldots f\sigma = \mathrm{S}\sigma\phi\sigma = 0, \quad f(\sigma, \sigma') = \mathrm{S}\sigma'\phi\sigma = 0, \quad f\sigma' = 1,$$

we have the ***imaginary normal*** ν, with (for the case of a ***real umbilic***) a ***real tensor***,

$$\text{XXX.} \ldots \nu = e'\phi\sigma + \phi\sigma' \perp \sigma, \quad \text{XXXI.} \ldots \mathrm{T}\nu = \pm \frac{(e - e')l^2}{abc};$$

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[*] *Dév. de Géométrie*, page 271, &c.

[†] The writer is not aware that this theorem, to which he was conducted by quaternions, has been enunciated before; but it has evidently an intimate connexion with a result of Professor Michael Roberts, cited in page 290 of Dr. Salmon's Treatise, respecting the ***imaginary geodetic tangents to a line of curvature***, drawn ***from an umbilicar point***, which are ***analogous*** to the ***imaginary tangents*** to a ***plane conic***, drawn from a ***focus*** of that curve. An illustration, which is almost a ***visible representation***, of the theorem (13.) is supplied by Plate II. to Liouville's Monge (and by the corresponding plate in an earlier edition), in which the ***prolonged*** and ***dotted parts*** of certain ***ellipses***, answering to the ***real projections*** of ***imaginary portions*** of the ***lines of curvature*** of the ***ellipsoid***, are ***seen*** to touch a system of ***four real right lines***, namely the ***projections*** (on the same plane of the greatest and least axes), of the ***four real umbilicar tangent planes***, and therefore also of what have been above called (408, (30.), (31.)) the ***eight (imaginary) umbilicar generatrices*** of the surface. Accordingly Monge observes (page 150 of Liouville's edition), that "toutes les ellipses, projections des lignes de courbure, seront inscrites dans ce parallélogramme dont chacune d'elles touchera les quatre côtés:" with a similar remark in his explanation of the corresponding Figure (page 160).

and we find, after reductions, the imaginary expression,

$$\text{XXXII.}\;.\;.\;\nu\sigma = \pm\sqrt{-1}\,\sigma\mathrm{T}\nu, \quad \text{whence} \quad \text{XXXIII.}\;.\;.\;\mathrm{S}\nu\sigma = 0, \quad \mathrm{S}\nu\sigma\phi\sigma = 0.$$

The *differential equations* V. VI. of a line of curvature are therefore *symbolically satisfied*, when we substitute, for the tangential vector τ, either the imaginary line σ *itself*, or the apparently *perpendicular* but in an imaginary sense *coincident*[*] vector $\nu\sigma$; and the recent assertions are justified.

(15.) As regards the *real* lines of curvature, on a central surface of the second order, we see by comparing the *general* differential equation II. with the expression 409, XXIII. for the differential of h, or of $P^{-2}D^{-2}$, that this latter product, or the product $P.D$ itself, is *constant*[†] for a *line of curvature*, as well as for a *geodetic line*, on such a surface, as indeed it is well known to be: although this *last* constant $(P.D)$ *may* become *imaginary*, for the case of a *single-sheeted*[‡] *hyperboloid*, and *must* be such for a line of curvature on an hyperboloid of *two sheets*.

(16.) And as regards the *general theory* of the *index surface* (9.), it is to be observed that this auxiliary surface depends *primarily* on the *scalar function* f, in the equation $f\rho = 1$, or generally $f\rho =$ const., of the *given surface;* and that it is *not entirely determined* by means of *that surface alone*. For if we write, for instance,

$$\text{XXXIV.}\;.\;.\;\mathfrak{f}f\rho = \mathfrak{f}1, \quad \text{with} \quad \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho \text{ as before,}$$

we shall have, as the new first differential equation of the same given surface, instead of III.,

$$\text{XXXV.}\;.\;.\;0 = \mathrm{d}\mathfrak{f}f\rho = 2\mathrm{S}n\nu\mathrm{d}\rho, \quad \text{with} \quad \text{XXXVI.}\;.\;.\;n = \mathfrak{f}'f\rho;$$

and if we then write, by analogy to IV.,

$$\text{XXXVII.}\;.\;.\;\mathrm{d}.n\nu = \Phi\mathrm{d}\rho = n\phi\mathrm{d}\rho + n'\nu\mathrm{S}\nu\mathrm{d}\rho, \quad \text{with} \quad \text{XXXVIII.}\;.\;.\;n' = 2\mathfrak{f}''f\rho,$$

the *new index surface*, constructed on the plan (9.), will have for its equation, analogous to XXIII., the following:

$$\text{XXXIX.}\;.\;.\;\mathrm{S}\rho'\Phi\rho' = n\mathrm{S}\rho'\phi\rho' + n'(\mathrm{S}\nu\rho')^2 = \text{const.}$$

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[*] As regards the paradox, of the *imaginary vector* σ being thus apparently *perpendicular to itself*, a similar one had occurred before, in the investigation 353, (17.), (18.), (19.); and it is explained, on the principles of modern geometry, by observing that this imaginary vector is *directed to the circle at infinity*. Compare 408, (31.), and the Note to page 459.

[†] Compare the first Note to page 667.

[‡] Although the writer has been content to employ, in the present work, some of these usual but rather long appellations, he feels the elegance of Dupin's phraseology, adopted also by Möbius, and by some other authors, according to which the two central hyperboloids are distinguished, as *elliptic* (for the case of *two* sheets), and *hyperbolic* (for the case of *one*). The phrase "*quadric*," for the *general* surface of the *second order* (or *second degree*), employed by Dr. Salmon and Mr. Cayley, is also very convenient. It may be here remarked, that Dupin was perfectly aware of, or rather appears to have first discovered, the existence of what have since his time come to be called the *focal conics;* which important curves were considered by him, as being at once *limits of confocal surfaces*, and also *loci of umbilics*. Comp. *Dév. de Géométrie*, pages 270, 277, 278, 279; see also page 390 of the *Aperçu Historique*, &c., by M. Chasles (Brussels, 1837).

(17.) But if we take this last constant $= n$, the *two index surfaces*, XXIII. and XXXIX., will have a *common diametral section*, made by the *given tangent plane*, namely the *index curve* (4.); and they will *touch each other*, in the *whole extent of that curve*. And it will be found that the *construction* (9.), for the *directions* of the *lines of curvature*, applies *equally well* to the *one* as to the *other*, of these *two* auxiliary surfaces: in fact, it is evident that the *differential equation* II., namely $S\nu d\nu d\rho = 0$, receives *no real alteration*, when ν is multiplied by *any scalar*, n, even if that scalar should be variable.

(18.) And instead of supposing that the *variable vector* ρ is thus obliged, as in 373, to satisfy a *given scalar equation*, of the form[*]

$$f\rho = \text{const.},$$

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[*] If $\rho = ix + jy + kz$, and $v = f\rho = F(x, y, z)$, and if we write,

$$dv = p dx + q dy + r dz, \quad dp = p' dx + r'' dy + q'' dz,$$
$$dq = q' dy + p'' dz + r'' dx, \quad dr = r' dz + q'' dx + p'' dy,$$

we may then write also, on the present plan, which gives $df\rho = 2S\nu d\rho$,

$$d\rho = i dx + j dy + k dz, \qquad \nu = -\tfrac{1}{2}(ip + jq + kr),$$
$$d\nu = -\tfrac{1}{2}(i dp + j dq + k dr), \quad Sd\rho d\nu = \tfrac{1}{2}(dx dp + dy dq + dz dr);$$

and the *index surface*, constructed as in (9.), and with ρ' changed to $\Delta\rho = i\Delta x + j\Delta y + k\Delta z$, will thus have the equation,

$$\text{(a)}. \ldots \tfrac{1}{2}p'\Delta x^2 + \tfrac{1}{2}q'\Delta y^2 + \tfrac{1}{2}r'\Delta z^2 + p''\Delta y\Delta z + q''\Delta z\Delta x + r''\Delta x\Delta y = 1,$$

or more generally = const.; so that it may be made in this way to depend upon, and be entirely determined by, the *six partial differential coefficients of the second order*, $p' \ldots p'' \ldots$, of the function v or $f\rho$, taken with respect to the *three rectangular co-ordinates*, xyz. And by comparing this equation (a) with the following equation of the same auxiliary surface, which results more directly from the principles employed in the text (comp. XVIII. XXIII.),

$$\text{(b)}. \ldots S\Delta\rho\phi\Delta\rho = g\Delta\rho^2 + S\lambda\Delta\rho\mu\Delta\rho = 1,$$

we can easily deduce *expressions* for those six partial *coefficients*, in terms of g, λ, μ. Thus, for example,

$$\tfrac{1}{2}D_x^2 v = \tfrac{1}{2}p' = -g + S\lambda i\mu i = S\lambda\mu - g + 2Si\lambda Si\mu;$$

but
$$Si\lambda Si\mu + Sj\lambda Sj\mu + Sk\lambda Sk\mu = -S\lambda\mu; \text{ therefore,}$$

$$\text{(c)}. \ldots \tfrac{1}{2}(D_x^2 v + D_y^2 v + D_z^2 v) = S\lambda\mu - 3g = c_1 + c_2 + c_3 = -m'',$$

if c_1, c_2, c_3 be the roots and m'' a coefficient of a certain cubic (354, III.), deduced from the linear and vector function $d\nu = \phi d\rho$, on a plan already explained. If then the function v satisfy, as in several physical questions, the *partial differential equation*,

$$\text{(d)}. \ldots D_x^2 v + D_y^2 v + D_z^2 v = 0,$$

the *sum* of these *three roots*, c_1, c_2, c_3, will *vanish*: and consequently, the *asymptotic cone* to the *index-surface*, found by changing 1 to 0 in the second member of (a), is *real*, and has (comp. 406, XXI., XXIX.) the property that

$$\text{(e)}. \ldots \cot^2 a + \cot^2 b = 1,$$

if a, b denote its two extreme semiangles. An entirely different method of trans-we may suppose, as in 372, that ρ is a *given vector function* of *two scalar variables*, x and y, between which there will then arise, by the same fundamental formula II., a *differential equation* of the *first order* and *second degree*, to be integrated (when possible) by known methods. For example, if we write,

$$\text{XL.}\ldots\ \rho = ix + jy + kz, \quad dz = pdx + qdy,$$

we shall satisfy the equation III. by assuming (with a constant factor understood),

$$\text{XLI.}\ldots\ \nu = ip + jq - k, \quad \text{whence} \quad \text{XLII.}\ldots\ d\nu = idp + jdq;$$

and thus the *general equation* II., for the lines of curvature on an arbitrary surface, receives (by the laws of *ijk*) the form,

$$\text{XLIII.}\ldots\ dp(dy + qdz) = dq(dx + pdz);$$

which last form has accordingly been assigned, and in several important questions employed, by Monge*: but which is now seen to be *included* in the still more concise (and more easily deduced and interpreted) *quaternion equation*,

$$S\nu d\nu d\rho = 0.$$

411. For a *central surface* of the *second order*, we have as usual $\nu = \phi\rho$, $\Delta\nu = \phi\Delta\rho$, and therefore (by 347, 348, and by the self-conjugate form of ϕ),

$$\text{I.}\ldots\ V\nu\Delta\nu = V\phi\rho\phi\Delta\rho = \psi V\rho\Delta\rho = m\phi^{-1}V\rho\Delta\rho;$$

the *general condition of intersection* 410, I. of *two normals*, at the extremities of a *finite chord* $\Delta\rho$, and the *general differential equation* 410, II. of the *lines of curvature*, may therefore for *such* a surface receive these *new* and *special forms*:

forming, by quaternions, the well known equation (d), occurred early to the present writer, and will be briefly mentioned somewhat farther on. In the mean time it may be remarked, that because $m'' = 0$ by (c), when the equation (d) is satisfied, we have then, by the general theory III. ii. 6 of linear and vector functions, and especially by the subarticles to 350, remembering that ϕ is here self-conjugate, the formulæ,

$$(\text{f})\ldots\ d\nu + \chi d\rho = 0, \quad \text{and} \quad (\text{g})\ldots\ \psi\sigma - \phi^2\sigma = m'\sigma,$$

χ, ψ being auxiliary functions, and m' another coefficient of the cubic, while σ is an arbitrary vector. For the same reason, and under the same condition (d), the function ϕ itself has the properties expressed by the equations,

$$(\text{h})\ldots\ \phi V\iota\kappa = \kappa\phi\iota - \iota\phi\kappa, \quad \text{and} \quad (\text{i})\ldots\ \phi^2 V\iota\kappa = V\phi\iota\phi\kappa - m'V\iota\kappa;$$

in which the *two vectors* ι, κ are *arbitrary*, and m' is the same *scalar coefficient* as before.

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* See the enunciation of the formula here numbered as XLIII., in page 133 of Liouville's Monge: compare also the applications of it, in pages 274, 303, 305, 357. (The corresponding pages of the Fourth Edition are, 115, 240, 265, 267, 312.) The quaternion equation, $S\nu d\nu d\rho = 0$, was published by the present writer, in a communication to the Philosophical Magazine, for the month of October, 1847 (page 289). See also the Supplement to the same Volume xxxi. (Third Series); and the Proceedings of the Royal Irish Academy for July, 1846.

$$\text{II.}\ \ldots\ S\Delta\rho\phi^{-1}V\rho\Delta\rho = 0, \quad \text{or} \quad \text{II}'.\ \ldots\ S\rho\Delta\rho\phi^{-1}\Delta\rho = 0;$$
$$\text{III.}\ \ldots\ Sd\rho\phi^{-1}V\rho d\rho = 0, \quad \text{or} \quad \text{III}'.\ \ldots\ S\rho d\rho\phi^{-1}d\rho = 0;$$

which admit of geometrical interpretations, and conduct to some new theorems, especially when they are transformed as follows:

$$\text{IV.}\ \ldots\ S\lambda\Delta\rho\,.\,S\rho\Delta\rho\phi^{-1}\mu + S\mu\Delta\rho\,.\,S\rho\Delta\rho\phi^{-1}\lambda = 0,$$
$$\text{V.}\ \ldots\ S\lambda d\rho\,.\,S\rho d\rho\phi^{-1}\mu + S\mu d\rho\,.\,S\rho d\rho\phi^{-1}\lambda = 0,$$

λ and μ being (as in 405, (5.), &c.) the *two real cyclic normals* of the surface: while the same equations may also be written under the still more simple forms,

$$\text{VI.}\ \ldots\ S\alpha\Delta\rho\,.\,S\alpha'\rho\Delta\rho + S\alpha'\Delta\rho\,.\,S\alpha\rho\Delta\rho = 0,$$
$$\text{VII.}\ \ldots\ S\alpha d\rho\,.\,S\alpha'\rho d\rho + S\alpha' d\rho\,.\,S\alpha\rho d\rho = 0,$$

α, α' being, as in several recent investigations, the *two real focal unit lines*, which are common to a whole *confocal system*.

(1.) The vector $\phi^{-1}V\rho\Delta\rho$ in II. has by I. the direction of $V\nu\Delta\nu$; whence, by 410, (6.), the interpretation of the recent equation II., or (for the present purpose) of the more general equation 410, I., is that *the chord* PP' *is perpendicular to its own polar, if the normals at its extremities intersect.* Accordingly, if their point of intersection be called N, the polar of PP' is perpendicular at once to PN and P'N, and therefore to PP' itself.

(2.) The equation II'. may be interpreted as expressing, that *when the normals* at P and P' thus *intersect in a point* N, *there exists a point* P'' *in the diametral plane* OPP', *at which the normal* P''N'' *is parallel to the chord* PP': a result which may be otherwise deduced, from elementary principles of the geometry of surfaces of the second order.

(3.) It is unnecessary to dwell on the *converse* propositions, that when *either* of these conditions is satisfied, there *is intersection* (or parallelism) of the *two normals* at P and P': or on the corresponding but *limiting* results, expressed by the equations III. and III'.

(4.) In order, however, to make any use in *calculation* of these new forms II., III., we must select some suitable expression for the self-conjugate function ϕ, and deduce a corresponding expression for the inverse function ϕ^{-1}. The *form*,*

$$\text{VIII.}\ \ldots\ \phi\rho = g\rho + V\lambda\rho\mu,$$

which has already several times occurred, has also been more than once inverted: but the following *new inverse*† *form*,

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* The *vector form* VIII. occurred, for instance, in pages 463, 469, 474, 484, 641, 669; and the connected *scalar form*,

$$f\rho = g\rho^2 + S\lambda\rho\mu\rho, \qquad 357,\ \text{II.}$$

has likewise been frequently employed.

† *Inverse forms*, for $\phi^{-1}\rho$ or $m^{-1}\psi\rho$, have occurred in pages 463, 484, 641 (the

$$\text{IX.} \ldots (g - S\lambda\mu) . \phi^{-1}\rho = \rho - \lambda S\rho\phi^{-1}\mu - \mu S\rho\phi^{-1}\lambda,$$

has an advantage, for our present purpose, over those assigned before. In fact, this form IX. gives at once the equation,

$$\text{X.} \ldots (g - S\lambda\mu) . \phi^{-1}V\rho\Delta\rho = V\rho\Delta\rho - \lambda S\rho\Delta\rho\phi^{-1}\mu - \mu S\rho\Delta\rho\phi^{-1}\lambda;$$

and so conducts immediately from II. to IV., or from III. to V. as a limit.

(5.) The equation IV. expresses *generally*, that the *chord* $\Delta\rho$, or PP′, is a *side* of a certain *cone* of the second order, which has its vertex at the point P of the given surface, and passes through all the points P′ for which the normals to that surface *intersect* the *given normal* at P; and the equation V. expresses *generally*, that the *two sides* of this last cone, in which it is *cut* by the *given tangent plane* at the same point P, are the *tangents to the lines of curvature*.

(6.) But if the surface be an *ellipsoid*, or a *double*-sheeted hyperboloid, then (comp. 408, (29.)) the *always real vectors*,* $\phi^{-1}\lambda$ and $\phi^{-1}\mu$, have the directions of *semidiameters* drawn to *two of the four real umbilics*; supposing then that ρ is *such* a semidiameter, and that it has the direction of $\pm\phi^{-1}\lambda$, the second term of the first member of the equation IV. vanishes, and the *cone* IV. breaks up into a *pair of planes*, of which the equations in ρ' are,

$$\text{XI.} \ldots S\lambda(\rho' - \rho) = 0, \quad \text{and} \quad \text{XII.} \ldots S\rho'\phi^{-1}\lambda\phi^{-1}\mu = 0;$$

whereof the *former* represents the *tangent plane at the umbilic* P, and the *latter* represents the *plane of the four real umbilics*.

(7.) It follows, then, that *the normal at the real umbilic* P *is not intersected by any real normal* to the surface, *except those which are drawn at points* P′ *of that principal section, on which all the real umbilics are situated*: *but* that *the same real umbilicar normal* PN *is, in an imaginary sense, intersected by all the imaginary normals, which are drawn from the imaginary points* P′ *of either of the two imaginary generatrices through* P.

(8.) In fact, the locus of the point P′, under the *condition of intersection* of its normal P′N′ with a *given normal* PN, is *generally* a *quartic curve*, namely the intersection of the *given surface* with the *cone* IV.; but when this cone *breaks up*, as in (6.), into *two planes*, whereof *one* is *normal*, and the *other tangential* to the surface, the general quartic is likewise *decomposed*, and becomes a system of a *real conic*, namely the *principal section* (7.), and *a pair of imaginary right lines*, namely the *two umbilicar generatrices* at P.

(9.) We see, at the same time, in a new way (comp. 410, (14.)), that *each such generatrix* is (in an imaginary sense) *a line of curvature*: because the (imaginary) *normals to the surface*, at all the points of that generatrix, are situated by (7.) in one *common* (imaginary) *normal plane*.

(10.) Hence *through a real umbilic*, on a surface of the second order, there pass

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correction in a Note to which last page should be attended to). In comparing these with the form IX., it will easily be seen (comp. page 661) that

$$\phi^{-1}\lambda = \frac{g\lambda - \lambda^2\mu}{g^2 - \lambda^2\mu^2}, \quad \phi^{-1}\mu = \frac{g\mu - \mu^2\lambda}{g^2 - \lambda^2\mu^2}.$$

* Compare the Note immediately preceding.

three lines of curvature: whereof *one* is a *real conic* (8.), and the *two others* are *imaginary right lines*, namely, the *umbilicar generatrices* as before.

(11.) If we prefer *differentials* to *differences*, and therefore use the equation V. of the lines of curvature, we find that this equation takes the form $0 = 0$, if the point P be an umbilic; and that if the normal at that point be parallel to λ, the *differential of the equation* V. breaks up into *two factors*, namely,

$$\text{XIII.} \ldots \mathrm{S}\lambda \mathrm{d}^2\rho = 0, \quad \text{and} \quad \text{XIV.} \ldots \mathrm{S}\mathrm{d}\rho\phi^{-1}\lambda\phi^{-1}\mu = 0;$$

whereof the former gives *two imaginary directions*, and the latter gives *one real direction*, coinciding precisely with the *three directions* (10.).

(12.) And if ρ, instead of being the vector of an *umbilic*, be only the vector of a point *on a generatrix* corresponding, we shall still *satisfy the differential equation* V., by supposing that $\mathrm{d}\rho$ belongs to the *same imaginary right line:* because we shall then have, as at the umbilic itself,

$$\text{XV.} \ldots \mathrm{S}\lambda\mathrm{d}\rho = 0, \quad \mathrm{S}\rho\mathrm{d}\rho\phi^{-1}\lambda = 0.$$

An *umbilicar generatrix* is therefore *proved anew* (comp. (9.)) to be, *in its whole extent*, a *line of curvature.*

(13.) The recent reasonings and calculations apply (6.), not only to an ellipsoid, but also to a double-sheeted hyperboloid, four umbilics for *each* of these two surfaces being *real.* But if for a moment we now consider specially the case of an *ellipsoid*, and if we denote for abridgment the real quotient $\frac{a-c}{a+c}$ by h, we may then substitute in IV. and V. for λ, μ, $\phi^{-1}\lambda$, $\phi^{-1}\mu$ the expressions,

$$\text{XVI.} \ldots \alpha - h\alpha' = \frac{2b\mathrm{U}\lambda}{a+c}; \quad h\alpha - \alpha' = \frac{2b\mathrm{U}\mu}{a+c};$$

$$\text{XVII.} \ldots \alpha + h\alpha' = \frac{-2b\phi^{-1}\mathrm{U}\lambda}{ac(a+c)}; \quad -h\alpha - \alpha' = \frac{-2b\phi^{-1}\mathrm{U}\mu}{ac(a+c)};$$

and then, after division by $h^2 - 1$, there remain only the two vector constants α α', the equation IV. reducing itself to VI., and V. to VII.

(14.) The simplified equations thus obtained are not however peculiar to *ellipsoids*, but extend to a whole *confocal system.* To prove this, we have only to combine the equations II. and III. with the *inverse form*,

$$\text{XVIII.} \ldots l^{-2}\phi^{-1}\rho = \alpha\mathrm{S}\alpha'\rho + \alpha'\mathrm{S}\alpha\rho - \rho(e + \mathrm{S}\alpha\alpha'),$$

which follows from 407, XV., and gives at once the equations VI. and VII., whatever the *species* of the surface may be.

(15.) The differential equation VII. must then be satisfied by the *three rectangular directions* of $\mathrm{d}\rho$, or of a *tangent to a line of curvature*, which answer to the *orthogonal intersections* (410, (12.)) of the *three confocals* through a given point P; it ought therefore, as a verification, to be satisfied *also*, when we substitute ν for $\mathrm{d}\rho$, ν being a *normal to a confocal* through that point: that is, we ought to have the equation,

$$\text{XIX.} \ldots \mathrm{S}\alpha\nu\mathrm{S}\alpha'\rho\nu + \mathrm{S}\alpha'\nu\mathrm{S}\alpha\rho\nu = 0.$$

And accordingly this is at once obtained from 407, XVI., by operating with $\mathrm{S}.\rho\nu$; so that the *three normals* ν are all *sides of this cone* XIX., or of the cone VII. with $\mathrm{d}\rho$ for a side, with which the cone V. is found to coincide (13.).

(16.) And because this last equation XIX., like VI. and VII., involves *only* the two *focal lines* α, α' as its *constants*, we may infer from it this *theorem*: "*If inde-*

*finitely many surfaces of the second order have only their asymptotic cones biconfocal,** *and pass through a given point, their normals at that point have a cone of the second order for their locus ;*" which latter *cone* is *also* the *locus of the tangents*, at the same point, *to all the lines of curvature which pass through it*, when different values are successively assigned to the scalar constant $a^2 - c^2$ (or $2l^2$) : that is, when the *asymptotes* α, α' to the *focal hyperbola* remain *unchanged in position*, but the *semiaxes* $(a^2 - b^2)^{\frac{1}{2}}$, $(b^2 - c^2)^{\frac{1}{2}}$ of that curve (here treated as both real) *vary together.*

(17.) The equation VI. of the *cone of chords* (5.) introduces the *fixed focal lines* α, α' by their *directions* only. But if we suppose that the *lengths* of those two lines are *equal*, without being *here* obliged to assume that each of those lengths is *unity*, we shall then have (comp. 407, (2.), (3.)), the following *rectangular system of unit lines*, in the directions of the *axes of the system*,

$$\text{XX.} \ldots \mathrm{U}(\alpha + \alpha'), \quad \mathrm{UV}\alpha\alpha', \quad \mathrm{U}(\alpha - \alpha'),$$

which obey in all respects the laws of ijk, and may often be conveniently *denoted* by those symbols, in investigations such as the present. And then, by decomposing the semidiameter ρ, and the chord $\Delta\rho$, in these three directions XX., we easily find the following *rectangular transformation*† of the foregoing equation VI.,

$$\text{XXI.} \ldots \frac{\mathrm{S}(\alpha + \alpha')^{-1}\rho}{\mathrm{S}(\alpha + \alpha')\Delta\rho} + \frac{\mathrm{S}(\alpha - \alpha')^{-1}\rho}{\mathrm{S}(\alpha - \alpha')\Delta\rho} = \frac{\mathrm{S}.(\mathrm{V}\alpha\alpha')^{-1}\rho}{\mathrm{S}.\mathrm{U}\alpha\alpha'\,\Delta\rho};$$

in which it is permitted to change $\Delta\rho$ to $d\rho$, in order to obtain a *new form* of the differential equation of the *lines of curvature ;* or else at pleasure to ν, and so to find, in a new way, a *condition* satisfied by the *three normals*, to the *three confocals* through P.

(18.) The *cone*, VI. or XXI., is *generally* the *locus of a system of three rectangular lines ;* each *plane through the vertex*, which is *perpendicular to any real side*, cutting it in a *real pair of mutually rectangular sides :* while, for the same reason, the *section* of the same cone, by any plane which does *not* pass through its *vertex* P, but cuts any *side* perpendicularly, is generally an *equilateral hyperbola*.

(19.) If, however, the *point* P be situated *in any one* of the *three principal planes*, perpendicular to the three lines XX., then the *cone* XXI. (as its equation shows) *breaks up* (comp. (6.)) into a *pair of planes*, of which *one* is that principal

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* That is, if the surfaces (supposed to have a common centre) be cut by the plane at infinity in biconfocal conics, real or imaginary.

† The corresponding *form*, in rectangular co-ordinates, of the *condition of intersection*, of *normals at two points* (xyz) and $(x'y'z')$, to the surface,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

is the equation (probably a known one, although the writer has not happened to meet with it),

$$\frac{(b^2 - c^2)x'}{x - x'} + \frac{(c^2 - a^2)y'}{y - y'} + \frac{(a^2 - b^2)z'}{z - z'} = 0;$$

in which it is evident that xyz and $x'y'z'$ may be interchanged.

plane itself, while the *other* is *perpendicular* thereto. And while the *former* plane *cuts* the surface in a *principal section*, which is *always* a *line of curvature* through P, the *latter* plane *usually* cuts the surface in *another conic*, which *crosses* the former section at *right angles*, and gives the *direction* of the second line of curvature.

(20.) *But* if we further suppose, as in (6.), that the point P is an *umbilic*, then (as has been seen) the *second plane* is a *tangent plane;* and the *second conic* (19.) is itself *decomposed*, into a *pair of imaginary right lines:* namely, as before, the *two umbilicar generatrices* through the point, which have been shown to be, in an imaginary sense, both *lines of curvature themselves*, and also a *portion* of the *envelope of all the others*.

(21.) We shall only here add, as *another transformation* of the general equation VI. of the *cone of chords*, which does not *even* assume $T\alpha = T\alpha'$, the following:

$$\text{XXII.} \ldots \mathrm{S}(\alpha+\alpha')\Delta\rho.\mathrm{S}(\alpha+\alpha')\rho\Delta\rho = \mathrm{S}(\alpha-\alpha')\Delta\rho.\mathrm{S}(\alpha-\alpha')\rho\Delta\rho;$$

where the *directions* of the *two new lines*, $\alpha+\alpha'$ and $\alpha-\alpha'$, are only obliged to be *harmonically conjugate* with respect to the directions of the *fixed focal lines* of the system: or in other words, are those of any two *conjugate semidiameters* of the *focal hyperbola*.

412. The subject of *Lines of Curvature* receives of course an additional illustration, when it is combined with the known conception of the corresponding *Centres of Curvature*. Without yet entering on the *general* theory of the *curvatures of sections* of an arbitrary surface, we may at least consider here the curvatures of those *normal sections*, which *touch* at any given point the *lines* of curvature. Denoting then by σ the vector of the *centre* S of curvature of *such* a section, and by R the *radius* PS, considered as a *scalar* which is positive when it has the direction of $+\nu$, it is easy to see that we have the *two fundamental equations :*

$$\text{I.} \ldots \sigma = \rho + R\mathrm{U}\nu; \qquad \text{II.} \ldots R^{-1}\mathrm{d}\rho + \mathrm{dU}\nu = 0;$$

whence follows this *new form* of the general differential equation 410, II. of the *lines of curvature*,

$$\text{III.} \ldots \mathrm{V}\mathrm{d}\rho\mathrm{dU}\nu = 0;$$

with several other combinations or transformations, among which the following may be noticed here:

$$\text{IV.} \ldots \frac{\mathrm{T}\nu}{R} + \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = 0.$$

(1.) The equation I. requires no proof; and from it the equation II. is obtained by merely differentiating* as if σ and R were constant: after which the formula III. follows at once, and IV. is easily deduced.

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* To students who are accustomed to *infinitesimals*, the *easiest* way is here to

(2.) To obtain from this last equation a more developed expression for R, we may assume for $\mathrm{d}\nu$, considered as a linear and self-conjugate function of $\mathrm{d}\rho$ (410, (1.)), the general form (comp. 410, XVIII.),

$$\text{V.} \ldots \mathrm{d}\nu = g\mathrm{d}\rho + \mathrm{V}\lambda\mathrm{d}\rho\mu,$$

in which g, λ, μ are independent of $\mathrm{d}\rho$; and then, while the *tangent* $\mathrm{d}\rho$ has (by 410, XXII.) one or other of the *two directions*,

$$\text{VI.} \ldots \mathrm{d}\rho \parallel \mathrm{UV}\nu\lambda \pm \mathrm{UV}\nu\mu,$$

the *curvature* R^{-1} receives one or other of the *two values* corresponding,

$$\text{VII.} \ldots R^{-1} = -\mathrm{T}\nu^{-1}(g + \mathrm{S}\lambda\mathrm{U}\nu.\mathrm{S}\mu\mathrm{U}\nu \pm \mathrm{TV}\lambda\mathrm{U}\nu.\mathrm{TV}\mu\mathrm{U}\nu).$$

(3.) One mode of arriving at this last transformation, or of showing that *if* (comp. again 410, XXII.) we assume,

$$\text{VIII.} \ldots \tau = (\text{or} \parallel)\, \mathrm{UV}\lambda\nu \pm \mathrm{UV}\mu\nu,$$

then $$\text{IX.} \ldots \mathrm{S}\lambda\tau\mu\tau^{-1} = \mathrm{S}\lambda\mathrm{U}\nu.\mathrm{S}\mu\mathrm{U}\nu \pm \mathrm{TV}\lambda\mathrm{U}\nu.\mathrm{TV}\mu\mathrm{U}\nu,$$

or $$\text{X.} \ldots 2\mathrm{S}\lambda\tau.\mathrm{S}\mu\tau^{-1} = \mathrm{S}(\mathrm{V}\lambda\mathrm{U}\nu.\mathrm{V}\mu\mathrm{U}\nu) \pm \mathrm{TV}\lambda\mathrm{U}\nu.\mathrm{TV}\mu\mathrm{U}\nu,$$

or finally, $$\text{XI.} \ldots 2\mathrm{SU}\lambda\tau.\mathrm{SU}\mu\tau^{-1} = \mathrm{S}(\mathrm{VU}\lambda\nu.\mathrm{VU}\mu\nu) \pm \mathrm{TVU}\lambda\nu.\mathrm{TVU}\mu\nu,$$

is to introduce the *auxiliary quaternion*,

$$\text{XII.} \ldots q = \mathrm{VU}\lambda\nu.\mathrm{VU}\mu\nu;$$

and to prove that, with the value (or direction) VIII. of τ, we have thus the equation (in which $\mathrm{V}q^2$, as usual, represents the square of $\mathrm{V}q$),

$$\text{XIII.} \ldots 2\mathrm{SU}\lambda\tau.\mathrm{SU}\mu\tau^{-1} = \mathrm{S}q \pm \mathrm{T}q = \frac{\mathrm{V}q^2}{\mathrm{S}q \mp \mathrm{T}q}.$$

(4.) And this may be done, by simply observing that we have thus (with the value VIII.) the expressions,

$$\text{XIV.} \ldots \mathrm{S}\tau\mathrm{U}\lambda = \frac{\pm\mathrm{SU}\lambda\mu\nu}{\mathrm{TVU}\mu\nu}, \quad \mathrm{S}\tau\mathrm{U}\mu = \frac{-\mathrm{SU}\lambda\mu\nu}{\mathrm{TVU}\lambda\nu};$$

$$\text{XV.} \ldots \mathrm{S}\tau\mathrm{U}\lambda.\mathrm{S}\tau\mathrm{U}\mu = \frac{\mp(\mathrm{SU}\lambda\mu\nu)^2}{\mathrm{TVU}\lambda\nu.\mathrm{TVU}\mu\nu} = \frac{\pm\mathrm{V}q^2}{\mathrm{T}q},$$

because $$\text{XVI.} \ldots \mathrm{V}q = -\mathrm{U}\nu.\mathrm{SU}\lambda\mu\nu;$$

and $$\text{XVII.} \ldots \tau^2 = -2 \pm 2\mathrm{SU}q = \pm\frac{2(\mathrm{S}q \mp \mathrm{T}q)}{\mathrm{T}q}.$$

(5.) Admitting then the expression VII., for the curvature R^{-1}, we easily see that it may be thus transformed:

$$\text{XVIII.} \ldots R^{-1} = -\mathrm{T}\nu^{-1}\left(g + \mathrm{T}\lambda\mu.\cos\left(\angle\frac{\nu}{\lambda} \mp \angle\frac{\nu}{\mu}\right)\right);$$

and that the *difference* of the *two* (principal) *curvatures*, of normal *sections* of an *arbitrary surface*, answering *generally* to the *two* (rectangular) *directions* of the

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conceive the *differentials* to be such. But it has already been abundantly shown, that this *view* of the latter is by no means *necessary*, in the treatment of them by quaternions. (Compare the second Note to page 667.)

lines of curvature through the particular point considered, *vanishes* when the *normal* ν has the *direction* of either of the two *cyclic* normals, λ, μ, of the *index surface* (410, (9.)); that is, when the *index curve* (410, (4.)), considered as a *section* of that index surface, is a *circle:* or finally, when the point in question is, in a received sense, an *umbilic** of the given surface.

(6.) That surface, although considered to be a *given* one, has hitherto (in these last sub-articles) been treated as quite *general.* But if we now suppose it to be a central surface of the *second order*, and to be represented by the equation,

$$\text{XIX.} \ldots f\rho = g\rho^2 + S\lambda\rho\mu\rho = 1,$$

which has already several times occurred, we see at once, from the formula VII. or XVIII. (comp. 410, (10.)), that the *difference of curvatures*, of the two principal normal sections of any such surface, *varies* proportionally to the *perpendicular* ($T\nu^{-1}$ or P) from the centre on the tangent plane, multiplied by the *product of the sines of the inclinations* of that plane, to the two *cyclic planes* of the surface.

(7.) In general (comp. 409, (3.)), it is easy to see that

$$\text{XX.} \ldots S\frac{d\nu}{d\rho} = S\tau^{-1}\phi\tau = -D^{-2},$$

if D denote the (scalar) *semidiameter* of the *index surface*, in the direction of $d\rho$ or of τ; but for the two directions of the *lines of curvature*, these semidiameters become (410, (3.), (4.)) the *semiaxes* of the *index curve.* Denoting then by a_1 and a_2 these last semiaxes, the *two principal radii of curvature* of *any surface* come by IV. to be thus expressed:

$$\text{XXI.} \ldots R_1 = \mathrm{a}_1^2 T\nu; \quad R_2 = \mathrm{a}_2^2 T\nu.$$

And if the surface be a central one, of the *second order*, then a_1, a_2 are the semiaxes of the *diametral section*, parallel to the *tangent plane;* while $T\nu$ is (comp. again 409, (3.)) the *reciprocal* P^{-1} of the *perpendicular*, let fall on that plane from the centre. Accordingly (comp. (6.), and 219, (4.)), it is known that the *difference* of the *inverse squares* of those *semiaxes* varies proportionally to the *product* of the *sines* of the *inclinations*, of the plane of the section to the two cyclic planes.

(8.) And as regards the *squares themselves*, it follows from 407, LXXI., that they may be thus expressed, in terms of the *principal semiaxes* of the *confocal surfaces*, and in agreement with known results:

$$\text{XXII.} \ldots \mathrm{a}_1^2 = a^2 - a_1^2; \quad \mathrm{a}_2^2 = a^2 - a_2^2;$$

being thus *both positive* for the case of an *ellipsoid; both negative*, for that of a *double-sheeted hyperboloid;* and *one* positive, but the *other* negative, for the case of an hyperboloid of *one sheet* (comp. 410, (15.)).

(9.) In *all* these cases, the *normal* $+\nu$ is drawn towards the *same side* of the *tangent plane*, as that on which the *centre* O of the *surface* is situated (because $S\nu\rho = 1$); hence (by I. and XXI.) *both* the *radii* of curvature R_1, R_2 are drawn in *this direction*, or towards *this side*, for the *ellipsoid;* but *one* such radius for the *single*-sheeted *hyperboloid*, and *both* radii for the hyperboloid of *two* sheets, are directed towards the *opposite side*, as indeed is evident from the forms of these surfaces.

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* Compare the second Note to page 669.

(10.) The following is another method of deducing *generally* the two principal curvatures of a surface, from the *self-conjugate function*,

$$\text{XXIII.} \ldots \mathrm{d}\nu = \phi \mathrm{d}\rho, \qquad 410, \text{IV}.$$

which affords some good practice in the processes of the present Calculus. Writing, for abridgment,

$$\text{XXIV.} \ldots r = \frac{\nu}{\sigma - \rho} = R^{-1}\mathrm{T}\nu = -\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = -\mathrm{S}\tau^{-1}\phi\tau,$$

where τ is still a tangent to a line of curvature, the equation II. is easily brought to the form,

$$\text{XXV.} \ldots -r\tau = \nu^{-1}\mathrm{V}\nu\phi\tau = \phi\tau - \nu^{-1}\mathrm{S}\tau\phi\nu = \Phi\tau,$$

where Φ denotes a *new linear and vector function*, which however is *not* in general *self-conjugate*, because we have not generally $\phi\nu \parallel \nu$. Treating then this *new function* on the plan of the Section III. ii. 6, we derive from it a *new cubic equation*, of the form,

$$\text{XXVI.} \ldots 0 = M + M'r + M''r^2 + r^3,$$

and with the coefficients,

$$\text{XXVII.} \ldots M = 0, \quad M' = \mathrm{S}\nu^{-1}\psi\nu, \quad M'' = m'' - \mathrm{S}\nu^{-1}\phi\nu;$$

ψ being a certain *auxiliary function* ($= m\phi^{-1}$), and m'' being the *coefficient** analogous to M'', in the cubic derived from the function ϕ *itself*. The root $r = 0$ is foreign to the present inquiry; but the *two curvatures*, R_1^{-1}, R_2^{-1}, are the *two roots* of the following *quadratic* in R^{-1}, obtained from the equation XXVI. by the rejection of that foreign root:

$$\text{XXVIII.} \ldots 0 = (R^{-1}\mathrm{T}\nu)^2 + M''R^{-1}\mathrm{T}\nu + M'.$$

(11.) As a first application of this general equation XXVIII., let $\phi\tau$ have again, as in V., the form $g\tau + \mathrm{V}\lambda\tau\mu$; we shall then have the values,

$$\text{XXIX.} \ldots M'' = 2(g + \mathrm{S}\lambda\mathrm{U}\nu\,.\,\mathrm{S}\mu\mathrm{U}\nu),$$

and $$\text{XXX.} \ldots M' = (g + \mathrm{S}\lambda\mathrm{U}\nu\,.\,\mathrm{S}\mu\mathrm{U}\nu)^2 - (\mathrm{V}\lambda\mathrm{U}\nu)^2(\mathrm{V}\mu\mathrm{U}\nu)^2,$$

= a great variety of transformed expressions; and the two resulting curvatures agree with those assigned by VII.

(12.) As a second application, let the surface be central of the second order, with *abc* for its scalar semiaxes (real or imaginary); then the *symbolical cubic* (350) in ϕ becomes,

$$\text{XXXI.} \ldots 0 = \phi^3 - m''\phi^2 + m'\phi - m = (\phi + a^{-2})(\phi + b^{-2})(\phi + c^{-2});$$

and the coefficients of the quadratic XXVIII. in R^{-1} take the values, in which N denotes the semidiameter of the surface in the direction of the normal:

$$\text{XXXII.} \ldots R_1^{-1} + R_2^{-1} = -M''\mathrm{T}\nu^{-1} = -(m'' + f\mathrm{U}\nu)P = (a^{-2} + b^{-2} + c^{-2} - N^{-2})P;$$

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* Compare the Note to page 673, continued in page 674. The reason of the evanescence of the coefficient M, or of the occurrence of a null root of the cubic, is that we have here $\Phi\phi^{-1}\nu = 0$, so that the symbol $\Phi^{-1}0$ may represent an actual vector (comp. 351). Geometrically, this corresponds to the circumstance that when we pass, along a semidiameter prolonged, from a surface of the second order to another surface of the same kind, concentric, similar, and similarly placed, the direction of the normal does not change.

$$\text{XXXIII.} \ldots R_1^{-1}R_2^{-1} = M''\mathrm{T}\nu^{-2} = -m\nu^{-4} = a^{-2}b^{-2}c^{-2}P^4;$$

both of which agree with known results, and admit of elementary verifications.*

(13.) *In general*, if we observe that $m'' - \phi = \chi$ (350, XVI.), we shall see that the quadratic XXVIII. in r (or in $R^{-1}\mathrm{T}\nu$) may be thus written:

$$\text{XXXIV.} \ldots 0 = \mathrm{S}\nu^{-1}(r^2\nu + r\chi\nu + \psi\nu);$$

or thus more briefly (comp. 398, LXXIX.),

$$\text{XXXV.} \ldots 0 = \mathrm{S}\nu^{-1}(\phi + r)^{-1}\nu.$$

(14.) Accordingly, the formula XXV. gives the expression,

$$\text{XXXVI.} \ldots \nu^2\tau = (\phi + r)^{-1}\nu . \mathrm{S}\tau\phi\nu;$$

from which, under the condition $\mathrm{S}\nu\tau = 0$, the equation XXXV. follows at once.

(15.) We have therefore *generally*, for the *product* of the two principal *curvatures of sections* of *any surface* at any point, the expression:

$$\text{XXXVII.} \ldots R_1^{-1}R_2^{-1} = r_1 r_2 \mathrm{T}\nu^{-2} = -\nu^{-4}\mathrm{S}\nu\psi\nu = -\mathrm{S}\,\frac{1}{\nu}\,\psi\,\frac{1}{\nu};$$

which contains an important theorem of Gauss, whereto we shall presently proceed.

(16.) Meanwhile we may remark that the recent analysis shows, that the squares a_1^2, a_2^2 (7.) of the semiaxes of the index-curve are *generally* the roots of the following equation,

$$\text{XXXVIII.} \ldots 0 = \mathrm{S}\nu(\phi + \mathrm{a}^{-2})^{-1}\nu,$$

when developed as a quadratic in a^2.

(17.) And that the same quadratic assigns the squares of the semiaxes of a diametral section, made by a plane $\perp \nu$, of the central surface of the second order which has $\mathrm{S}\rho\phi\rho = 1$ for its equation.

(18.) Accordingly, $\mathrm{V}\rho\phi\rho$ has the direction of a tangent to this surface, which is perpendicular to ρ at its extremity; and therefore the vector,

$$\text{XXXIX.} \ldots \sigma = \rho^{-1}\mathrm{V}\rho\phi\rho = \phi\rho - \rho^{-1} = (\phi - \rho^{-2})\rho,$$

is perpendicular to the plane of the diametral section, which has the *semidiameter* ρ for a *semiaxis*: so that it is perpendicular also to ρ itself. The equation,

$$\text{XL.} \ldots \mathrm{S}\sigma(\phi - \rho^{-2})^{-1}\sigma = 0,$$

assigns therefore the values of the squares $(-\rho^2)$ of the scalar semiaxes of the central section $\perp \sigma$; which agrees with the formula XXXVIII.

(19.) If then a *surface* be *derived* from a *given* central surface of the *second order*, as the *locus of the extremities of normals* (erected at the centre) *to the diametral sections* of the given surface, *each such normal* (when real) having the *length* of *one of the semiaxes* of that section, the *equation of this new surface*† (or locus) will admit of being written thus:

$$\text{XLI.} \ldots \mathrm{S}\rho(\phi - \rho^{-2})^{-1}\rho = 0.$$

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* As an easy verification by quaternions of the expression XXXII., it may be remarked (comp. 408, (27.)), that if α, β, γ be *any three rectangular unit lines*, then

$$f\alpha + f\beta + f\gamma = \text{const.} = c_1 + c_2 + c_3 = a^{-2} + b^{-2} + c^{-2}.$$

† When the given surface is an *ellipsoid*, this *derived* surface XLI. is therefore the celebrated *Wave Surface* of Fresnel, which will be briefly mentioned somewhat farther on.

(20.) The first of the values XXIV., for the auxiliary scalar r, gives the expression (if $\nu = \phi\rho$, as it is for a central surface of the second order),

$$\text{XLII.} \ldots \sigma = \rho + r^{-1}\nu = (1 + r^{-1}\phi)\rho = r^{-1}(\phi + r)\rho ;$$

whence, by inversion, and operation with ϕ,

$$\text{XLIII.} \ldots \rho = r(\phi + r)^{-1}\sigma ; \qquad \text{XLIV.} \ldots \nu = r(\phi + r)^{-1}\phi\sigma ;$$

and therefore, because $S\rho\nu = 1$,

$$\text{XLV.} \ldots r^{-2} = S((\phi + r)^{-1}\sigma . (\phi + r)^{-1}\phi\sigma) = S . \sigma(\phi + r)^{-2}\phi\sigma .$$

(21.) The following is a quite different way of arriving at this result, which is also useful for other purposes. Considering σ as the vector OS of a point S on the *Surface of Centres*, that is, on the *locus* of all the centres of curvature of principal normal sections, the vector (say v) of the *Reciprocal Surface* is connected with σ (comp. 373, (21.)) by the *equations of reciprocity*,*

$$\text{XLVI.} \ldots S\sigma v = Sv\sigma = 1 ; \qquad \text{XLVII.} \ldots Sv d\sigma = 0 ; \qquad \text{XLVIII.} \ldots S\sigma dv = 0 ;$$

which are all satisfied by the vector expression,

$$\text{XLIX.} \ldots v = \frac{\tau}{S\rho\tau},$$

where τ is, as before, a tangent to the line of curvature: so that, if ω denote the variable vector of the normal plane to this last curve, the equation of that plane (comp. 369, IV.) may be thus written,

$$\text{L.} \ldots Sv(\omega - \rho) = 0.$$

This *normal plane*, to the *line of curvature* at P, is therefore at the same time the *tangent plane* to the *surface of centres* at S, as indeed it is known to be, from simple geometrical considerations, independently of the *form* of the *given* surface, which remains here entirely arbitrary.

(22.) The expression XLIX. for v gives *generally* the relation,

$$\text{LI.} \ldots S\rho v = 1 ;$$

giving also, by 410, V. and VI., these two other equations,

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* It is understood that $d\sigma$ and dv, in the differential equations XLVII., XLVIII., are in general only obliged to have directions *tangential* to the surface of centres, and to its reciprocal, at corresponding points: so that the equations might be in some respects more clearly written thus, $Sv\delta\sigma = 0$, $S\sigma\delta v = 0$, the mark d being reserved to indicate changes which arise from motion along a *given line* of curvature, while δ should have a more general signification. Accordingly if, in particular, we write $\delta\rho = \nu d\rho$, for a variation answering to motion along the *other line*, and denote the two radii of curvature for the two directions $d\rho$ and $\delta\rho$ by R_1 and R_2, we shall have by II., $R_1^{-1}d\rho + dU\nu = 0$, $R_2^{-1}\delta\rho + \delta U\nu = 0$, and therefore by I.,

$$d\sigma = dR_1 . U\nu, \quad \delta\sigma = \delta\rho + \delta(R_1 U\nu) = (1 - R_1 R_2^{-1})\nu d\rho + \delta R_1 . U\nu ;$$

so that we have *both* $Sd\rho d\sigma = 0$, and $Sd\rho\delta\sigma = 0$, and therefore the *tangent* $d\rho$ or τ to the *given line* of curvature has the direction of the *normal* v to the corresponding *sheet* of the surface of centres, as is otherwise visible from geometry. And when we have thus found an equation of the form $tv = \tau$, operation with $S . \sigma$ gives by XLVI. the value $t = S\rho\tau$, as in XLIX., because $\sigma - \rho \parallel \nu \perp \tau$.

$$\text{LII.}\ldots S\nu\upsilon = 0, \quad \text{and} \quad \text{LIII.}\ldots S\nu\upsilon\phi\upsilon = 0,$$

which are still independent of the *form* of the given surface.

(23.) But if that surface be a *central quadric*,[*] then the equation LI. may be thus written,

$$\text{LIV.}\ldots 1 = S\upsilon\phi^{-1}\nu = S\nu\phi^{-1}\upsilon;$$

combining which with LII. and LIII., we derive the expressions:

$$\text{LV.}\ldots \nu = \frac{\upsilon^2\phi\upsilon - \upsilon f\upsilon}{\upsilon^4 - f\upsilon . F\upsilon}; \qquad \text{LVI.}\ldots \rho = \phi^{-1}\nu = \frac{\upsilon^3 - \phi^{-1}\upsilon f\upsilon}{\upsilon^4 - f\upsilon . F\upsilon};$$

wherein $f\upsilon = S\upsilon\phi\upsilon$, and $F\upsilon = S\upsilon\phi^{-1}\upsilon$, as usual.

(24.) Operating with $S.\nu$ on this last expression for ρ, and attending to LII. and LIV., we find the following *quaternion forms* of the *Equation of the Reciprocal of the Surface of Centres:*

$$\text{LVII.}\ldots 1 = (S\nu\rho =)\ \frac{-f\upsilon}{\upsilon^4 - f\upsilon . F\upsilon}; \quad \text{or} \quad \text{LVIII.}\ldots \upsilon^4 = (F\upsilon - 1) f\upsilon;$$

or

$$\text{LIX.}\ldots 1 = (F\upsilon - 1) f\frac{1}{\upsilon}; \quad \text{or} \quad \text{LX.}\ldots F\upsilon - \frac{1}{f\frac{1}{\upsilon}} = 1;\ \&c.,$$

whereof the second, when translated into co-ordinates, is found to agree perfectly with a known[†] equation of the same reciprocal surface.

(25.) Differentiating the form LX., and observing that

$$\text{LXI.}\ldots \left(f\frac{1}{\upsilon}\right)^{-1} = \frac{\upsilon^4}{f\upsilon}, \quad d.\upsilon^4 = 4S\upsilon^3 d\upsilon, \quad df\upsilon = 2S\phi\upsilon d\upsilon, \quad dF\upsilon = 2S\phi^{-1}\upsilon d\upsilon,$$

we find, by comparison with XLVI. and XLVIII., the expression:

$$\text{LXII.}\ldots \sigma = \phi^{-1}\upsilon - \frac{2\upsilon^3}{f\upsilon} + \frac{\upsilon^4\phi\upsilon}{(f\upsilon)^2}; \quad \text{or} \quad \text{LXIII.}\ldots \sigma = \phi^{-1}\upsilon + \frac{2\upsilon}{fU\upsilon} + \frac{\phi\upsilon}{(fU\upsilon)^2};$$

or finally by XLIX., with the recent signification XXIV. of r,

$$\text{LXIV.}\ldots \sigma = r^{-2}(\phi + r)^2\phi^{-1}\upsilon, \quad \text{because} \quad \text{LXV.}\ldots r = fU\tau = fU\upsilon:$$

and, for the same reason, the equation LX. of the *reciprocal surface* may be thus briefly written,

$$\text{LXVI.}\ldots F\upsilon + r^{-1}\upsilon^2 = 1, \quad \text{while} \quad \text{LXVI}'.\ldots f\upsilon + r\upsilon^2 = 0.$$

(26.) Inverting the last form for σ, and using again the relation XLVI., we first find for υ the expression,

$$\text{LXVII.}\ldots \upsilon = r^2(\phi + r)^{-2}\phi\sigma;$$

and then are conducted anew to the equation XLV., or to the following,

$$\text{LXVIII.}\ldots 1 = S.\sigma(1 + r^{-1}\phi)^{-2}\phi\sigma.$$

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[*] Compare the last note to page 672; see also the use made of this known name "quadric," for a surface of the second order (or degree), in the sub-articles to 399 (pages 614, &c.).

[†] The equation alluded to, which is one of the *fourth degree*, appears to have been first assigned by Dr. Booth, in a Tract on *Tangential Co-ordinates* (1840), cited in page 168 of Dr. Salmon's Treatise. See also the Abstract of a Paper by Dr. Booth, in the Proceedings of the Royal Society for April, 1858.

(27.) This last equation may also be thus written,

$$\text{LXIX.}\ldots 1 = \mathrm{S}.\sigma(1+r^{-1}\phi)^{-3}(\phi+r^{-1}\phi^2)\sigma;$$

but by combining XLIII. LI. LXVII. we have,

$$\text{LXX.}\ldots 1 = (\mathrm{S}\rho\upsilon =)\,\mathrm{S}.\sigma(1+r^{-1}\phi)^{-3}\phi\sigma;$$

hence

$$\text{LXXI.}\ldots 0 = \mathrm{S}.\sigma(1+r^{-1}\phi)^{-3}\phi^2\sigma,$$

a result which may be otherwise and more directly deduced, under the form $\mathrm{S}\nu\upsilon = 0$ (LII.), from the expressions XLIV. LXVII. for ν and υ.

(28.) If we write,

$$\text{LXXII.}\ldots \tau = \mathrm{Ud}\rho, \quad \tau' = \mathrm{U}(\nu\mathrm{d}\rho), \quad \text{and therefore} \quad \text{LXXIII.}\ldots \tau\tau' = \mathrm{U}\nu,$$

τ and τ' being thus ***unit-tangents*** to the lines of curvature, the equation III. gives, generally,

$$\text{LXXIV.}\ldots 0 = \mathrm{V}\tau\mathrm{d}(\tau\tau') = -\mathrm{d}\tau' + \tau\mathrm{S}\tau'\mathrm{d}\tau, \quad \text{whence} \quad \text{LXXIV}'.\ldots \mathrm{d}\tau' \parallel \tau;$$

of which ***general parallelism*** of $\mathrm{d}\tau'$ to τ, the geometrical reason is (comp. again III.) that a ***line of curvature*** on an ***arbitrary surface*** is, at the same time, a line of curvature on the ***developable normal surface*** which ***rests*** upon that line, and ***to which*** the vectors τ' or $\nu\mathrm{d}\rho$ are ***normals***.

(29.) The same substitution LXXIII. for $\mathrm{U}\nu$ gives by II., if we denote by s the ***arc*** of a line of curvature, measured from any fixed point thereof, so that (by 380, (7.), &c.),

$$\text{LXXV.}\ldots \mathrm{Td}\rho = \mathrm{d}s, \quad \mathrm{d}\rho = \tau\mathrm{d}s, \quad \mathrm{D}_s\rho = \tau,$$

the following ***general expression*** for the curvature of the given surface, in the direction τ of the given line, which by LXXIV′. is also that of $\mathrm{d}\tau'$:

$$\text{LXXVI.}\ldots R^{-1} = \mathrm{S}.\tau\mathrm{D}_s(\tau\tau') = -\mathrm{S}.\tau\tau'\mathrm{D}_s\tau = \mathrm{S}(\mathrm{U}\nu^{-1}.\mathrm{D}_s^2\rho);$$

but $\mathrm{D}_s^2\rho$ is (by 389, (4.)) what we have called the ***vector of curvature*** of the line of curvature, considered as a ***curve in space***, and $R^{-1}\mathrm{U}\nu$ is the corresponding vector of curvature of the ***normal section*** of the given surface, which has the same tangent τ at the given point: hence ***the latter vector of curvature is*** (generally) ***the projection of the former, on the normal ν to the given surface.***

(30.) In like manner, if we denote for a moment by $R_{\prime}^{-1}$ the curvature of the developable normal surface (28.), for the same direction τ, the general formula II. gives, by LXXIV.,

$$\text{LXXVII.}\ldots R_{\prime}^{-1} = \tau\mathrm{D}_s\tau' = -\mathrm{S}\tau'\mathrm{D}_s\tau = \mathrm{S}.\tau'^{-1}\mathrm{D}_s^2\rho;$$

the ***vector*** $R_{\prime}^{-1}\tau'$ of this ***new curvature*** is therefore the ***projection on the new normal*** τ', of the ***vector of curvature*** $\mathrm{D}_s^2\rho$ of the ***given line*** of curvature. But we shall soon see that these two last results are included in one more general,* respecting ***all plane sections*** of an arbitrary surface.

(31.) The general parallelism LXXIV′. conducts easily, for the case of a central quadric, to a known and important theorem, which may be thus investigated. Writing, for such a surface,

$$\text{LXXVIII.}\ldots r = f\tau, \quad r' = f\tau',$$

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* Namely in Meusnier's Theorem, which can be proved ***generally*** by quaternions with about the same ease as the two foregoing ***cases*** of it.

so that r retains here its recent signification LXV., and r' is the analogous scalar for the other direction of curvature, we have by LXXIV. the differential,

$$\text{LXXIX.} \ldots \mathrm{d}r' = 2\mathrm{S}\phi\tau'\mathrm{d}\tau' = 2\mathrm{S}\tau\phi\tau'\mathrm{S}\tau'\mathrm{d}\tau = 0,$$

because $\mathrm{S}\tau\phi\tau' = 0$, by 410, XI.

(32.) We have then the relation,

$$\text{LXXX.} \ldots f\mathrm{U}(\nu\mathrm{d}\rho) = f\tau' = r' = \text{const.};$$

that is to say, the *square* (r'^{-1}) of the scalar *semidiameter* (D') of the surface, which is *parallel to the second tangent* (τ'), is *constant for any one line of curvature* (τ); and accordingly (comp. XXII., and the expression 407, LXXI. for $f\mathrm{U}\nu_1$), the value of this square is,

$$\text{LXXXI.} \ldots (f\mathrm{U}\nu\mathrm{d}\rho)^{-1} = r'^{-1} = a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2,$$

if a', b', c' be the scalar semiaxes of the confocal, which cuts the given quadric (abc) along the line of curvature, whereof the variable tangent is τ.

(33.) This constancy of $f\mathrm{U}\nu\mathrm{d}\rho$ may be proved in other ways; for instance, the general equation $\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho = 0$ gives, for a line of curvature on an *arbitrary surface*,

$$\text{LXXXII.} \ldots \mathrm{d}\nu = \nu\mathrm{S}\nu^{-1}\mathrm{d}\nu + \mathrm{d}\rho\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho}; \qquad \text{LXXXIII.} \ldots \mathrm{V}\mathrm{d}\nu\mathrm{d}\rho = \nu\mathrm{d}\rho\mathrm{S}\nu^{-1}\mathrm{d}\nu;$$

and $$\text{LXXXIV.} \ldots \mathrm{S}.\mathrm{d}\rho\phi(\nu\mathrm{d}\rho) = 0, \quad \text{because} \quad \mathrm{d}\nu = \phi\mathrm{d}\rho;$$

while for a *central quadric* $(f\rho = 1,\ \phi\rho = \nu)$ it is easy to show that we have also,

$$\text{LXXXV.} \ldots \phi(\nu\mathrm{d}\rho) = \mathrm{V}\rho\mathrm{d}\rho f(\nu\mathrm{U}\mathrm{d}\rho);$$

hence, for such a surface, if we suppose for simplicity that $\mathrm{d}s$ or $\mathrm{T}\mathrm{d}\rho$ is constant, which gives $\mathrm{V}\nu\mathrm{d}^2\rho \parallel \mathrm{d}\rho$, we have,

$$\text{LXXXVI.} \ldots \mathrm{d}f(\nu\mathrm{d}\rho) = 2\mathrm{S}(\phi(\nu\mathrm{d}\rho).\mathrm{d}(\nu\mathrm{d}\rho)) = 2\mathrm{S}\nu^{-1}\mathrm{d}\nu.f(\nu\mathrm{d}\rho),$$

a differential equation of the *second order*, of which a *first integral* is evidently,

$$\text{LXXXVII.} \ldots f(\nu\mathrm{d}\rho) = C\nu^2\mathrm{d}\rho^2, \quad \text{or} \quad \text{LXXXVII}'. \ldots f\mathrm{U}(\nu\mathrm{d}\rho) = C = \text{const.}$$

(34.) But we see that the *lines of curvature* on a central quadric are thus *included* in a *more general system* of curves on the same surface, represented by the differential equation LXXXVI., of which the *complete integral* would involve *two constants*: and which expresses that the *semidiameters* parallel to those *tangents* to the *surface*, which *cross* any *one* such curve at *right angles*, have a *common square*, and therefore (if real) a *common length*, so that (in this case) they *terminate on a sphero-conic.**

(35.) Admitting however, as a *case* of this property, the *constancy* LXXX. of the scalar lately called r', namely the *second root* of the quadratic XXXIV. or XXXV., of which the coefficients and the first root r *vary*, in passing from one point to another of what we may call for the moment a *line of first curvature*, we have only to conceive r and v to be accented in the equations LXVI. LXVI'., in order to perceive this *theorem*, which perhaps is new:

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* Compare the sub-articles (6.) (7.) (8.) to 219, in page 231.

The Curve on the Reciprocal* (24.) *of the Surface of Centres* of curvature of a central quadric, which answers to the *second curvature* of that given surface for all the points of a *given line* of *first* curvature, or which is *itself* in a known sense the *reciprocal* (with respect to the given centre) of the *developable normal surface* (28.) which *rests* upon that *line*, is the *intersection* of *two quadrics;* whereof *one* (LXVI'.) is a *cone, concyclic with the given surface* ($f\rho = 1$); while the *other* (LXVI.) is a surface *concyclic with the reciprocal* of that given quadric ($F\nu = 1$).

(36.) Again, the scalar *Equation of the Surface of Centres* (21.) may be said to be the result of the *elimination* of r^{-1} between the equations LXVIII. and LXXI., whereof the latter is the *derivative*† of the former with respect to that scalar; we have therefore this theorem:

An Auxiliary Quadric (LXVIII. or XLV.) *touches the Second Sheet of the Surface of Centres* of a given quadric, *along a Quartic Curve*, which is the *locus of the centres of Second Curvature* for all the points of a *Line of First Curvature* (35.); and (for the same reason) the same auxiliary quadric is *circumscribed*, along the same quartic, by the *Developable Normal Surface* (28.), which *rests* on that *first line:* with permission, of course, to interchange the words *first* and *second*, in this enunciation.

(37.) When the arbitrary constant r is thus allowed to take successively all values, corresponding to *both systems* of lines of curvature, the *Surface of Centres* is therefore at once the *Envelope*‡ *of the Auxiliary Quadric* LXVIII., and the *Locus of the Quartic Curve* (36.), in which one or other of its *two sheets* is *touched*, by that auxiliary quadric in one of its successive states, and also by one of the developable surfaces of normals to the given surface.

(38.) To obtain the *vector equation* of that envelope or locus, we may proceed

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* The *variable vector* of this *curve* is easily seen (comp. XLIX.) to be,

$$v' = \frac{\tau'}{S\tau'\rho} = \frac{\nu\tau}{S\nu\tau\rho};$$

and the *reciprocal surface* (21.) or (24.) is by (25.) the *locus of this quartic* (35.).

† The analogous relation, between the co-ordinate forms of the equations, was perhaps thought too obvious to be mentioned, in page 161 of Dr. Salmon's Treatise; or possibly it may have escaped notice, since the quartic curve (36.) is only mentioned there as an *intersection of two quadrics*, which is *on the surface of centres*, and answers to points of a *line of curvature* upon the given surface. But as regards the *possible novelty*, even *in part*, of any such *geometrical deductions* as those given in the text from the *quaternion analysis* employed, the writer wishes to be understood as expressing himself with the utmost diffidence, and as most willing to be corrected, if necessary. The power of *derivating* (or differentiating) any *symbolical expression* of the form LXVIII., or of any analogous *form*, with respect to *any scalar* which it involves *explicitly*, as if the expression were *algebraical*, is an important but an easy consequence from the principles of the Section III. ii. 6, which has been so often referred to.

‡ Compare the Note immediately preceding.

as follows, using a new expression for σ, in terms of ν or of ρ, which may then be transformed into a function of two independent and scalar variables. Denoting (comp. (32.)) by a_1, b_1, c_1 the semiaxes of the confocal which cuts the given surface in the given line of curvature, and by a_2, b_2, c_2 those of the other confocal, so that the *normals* ν_1, ν_2 to these two confocals have the directions of the *tangents* τ', τ lately considered, we have not only the expressions LXXXI. for r'^{-1}, with $a'b'c'$ changed to a_1, b_1, c_1, but also the analogous expressions (comp. 407, LXXI.),

$$\text{LXXXVIII.} \ldots r^{-1} = a^2 - a_2^2 = b^2 - b_2^2 = c^2 - c_2^2.$$

We have therefore by XLII., combined with 407, XVI., this very simple expression for σ:

$$\text{LXXXIX.} \ldots \sigma = (\phi^{-1} + r^{-1})\nu = \phi_2^{-1}\nu = \phi_2^{-1}\phi\rho\,;$$

containing, in the present notation, and as a result of the present analysis, a known and interesting theorem,* on which however we cannot here delay.

(39.) It follows from this last value of σ, combined with the expression 408, LXXXII. for ρ, that we may write,

$$\text{XC.} \ldots \sigma = l^{-2}\left(\frac{a^{-1}a_1a_2^3}{a+a'} + \frac{\sqrt{-1}\,b^{-1}b_1b_2^3}{\mathrm{V}aa'} + \frac{c^{-1}c_1c_2^3}{a-a'}\right),$$

as the sought *Vector Equation of the Surface of Centres* of curvature of a given quadric (*abc*); ambiguous signs being virtually included in these three terms, because in the subsequent eliminations† the semiaxes enter only by their squares: while l, a, a' are *constants*, as in 407, &c., for the whole confocal *system*, and *abc* are also constant *here*, but $a^2 - a_1^2$ and $a^2 - a_2^2$, or r'^{-1} and r^{-1} (38.), are *variable*, and may be considered to be the *two independent scalars* of which σ is a *vector function*.

413. Some brief remarks may here be made, on the connexion of the general formula,

$$\text{I.} \ldots \mathrm{S}\nu^{-1}(\phi + r)^{-1}\nu = 0, \qquad 412, \text{XXXV.}$$

in which $r = R^{-1}\mathrm{T}\nu$ (412, XXIV.), and which when developed by the rules of the Section III. ii. 6 takes (comp. 398, LXXIX.) the form of the quadratic,

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* Namely Dr. Salmon's theorem (page 161 of his Treatise), that the *centres of curvature* of a given quadric at a given point are the *poles of the tangent plane*, with respect to the two confocals. The connected theorem (page 136), respecting the *rectilinear locus* of the *poles* of a *given plane*, with respect to the surfaces of a *confocal system*, is at once deducible from the quaternion expression 407, XVI. for $\phi^{-1}\nu$, although the theorem did not happen to be known to the present writer, or at least remembered by him, when he investigated that *formula of inversion* for other applications, of which some have been already given.

† The corresponding elimination in co-ordinates was first effected by Dr. Salmon, who thus determined the equation of the surface of centres of curvature of a quadric to be one of the *twelfth degree*. (Compare pages 161, 162 of his already cited Treatise.)

$$\text{II.} \ldots r^2 + r\mathrm{S}\nu^{-1}\chi\nu + \mathrm{S}\nu^{-1}\psi\nu = 0, \qquad 412,\ \text{XXXIV}.$$

with Gauss's* theory of the *Measure of Curvature of a Surface;* and especially with his fundamental result, that this *measure* is equal to the *product of the two principal curvatures of sections* of that surface: a relation which, in our notations, may be thus expressed,

$$\text{III.} \ldots \mathrm{V}.\mathrm{dU}\nu\, \delta\mathrm{U}\nu = R_1^{-1}R_2^{-1}\mathrm{V}\mathrm{d}\rho\delta\rho.$$

(1.) As regards the deduction, by quaternions, of the equation III., in which d and δ may be regarded as two† distinct symbols of differentiation, performed with respect to two independent scalar variables, we may observe that, by principles and rules already established,

$$\text{IV.} \ldots \mathrm{dU}\nu = \mathrm{V}\frac{\mathrm{d}\nu}{\nu}.\mathrm{U}\nu, \quad \delta\mathrm{U}\nu = \mathrm{V}\frac{\delta\nu}{\nu}.\mathrm{U}\nu = -\mathrm{U}\nu.\mathrm{V}\frac{\delta\nu}{\nu};$$

and that therefore the first member of III. may be thus transformed:

$$\text{V.} \ldots \mathrm{V}.\mathrm{dU}\nu\,\delta\mathrm{U}\nu = \mathrm{V}\left(\mathrm{V}\frac{\mathrm{d}\nu}{\nu}.\mathrm{V}\frac{\delta\nu}{\nu}\right) = -\nu^{-1}\mathrm{S}\nu^{-1}\mathrm{d}\nu\delta\nu.$$

(2.) Again, since we have $\mathrm{d}\nu = \phi\mathrm{d}\rho$ (410, IV., &c.), and in like manner $\delta\nu = \phi\delta\rho$, the relations $\mathrm{S}\nu\mathrm{d}\rho = 0$, $\mathrm{S}\nu\delta\rho = 0$, and the self-conjugate property of ϕ, allow us to write,

$$\text{VI.} \ldots \mathrm{V}\mathrm{d}\nu\delta\nu = \psi\mathrm{V}\mathrm{d}\rho\delta\rho, \quad \text{and} \quad \text{VII.} \ldots \mathrm{V}\mathrm{d}\rho\delta\rho = \nu^{-1}\mathrm{S}\nu\mathrm{d}\rho\delta\rho;$$

whence follows at once by V. the formula III., if we remember the general expression, deduced from the quadratic II.,

$$\text{VIII.} \ldots R_1^{-1}R_2^{-1} = -\nu^{-2}r_1r_2 = -\mathrm{S}\frac{1}{\nu}\psi\frac{1}{\nu}. \qquad 412,\ \text{XXXVII}.$$

(3.) If then we suppose that P, P_1, P_2 are ***any three near points*** on an ***arbitrary surface***, and that R, R_1, R_2 are three near and ***corresponding points*** on the ***unit sphere***, determined by the condition of parallelism of the ***radii*** OR, OR_1, OR_2 to the ***normals*** PN, $\mathrm{P}_1\mathrm{N}_1$, $\mathrm{P}_2\mathrm{N}_2$, the ***two small triangles*** thus formed will bear to each other the ***ultimate ratio***,

$$\text{IX.} \ldots \lim. \frac{\Delta\mathrm{RR}_1\mathrm{R}_2}{\Delta\mathrm{PP}_1\mathrm{P}_2} = R_1^{-1}R_2^{-1};$$

a result which justifies (although by an entirely new analysis) the adoption by Gauss of this *product** of curvatures of *sections*, as the *measure* of the curvature of the *surface*, with his signification of the phrase.

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* The reader is referred to the Additions to Liouville's Monge (pages 505, &c.), in which the beautiful Memoir by Gauss, entitled: ***Disquisitiones generales circa superficies curvas***, is with great good taste reprinted in the Latin, from the ***Commentationes recentiores*** of the Royal Society of Göttingen. He is also supposed to look back, if necessary, to the Section III. ii. 6 of these ***Elements*** (pages 435, &c.), and especially to the deduction in page 437 of ψ from ϕ, remembering that the latter function (and therefore also the former) is here self-conjugate.

† Compare page 487, and the Note to page 684.

(4.) As another form of this important product or measure, if we conceive that the vector ρ of the surface is expressed as a function (372) of two independent scalars, t and u, and if we write for abridgment,

$$\text{X.} \ldots \mathrm{D}_t\rho = \rho', \quad \mathrm{D}_u\rho = \rho_{,}, \quad \mathrm{D}_t^2\rho = \rho'', \quad \mathrm{D}_t\mathrm{D}_u\rho = \rho_{,}', \quad \mathrm{D}_u^2\rho = \rho_{,,}$$

which will allow us (comp. 372, V.) to assume for the normal vector ν the expression,

$$\text{XI.} \ldots \nu = \mathrm{V}\rho'\rho_{,}$$

it is easy to prove† that we have generally,

$$\text{XII.} \ldots R_1^{-1}R_2^{-1} = \mathrm{S}\frac{\rho''}{\nu}\,\mathrm{S}\frac{\rho_{,,}}{\nu} - \left(\mathrm{S}\frac{\rho_{,}'}{\nu}\right)^2;$$

which takes as a verification the well-known form,

$$\text{XIII.} \ldots R_1^{-1}R_2^{-1} = \frac{rt - s^2}{(1 + p^2 + q^2)^2},$$

when we write (comp. 410, (18.)),

$$\text{XIV.} \ldots \rho = ix + jy + kz, \quad \rho' = \mathrm{D}_x\rho = i + kp, \quad \rho_{,} = \mathrm{D}_y\rho = j + kq;$$
$$\text{XV.} \ldots \nu = \mathrm{V}\rho'\rho_{,} = k - ip - jq, \quad \rho'' = kr, \quad \rho_{,}' = ks, \quad \rho_{,,} = kt.$$

(5.) In general, the equation XII. may be thus transformed,

$$\text{XVI.} \ldots \nu^4R_1^{-1}R_2^{-1} = \mathrm{S}(\mathrm{V}\nu\rho''.\mathrm{V}\nu\rho_{,,}) - (\mathrm{V}\nu\rho_{,}')^2 + \nu^2(\mathrm{S}\rho''\rho_{,,} - \rho_{,}'^2);$$

also $$\text{XVII.} \ldots \mathrm{T}d\rho^2 = e\,dt^2 + 2f\,dt\,du + g\,du^2,$$

if $\text{XVIII.} \ldots e = -\rho'^2, \quad f = -\mathrm{S}\rho'\rho_{,}, \quad g = -\rho_{,}^2,$ whence $\text{XIX.} \ldots \nu^2 = f^2 - eg$

and if we still denote, as in X., derivations relatively to t and u by upper and lower accents, we may substitute in the quadruple of the equation XVI. the values,

$$\text{XX.} \ldots 2\mathrm{V}\nu\rho'' = (e_{,} - 2f')\rho' + e'\rho_{,}, \quad 2\mathrm{V}\nu\rho_{,}' = -g'\rho' + e_{,}\rho_{,}, \quad 2\mathrm{V}\nu\rho_{,,} = -g_{,}\rho' + (2f_{,} - g')\rho_{,}$$

and $$\text{XXI.} \ldots 2(\mathrm{S}\rho''\rho_{,,} - \rho_{,}'^2) = e_{,,} - 2f_{,}' + g'';$$

hence *the measure of curvature is an explicit function of the ten scalars*,

$$\text{XXII.} \ldots e, f, g; \quad e', f', g'; \quad e_{,}, f_{,}, g_{,}; \quad \text{and} \quad e_{,,} - 2f_{,}' + g'':$$

and therefore, as was otherwise proved by Gauss, *this measure depends only*‡ *on the*

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* If it be supposed to be in any manner known that a *limit* such as IX. *exists*, or that the *quotient* of the two *vector areas* in III. is a scalar *independent of the directions* of PP_1, PP_2, or of $d\rho$, $\delta\rho$, we have only to assume that these are the directions of the *lines of curvature*, in order to obtain *at once*, by 412, II., the *product* $R_1^{-1}R_2^{-1}$ as the *value* of this quotient or limit.

† The quadratic in R^{-1} may be formed by operating on 412, II. with $\mathrm{S}.\rho'$ and $\mathrm{S}.\rho_{,}$, and then eliminating $dt : du$.

‡ The proof by quaternions, above given, of this exclusive dependence, is perhaps as simple as the subject will allow, and is somewhat shorter than the corresponding proof in the *Lectures*: in page 605 of which is given however the equation,

expression (XVII.) *of the square of a linear element*, in terms of two independent scalars (t, u), and of their differentials ($\mathrm{d}t$, $\mathrm{d}u$).

(6.) Hence follow also these two other theorems* of Gauss:—

If a *surface* be considered as an *infinitely thin solid*, and supposed to be *flexible* but *inextensible*, then every *deformation* of it, as such, will leave *unaltered*, Ist, the *Measure of Curvature at any Point*, and IInd, the *Total Curvature of any Area*; that is, the area of the corresponding portion of the *unit sphere*, determined as in (3.) by radii parallel to normals.

(7.) Supposing now that t and u are *geodetic co-ordinates*, whereof the former represents the *length* of a geodetic AP from a *fixed point* A of the surface, and the latter represents the *angle* BAP which this variable geodetic makes at A with a *fixed geodetic* AB, it is easy to see that the general expression XVII. takes the shorter form,

$$\text{XXIII.} \ldots \mathrm{Td}\rho^2 = \mathrm{d}t^2 + n^2\mathrm{d}u^2, \quad \text{in which} \quad \text{XXIV.} \ldots n = \mathrm{T}\rho_{,} = \mathrm{T}\nu;$$

so that we have now the values,

$$\text{XXV.} \ldots e = 1, \quad f = 0, \quad g = n^2, \quad g' = 2nn', \quad g'' = 2nn'' + 2n'^2,$$

and the derivatives of e and f all vanish. And thus the general expression XII. for the *measure of curvature* reduces itself by (5.) to the very simple form,

$$\text{XXVI.} \ldots R_1^{-1}R_2^{-1} = -n^{-1}n'' = -n^{-1}\mathrm{D}_t^2 n;$$

in which n is generally a function of both t and u, although here twice derivated with respect to the former only.

(8.) The point P being denoted by the symbol (t, u), and any other point P′ of the surface by ($t + \Delta t$, $u + \Delta u$), we may consider the two connected points P_1, P_2, of which the corresponding symbols are ($t + \Delta t$, u) and (t, $u + \Delta u$); and then the *quadrilateral* $\mathrm{PP_1P'P_2}$, bounded by two portions $\mathrm{PP_1}$, $\mathrm{P_2P'}$ of *geodetic lines* from A, and (as we may suppose) by two arcs $\mathrm{PP_2}$, $\mathrm{P_1P'}$ of *geodetic circles* round the same fixed point, will have its *area* ultimately $= n\Delta t\Delta u$ (by XXIII.), and therefore (by XXVI., comp. (3.), (6.)) its *total curvature* ultimately $= -n''\Delta t\Delta u$, or $= -\Delta_t n'.\Delta u$, when Δt and Δu diminish together, by an approach of P′ to P.

(9.) Again, in the immediate neighbourhood of A, we have $n = t$, $n' = 1$; changing then $-\Delta_t n'$ to $-\mathrm{d}_t n'$, and integrating with respect to t from $t = 0$, we obtain $1 - n'$ as the coefficient of Δu in the result, and are thus conducted to the expression:

XXVII. . . *Total Curvature of Triangle* APP′ $= (1 - n')\,\Delta u$, ultimately,

if AP, AP′ be *any two geodetic lines*, making with each other a *small angle* $= \Delta u$, and if PP′ be *any small arc* (geodetic or not) on the same surface.

$$\begin{aligned} 4(eg - f^2)^2 R_1^{-1}R_2^{-1} = e\,&(g'^2 - 2g_{,}f' + g_{,}e_{,}) \\ &+ f(e'g_{,} - e_{,}g' - 2e_{,}f_{,} - 2g'f' + 4f'f_{,}) \\ &+ g(e_{,}^2 - 2e'f_{,} + e'g') - 2(eg - f^2)(e_{,,} - 2f_{,}' + g''), \end{aligned}$$

which may now be deduced *at sight* from XVI., by the substitutions XIX. XX. XXI., and differs only in notation from the equation of Gauss (Liouville's Monge, page 523, or Salmon, page 309).

* See page 524 of Liouville's Monge.

(10.) Conceive then that PQ is a *finite arc* of *any curve* upon the surface, for which therefore t, and consequently n', may be conceived to be a function of u; we shall have this other expression of the same kind,

XXVIII. . . *Total Curvature of Area* APQ $= \int(1-n')\mathrm{d}u = \Delta u - \int n'\mathrm{d}u$;

the *area* here considered being bounded by the *two geodetic lines* AP, AQ, which make with each other the finite angle Δu, and by the *arc* PQ of the *arbitrary curve*.

(11.) If this curve be *itself* a *geodetic*, and if we treat its co-ordinates t, u, and its vector ρ, as functions of its arc, s, then the second differential of ρ, namely,

XXIX. . . $\mathrm{d}^2\rho = \rho'\mathrm{d}^2t + \rho_{,}\mathrm{d}^2u + \rho''\mathrm{d}t^2 + 2\rho_{,}'\mathrm{d}t\mathrm{d}u + \rho_{,,}\mathrm{d}u^2$,

must be normal to the surface at P, and consequently perpendicular to ρ' and $\rho_{,}$. Operating[*] therefore with $\mathrm{S}.\rho'$, and attending to the relations XVIII. and XXV., which give

XXX. . . $\rho'^2 = -1, \quad \mathrm{S}\rho'\rho_{,} = \mathrm{S}\rho'\rho'' = \mathrm{S}\rho'\rho'_{,} = 0, \quad \mathrm{S}\rho'\rho_{,,} = -\mathrm{S}\rho_{,}\rho_{,}' = nn'$,

we obtain the differential equation,

XXXI. . . $\mathrm{d}^2t = nn'\mathrm{d}u^2$, or XXXII. . . $\mathrm{d}v = -n'\mathrm{d}u$,

if we observe that we may write,

XXXIII. . . $\mathrm{d}t = \cos v\mathrm{d}s, \quad n\mathrm{d}u = \sin v\mathrm{d}s$, because XXXIV. . . $\mathrm{d}t^2 + n^2\mathrm{d}u^2 = \mathrm{d}s^2$;

v being here the variable angle, which the geodetic PQ makes at P with AP prolonged.

(12.) Substituting then for $-n'\mathrm{d}u$, in XXVIII., its value $\mathrm{d}v$ given by XXXII., the integration becomes possible, and the result is $\Delta u + \Delta v$; where Δu is still the angle at A, and $\pi + \Delta v = (\pi - v) + (v + \Delta v)$ is the sum of the angles at P and Q, in the *geodetic triangle* APQ.

(13.) Writing then B and C instead of P and Q, we thus arrive at another most remarkable Theorem[†] of Gauss, which may be expressed by the formula:

XXXV. . . *Total Curvature of a Geodetic Triangle* ABC $= \mathrm{A} + \mathrm{B} + \mathrm{C} - \pi$,

= what may be called the *Spheroidal Excess*; A, B, C, in the second member, being used to denote the *three angles* of the triangle: and the *total surface* of the *unit sphere* $(= 4\pi)$ being represented by 720°, when the *part* corresponding to the *geodetic triangle* is thus represented by the *angular excess*, $\mathrm{A} + \mathrm{B} + \mathrm{C} - 180°$.

(14.) And it is easy to perceive, on the one hand, how this theorem admits of being *extended*, as it was by Gauss, to *all geodetic polygons*: and on the other hand, how it may require to be *modified*, as it was by the same eminent geometer, so as to give what would on the same plan be called a *spheroidal defect*, when the *measure of curvature* is *negative*, as it is for surfaces (or parts of surfaces) of which the principal sections have their curvatures *oppositely directed*.

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[*] To operate with $\mathrm{S}.\rho_{,}$ would give a result not quite so simple, but reducible to the form XXXI., with the help of $\mathrm{d}^2s = 0$.

[†] The enunciation of this theorem, respecting which its illustrious discoverer justly says, "Hoc theorema, quod, ni fallimur, ad elegantissima in theoria superficierum curvarum referendum esse videtur," . . . is given in page 533 of the Additions to

414. The only sections of a surface, of which the curvatures have been above determined, are the *two principal normal sections* at any proposed point; but the general expressions of III. iii. 6 may be applied to find the curvature of *any plane section*, normal or oblique, and therefore also of *any curve* on a *given surface*, when only its *osculating plane* is known. Denoting (as in 389, &c.) by ρ and κ the vectors of the given point P, and of the *centre* K of the *osculating circle* at that point, and by s the *arc* of the curve, we have generally (by 389, XII. and VI.),

$$\text{I. . . } \textit{Vector of Curvature of Curve} = \mathrm{KP}^{-1} = (\rho - \kappa)^{-1} = \mathrm{D}_s^2\rho = \frac{1}{\mathrm{d}\rho}\mathrm{V}\frac{\mathrm{d}^2\rho}{\mathrm{d}\rho};$$

the independent variable in the last expression being arbitrary. And if we denote by σ and ξ the vectors of the points S and X, in which the *axis* of the osculating circle meets respectively the *normal* and the *tangent plane* to the given surface, we shall have also, by the right-angled triangles, the general decomposition, $\mathrm{KP}^{-1} = \mathrm{SP}^{-1} + \mathrm{XP}^{-1}$ (as vectors), or

$$\text{II. . . } \mathrm{D}_s^2\rho = (\rho - \kappa)^{-1} = (\rho - \sigma)^{-1} + (\rho - \xi)^{-1};$$

where the two components admit of being transformed as follows:

$$\text{III. . . } \textit{Normal Component of Vector of Curvature of Curve (or Section)} = (\rho - \sigma)^{-1} = \nu^{-1}\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = (\rho - \sigma_1)^{-1}\cos^2 v + (\rho - \sigma_2)^{-1}\sin^2 v$$

$$= \textit{Vector of Normal Curvature of Surface} \text{ for the direction of the given tangent;}$$

σ_1, σ_2 being the vectors of the *centres* S_1, S_2 (comp. 412) of the *two principal curvatures*, and v being the *angle* at which the curve (or its tangent $\mathrm{d}\rho$) crosses the *first line of curvature* (or its tangent τ_1), while σ is the vector of the centre S of the *sphere* which is said to *osculate* to the *surface*, in the *given direction* (of $\mathrm{d}\rho$); and

$$\text{IV. . . } \textit{Tangential Component of Vector of Curvature} = (\rho - \xi)^{-1} = \nu^{-1}\mathrm{d}\rho^{-1}\mathrm{S}\nu\mathrm{d}\rho^{-1}\mathrm{d}^2\rho$$

$$= \textit{Vector of Geodetic Curvature of Curve (or Section)};$$

this latter *vector* being here so called, because in fact its *tensor* re-

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Liouville's Monge. A proof by quaternions was published in the *Lectures* (pages 606–609, see also the few preceding pages), but the writer conceives that the one given above will be found to be not only shorter, but more clear.

presents what is known by the name of the *geodetic*[*] *curvature* of a *curve upon a surface:* the independent variable being still arbitrary.

(1.) As regards the decomposition II., if α, β be any two rectangular vectors OA, OB, and if $\gamma = \text{OC} =$ the perpendicular from O on AB, then (comp. 316, L., and 408, XLI.),

$$\text{V.} \ldots \gamma^{-1} = \frac{\beta}{\mathrm{V}\alpha\beta} + \frac{\alpha}{\mathrm{V}\beta\alpha} = \alpha^{-1} + \beta^{-1}.$$

(2.) To prove the first transformation III., we have, by I. and II., observing that $\mathrm{dS}\nu\mathrm{d}\rho = 0$,

$$\text{VI.} \ldots \frac{\nu}{\rho - \sigma} = \mathrm{S}\frac{\nu}{\rho - \kappa} = \mathrm{S}.\frac{\nu}{\mathrm{d}\rho}\mathrm{V}\frac{\mathrm{d}^2\rho}{\mathrm{d}\rho} = \frac{-\mathrm{S}\nu\mathrm{d}^2\rho}{\mathrm{d}\rho^2} = \frac{\mathrm{Sd}\nu\mathrm{d}\rho}{\mathrm{d}\rho^2} = \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho}.$$

(3.) Hence, by 412, (7.), if we denote the vector III. of normal curvature by $R^{-1}\mathrm{U}\nu$, we have the general expressions (comp. 412, I. XXI.),

$$\text{VII.} \ldots \sigma = \rho + R\mathrm{U}\nu, \quad R = D^2.\mathrm{T}\nu, \quad \text{with} \quad \text{VIII.} \ldots \mathrm{T}\nu = P^{-1},$$

for the case of a *central quadric; D* being *generally* the *semidiameter* of the *index surface* (410, (9.), &c.), or for a quadric the semidiameter of that surface *itself*, which has the direction of the *tangent* (or of $\mathrm{d}\rho$): and P being, for the latter surface, the *perpendicular* from the centre on the *tangent plane*, as in some earlier formulæ.

(4.) To deduce the second transformation III., which contains a theorem of Euler, let τ, τ_1, τ_2 denote unit tangents to the section and the two lines of curvature, so that

$$\text{IX.} \ldots \tau = \tau_1 \cos v + \tau_2 \sin v, \quad \text{and} \quad \tau^2 = \tau_1{}^2 = \tau_2{}^2 = -1;$$

we may then write generally (comp. 412, IV.),

$$\text{X.} \ldots R^{-1}\mathrm{T}\nu = \frac{\nu}{\sigma - \rho} = -\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = -\mathrm{S}\tau^{-1}\phi\tau = \mathrm{S}\tau\phi\tau,$$

and shall have the values (comp. 410, XI.),

$$\text{XI.} \ldots \mathrm{S}\tau_1\phi\tau_1 = R_1{}^{-1}\mathrm{T}\nu, \quad \mathrm{S}\tau_2\phi\tau_2 = R_2{}^{-1}\mathrm{T}\nu, \quad \mathrm{S}\tau_1\phi\tau_2 = \mathrm{S}\tau_2\phi\tau_1 = 0;$$

whence

$$\text{XII.} \ldots R^{-1} = R_1{}^{-1}\cos^2 v + R_2{}^{-1}\sin^2 v,$$

and the required transformation is accomplished.

(5.) The theorem of Meusnier may be considered to be a result of the *elimination* (2.) of $\mathrm{d}^2\rho$ from the expressions for the *normal component* III. of what we may call the *Vector* $\mathrm{D}_s{}^2\rho$ *of Oblique Curvature;* and it may be expressed by the equation,

$$\text{XIII.} \ldots \mathrm{S}\frac{\rho - \sigma}{\rho - \kappa} = 1, \quad \text{or} \quad \text{XIII}'. \ldots \mathrm{S}\frac{\sigma - \kappa}{\rho - \kappa} = 0, \text{ which gives } \text{XIII}''. \ldots \text{PKS} = \frac{\pi}{2},$$

if it be now understood that the point S, of which σ is the vector, is the *centre* of the

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[*] The name, "*courbure géodésique*," was introduced by M. Liouville, and has been adopted by several other mathematical writers. Compare pages 568, 575, &c. of his *Additions* to Monge.

circle which *osculates* to the *normal section*; or of the *sphere* which osculates in the *same direction* to the *surface*, as will be more clearly seen by what follows.

(6.) In general, if $\rho + \Delta\rho$ be the vector of *any second point* P′ of the given surface, the equation

$$\text{XIV.}\ldots \mathrm{S}\frac{\nu}{\omega - \rho} = \mathrm{S}\frac{\nu}{\Delta\rho}, \text{ with } \omega \text{ for a variable vector,}$$

represents rigorously the sphere which *touches* the surface at the given point P, and passes *through* the second point P′; conceiving then that the latter point *approaches* to the former, and observing that the development* by Taylor's Series of the equation $f\rho = \text{const.}$ gives (if $\mathrm{d}f\rho = 2\mathrm{S}\nu \mathrm{d}\rho$, and $\mathrm{d}\nu = \phi \mathrm{d}\rho$),

$$\text{XV.}\ldots 0 = \Delta\rho^{-2}\Delta f\rho = 2\mathrm{S}\frac{\nu}{\Delta\rho} + \mathrm{S}\frac{\phi\Delta\rho}{\Delta\rho} + \text{terms which vanish } \textit{generally} \text{ with } \Delta\rho,$$

even if they be not *always null*, we are conducted in a new way, by the known conception of the *Osculating Sphere* for a given *direction* to a *surface*, to the same *centre* S, and *radius* R, as before: the equation of *this* sphere being,

$$\text{XVI.}\ldots \mathrm{S}\frac{2\nu}{\omega - \rho} = \left(\lim.\,\mathrm{S}\frac{2\nu}{\Delta\rho} = -\lim.\,\mathrm{S}\frac{\phi\Delta\rho}{\Delta\rho} = \right) - \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho}.$$

(7.) Conversely, if we *assume* a *radius* R, such that R^{-1} is algebraically *intermediate* between R_1^{-1} and R_2^{-1}, the *tangent sphere*,

$$\text{XVII.}\ldots \mathrm{S}\frac{2\nu}{\omega - \rho} = \frac{\mathrm{T}\nu}{R}, \quad \text{or} \quad \text{XVII}'.\ldots \mathrm{S}\frac{2\mathrm{U}\nu}{\omega - \rho} = R^{-1},$$

will *cut the surface* in *two directions of osculation*, assigned by the formula XII., but if R^{-1} be *outside* those *limits*, there will be *only contact*, and *not* any (real) *intersection*, at least in the vicinity of P.

(8.) If P′ be again, as in (6.), *any second point* of the surface, and if we denote for a moment by (Π) and (Σ) the normal *plane* PNP′ and the normal *section* corresponding, we may suppose that N is the point in which the normals *to the plane curve* (Σ) at P and P′ intersect; and if we then erect a perpendicular at N to the plane (Π), it will be crossed by *every* perpendicular at P′ to the tangent P′T′ to the section, and therefore in particular by the *normal* at P′ to the *surface*, in a point which we may call N′: so that the line P′N is the *projection*, on the plane PP′N, of this *second normal* P′N′ to the surface. Conceiving then the *plane* (Π) to be *fixed*, but the *point* P′ to *approach* indefinitely to P, we see that the *centre* S of *curvature* of the *normal section* (Σ), which is also by (6.) the centre of the *osculating sphere* to the surface for the *same direction*, is the *limiting position* of the point N, in which the *given normal* at P is *intersected by the projection*[*] *of the near normal* P′N′, *on the given normal plane.*

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* Compare Art. 374, and the Second Note to page 508. The occasional use, there mentioned, of the differential symbol $\mathrm{d}\rho$ as signifying a finite and *chordal vector*, in the development of $f(\rho + \mathrm{d}\rho)$, has appeared obscure, in the *Lectures*, to some friends of the writer; and he has therefore aimed, for the sake of clearness, in at least the *text* of these *Elements*, and especially in the geometrical applications, to confine that symbol to its *first* signification (100, 369, 373, &c.), as denoting a *tangential vector* (finite or infinitely small, and to a curve or surface): ρ itself being generally regarded as a *vector function*, and *not* as an independent variable (comp. 362, (3.)).

(9.) The two components III. and IV. are included in the *binomial expression,*

$$\text{XVIII.}\ldots \textit{Vector of Oblique Curvature} \text{ (or of } \textit{Curvature of Oblique Section}\text{)}$$
$$= (\rho - \kappa)^{-1} = \nu^{-1}\mathrm{S}d\nu d\rho^{-1} + \nu^{-1}d\rho^{-1}\mathrm{S}\nu d\rho^{-1}d^2\rho,$$

which is obtained by substituting in I. the general equivalent 409, XXI. for $d^2\rho$, and in which (as before) the independent variable is arbitrary; and the *tangential component* IV. may be otherwise found by observing that, by I. and II.,

$$\text{XIX.}\ldots \frac{\nu d\rho}{\rho - \xi} = \mathrm{S}\frac{\nu d\rho}{\rho - \kappa} = \mathrm{S}\frac{\nu d^2\rho}{d\rho} = -\mathrm{S}\nu d\rho^{-1}d^2\rho,$$

and that $-(\nu d\rho)^{-1} = \nu^{-1}d\rho^{-1}$, because $\mathrm{S}\nu d\rho = 0$.

(10.) Another way of deducing the same component IV., is to resolve the following system of three scalar equations, which by the geometrical definition of the point X the vector ξ must satisfy:

$$\text{XX.}\ldots \mathrm{S}(\xi - \rho)\nu = 0; \quad \mathrm{S}(\xi - \rho)d\rho = 0; \quad \mathrm{S}(\xi - \rho)d^2\rho = d\rho^2;$$

and which give,

$$\text{XXI.}\ldots \xi - \rho = \frac{\nu d\rho^3}{\mathrm{S}\nu d\rho d^2\rho} = \frac{\nu d\rho}{\mathrm{S}\nu d\rho^{-1}d^2\rho},$$

or $(\rho - \xi)^{-1} = \&c.$, as before. We have also the transformations,

$$\text{XXII.}\ldots \textit{Vector of Geodetic Curvature} = (\rho - \xi)^{-1}$$
$$= (\nu d\rho)^{-1}\,\mathrm{S}(\nu \mathrm{U}d\rho \,.\, d\mathrm{U}d\rho) = -\nu d\rho\,\mathrm{S}\frac{d\rho^{-2}d^2\rho}{\nu d\rho} = \&c.$$

(11.) The definition of the point X shows also easily, that *if a developable surface* (D) *be circumscribed to a given surface* (S), *along a given curve* (C), *and if, in the unfolding of the former surface, the point* X *be carried with the tangent plane, originally drawn to the latter surface at* P, *it will become the centre of curvature, at the new point* (P), *to the new or plane curve* (C′) obtained by this development: so that the *radius* (PX) *of geodetic curvature* is equal, as indeed it is known[†] to be, to the *radius of plane curvature* of the *developed curve.*

(12.) This *plane* curve (C′) is therefore a *circle*[‡] (or part of one) if the condition,

$$\text{XXIII.}\ldots \overline{\mathrm{PX}} = \mathrm{T}(\xi - \rho) = \text{const.},$$

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[*] The reader may compare the calculations and constructions, in pages 600, 601 of the *Lectures.* In the language of infinitesimals, an *infinitely near normal* P′N′ *intersects the axis of the osculating circle,* to the given normal section.

[†] Compare page 576 of the *Additions* to Liouville's Monge.

[‡] The *curves* on any given surface, which thus *become circles* by development, have also the *isoperimetrical property* expressed in quaternions (comp. the first Note to page 530) by the formula,

$$\text{XXVI.}\ldots \delta\!\int \mathrm{S}(\mathrm{U}\nu \,.\, d\rho\delta\rho) + c\delta\!\int \mathrm{T}d\rho = 0,$$

which conducts to the differential equation,

$$\text{XXVII.}\ldots c^{-1}d\rho = \mathrm{V}.\mathrm{U}\nu\, d\mathrm{U}d\rho \text{ (comp. 380, IV.)},$$

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be satisfied; but it degenerates into a *right line*, if this radius of *geodetic* curvature be *infinite*, that is, if

$$\text{XXIV.} \ldots \mathrm{T}(\rho-\xi)^{-1}=0, \quad \text{or} \quad \text{XXV.} \ldots \mathrm{S}\nu d\rho d^2\rho=0,$$

or finally (by 380, II., comp. 409, XXV.), if the original curve (c) be a *geodetic line* on the *given surface* (s), and therefore also on the *developable* (d): which agrees with the fundamental property (382, 383) of geodetics on a developable surface.

(13.) Accordingly it may be here observed that the general formula IV., combined with the notations and calculations of 382, conducts to the expression $(z+v')\mathrm{T}\rho'^{-1}$, or $\frac{zdx+dv}{ds}$, for the geodetic curvature of *any curve* on a developable surface, whereof the element ds crosses a generating line at the variable angle v, while zdx is the angle between two such consecutive lines: a result easily confirmed by geometrical considerations, and agreeing with the differential equation $z+v'=0$ (382, IX.) of *geodetics* on a developable.

415. We shall conclude the present Section with a few supplementary remarks, including a new and simplified proof of an important *theorem* (354), which we have had frequent occasion to employ for purposes of *geometry*, and which presents itself often in *physical* applications of quaternions also: namely, that *if the linear and vector function* ϕ *be self-conjugate*, then the *Vector Quadratic*,

$$\text{I.} \ldots \mathrm{V}\rho\phi\rho=0, \qquad 354,\ \text{I.}$$

represents generally a *System of Three Real and Rectangular Directions;* and that these (comp. 405, (1.), (2.), &c.) are the directions of the *Axes of the Central Surfaces of the Second Order*, which are represented by the scalar equation,

$$\text{II.} \ldots \mathrm{S}\rho\phi\rho=\text{const.};$$

or more generally,

$$\text{III.} \ldots \mathrm{S}\rho\phi\rho=C\rho^2+C', \text{ where } C \text{ and } C' \text{ are any two scalar constants.}$$

(1.) It is an easy consequence of the theory (350) of the *symbolic and cubic equation* in ϕ, that if c be a root of the derived *algebraical cubic* $M=0$ (354), and if we write $\Phi=\phi+c$ (as in that Article), the *new* linear and vector function $\Phi\rho$ must be reducible to the *binomial form* (351),

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and in which the scalar constant c can be shown to have the value,

$$\text{XXVIII.} \ldots c=(\xi-\rho)\,\mathrm{U}.\nu d\rho=\pm\mathrm{T}(\xi-\rho)=\textit{Radius of Geodetic Curvature},$$

= radius of developed circle; and each such curve includes, by XXVI., on the given surface, a *maximum area* with a *given perimeter:* on which account, and in allusion to a well-known classical story, the writer ventured to propose, in page 582 of the *Lectures*, the *name* "Didonia" for a curve of this kind, while acknowledging that the *curves themselves* had been discovered and discussed by M. Delaunay.

IV. . . $\Phi\rho = \phi\rho + c\rho = \beta S\alpha\rho + \beta' S\alpha'\rho$, with V. . . $V\beta\alpha + V\beta'\alpha' = 0$,

as the *condition* (353, XXXVI.) of *self-conjugation*. With this condition we may then write,

VI. . . $\beta = A\alpha + B\alpha'$, $\beta' = A'\alpha' + B\alpha$;

and it is easy to see that no essential generality is lost, by supposing that α and α' are two rectangular vector units, which may be turned about in their own plane, if β and β' be suitably modified: so that we may assume,

VII. . . $\alpha^2 = \alpha'^2 = -1$, $S\alpha\alpha' = 0$; whence VIII. . . $\Phi\alpha = -\beta$, $\Phi\alpha' = -\beta'$,

and IX. . . $V\beta'\alpha' = B\alpha\alpha' = -V\beta\alpha$, $V\beta\alpha' = A\alpha\alpha'$, $V\beta'\alpha = -A'\alpha\alpha'$.

(2.) The equation I., under the form,

X. . . $V\rho\Phi\rho = 0$, is satisfied by XI. . . $\Phi\rho = 0$, or XII. . . $V\alpha\alpha'\rho = 0$;

and it cannot be satisfied otherwise, unless we suppose,

XIII. . . $\rho = x\alpha + x'\alpha'$, and XIV. . . $V(x\beta + x'\beta')(x\alpha + x'\alpha') = 0$;

that is, by IX.,

XV. . . $B(x'^2 - x^2) + (A - A')xx' = 0$:

while conversely the expression XIII. will satisfy I., under this condition XV. But this quadratic in $x' : x$, of which the coefficients B and $A - A'$ do not *generally* vanish, has necessarily *two real roots*, with a product $= -1$; hence there *always exists*, as asserted, a system of *three real and rectangular directions*, such as the following,

XVI. . . $x\alpha + x'\alpha'$, $x'\alpha - x\alpha'$, and $\alpha\alpha'$ (or $V\alpha\alpha'$),

which satisfy the equation I.; and this system is *generally definite:* which proves the *first part* of the Theorem.

(3.) The lines α, α' may be made by (1.) to *turn* in their own plane, till they coincide with the two first directions XVI.; which will give,

XVII. . . $B = 0$, $\beta = A\alpha$, $\beta' = A'\alpha'$,

and therefore,

XVIII. . . $\phi\rho = -c\rho + A\alpha S\alpha\rho + A'\alpha' S\alpha'\rho$
$= (c + A)\alpha S\alpha\rho + (c + A')\alpha' S\alpha'\rho + c\alpha\alpha' S\alpha\alpha'\rho$;

and thus the scalar equation II. will take the form,

XIX. . . $S\rho\phi\rho = (c + A)(S\alpha\rho)^2 + (c + A')(S\alpha'\rho)^2 + c(S\alpha\alpha'\rho)^2 =$ const.,

which represents generally a *central surface* of the *second order*, with its *three axes* in the *three directions* α, α', $\alpha\alpha'$ of ρ; and does not cease to represent such a surface, and with such axes, when for $S\rho\phi\rho$ we substitute, as in III., this new expression:

XX. . . $S\rho\phi\rho - C\rho^2 = S\rho\phi\rho + C((S\alpha\rho)^2 + (S\alpha'\rho)^2 + (S\alpha\alpha'\rho)^2) = C' =$ const.;

the *second surface* being in fact *concyclic* (or having the same cyclic planes) with the first, and the *new term*, $-C\rho$, in $\phi\rho$, disappearing under the sign $V.\rho$: so that the *second part* of the Theorem is proved anew.

(4.) It would be useless to dwell here on the *cases*, in which the *surfaces* XIX., XX. come to be of *revolution*, or even to be *spheres*, and when consequently the *directions* of their *axes*, or of ρ in I., become partially or even wholly *indeterminate*. But as an example of the *reduction* of an equation in quaternions to the *form* I., without its *at first* presenting itself under that form, we may take the very simple equation,

$$\text{XXI.} \ldots \rho\iota\rho\kappa = \iota\rho\kappa\rho, \text{ with } \kappa \textit{ not } \parallel \iota,$$

which may be reduced (comp. 354, (12.)) to

$$\text{XXII.} \ldots \mathrm{V}.\rho\mathrm{V}\iota\rho\kappa = 0\,;$$

and which is accordingly satisfied (comp. 373, XXIX.) by the *three rectangular directions*,

$$\text{XXIII.} \ldots \mathrm{U}\iota - \mathrm{U}\kappa, \quad \mathrm{V}\iota\kappa, \quad \mathrm{U}\iota + \mathrm{U}\kappa,$$

of the *axes* (*abc*) of the *ellipsoid*,

$$\text{XXIV.} \ldots \mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad 282, \text{ XIX.}$$

which is *one* of the surfaces of the *concyclic system* (comp. III.),

$$\text{XXV.} \ldots \mathrm{S}\iota\rho\kappa\rho = C\rho^2 + C',$$

as appears from the transformations 336, XI., &c.

(5.) In applying the theorem thus recently proved anew, we have on several occasions used the expression,

$$\text{XXVI.} \ldots \mathrm{d}\nu = \phi \mathrm{d}\rho, \qquad 410, \text{ IV.}$$

in which ν is a vector normal to a surface whereof ρ is the variable vector, and the function ϕ is treated as *self-conjugate* (363).

(6.) It is, however, important to remark that, in order to justify the assertion of this last property, the following *expression of integral form*,

$$\text{XXVII.} \ldots \int \mathrm{S}\nu \mathrm{d}\rho,$$

must admit of being equated to *some scalar function* of ρ, such as $\frac{1}{2}f\rho$ + const., *without* its being assumed that ρ *itself* is a *function*, of any determinate form, of a scalar variable, t. The *self-conjugation* of the linear and vector function ϕ in XXVI., is the *condition of the existence of the integral* XXVII., considered as representing *such* a *scalar function* (comp. again 363).

(7.) There are indeed several investigations, in which it is sufficient to regard ν as denoting *some normal vector*, of which only the *direction* is important, and which may therefore be multiplied by an *arbitrary scalar coefficient*, constant or variable, without any change in the results (comp. the calculations respecting *geodetic lines*, in the Section III. iii. 5, and many others which have already occurred).

(8.) And there have been other general investigations, such as those regarding the *lines of curvature* on an arbitrary surface, in which $\mathrm{d}\nu$ was treated as a self-conjugate function of $\mathrm{d}\rho$, while yet (comp. 410, (17.)) the fundamental differential equation $\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho = 0$ was not affected by any such multiplication of ν by n.

(9.) But there are questions in which a factor of this sort may be introduced, with advantage for *some* purposes, while yet it is inconsistent with the *self-conjugation* above mentioned, unless the multiplier n be such as to render the *new expression* $\mathrm{S}n\nu\mathrm{d}\rho$ (comp. XXVII.) an *exact differential* of some scalar function of ρ.

(10.) For example, in the theory of *Reciprocal Surfaces* (comp. 412, (21.)), it is convenient to employ the system of the *three* connected equations,

$$\text{XXVIII.} \ldots \mathrm{S}\nu\rho = 1, \quad \mathrm{S}\nu\mathrm{d}\rho = 0, \quad \mathrm{S}\rho\mathrm{d}\nu = 0\,; \qquad 373, \text{ L. LI.}$$

but when the *length* of ν is determined so as to satisfy the *first* of these equations, ν^{-1} being then the *vector perpendicular* from the origin *on the tangent plane* to the *given* but *arbitrary surface* of which ρ is the vector, while ρ^{-1} is the *corresponding perpendicular* for the *reciprocal surface* with ν for vector, the *differential* $d\nu$ *loses generally its self-conjugate character*, as a linear and vector function of $d\rho$: although it *retains* that character if the scalar function $f\rho$ be *homogeneous*, in the equation $f\rho$ = const. of the original surface, as it is for the case of a *central quadric*,* for which $\nu = \phi\rho$, $d\nu = \phi d\rho$, &c., as in former Articles.

(11.) In fact, the introduction of the first equation XXVIII. is equivalent to the multiplication of ν by the factor $n = (S\nu\rho)^{-1}$; and if we write (comp. 410, (16.)),

$$\text{XXIX.} \ldots d f\rho = 2S\nu d\rho, \quad d\nu = \phi d\rho, \quad dn = S\sigma d\rho,$$

we shall have this new pair of conjugate linear and vector functions,

$$\text{XXX.} \ldots d.n\nu = \Phi d\rho = n\phi d\rho + \nu S\sigma d\rho, \qquad \text{XXXI.} \ldots \Phi' d\rho = n\phi d\rho + \sigma S\nu d\rho;$$

and these will not be *equal generally*, because we shall not in general have $\sigma \parallel \nu$. But this last parallelism exists in the *case of homogeneity* (10.), because we have then the relations,

$$\text{XXXII.} \ldots 2S\nu\rho = rf\rho, \quad d.n^{-1} = dS\nu\rho = rS\nu d\rho,$$

if r be the number which represents the *dimension* of $f\rho$ (supposed to be *whole*).

(12.) On the other hand it may happen, that the *differential equation* $S\nu d\rho = 0$ represents a *surface*, or rather a set of surfaces, without the *expression* $S\nu d\rho$ being an *exact differential*, as in (6.); and *then* there necessarily exists a scalar *factor*, or multiplier, n, which *renders* it such a differential.

(13.) For example the differential equation,

$$\text{XXXIII.} \ldots S\gamma\rho d\rho = S\nu d\rho = 0, \quad \text{with} \quad \text{XXXIV.} \ldots \nu = V\gamma\rho, \quad d\nu = V\gamma d\rho = \phi d\rho,$$

represents an *arbitrary plane* (or a *set of planes*), drawn *through a given line* γ; but the *expression* $S\gamma\rho d\rho$ *itself is not* an exact differential, and the *integral* XXVII. represents *no scalar function of* ρ, with the present form of ν, of which the differential $d\nu$ is accordingly a linear function $\phi d\rho$, which is *not conjugate to itself, but to its opposite* (comp. 349, (4.)), so that we have *here* $\phi' d\rho = -\phi d\rho$.

(14.) But if we multiply ν by the *factor*,

$$\text{XXXV.} \ldots n = \nu^{-2} = (V\gamma\rho)^{-2}, \text{ which gives } \text{XXXVI.} \ldots dn = S\sigma d\rho, \quad \sigma = 2n^2\gamma V\gamma\rho,$$

and therefore $S\gamma\sigma = 0$, $S\rho\sigma = -2n$, then the *new normal vector* $n\nu$, or ν^{-1}, is found to have the *self-conjugate differential*,

$$\text{XXXVII.} \ldots d.n\nu = d.\nu^{-1} = -\nu^{-1}V\gamma d\rho.\nu^{-1} = \Phi d\rho = \Phi' d\rho;$$

and accordingly the *new expression*,

$$\text{XXXVIII.} \ldots Sn\nu d\rho = S\nu^{-1}d\rho = S\frac{d\rho}{V\gamma\rho}, \text{ with } \gamma \text{ constant},$$

is easily seen to be an *exact differential*, namely (if $T\gamma = 1$), that of the *angle* which the *plane* of γ and ρ makes with a *fixed plane* through γ: so that, when ν is thus changed to $n\nu$, the *integral* in XXVII. acquires a *geometrical signification*, which is often useful in *physical applications*, since it then represents the *change* of this *angle*, in passing from one position of ρ to another; or the angle through which the *variable plane* of $\gamma\rho$ has *revolved*.

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* It was for this reason that the symbol $T\nu$ was not interpreted *generally* as denoting the reciprocal, P^{-1}, of the length of the perpendicular from the origin on the tangent plane, in the formulæ of 410, 412, 414: although, in several of those formulæ, as in an equation of 409, (3.), that symbol *was* so interpreted, for the *case* of a central surface of the second order.

(15.) In fact, the general formula 335, XV. for the *differential of the angle of a quaternion* gives, if we write

$$\text{XXXIX.} \ldots q = \frac{\mathrm{V}\gamma\rho}{\mathrm{V}\gamma\rho_0}, \quad \gamma = \text{const.}, \quad \rho_0 = \text{const.}, \quad \mathrm{T}\gamma = 1,$$

the two connected expressions:

$$\text{XL.} \ldots \mathrm{d}\angle q = \pm \mathrm{S}\frac{\mathrm{d}\rho}{\mathrm{V}\gamma\rho}; \qquad \text{XLI.} \ldots \int \mathrm{S}\frac{\mathrm{d}\rho}{\mathrm{V}\gamma\rho} = \pm \Delta \angle (\mathrm{V}\gamma\rho : \mathrm{V}\gamma\rho_0);$$

which contain the above-stated result, and can easily be otherwise established.

(16.) *In general*, if the linear and vector function $\mathrm{d}\nu = \varphi\mathrm{d}\rho$ be *not* self-conjugate, and if the function $\mathrm{d}.n\nu = \phi\mathrm{d}\rho$ be formed from it as in (11.), it results from that sub-article, and from 349, (4.), that we may write,

$$\text{XLII.} \ldots (\varphi - \varphi')\mathrm{d}\rho = 2\mathrm{V}\gamma\mathrm{d}\rho, \quad (\phi - \phi')\mathrm{d}\rho = 2\mathrm{V}\gamma_{,}\mathrm{d}\rho,$$

with the relation,

$$\text{XLIII.} \ldots 2\gamma_{,} = 2n\gamma + \mathrm{V}\nu\sigma;$$

where γ, $\gamma_{,}$ are *independent of* $\mathrm{d}\rho$, although they *may* depend on ρ *itself*. If then the *new* linear function $\phi\mathrm{d}\rho$ is to be *self-conjugate*, so that $\gamma_{,} = 0$, we must have

$$\text{XLIV.} \ldots 2n\gamma + \mathrm{V}\nu\sigma = 0, \quad \text{and therefore} \quad \text{XLV.} \ldots \mathrm{S}\gamma\nu = 0;$$

which latter very simple equation, not involving either n or σ, is thus a form, in quaternions, of the *Condition of Integrability*[*] *of the differential equation* $\mathrm{S}\nu\mathrm{d}\rho = 0$, if the vector γ be deduced from ν as above.

(17.) The *Bifocal Transformation* of $\mathrm{S}\rho\phi\rho$, in 360, (2.), has been sufficiently considered in the present Section (III. iii. 7); but it may be useful to remark here, that the *Three Mixed Transformations* of the same scalar function $f\rho$, in the same series of sub-articles, include virtually the whole known theory of the *Modular and Umbilicar Generations* of *Surfaces of the Second Order*.

(18.) Thus, in the formulæ of 360, (4.), if we make $e = 1$, ϵ is the vector of an *Umbilicar Focus* of the surface $f\rho = 1$, and ζ is the vector of a point on the *Umbilicar Directrix* corresponding; whence the *umbilicar focal conic* and *dirigent cylinder* (real or imaginary) can be deduced, as the *loci* of this *point* and *line*.

(19.) Again, by making e_1 and e_3 each $= 1$, in the formulæ of 360, (6.), we obtain *Two Modular Transformations* of the equation of the same surface; ϵ_1, ϵ_3 being

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[*] If the proposed equation be

$$\mathrm{S}\nu\mathrm{d}\rho = p\mathrm{d}x + q\mathrm{d}y + r\mathrm{d}z = 0, \quad \text{so that} \quad \nu = -(ip + jq + kr),$$

we easily find that $2\gamma = iP + jQ + kR$, where

$$P = \mathrm{D}_z q - \mathrm{D}_y r, \quad Q = \mathrm{D}_x r - \mathrm{D}_z p, \quad R = \mathrm{D}_y p - \mathrm{D}_x q;$$

the condition of integrability XLV. becomes therefore here,

$$pP + qQ + rR = 0, \text{ which agrees with known results.}$$

vectors of *Modular Foci*, in two distinct planes, and ζ_1, ζ_3 being vectors of points upon the *Modular Directrices* corresponding: whence the ***modular focal conics***, and ***dirigent cylinders*** (real or imaginary), are found by easy eliminations.

(20.) Thus, by assuming that either

$$\text{XLVI.} \ldots \mathrm{S}\lambda(\rho - \zeta_1) = 0, \quad \mathrm{S}\lambda(\rho - \zeta_3) = 0,$$

or

$$\text{XLVII.} \ldots \mathrm{S}\mu(\rho - \zeta_1) = 0, \quad \mathrm{S}\mu(\rho - \zeta_3) = 0,$$

the equations 360, XVI., XVII. may be brought to the forms,

$$\text{XLVIII.} \ldots (\rho - \epsilon_1)^2 = m_1^2(\rho - \zeta_1)^2, \qquad \text{XLIX.} \ldots (\rho - \epsilon_3)^2 = m_3^2(\rho - \zeta_3)^2,$$

with the values,

$$\text{L.} \ldots m_1^2 = 1 - \frac{c_2}{c_1}, \quad \text{and} \quad \text{LI.} \ldots m_3^2 = 1 - \frac{c_2}{c_3};$$

in which c_1, c_2, c_3 are the ***three roots*** of a certain ***cubic*** ($M = 0$), or the ***inverse squares*** of the three scalar ***semiaxes*** (real or imaginary) of the surface, arranged in algebraically ascending order (357, IX., XX.; 405, (6.), &c.): and m_1, m_3 are the *two* (real or imaginary) *Moduli*, or represent the ***modular ratios***, in the ***two modes of Modular Generation**** corresponding.

(21.) It is obvious that an equation of the form,

$$\text{LII.} \ldots \mathrm{T}\phi\rho = C = \text{const.},$$

represents a ***central quadric***, if $\phi\rho$ be ***any linear***† and vector function of ρ, of the

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* Mac Cullagh's rule of modular generation, which includes both those modes, was expressed in page 437 of the *Lectures* by an equation of the form,

$$\mathrm{T}(\rho - \alpha) = \mathrm{TV}.\gamma\mathrm{V}\beta\rho;$$

in which the origin is on a directrix, β is the vector of another point of that right line, α is the vector of the corresponding focus, γ is perpendicular to a directive (that is, generally, to a cyclic) plane, ρ is the vector of any point P of the surface, and $\pm \mathrm{S}\beta\gamma$ is the constant modular ratio, of the distance $\overline{\mathrm{AP}}$ of P from the focus, to the distance of the same point P from the directrix OB, measured parallel to the directive plane. The new forms (360), above referred to, are however much better adapted to the working out of the various consequences of the construction; but it cannot be necessary, at this stage, to enter into any details of the quaternion transformations: still less need we here pause to give references on a subject so interesting, but by this time so well known to geometers, as that of the modular and umbilicar generations of surfaces of the second order. But it may just be noted, in order to facilitate the applications of the formulæ L. and LI., that if we write, as usual, for ***all*** the central quadrics, $a^2 > b^2 > c^2$, whether b^2 and c^2 be positive or negative, then the roots c_1, c_2, c_3 coincide, for the ***ellipsoid***, with a^{-2}, b^{-2}, c^{-2}; for the ***single***-sheeted ***hyperboloid***, with c^{-2}, a^{-2}, b^{-2}; and for the ***double***-sheeted hyperboloid with b^{-2}, c^{-2}, a^{-2}, (comp. page 651).

† In page 664 the notation,

$$d\rho = 2\mathrm{S}\nu d\rho = 2\mathrm{S}\phi\rho d\rho, \qquad 409, \text{IV}.$$

was employed for an ***arbitrary surface;*** but with the understanding that ***this function*** $\phi\rho$ (comp. 363) was ***generally non-linear***. It may be better, however, as a kind considered in the Section III. ii. 6, whether self-conjugate or not; but it requires a little more attention to perceive, that an equation of this *other form*,

$$\text{LIII.} \ldots \mathrm{T}(\rho - \mathrm{V}.\beta\mathrm{V}\gamma\alpha) = \mathrm{T}(\alpha - \mathrm{V}.\gamma\mathrm{V}\beta\rho),$$

represents *such* a surface, whatever the *three vector constants* α, β, γ may be. The discussion of this last *form* would present some circumstances of interest, and might be considered to supply a *new mode of generation*, on which however we cannot enter here.

(22.) The surfaces of the second order, considered hitherto in the present Section, have all had the *origin* for *centre*. But if, retaining the significations of ϕ, f, and F, we compare the two equations,

$$\text{LIV.} \ldots f(\rho - \kappa) = C, \quad \text{and} \quad \text{LV.} \ldots f\rho - 2\mathrm{S}\epsilon\rho = C',$$

we shall see (by 362, &c.) that the constants are connected by the two relations,

$$\text{LVI.} \ldots \epsilon = \phi\kappa, \quad C' = C - f\kappa = C - \mathrm{S}\epsilon\kappa = C - F\epsilon;$$

so that the equation,

$$\text{LVII.} \ldots f\rho - 2\mathrm{S}\epsilon\rho = f(\rho - \phi^{-1}\epsilon) - F\epsilon,$$

is an *identity*.

(23.) If then we meet an equation of the form LV., in which (as has been usual) we have still $f\rho = \mathrm{S}\rho\phi\rho =$ a scalar and *homogeneous* function of ρ, of the *second* dimension, we shall know that it represents *generally* a surface of *that* order, with the expression (comp. 347, IX., &c.),

$$\text{LVIII.} \ldots \kappa = \phi^{-1}\epsilon = m^{-1}\psi\epsilon = \textit{Vector of Centre}.$$

(24.) It may happen, however, that the two relations,

$$\text{LIX.} \ldots m = 0, \quad \mathrm{T}\psi\epsilon > 0,$$

exist together; and *then* the *centre* may be said to be at an *infinite distance*, but in a *definite direction*: and the surface becomes a *Paraboloid*, elliptic or hyperbolic, according to conditions which are easy consequences from what has been already shown.

(25.) On the other hand it may happen that the two equations,

$$\text{LX.} \ldots m = 0, \quad \psi\epsilon = 0,$$

are satisfied together; and then the vector κ of the centre acquires, by LVIII., an *indeterminate value*, and the surface becomes a *Cylinder*, as has been already sufficiently exemplified.

(26.) It would be tedious to dwell here on such details; but it may be worth

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general rule, to *avoid* writing $\nu = \phi\rho$, *except* for central quadrics; and to confine ourselves to the notation $\mathrm{d}\nu = \phi\mathrm{d}\rho$, as in some recent and several earlier sub-articles, when we wish, for the sake of *association* with other investigations and results, to treat the *function* ϕ as *linear* (or *distributive*); because we shall thus be at liberty to treat the *surface* as *general*, notwithstanding this property of ϕ. As regards the methods of *generating* a quadric, it may be worth while to look back at the Note to page 649, respecting the *Six Generations of the Ellipsoid*, which were given by the writer in the *Lectures*, with suggestions of a few others, as interpretations of quaternion equations.

while to observe, that the *general equation of a Surface of the Third Degree* may be thus written:

$$\text{LXI.} \ldots \mathrm{S}q\rho q'\rho q''\rho + \mathrm{S}\rho\phi\rho + \mathrm{S}\gamma\rho + C = 0;$$

C and γ being any scalar and vector constants; $\phi\rho$ any linear, vector, and self-conjugate function; and q, q', q'' any three constant quaternions: while ρ is, as usual, the variable vector of the surface.

(27.) In fact, besides the ***one scalar constant***, C, ***three*** are included in the ***vector*** γ, and ***six others*** in the function ϕ (comp. 358); and of the ***ten*** which remain to be introduced, for the expression of a scalar and ***homogeneous*** function of ρ, of the ***third degree***, the ***three versors*** $\mathrm{U}q$, $\mathrm{U}q'$, $\mathrm{U}q''$ supply ***nine*** (comp. 312), and the ***tensor*** $\mathrm{T}\,.\,qq'q''$ is the ***tenth***.

(28.) And for the same reason the ***monomial equation***,

$$\text{LXII.} \ldots \mathrm{S}q\rho q'\rho q''\rho = 0,$$

with the same significations of q, q', q'', represents the *general Cone of the Third Degree*, or *Cubic Cone*, which has its vertex at the origin of vectors.

(29.) If then we combine this last equation with that of a ***secant plane***, such as $\mathrm{S}\epsilon\rho + 1 = 0$, we shall get a quaternion expression for a *Plane Cubic*, or *plane curve of the third degree*: and if we combine it with the equation $\rho^2 + 1 = 0$ of the ***unit-sphere***, we shall obtain a corresponding expression for a *Spherical Cubic*,* or for a curve upon a spheric surface, which is cut by an arbitrary great circle in ***three pairs of opposite points***, real or imaginary.

(30.) Finally, as an example of ***sections of surfaces***, represented by ***transcendental equations***, let us consider the *Screw Surface*, or *Helicoid*,† of which the vector equation may be thus written (comp. the sub-arts. to 314):

$$\text{LXIII.} \ldots \rho = c\,(x + a)\,\alpha + y\alpha^x\gamma, \quad \text{with} \quad \mathrm{T}\alpha = 1, \quad \gamma = \mathrm{V}\alpha\beta, \quad \text{and} \quad y > 0;$$

α being the ***unit axis***, while β, γ are two other constant vectors, a, c two scalar constants, and x, y two variable scalars.

(31.) Cutting this surface by the plane of $\beta\gamma$, or supposing that

$$\text{LXIV.} \ldots 0 = \mathrm{S}\gamma\beta\rho = \beta^2\mathrm{S}\alpha\rho - \mathrm{S}\alpha\beta\mathrm{S}\beta\rho, \quad \text{and writing} \quad \text{LXV.} \ldots c = b\mathrm{S}\alpha\beta,$$

we easily find that the scalar and vector equations of what we may call the *Screw Section* may be thus written:

$$\text{LXVI.} \ldots b\,(x + a) = y\mathrm{S}\,.\,\alpha^{x-1}; \qquad \text{LXVII.} \ldots \rho = y(\gamma\mathrm{S}\,.\,\alpha^x - \beta\mathrm{S}\,.\,\alpha^{x-1}).$$

(32.) Derivating these with respect to x, and eliminating β and y', we arrive at the equation,

$$\text{LXVIII.} \ldots \rho = (x + a)\,\rho' + z\gamma, \quad \text{if} \quad \text{LXIX.} \ldots 2bz = \pi y^2;$$

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* Compare the Note to page 48; see also the *theorem* in that page, which contains perhaps a new mode of *generation* of cubic curves in a given plane: or, by an easy modification, of the corresponding curves upon a sphere.

† Already mentioned in pages 383, 502, 514, 557. The condition $y > 0$ answers to the supposition that, in the generation of the surface, the perpendiculars from a given helix on the axis of the cylinder are not prolonged beyond that axis.

but $z\gamma$ in LXVIII. is the vector of the point, say G, in which the ***tangent to the section*** at the point (x, y), or P, ***intersects the given line*** γ, namely the line in the plane of that section which is ***perpendicular to the axis*** α: we see then, by LXIX., that ***this point of intersection depends only on the constant, b, and on the variable, y,*** being ***independent of the constant, a, and of the variable, x.***

(33.) To interpret this result of calculation, which might have been otherwise found with the help of the expression 372, XII. (with β changed to γ) for the ***normal*** ν to a screw-surface, we may observe, first, that the equation LXVII., which may be written as follows,

$$\text{LXX.} \ldots \rho = y\mathrm{V}.\alpha^{x+1}\beta, \quad \text{and gives} \quad \text{LXXI.} \ldots \mathrm{TV}\alpha\rho = y\mathrm{T}\gamma,$$

would represent an *ellipse*, if the coefficient y were treated as *constant*; namely, the section of the right *cylinder* LXXI. by the *plane* LXIV.; the *vector semiaxes* (major and minor) of this ellipse being $y\beta$ and $y\gamma$ (comp. 314, (2.)).

(34.) By assigning a new value to the constant a, we pass to a ***new screw surface*** (30.), which differs only in ***position*** from the former, and may be conceived to be formed from it by ***sliding*** along the ***axis*** α; while the value of x, corresponding to a ***given*** y, will ***vary*** by LXVI., and thus we shall have a ***new screw section*** (31.), which will ***cross the ellipse*** (33.) in a ***new point*** Q: but the ***tangent*** to the ***section*** at this point will ***intersect*** by (32.) the ***minor axis*** of the ellipse in the ***same point*** G as before.

(35.) We shall thus have a *Figure** such as the following (Fig. 85); in which if F be a ***focus*** of the ellipse BC, and G (as above) the ***point of convergence*** of the ***tangents*** to the ***screw sections*** at the points P, Q, &c., of that ellipse, it is easy to prove, by pursuing the same analysis a little farther, Ist, that the ***angle*** (g), subtended at this focus F by the minor semiaxis OC, which is also a ***radius*** (r) of the ***cylinder*** LXXI., is equal to the ***inclination*** of the ***axis*** (α) of that cylinder to the ***plane*** of the ellipse, as may indeed be inferred from elementary principles; and IInd, what is less obvious, that the ***other angle*** (h), subtended at the same focus (F) by the interval OG, or by what may be called (with reference to the present construction, in which it is supposed that $b < 0$, or that the angles made by $\mathrm{D}_x\rho$ and β with α are either ***both acute***, or ***both obtuse***) the ***Depression*** (s) ***of the Skew Centre*** (G), is equal to the inclination of the same axis (α) to the ***helix*** on the same cylinder, which is obtained (comp. 314, (10.)) by treating y as constant, in the equation LXIII. of the *Screw Surface*.

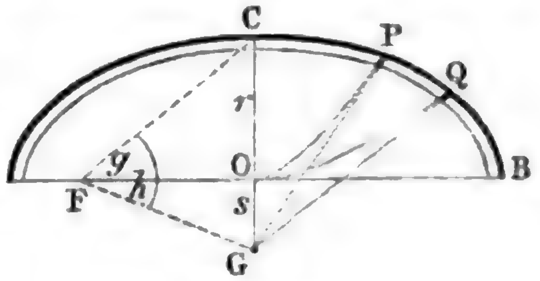

Fig. 85.

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* Those who are acquainted, even slightly, with the theory of *Oblique Arches* (or *skew bridges*), will at once see that this Figure 85 may be taken as representing rudely such an arch: and it will be found that the *construction* above deduced agrees with the celebrated *Rule of the Focal Excentricity*, discovered practically by the late Mr. Buck. This application of Quaternions was alluded to, in page 620 of the *Lectures*.

SECTION 8.—*On a few Specimens of Physical Application of Quaternions, with some Concluding Remarks.*

416. It remains to give, according to promise (368), before concluding this work, some examples* of *physical applications* of the present Calculus: and as a first specimen, we shall take the *Statics of a Rigid Body*.

(1.) Let $\alpha_1, .. \alpha_n$ be n *Vectors of Application*, and let $\beta_1, .. \beta_n$ be n corresponding *Vectors of Force*, in the sense that n forces are applied at the points $A_1, .. A_n$ of a *free* but *rigid system*, and are represented as usual by so many right lines from those points, to which *lines* the *vectors* $OB_1, .. OB_n$ are *equal*, though drawn from a common origin; and let $\gamma (= OC)$ be the vector of an *arbitrary point* C of space. Then the *Equation*† *of Equilibrium* of the system or body, under the action of these n applied forces, may be thus written:

$$\text{I.} \ldots \Sigma V(\alpha - \gamma)\beta = 0; \quad \text{or thus,} \quad \text{I}'. \ldots V\gamma\Sigma\beta = \Sigma V\alpha\beta.$$

(2.) The supposed *arbitrariness* (1.) of γ enables us to break up the formula I. or I'., into the *two* vector equations:

$$\text{II.} \ldots \Sigma\beta = 0; \qquad \text{III.} \ldots \Sigma V\alpha\beta = 0;$$

of *each* of which it is easy to assign, as follows, the *physical signification*.

(3.) The equation II. expresses that if the forces, which are applied at the points A_1.. of the body, were all *transported* to the origin O, their *statical resultant*, or *vector sum*, would be *zero*.

(4.) The equation III. expresses that the resultant of all the *couples*, produced in the usual way by such a transference of the applied forces to the assumed origin, is *null*.

(5.) And the equation I., which as above includes *both* II. and III., expresses that if all the given forces be transported to *any common point* C, the *couples* hence arising will *balance each other*: which is a sufficient condition of equilibrium of the system.

(6.) When we have only the *relation*,

$$\text{IV.} \ldots S(\Sigma\beta . \Sigma V\alpha\beta) = 0,$$

without $\Sigma\beta$ vanishing, the applied forces have then an *Unique Resultant* $= \Sigma\beta$, acting along the line of which I. or I'. is the equation, with γ for its variable vector.

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* The reader may compare the remarks on *hydrostatic pressure*, in pages 434, 435.

† We say here, "*equation*:" because the *single quaternion formula*, I. or I'., contains virtually the *six usual scalar equations*, or conditions, of the equilibrium at present considered.

(7.) And the ***physical interpretation*** of this condition IV. is, that when the forces are transported to O, as in (3.) and (4.) the ***resultant force*** is ***in the plane*** of the ***resultant couple***.

(8.) When the equation II., but not III., is satisfied, the applied forces compound themselves into *One Couple*, of which the *Axis* = $\Sigma V\alpha\beta$, whatever may be the position of the *origin*.

(9.) When ***neither*** II. nor III. is satisfied, we may still propose so to place the ***auxiliary point*** C, that when the given forces are transferred ***to it***, as in (5.), the ***resultant force*** $\Sigma\beta$ may have the ***direction*** of the ***axis*** $\Sigma V(\alpha-\gamma)\beta$ of the ***resultant couple***, or else the ***opposite*** of that direction; so that, in each case, the condition,*

$$\text{V.}\ldots\ V\frac{\Sigma V(\alpha-\gamma)\beta}{\Sigma\beta}=0,$$

shall be satisfied by a suitable limitation of the auxiliary vector γ.

(10.) This last equation V. represents therefore the *Central Axis* of the given system of applied forces, with γ for the variable vector of that right line: or the *axis* of the ***screw-motion*** which those forces ***tend*** to produce, when they are ***not in balance***, as in (1.), and neither tend to produce ***translation alone***, as in (6.), nor ***rotation alone***, as in (8.).

(11.) In general, if q be an ***auxiliary quaternion***, such that

$$\text{VI.}\ldots\ q\Sigma\beta=\Sigma V\alpha\beta,$$

its ***vector part***, Vq, is equal by (V.) to the *Vector-Perpendicular*, let fall from the origin ***on the central axis***; while its ***scalar part***, Sq, is easily proved to be the ***quotient***, of what may be called the *Central Moment*, divided by the *Total Force*: so that $Vq=0$ when the central axis passes ***through the origin***, and $Sq=0$ when there exists an ***unique resultant***.

(12.) When the total force $\Sigma\beta$ does not vanish, let Q be a ***new auxiliary quaternion***, such that

$$\text{VII.}\ldots\ Q=\frac{\Sigma\alpha\beta}{\Sigma\beta}=q+\frac{\Sigma S\alpha\beta}{\Sigma\beta},$$

with $\qquad \text{VIII.}\ldots\ c=SQ=Sq, \quad$ and $\quad \text{IX.}\ldots\ \gamma=\text{OC}=VQ,$

for its scalar and vector parts; then $c\Sigma\beta$ represents, both in quantity and in direction, the *Axis of the Central Couple* (9.), and γ is the vector of a point C which is ***on the central axis*** (10.), considered as a right line having ***situation in space***: while the ***position*** of this ***point*** on this ***line*** depends only on the ***given system*** of applied forces, and does not vary with the assumed ***origin*** O.

(13.) Under the same conditions, we have the transformations,

$$\text{X.}\ldots\ \Sigma\alpha\beta=(c+\gamma)\Sigma\beta; \qquad \text{XI.}\ldots\ T\Sigma\alpha\beta=(c^2-\gamma^2)^{\frac{1}{2}}T\Sigma\beta;$$

$$\text{XII.}\ldots\ \Sigma V\alpha\beta=c\Sigma\beta+V\gamma\Sigma\beta; \qquad \text{XIII.}\ldots\ (\Sigma V\alpha\beta)^2=c^2(\Sigma\beta)^2+(V\gamma\Sigma\beta)^2;$$

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* The equation V. may also be obtained from the condition,

$$\text{V}'.\ldots\ T\Sigma V(\alpha-\gamma)\beta=\text{a } \textit{minimum},$$

when γ is treated as the only variable vector; which answers to a known property of the *Central Moment*.

whereof XII. contains the known *law*, according to which the *axis* of the *couple* (4.), obtained by transferring all the forces to an assumed point O, *varies* generally in *quantity* and in *direction* with the *position* of that point: while XIII. expresses the known corollary from that law, in virtue of which the *quantity alone*, or the *energy* ($T\Sigma V\alpha\beta$) of the couple here considered, is the *same* for all the points O of any one *right cylinder*, which has the *central axis* of the system for its *axis of revolution*.

(14.) If we agree to call the *quaternion product* PA.AA′ the *quaternion moment*, or simply the *Moment*, of the *applied force* AA′ at A, *with respect to the Point* P, the *quaternion sum* $\Sigma\alpha\beta$ in X. may then be said to be the *Total Moment* of the given system of forces, with respect to the assumed *origin* O; and the formula XI. expresses that the *tensor* of this *sum*, or what may be called the *quantity* of this total moment, is *constant* for all points O which are situated *on any one spheric surface*, with the point C determined in (12.) for its *centre*: being also a *minimum* when O is placed at that point C *itself*, and being then equal to what has been already called the *central moment*, or the energy of the central couple.

(15.) For these and other reasons, it appears not improper to call *generally* the point C, above determined, the *Central Point*, or simply the *Centre*, of the given system of applied forces, when the total force does not vanish; and accordingly in the particular but important case, when all those forces are *parallel*, without their *sum* being *zero*, so that we may write,

$$\text{XIV.} \ldots \beta_1 = b_1\beta, \ldots \quad \beta_n = b_n\beta, \quad T\Sigma\beta > 0,$$

the scalar c in (12.) vanishes, and the vector γ becomes (comp. Art. 97 on *barycentres*),

$$\text{XV.} \ldots \text{OC} = \gamma = \frac{b_1\alpha_1 + \ldots + b_n\alpha_n}{b_1 + \ldots + b_n} = \frac{\Sigma b\alpha}{\Sigma b};$$

so that the *point* C, thus determined, is *independent of the common direction* β, and coincides with what is usually called the *Centre of Parallel Forces*.

(16.) The conditions of equilibrium (1.), which have been already expressed by the formula I., may also be included in this other quaternion equation,

$$\text{XVI.} \ldots \textit{Total Moment} = \Sigma\alpha\beta = \textit{a scalar constant},$$

of which the *value* is *independent of the origin*; and which, with its sign changed, represents what may perhaps be called the *Total Tension* of the system.

(17.) *Any infinitely small change*, in the *position of a rigid body*, is equivalent to the alteration of each of its vectors α to another of the form,

$$\text{XVII.} \ldots \alpha + \delta\alpha = \alpha + \epsilon + V\iota\alpha,$$

ϵ and ι being *two* arbitrary but infinitesimal vectors, which do not vary in the passage from one point A of the body to another: and thus the *conditions of equilibrium* (1.) may be expressed by this other formula,

$$\text{XVIII.} \ldots \Sigma S\beta\delta\alpha = 0,$$

which contains, for the case here considered, the *Principle of Virtual Velocities*, and admits of being extended easily to other cases of Statics.

417. The general *Equation of Dynamics* may be thus written,

$$\text{I.} \ldots \Sigma m S(D_t^2\alpha - \xi)\delta\alpha = 0,$$

with significations of the symbols which will soon be stated; but as we only propose (416) to give here some *specimens* of physical application, we shall aim chiefly, in the following sub-articles, at the deduction of a few formulæ and theorems, respecting *Axes* and *Moments* of *Inertia*, and subjects therewith connected.

(1.) In the formula I., α is the vector of position, at the time t, of an element m of the system; $\delta\alpha$ is any variation of that vector, geometrically compatible with the mutual connexions between the parts of that system; the vector $m\xi$ represents a moving force, or ξ an accelerating force, which acts on the element m of mass; D and S are marks, as usual, of derivating and taking the scalar; and the summation denoted by Σ extends to all the elements, and is generally equivalent to a triple integration, or to an addition of triple integrals in space. And the formula is obtained (comp. 416, (17.)), by a combination of D'Alembert's principle with the principle of virtual velocities, which is analogous to that employed in the *Mécanique Analytique* by Lagrange.

(2.) For the case of a *free* but *rigid body*, we may substitute for $\delta\alpha$ the expression $\epsilon + \mathrm{V}\iota\alpha$, assigned by 416, XVII.; and then, on account of the arbitrariness of the two infinitesimal vectors ϵ and ι, the formula I. breaks up into the two following,

$$\text{II.} \ldots \Sigma m(\mathrm{D}_t^2\alpha - \xi) = 0; \qquad \text{III.} \ldots \Sigma m\mathrm{V}\alpha(\mathrm{D}_t^2\alpha - \xi) = 0;$$

which correspond to the two statical equations 416, II. and III., and contain respectively the law of motion of the centre of gravity, and the law of description of areas.

(3.) If the body have a *fixed point*, which we may take for the origin o, we eliminate the reaction at that point, by attending only to the equation III.; and may then express the connexions between the elements m by the formula,

$$\text{IV.} \ldots \mathrm{D}_t\alpha = \mathrm{V}\iota\alpha, \quad \text{whence} \quad \text{V.} \ldots \mathrm{D}_t^2\alpha = \iota\mathrm{V}\iota\alpha - \mathrm{V}\alpha\mathrm{D}_t\iota;$$

ι being the *Vector-Axis of instantaneous Rotation* of the body, in the sense that its *versor* $\mathrm{U}\iota$ represents the *direction* of the *axis*, and that its *tensor* $\mathrm{T}\iota$ represents the *angular velocity*, of such rotation at the time t.

(4.) By V., the equation III. becomes,

$$\text{VI.} \ldots \Sigma m\alpha\mathrm{V}\alpha\mathrm{D}_t\iota = \Sigma m(\mathrm{V}\iota\alpha\mathrm{S}\iota\alpha - \mathrm{V}\alpha\xi);$$

and other easy combinations give the laws of areas and living force, under the forms,

$$\text{VII.} \ldots \Sigma m\alpha\mathrm{D}_t\alpha - \Sigma m\mathrm{V}\int\alpha\xi dt = \gamma = \text{a constant vector};$$
$$\text{VIII.} \ldots \tfrac{1}{2}\Sigma m(\mathrm{D}_t\alpha)^2 - \Sigma m\mathrm{S}\int\iota\alpha\xi dt = c = \text{a constant scalar}.$$

(5.) When the applied forces vanish, or balance each other, or more generally when they compound themselves into a single force acting at the fixed point, so that in each case the condition

$$\text{IX.} \ldots \Sigma m\mathrm{V}\alpha\xi = 0$$

is satisfied, the equations (4.) are simplified; and if we introduce a linear, vector, and self-conjugate function ϕ, such that

$$\text{X.} \ldots \phi\iota = \Sigma m\alpha\mathrm{V}\alpha\iota = \iota\Sigma m\alpha^2 - \Sigma m\alpha\mathrm{S}\alpha\iota,$$

and write h^2 for $-2c$, they take the forms,

$$\text{XI.} \ldots \therefore \phi D_t\iota + V\iota\phi\iota = 0; \qquad \text{XII.} \ldots \phi\iota + \gamma = 0; \qquad \text{XIII.} \ldots S\iota\phi\iota = h^2;$$

γ and h being two real constants, of the vector and scalar kinds, connected with each other and with ι by the relation,

$$\text{XIV.} \ldots S\iota\gamma + h^2 = 0; \quad \text{also} \quad \text{XV.} \ldots \phi D_t\iota = V\iota\gamma.$$

It may be added that γ is now the *vector sum* of the doubled *areal velocities* of all the elements of the body, multiplied each by the mass m of that element, and each represented by a *right line* $\alpha D_t\alpha$ perpendicular to the plane of the area described round the fixed point O in the time dt; and that h^2 is the *living force*, or *vis viva* of the body, namely the *positive sum* of all the products obtained by multiplying each element m by the *square* of its *linear velocity*, regarded as a *scalar* ($TD_t\alpha$).

(6.) When ι is regarded as a variable vector, the equation XIII. represents an *ellipsoid*, which is *fixed in the body*, but *moveable with it;* and the equation XIV. represents a *tangent plane* to this ellipsoid, which plane is *fixed in space*, but *changes* in general its position relatively to the *body*. And thus the *motion* of that body may generally be conceived, as was shown by Poinsot, to be performed by the *rolling* (*without gliding*) of an *ellipsoid upon a plane;* the former *carrying the body* with it, while its *centre* O remains *fixed:* and the *semidiameter* (ι) *of contact* being the *vector-axis* (3.) of *instantaneous rotation.*

(7.) The ellipsoid XIII. may be called, perhaps, the *Ellipsoid of Living Force*, on account of the signification (5.) of the constant h^2 in its equation; and the fixed plane XIV., on which it rolls, is parallel to what may be called the *Plane of Areas* ($S\iota\gamma = 0$): no use whatever having hitherto been made, in this investigation, of any *axes* or *moments* of *inertia*. But if we here admit the usual definition of such a moment, we may say that the *Moment of Inertia of the body*, with respect to *any axis* ι through the fixed point, is equal to the *living force* h^2 *divided by the square** *of the semidiameter* $T\iota$ of the ellipsoid XIII.; because this moment is,

$$\text{XVI.} \ldots \Sigma m(TV\alpha U\iota)^2 = \iota^{-2}\Sigma m(V\iota\alpha)^2 = -S\iota^{-1}\phi\iota = h^2 T\iota^{-2}.$$

(8.) The equations XII. and XIII. give,

$$\text{XVII.} \ldots \therefore 0 = \gamma^2 S\iota\phi\iota - h^2(\phi\iota)^2 = S\iota\nu, \quad \text{if} \quad \text{XVIII.} \ldots \nu = \gamma^2\phi\iota - h^2\phi^2\iota;$$

and this equation XVII. represents a *cone* of the second degree, fixed in the body (comp. (6.)), but moveable with it, of which the axis ι is always a *side*, and to which the *normal*, at any point of that side, has the direction of the line ν. But it follows from XI., or from XII. XV., and from the properties of the function ϕ, that $D_t\iota$ is perpendicular to both $\phi\iota$ and $\phi^2\iota$, and therefore also by XVIII. to ν; the cone XVII. is therefore *touched*, along the side ι, by that *other cone*, which is the *locus in space* of the *instantaneous axis* of rotation. We are then led, by this simple quaternion analysis, to a *second representation* of the *motion of the body*, which also was proposed by Poinsot: namely, as the *rolling of one cone on another*.

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* Hence it may easily be inferred, with the help of the general *construction of an ellipsoid* (217, (6.)), illustrated by Figure 53 in page 226, that for *any solid body*, and *any given point* A thereof, there can always be found (indeed in more ways than one) *two other points*, B and C, which are likewise *fixed in the body*, and are such that the *square-root of the moment of inertia*, round *any axis* AD, is geometrically *constructed by the line* BD, if the point D be determined on the axis, by the condition that A and D shall be *equally distant* from C. This theorem, with some others here reproduced, was given in the Abstract of a Paper read before the Royal Irish Academy on the 10th of January, 1848, and was published in the *Proceedings* of that date.

(9.) To treat briefly by quaternions some of Mac Cullagh's results on this subject, it may be noted that the *line* γ, though *fixed in space*, describes *in the body* a *cone* of the second degree, of which the equation is, by what precedes,

$$\text{XIX.} \ldots g^2 S\gamma\phi^{-1}\gamma + h^2\gamma^2 = 0, \quad \text{if} \quad \text{XX.} \ldots g = T\gamma, \quad \text{or} \quad \text{XXI.} \ldots \gamma^2 + g^2 = 0;$$

while, if we write $\gamma = \text{OC}$, the *point* C is indeed fixed in *space*, but describes a *sphero-conic* in the *body*, which is part of the common intersection of the *cone* XIX., the *sphere* XXI., and the *reciprocal ellipsoid* (comp. XIII.),

$$\text{XXII.} \ldots S\gamma\phi^{-1}\gamma = h^2.$$

(10.) Also, the *normal* to the *new cone* (9.), at any point of the side γ, has the direction of $g^2\phi^{-1}\gamma + h^2\gamma$, or of $\iota + h^2\gamma^{-1}$ (comp. XIV.); and if a line in this direction be drawn through the fixed point O, it will be the *side of contact* of the *plane of areas* (7.), with the *cone of normals* at O to the cone XIX.; which *last* (or *reciprocal*) *cone rolls on that plane of areas*.

(11.) As regards the *Axes of Inertia*, it may be sufficient here to observe that if the body revolve round a *permanent axis*, and with a *constant velocity*, the *vector axis* ι is constant; and must therefore satisfy the equation,

$$\text{XXIII.} \ldots V\iota\phi\iota = 0, \quad \text{because XXIV.} \ldots D_t\iota = 0;$$

it has therefore in general (comp. 415) one or other of *Three Real and Rectangular Directions*, determined by the condition XXIII.: namely, those of the *Axes of Figure* of either of the two *Reciprocal Ellipsoids*, XIII. XXII.

(12.) And the *Three Principal Moments*, say A, B, C, corresponding to those *three principal axes*, are by XVI. the three scalar values of $-\iota^{-1}\phi\iota$; so that the *symbolical cubic* (350) in ϕ may be thus written,

$$\text{XXV.} \ldots (\phi + A)(\phi + B)(\phi + C) = 0.$$

(13.) Forming then this symbolical cubic by the general method of the Section III. ii. 6, we find that the *three moments* A, B, C, are the *three roots* (always *real*, by this analysis) of the *algebraic* and *cubic equation*,

$$\text{XXVI.} \ldots A^3 - 2n^2A^2 + (n^4 + n'^2)A - (n^2n'^2 - n''^2) = 0;$$

in which, n^2, n'^2, n''^2 are three positive scalars, namely,

$$\text{XXVII.} \ldots n^2 = -\Sigma m\alpha^2; \quad n'^2 = -\Sigma mm'(V\alpha\alpha')^2; \quad n''^2 = \Sigma mm'm''(S\alpha\alpha'\alpha'')^2;$$

and the combination $n^2n'^2 - n''^2$ is another positive scalar, of which the value may be thus expressed,

$$\text{XXVIII.} \ldots ABC = n^2n'^2 - n''^2 = \Sigma m^2m'\alpha^2(V\alpha\alpha')^2$$
$$+ 2\Sigma mm'm''(T\alpha\alpha'T\alpha'\alpha''T\alpha''\alpha + S\alpha\alpha'S\alpha'\alpha''S\alpha''\alpha),$$

if α, α', α'', &c. be the vectors of the mass-elements m, m', m'', &c.

(14.) And because the equation XXV. gives this other symbolical result,

$$\text{XXIX.} \ldots -ABC\phi^{-1} = \phi^2 + (A + B + C)\phi + BC + CA + AB,$$

it follows that

$$\text{XXX.} \ldots \phi^{-1}0 = 0;$$

and therefore, by XV., &c., that if a body, with a fixed point, &c., *begin* to revolve round one of its three principal axes of inertia, it will *continue* to revolve round that axis, with an *unchanged velocity* of rotation.

(15.) It has hitherto been supposed, that all the moments of inertia are referred to axes passing through *one point* O of the body; but it is easy to remove this restriction. For example, if we denote the moment XVI. by I_0, and if I_ω be the corresponding moment for an axis *parallel* to ι, but drawn through a *new point* Ω, of which the vector is ω, then

$$\begin{aligned}\text{XXXI.} \ldots I_\omega &= \iota^{-2}\Sigma m(\mathrm{V}\iota(\alpha - \omega))^2 \\ &= I_0 + 2\Sigma m . \mathrm{S}(\omega\iota^{-1}\mathrm{V}\iota\kappa) + p^2\Sigma m,\end{aligned}$$

if

$$\text{XXXII.} \ldots \kappa\Sigma m = \Sigma m\alpha, \quad \text{and} \quad \text{XXXIII.} \ldots p = \mathrm{TV}\omega\mathrm{U}\iota,$$

so that κ is the vector of the *centre of inertia* (or of gravity) of the body, and p is the *distance* between the two parallel axes.

(16.) If then we suppose that the condition

$$\text{XXXIV.} \ldots \mathrm{V}\iota\kappa = 0$$

is satisfied, that is, if the axis ι pass through the centre of inertia, we shall have the very simple relation,

$$\text{XXXV.} \ldots I_\omega = I_0 + p^2\Sigma m;$$

which agrees with known results.

418. As a *third specimen* of physical applications of quaternions, we propose to consider briefly the motions of a *System of Bodies*, $m, m', m'', \ldots$ regarded as free material points, of which the variable vectors are $\alpha, \alpha', \alpha'', \ldots$ and which are supposed to attract each other according to the law of the inverse square: the fundamental formula employed being the following,

$$\text{I.} \ldots \Sigma m\mathrm{S}\mathrm{D}_t^2\alpha\delta\alpha + \delta P = 0, \quad \text{if} \quad \text{II.} \ldots P = \Sigma\frac{mm'}{\mathrm{T}(\alpha - \alpha')}:$$

P thus denoting the *Potential* (or *force-function*) of the system, and the variations $\delta\alpha, \delta\alpha', \ldots$ being infinitesimal, but otherwise arbitrary.

(1.) To deduce the formula I., with the signification II. of P, from the general equation 417, I. of dynamics, we have first, for the case of two bodies, the following expressions for the accelerating forces,

$$\text{III.} \ldots \xi = \frac{m'}{(\alpha - \alpha')r}, \quad \xi' = \frac{m}{(\alpha' - \alpha)r}, \quad \text{if} \quad r = \mathrm{T}(\alpha - \alpha');$$

whence follows the transformation,[*]

$$\text{IV.}\ldots -\mathrm{S}(m\xi\delta\alpha + m'\xi'\delta\alpha') = \frac{-mm'}{r}\,\mathrm{S}\,\frac{\delta(\alpha-\alpha')}{\alpha-\alpha'} = \delta\,\frac{mm'}{r};$$

a result easily extended, as above. If the *law* of attraction were supposed different, there would be no difficulty in modifying the expression for the potential accordingly.

(2.) *In general*, when a *scalar*, f (as here P), is a *function* of one or more *vectors*, α, α', . . . its *variation* (or differential) can be expressed as a *linear and scalar function* of *their* variations (or differentials), of the form $\mathrm{S}\beta\delta\alpha + \mathrm{S}\beta'\delta\alpha' + \ldots$ (or $\Sigma\mathrm{S}\beta\mathrm{d}\alpha$); in which β, β' . . . are certain *new* and *finite vectors*, and are themselves generally *functions* of α, α', . . ., *derived* from the *given* scalar function f. And we shall find it convenient to *extend* the *Notation*[†] *of Derivatives*, so as to denote these *derived vectors* β, β', &c., by the *symbols*, $\mathrm{D}_\alpha f$, $\mathrm{D}_{\alpha'} f$, &c. In this manner we shall be able to write,

$$\text{V.}\ldots \delta P = \Sigma\mathrm{S}(\mathrm{D}_\alpha P \,.\, \delta\alpha);$$

and the differential equations of motion of the bodies m, m', m'', . . will take by I. the forms:

$$\text{VI.}\ldots m\mathrm{D}_t^2\alpha + \mathrm{D}_\alpha P = 0, \quad m'\mathrm{D}_t^2\alpha' + \mathrm{D}_\alpha' P = 0, \text{ \&c.};$$

or more fully,

$$\text{VII.}\ldots \mathrm{D}_t^2\alpha = \frac{m'}{(\alpha-\alpha')\,\mathrm{T}(\alpha-\alpha')} + \frac{m''}{(\alpha-\alpha'')\,\mathrm{T}(\alpha-\alpha'')}) + \ldots; \text{ \&c.}$$

(3.) The laws of the centre of gravity, of areas, and of living force, result immediately from these equations, under the forms,

$$\text{VIII.}\ldots \Sigma m\mathrm{D}_t\alpha = \beta; \qquad \text{IX.}\ldots \Sigma m\mathrm{V}\alpha\mathrm{D}_t\alpha = \gamma;$$

and

$$\text{X.}\ldots T = -\tfrac{1}{2}\Sigma m(\mathrm{D}_t\alpha)^2 = P + H;$$

in which β, γ are constant vectors, H is a constant scalar, and $2T$ is the living force of the system (comp. 417, (5.)).

(4.) One mode (comp. 417, (2.)) of deducing the three equations, of which these are the first integrals, is the following. To obtain VIII., change every variation $\delta\alpha$ in I. to one *common* but *arbitrary* infinitesimal vector, ϵ. For IX., change $\delta\alpha$ to $\mathrm{V}\iota\alpha$, $\delta\alpha'$ to $\mathrm{V}\iota\alpha'$, &c.; ι being *another* arbitrary and infinitesimal vector. Finally, to arrive at X., change *variations* to *differentials* ($\delta\alpha$ to $\mathrm{d}\alpha$, &c.), and integrate once, as for the two former equations, with respect to the time t.

(5.) The formula I. admits of being *integrated by parts*, without any restriction on the *variations* $\delta\alpha$, by means of the general transformation,

$$\text{XI.}\ldots \mathrm{S}(\mathrm{D}_t^2\alpha \,.\, \delta\alpha) = \mathrm{D}_t\mathrm{S}(\mathrm{D}_t\alpha \,.\, \delta\alpha) - \tfrac{1}{2}\delta \,.\, (\mathrm{D}_t\alpha)^2,$$

combined with the introduction of the following definite integral (comp. X.),

$$\text{XII.}\ldots F = \int_0^t (P + T)\,\mathrm{d}t.$$

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[*] It may not be useless here to compare the expression in page 417, for the *differential of a proximity*.

[†] In this extended notation, such a formula as $\mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho$ would give,

$$\nu = \tfrac{1}{2}\mathrm{D}_\rho f\rho.$$

(6.) In fact, if we denote by α_0, α'_0, .. the *initial values* of the vectors α, α', .. or their values when $t = 0$, and by $D_0\alpha$, $D_0\alpha'$, .. the corresponding values of $D_t\alpha$, $D_t\alpha'$, .., we shall thus have, as a first integral of the equation I., the formula,

$$\text{XIII.} \ldots \Sigma m S (D_t\alpha \,.\, \delta\alpha - D_0\alpha \,.\, \delta\alpha_0) + \delta F = 0\,;$$

in which no variation δt is assigned to t, and which conducts to important consequences.

(7.) To draw from it some of these, we may observe that if the masses m, m', .. be treated as constant and known, the complete integrals of the equations VI. or VII. must be conceived to give what may be called the *final vectors* of *position* α, α', .. and of *velocity* $D_t\alpha$, $D_t\alpha'$, .. in terms of the *initial vectors* α_0, α'_0, .. $D_0\alpha$, $D_0\alpha'$, .. and of the *time*, t: whence, conversely, we may conceive the initial vectors of velocity to be expressible as functions of the initial and final vectors of position, and of the time. In this way, then, we are led to consider P, T, and F as being *scalar functions* (whether we are or are not prepared to *express* them as such), of α, α', .. α_0, α'_0, .. and t; and thus, by (2.), the recent formula XIII. breaks up into the two following systems of equations:

$$\text{XIV.} \ldots mD_t\alpha + D_\alpha F = 0, \quad m'D_t\alpha' + D_{\alpha'}F = 0, \ \&\text{c.}\,;$$

and

$$\text{XV.} \ldots -mD_0\alpha + D_{\alpha_0}F = 0, \quad -m'D_0\alpha' + D_{\alpha'_0}F = 0, \ \&\text{c.}\,;$$

whereof the *former* may be said to be *intermediate integrals*, and the *latter* to be *final integrals*, of the *differential equations of motion* of the system, which are included in the formula I.

(8.) In fact, the equations XIV. do not involve the final *vectors of acceleration* $D_t^2\alpha$, .. as the differential equations VI. or VII. had done; and the equations XV. express, at least theoretically, the dependence of the *final* vectors of *position* α, .. on the *time*, t, and on the *initial* vectors of position α_0, .. and of *velocity* $D_0\alpha$, .. as by (7.) the *complete* integrals ought to do. And on account of these and other important properties, the function here denoted by F may be called the *Principal** *Function of Motion of the System.*

(9.) If the initial vectors α_0, .. and $D_0\alpha$, .. be *given*, that is, if we consider the actual *progress* in space of the mutually attracting system of masses m, .. from one set of positions to another, then the function F depends upon the *time* alone; and by its definition XII., its *rate* or *velocity* of increase, or its *total derivative* with respect to t, is thus expressed,

$$\text{XVI.} \ldots D_tF = P + T.$$

(10.) But we may inquire what is the *partial derivative*, say (D_tF), of the same definite integral F, when regarded (7.) as a function of the final and initial vectors of position α, .. α_0, .. which involves *also* the *time explicitly*, and is *now* to be derivated with respect *only* to *that* variable t, *as if* the final vectors α, .. were *constant*: whereas in fact those vectors *alter* with the time, in the course of any actual *motions* of the system.

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* This function was in fact so called, in two Essays by the present writer, "On a General Method in Dynamics," published in the *Philosophical Transactions* (London), for the years 1834 and 1835; although of course *coordinates*, and not *quaternions*, were *then* employed, the latter not having been discovered until 1843: and the notation S, since adopted for *scalar*, was then used instead of F.

(11.) For this purpose, it is sufficient to observe that the *part* of the *total derivative* D_tF, which arises from the last mentioned changes of $\alpha, \ldots$ is (by XIV. and X.),

$$\text{XVII.} \ldots \Sigma S(D_\alpha F . D_t\alpha) = 2T;$$

and therefore (by XVI. and X.), that the *remaining part* must be,

$$\text{XVIII.} \ldots (D_tF) = P - T = -H.$$

(12.) The *complete variation* of the function F is therefore (comp. XIII.), when t as well as $\alpha, \ldots$ and $\alpha_0, \ldots$ is treated as *varying*,

$$\text{XIX.} \ldots \delta F = -H\delta t - \Sigma m S D_t\alpha\delta\alpha + \Sigma m S D_0\alpha\delta\alpha_0.$$

(13.) And hence, with the help of the equations X. XIV. XV., it is easy to infer that the principal function F must satisfy the two following *Partial Differential Equations in Quaternions:*

$$\text{XX.} \ldots (D_tF) - \tfrac{1}{2}\Sigma m^{-1}(D_\alpha F)^2 = P;$$
$$\text{XXI.} \ldots (D_tF) - \tfrac{1}{2}\Sigma m^{-1}(D_{\alpha_0} F)^2 = P_0;$$

in which P_0 denotes the initial value of the potential P.

(14.) If we write

$$\text{XXII.} \ldots V = \int_0^t 2T dt,$$

so that V represents what is called the *Action*, or the accumulated living force, of the system during the time t, then by X. and XII. the two definite integrals F and V are connected by the very simple relation,

$$\text{XXIII.} \ldots V = F + tH;$$

whence by XIX. the *complete variation* of V, considered as a function of the final and initial vectors of position, and of the constant H of living force, which does not explicitly involve the time, may be thus expressed,

$$\text{XXIV.} \ldots \delta V = t\delta H - \Sigma m S D_t\alpha\delta\alpha + \Sigma m S D_0\alpha\delta\alpha_0.$$

(15.) The *partial derivatives* of this new function V, which is for *some* purposes more useful than F, and may be called, by way of distinction from it, the *Characteristic* Function* of the motion of the system, are therefore,

$$\text{XXV.} \ldots D_\alpha V = -mD_t\alpha, \&c.;$$
$$\text{XXVI.} \ldots D_{\alpha_0} V = +mD_0\alpha, \&c.;$$
$$\text{and XXVII.} \ldots D_H V = t.$$

(16.) The *intermediate integrals* (7.) of the differential equations of motion, which were before expressed by the formulæ XIV., may now, somewhat less simply, be regarded as the result of the elimination of H between the formulæ XXV. XXVII.; and the *final integrals* of those equations VI. or VII., which were expressed by XV., are now to be obtained by eliminating the same constant H between the recent equations XXVI. XXVII.

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* The *Action*, V, was in fact so called, in the two Essays mentioned in the preceding Note. The properties of this *Characteristic Function* had been perceived by the writer, before those of that which he came afterwards to call the *Principal Function*, as above.

(17.) The *Characteristic Function*, V, is obliged (comp. (13.)) to satisfy the two following *partial differential equations*,

$$\text{XXVIII.} \ldots \tfrac{1}{2}\Sigma m^{-1}(\mathrm{D}_\alpha V)^2 + P + H = 0;$$
$$\text{XXIX.} \ldots \tfrac{1}{2}\Sigma m^{-1}(\mathrm{D}_{\alpha_0} V)^2 + P_0 + H = 0;$$

it vanishes, like F, when $t = 0$, at which epoch $\alpha = \alpha_0$, $\alpha' = \alpha'_0$, &c.; *each* of these two functions, F and V, depends ***symmetrically*** on the initial and final vectors of position: and each does so, only by depending on the mutual ***configuration*** of ***all*** those initial and final ***positions***.

(18.) It follows (comp. (4.), see also 416, (17.), and 417, (2.)), that the function F must satisfy the two conditions,

$$\text{XXX.} \ldots \Sigma(\mathrm{D}_\alpha F + \mathrm{D}_{\alpha_0} F) = 0; \qquad \text{XXXI.} \ldots \Sigma \mathrm{V}(\alpha \mathrm{D}_\alpha F + \alpha_0 \mathrm{D}_{\alpha_0} F) = 0;$$

which accordingly are forms, by XIV. XV., of the equations VIII. and IX., and therefore are expressions for the law of motion of the centre of gravity, and the law of description of areas. And, in like manner, the function V is obliged to satisfy these two analogous conditions,

$$\text{XXXII.} \ldots \Sigma(\mathrm{D}_\alpha V + \mathrm{D}_{\alpha_0} V) = 0; \qquad \text{XXXIII.} \ldots \Sigma \mathrm{V}(\alpha \mathrm{D}_\alpha V + \alpha_0 \mathrm{D}_{\alpha_0} V) = 0;$$

which accordingly, by XXV. XXVI., are new forms of the same equations VIII. IX., and consequently are new expressions of the same two laws.

(19.) All the foregoing conditions are satisfied when t is ***small***, that is, when the ***time*** of motion of the system is ***short***, by the following ***approximate expressions*** for the functions F and V, with the respectively derived and mutually connected expressions for H and t:

$$\text{XXXIV.} \ldots F = \frac{t}{2}(P + P_0) + \frac{s^2}{2t};$$

$$\text{XXXV.} \ldots V = s(P + P_0 + 2H)^{\frac{1}{2}};$$

$$\text{XXXVI.} \ldots H = -(\mathrm{D}_t F) = -\tfrac{1}{2}(P + P_0) + \frac{s^2}{2t^2};$$

$$\text{XXXVII.} \ldots t = \mathrm{D}_H V = s(P + P_0 + 2H)^{-\frac{1}{2}};$$

in which s denotes a real and positive scalar, such that

$$\text{XXXVIII.} \ldots s^2 = -\Sigma m(\alpha - \alpha_0)^2, \quad \text{or} \quad \text{XXXIX.} \ldots s = \sqrt{\Sigma m \mathrm{T}(\alpha - \alpha_0)^2}.$$

419. As a *fourth specimen*, we shall take the case of a free point or particle, attracted to a fixed centre* o, from which its variable vector is α, with an accelerating force $= Mr^{-2}$, if $r = \mathrm{T}\alpha =$ the distance

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* When ***two free masses***, m and m', with variable vectors α and α', attract each other according to the law of the inverse square, the differential equation of the ***relative motion*** of m about m' is, by 418, VII.,

$$\text{I}'. \ldots \mathrm{D}^2(\alpha - \alpha') = (m + m')(\alpha - \alpha')^{-1} r^{-1}, \quad \text{if} \quad r = \mathrm{T}(\alpha - \alpha');$$

and this equation I′. reduces itself to I., when we write α for $\alpha - \alpha'$, and M for $m + m'$.

of the point from the centre, while M is the attracting mass: the differential equation of the motion being,

$$\text{I.} \ldots D^2\alpha = M\alpha^{-1}r^{-1},$$

if D (abridged from D_t) be the sign of derivation, with respect to the time t.

(1.) Operating on I. with $V.\alpha$, and integrating, we obtain immediately the equation (comp. 338, (5.)),

$$\text{II.} \ldots V\alpha D\alpha = \beta = \text{const.};$$

which expresses at once that the *orbit is plane*, and also that the *area* described in it is *proportional to the time;* $U\beta$ being the fixed *unit-normal* to the *plane*, round which the point, in its angular motion, *revolves positively;* and $T\beta$ representing in quantity the *double areal velocity*, which is often denoted by c.

(2.) And it is important to remark, that these conclusions (1.) would have been obtained by the same analysis, if r^{-1} in I. had been replaced by *any other scalar function*, $f(r)$, of the *distance;* that is, for *any other law of central force*, instead of the law of the *inverse square*.

(3.) *In general*, we have the transformation,

$$\text{III.} \ldots \alpha^{-1}T\alpha^{-1} = dU\alpha : V\alpha d\alpha,$$

because, by 334, XV., &c., we have,

$$\text{IV.} \ldots dU\alpha = V(d\alpha.\alpha^{-1}).U\alpha = \alpha^{-2}U\alpha.V\alpha d\alpha = \alpha^{-1}T\alpha^{-1}.V\alpha d\alpha;$$

the equation I. may therefore by II. be transformed as follows,

$$\text{V.} \ldots D^2\alpha = \gamma DU\alpha, \quad \text{if} \quad \text{VI.} \ldots \gamma = -M\beta^{-1};$$

and thus it gives, by an immediate integration,

$$\text{VII.} \ldots D\alpha = \gamma(U\alpha - \epsilon), \quad \text{or} \quad \text{VII}'. \ldots D\alpha = (\epsilon - U\alpha)\gamma,$$

ϵ being a *new constant vector*, but one situated *in the plane* of the orbit, to which plane β and γ are *perpendicular*.

(4.) But α, $D\alpha$, $D^2\alpha$ are here (comp. 100, (5.) (6.) (7.)) the vectors of *position*, *velocity*, and *acceleration* of the moving point; and it has been defined (100, (5.)) that if, for *any motion* of a point, the *vectors of velocity* be set off from any *common origin*, the *curve* on which they *terminate* is the *Hodograph*[*] of that motion.

(5.) Hence α and $D\alpha$, if the latter like the former be drawn from the fixed point O, are the vectors of *corresponding points* of *orbit* and *hodograph;* and because the formula VII. gives,

$$\text{VIII.} \ldots S\gamma D\alpha = 0, \quad \text{and} \quad \text{IX.} \ldots (D\alpha + \gamma\epsilon)^2 = \gamma^2,$$

it follows that the hodograph is, in the present question, a *Circle*, in the plane of the orbit, with $-\gamma\epsilon$ (or $+\epsilon\gamma$) for the vector of its *centre*, and with $T\gamma = MT\beta^{-1}$ for its *radius*, which radius we shall also denote by h.

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[*] Compare Fig. 32, p. 98; see also pages 100, 515, 578, from the two latter of which it may be perceived, that the *conception* of the *hodograph* admits of some purely *geometrical* applications.

(6.) The *Law of the Circular* Hodograph* is therefore a mathematical consequence of the *Law of the Inverse Square;* and conversely it will soon be proved, that *no other law* of *central force* would allow generally the *hodograph* to be a *circle.*

(7.) For the law of nature, the *Radius* (h) *of the Hodograph* is equal, by (1.) and (5.), to the quotient of the *attracting mass* (M), *divided by the double areal velocity* ($T\beta$ or c) in the *orbit;* and if we write

$$\text{X.} \ldots e = T\epsilon,$$

this positive scalar e may be called the *Excentricity* of the hodograph, regarded as a *circle excentrically situated*, with respect to the *fixed centre of force*, O.

(8.) Thus, if $e < 1$, the fixed point O is *interior* to the hodographic circle; if $e = 1$, the point O is *on the circumference;* and if $e > 1$, the centre O of force is then *exterior* to the hodograph, being however, in *all* these cases, situated *in its plane.*

(9.) The equation VII. gives,

$$\text{XI.} \ldots \epsilon - U\alpha = -\gamma^{-1}D\alpha = D\alpha \,.\, \gamma^{-1};$$

operating then on this with S.α, and writing for abridgment,

$$\text{XII.} \ldots p = \beta\gamma^{-1} = M^{-1}T\beta^2 = c^2M^{-1}, \quad \text{and} \quad \text{XIII.} \ldots SU\alpha\epsilon = \cos v,$$

so that p is a constant and positive scalar, while v is the inclination of α to $-\epsilon$, we find,

$$\text{XIV.} \ldots r + S\alpha\epsilon = p; \quad \text{or} \quad \text{XV.} \ldots r = \frac{p}{1 + e\cos v};$$

the *orbit* is therefore a *plane conic*, with the centre of force O for a *focus*, having e for its *excentricity*, and p for its *semiparameter.*

(10.) And we see, by XII., that if this semiparameter p be *multiplied* by the attracting mass M, the *product* is the *square* of the double areal velocity c; so that this constant c may be denoted by $(Mp)^{\frac{1}{2}}$, which agrees with known results.

(11.) If, on the other hand, we *divide* the mass (M) by the semiparameter (p), the *quotient* is by XII. the *square* of the *radius* ($MT\beta^{-1}$ or h) of the *hodograph.*

(12.) And if we *multiply* the same semiparameter p by this radius $MT\beta^{-1}$ of the *hodograph*, the *product* is then, by the same formula XII., the constant $T\beta$ or c of double areal velocity in the *orbit*, so that $h = Mc^{-1} = cp^{-1}$.

(13.) If we had operated with V.α on VII'., we should have found,

$$\text{XVI.} \ldots \beta = V.\alpha(\epsilon - U\alpha)\gamma = (S\alpha\epsilon + r)\gamma;$$

which would have conducted to the same equations XIV. XV. as before.

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* This *law of the circular hodograph* was deduced *geometrically*, in a paper read before the Royal Irish Academy, by the present author, on the 14th of December, 1846; but it was virtually contained in a *quaternion formula*, equivalent to the recent equation VII., which had formed part of an earlier communication, in July, 1845. (See the *Proceedings* for those dates; and especially pages 345, 347, and xxxix., xlix., of Vol. III.)

(14.) If we operate on VII. with $\mathrm{S}.\alpha$, we find this other equation,

$$\text{XVII.}\ldots -r\mathrm{D}r = \mathrm{S}\alpha\mathrm{D}\alpha = \gamma\mathrm{V}\alpha\iota\,;$$

but $$\text{XVIII.}\ldots -\gamma^2 = h^2 = \frac{M}{p} \text{ (by VI. and XII., comp. (11.))},$$

and $$\text{XIX.}\ldots -(\mathrm{V}\alpha\iota)^2 = e^2r^2 - (p-r)^2 = p\,(2r - p - r^2a^{-1}),$$

if we write $$\text{XX.}\ldots a = \frac{p}{1-e^2}\,;$$

hence squaring XVII., and dividing by r^2, we obtain the equation,

$$\text{XXI.}\ldots \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = M\left(\frac{2}{r} - \frac{1}{a} - \frac{p}{r^2}\right).$$

(15.) It is obvious that this last equation, XXI., connects the *distance*, r, with the *time*, t, as the formula XV. connects the same distance r with the *true anomaly*, v; that is, with the *angular elongation* in the orbit, from the *position of least distance*. But it would be improper here to delay on any of the elementary consequences of these two known equations: although it seemed useful to show, as above, how the equations themselves might easily be deduced by *quaternions*, and be connected with the theory of the *hodograph*.

(16.) The equation II. may be interpreted as expressing, that the *parallelogram* (comp. Fig. 82) under the vectors α and $\mathrm{D}\alpha$ of position and velocity, or under any two *corresponding vectors* (5.) of the *orbit* and *hodograph*, has a *constant plane and area*, represented by the constant vector β, which is *perpendicular* (1.) to that plane. But it is to be observed that, by (2.), these *constancies*, and this *representation*, are *not peculiar* to the law of the *inverse square*, but exist for *all other laws* of *central force*.

(17.) In general, if any scalar function R (instead of Mr^{-2}) represent the accelerating force of attraction, at the distance r from the fixed centre O, the differential equation of motion will be (instead of I.),

$$\text{XXII.}\ldots \mathrm{D}^2\alpha = Rr\alpha^{-1} = -R\mathrm{U}\alpha\,;$$

and if we still write $\mathrm{V}\alpha\mathrm{D}\alpha = \beta$, as in II., the formula IV. will give,

$$\text{XXIII.}\ldots \mathrm{D}^3\alpha = -\mathrm{D}R.\mathrm{U}\alpha - Rr^{-2}\beta\mathrm{U}\alpha, \quad \text{and XXIV.}\ldots \mathrm{V}\frac{\mathrm{D}^3\alpha}{\mathrm{D}^2\alpha} = r^{-2}\beta\,;$$

in which $$\beta = c\mathrm{U}\beta, \quad \text{if } c = \mathrm{T}\beta, \text{ as before.}$$

(18.) Applying then the general formula 414, I., we have, for *any law*[*] *of force*, the expressions,

$$\text{XXV.}\ldots \textit{Vector of Curvature of Hodograph} = \frac{1}{\mathrm{D}^2\alpha}\mathrm{V}\frac{\mathrm{D}^3\alpha}{\mathrm{D}^2\alpha} = \frac{c}{Rr^2}\mathrm{U}\alpha\beta\,;$$

$$\text{XXVI.}\ldots \textit{Radius } (h) \textit{ of Curvature of Hodograph} = Rr^2c^{-1}$$
$$= \frac{\textit{Force} \times \textit{Square of Distance}}{\textit{Double Areal Velocity in Orbit}}\,;$$

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[*] The *general value* XXVI., of the *radius of curvature of the hodograph*, was *geometrically* deduced in the Paper of 1846, referred to in a recent Note.

of which the last not only conducts, in a new way, for the *law of nature*, to the constant value (7.), $h = Mc^{-1}$, but also proves, as stated in (6.), that for *any other law* of central force *the hodograph cannot be a circle*, unless indeed the *orbit* happens to be such, and to have moreover the centre of force at *its* centre.

(19.) Confining ourselves however at present to the law of the inverse square, and writing for abridgment (comp. (5.)),

$$\text{XXVII.} \ldots \kappa = \text{OH} = \epsilon\gamma = \textit{Vector of Centre H of Hodograph},$$

which gives, by (5.) and (7.),

$$\text{XXVIII.} \ldots \mathrm{T}\kappa = eh,$$

the *origin* O of vectors being still the centre of *force*, we see by the properties of the *circle*, that the *product* of any two *opposite velocities* in the *orbit* is *constant*; and that this constant product* may be expressed as follows,

$$\text{XXIX.} \ldots (e-1)\,h\mathrm{U}\kappa \,.\, (e+1)\,h\mathrm{U}\kappa = h^2(1-e^2) = Ma^{-1},$$

by XVIII. and XX.

(20.) The expression XXIX. may be otherwise written as $\kappa^2 - \gamma^2$; and if υ be the vector of any point U *external* to the circle, but in its plane, and u the length of a *tangent* UT from that point, we have the analogous formula,

$$\text{XXX.} \ldots u^2 = \gamma^2 - (\upsilon - \kappa)^2 = \mathrm{T}\,(\upsilon - \kappa)^2 - h^2.$$

(21.) Let τ and τ' be the vectors OT, OT′ of the *two* points of contact of tangents thus drawn to the hodograph, from an external point U in its plane; then *each* must satisfy the system of the *three* following scalar equations,

$$\text{XXXI.} \ldots \mathrm{S}\gamma\tau = 0;\quad \text{XXXII.} \ldots (\tau - \kappa)^2 = \gamma^2;\quad \text{XXXIII.} \ldots \mathrm{S}\,(\tau - \kappa)(\upsilon - \kappa) = \gamma^2;$$

whereof the *first alone* represents the *plane*; the *two first* jointly represent (comp. (5.)) the *circle*; and the *third* expresses the *condition of conjugation* of the points T and U, and may be regarded as the *scalar equation of the polar* of the latter point. It is understood that $\mathrm{S}\gamma\upsilon = 0$, as well as $\mathrm{S}\gamma\kappa = 0$, &c., because γ is perpendicular (3.) to the plane.

(22.) Solving this system of equations (21.), we find the two expressions,

$$\text{XXXIV.} \ldots \tau = \kappa + \gamma\,(\gamma + u)\,(\upsilon - \kappa)^{-1};\quad \text{XXXIV}'. \ldots \tau' = \kappa + \gamma\,(\gamma - u)\,(\upsilon - \kappa)^{-1};$$

in which the scalar u has the same value as in (20.). As a verification, these expressions give, by what precedes,

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* In strictness, it is only for a *closed orbit*, that is, for the case (8.) of the centre of force being *interior* to the *hodograph* ($e < 1$), that two velocities *can* be *opposite*; their *vectors* having then, by the fundamental rules of quaternions, a *scalar* and *positive product*, which is here found to be $= Ma^{-1}$, by XXIX., in consistency with the known theory of *elliptic motion*. The result however admits of an *interpretation*, in other cases also. It is obvious that when the centre O of force is *exterior* to the hodograph, the *polar* of that point divides the *circle* into *two parts*, whereof one is *concave*, and the other *convex*, towards O; and there is no difficulty in seeing, that the *former* part corresponds to the *branch* of an *hyperbolic orbit*, which can be described under the influence of an *attracting force*: while the *latter* part answers to that *other branch* of the same complete hyperbola, whereof the description would require the force to be *repulsive*.

$$\text{XXXV.}\ldots\ \mathrm{S}(\tau-\kappa)(\tau-\upsilon)=0; \quad \text{XXXV}'.\ldots\ \mathrm{S}(\tau'-\kappa)(\tau'-\upsilon)=0;$$

and $$\text{XXXVI.}\ldots\ (\tau-\upsilon)^2=(\tau'-\upsilon)^2=-u^2.$$

In fact it is found that

$$\text{XXXVII.}\ldots\ \tau-\upsilon=u(u+\gamma)(\upsilon-\kappa)^{-1}; \quad \text{XXXVIII.}\ldots\ \mathrm{T}(u+\gamma)=\mathrm{T}(\upsilon-\kappa);$$

and $$\text{XXXIX.}\ldots\ (\tau-\upsilon)(\tau-\kappa)=u\gamma;$$

$u+\gamma$ being here a quaternion.

(23.) If υ' be the vector OU′ of *any point* U′, on the *polar* of the point U with respect to the circle, then changing τ to υ', and u to z, in XXXIV., we find this *vector form* (comp. (21.)) of the *equation of that polar*,

$$\text{XL.}\ldots\ \upsilon'=\kappa+\gamma(\gamma+z)(\upsilon-\kappa)^{-1},$$

or, by an easy transformation,

$$\text{XLI.}\ldots\ (h^2+u^2)\,\upsilon'=h^2\upsilon+u^2\kappa+z\gamma(\kappa-\upsilon),$$

in which z is an arbitrary scalar.

(24.) If then we suppose that U′ is the *intersection* of the chord TT′ with the right line OU, the condition

$$\text{XLII.}\ldots\ \mathrm{V}\upsilon'\upsilon=0 \quad \text{will give} \quad \text{XLIII.}\ldots\ z\gamma=\frac{u^2\mathrm{V}\kappa\upsilon}{\upsilon^2-\mathrm{S}\kappa\upsilon};$$

but $$\text{XLIV.}\ldots\ \mathrm{V}\kappa\upsilon\,.\,(\kappa-\upsilon)=\kappa\mathrm{S}(\kappa\upsilon-\upsilon^2)+\upsilon\mathrm{S}(\kappa\upsilon-\kappa^2);$$

the coefficient then of κ, in the expanded expression for υ', disappears as it ought to do: and we find, after a few reductions,

$$\text{XLV.}\ldots\ \upsilon'=\upsilon\left(1+\frac{u^2}{\upsilon^2-\mathrm{S}\kappa\upsilon}\right)=\frac{\gamma^2-\kappa^2+\mathrm{S}\kappa\upsilon}{\upsilon-\upsilon^{-1}\mathrm{S}\kappa\upsilon},$$

a result which might have been otherwise obtained, by eliminating a new scalar y between the two equations,

$$\text{XLVI.}\ldots\ \upsilon'=y\upsilon, \quad \mathrm{S}(y\upsilon-\kappa)(\upsilon-\kappa)=\gamma^2.$$

(25.) Introducing then two auxiliary vectors, λ, μ, such that

$$\text{XLVII.}\ldots\ \lambda=\upsilon^{-1}\mathrm{S}\kappa\upsilon, \quad \text{or} \quad \mathrm{S}\kappa\upsilon=\upsilon\lambda=\lambda\upsilon,$$

and therefore $$\text{XLVII}'.\ldots\ \lambda-\kappa=\upsilon^{-1}\mathrm{V}\kappa\upsilon, \quad \mathrm{S}\kappa\lambda=\lambda^2, \quad (\lambda-\kappa)^2=\kappa^2-\lambda^2,$$

and $$\text{XLVIII.}\ldots\ \mu=\lambda\left(1+\left(1+\frac{\gamma^2-\kappa^2}{\lambda^2}\right)^{\frac{1}{2}}\right), \quad \text{whence} \quad \mu \parallel \lambda, \quad (\mu-\kappa)^2=\gamma^2,$$

we have the very simple relation,

$$\text{XLIX.}\ldots\ (\upsilon-\lambda)(\upsilon'-\lambda)=(\mu-\lambda)^2, \quad \text{or} \quad \text{L.}\ldots\ \mathrm{LU}\,.\,\mathrm{LU}'=\mathrm{LM}^2,$$

if $\lambda=$ OL, and $\mu=$ OM. Accordingly, the point L is the foot of the perpendicular let fall from the centre H on the right line OU, while M is one of the two points M, M′ of intersection of that line with the circle; so that the equation L. expresses, that the points U, U′ are *harmonically conjugate*, with respect to the *chord* MM′, of which L is the middle point, as is otherwise evident from geometry.

(26.) The vector α of the *orbit* (or of *position*), which *corresponds* to the vector $\tau\,(=\mathrm{D}\alpha)$ of the *hodograph* (or of *velocity*), and of which the length is $\mathrm{T}\alpha=r=$ the *distance*, may be deduced from τ by the equations,

$$\text{LI.}\ldots\ \alpha=r(\kappa-\tau)\gamma^{-1}, \quad \text{and} \quad \text{LII.}\ldots\ \mathrm{V}\tau\alpha=-\beta=M\gamma^{-1};$$

whence follow the expressions,

$$\text{LIII.}\ldots\ \textit{Potential}=Mr^{-1}=(\text{say})\ P=\mathrm{S}\tau(\kappa-\tau)=\mathrm{S}\upsilon(\kappa-\tau);$$

the second expression for P being deduced from the first, by means of the relation XXXV.

(27.) The *first* expression LIII. for P shows that the potential is equal, Ist, to the *rectangle* under the *radius* of the hodograph, and the *perpendicular* from the centre O of force, on the tangent at T to that circle; and IInd, to the *square* of the *tangent* from the same point T of the hodograph, to what may be called the *Circle of Excentricity*, namely to that new circle which has OH for a diameter. And the first of these values of the potential may be otherwise deduced from the equality (7.) of the mass M, to the product hc of the radius h of the hodograph, multiplied by the constant c of double areal velocity, or by the *constant parallelogram* (16.) under any two corresponding vectors.

(28.) The *second* expression LIII. for the potential P, corresponding to the point T of the hodograph, may (by XXXIV., &c.) be thus transformed, with the help of a few reductions of the same kind as those recently employed:

$$\text{LIV.} \ldots P = \frac{M}{r} = \frac{h^2 \mathrm{S}q + u\gamma \mathrm{V}q}{h^2 + u^2}, \quad \text{if} \quad \text{LV.} \ldots q = \upsilon(\kappa - \upsilon),$$

q being thus an auxiliary quaternion; and in like manner, for the other point T′ lately considered, we have the analogous value,

$$\text{LVI.} \ldots P' = \frac{M}{r'} = \frac{h^2 \mathrm{S}q - u\gamma \mathrm{V}q}{h^2 + u^2};$$

whence

$$\text{LVII.} \ldots P.P' = \frac{h^2 (\mathrm{S}q^2 + u^2 \upsilon^2)}{h^2 + u^2};$$

and therefore,

$$\text{LVIII.} \ldots \frac{r}{M} = P^{-1} = \frac{\mathrm{S}q + u\gamma^{-1}\mathrm{V}q}{\mathrm{S}q^2 + u^2\upsilon^2}.$$

$$\text{LIX.} \ldots \frac{r'}{M} = P'^{-1} = \frac{\mathrm{S}q - u\gamma^{-1}\mathrm{V}q}{\mathrm{S}q^2 + u^2\upsilon^2};$$

and finally,

$$\text{LX.} \ldots \frac{2M}{r + r'} = \frac{2PP'}{P + P'} = \mathrm{S}q + \frac{u^2\upsilon^2}{\mathrm{S}q} = \upsilon(\lambda - \upsilon') = \text{OU} . \text{U}'\text{L}.$$

(29.) In fact, the same second expression LIII. shows, that if V and V′ be the feet of perpendiculars from T and T′ on HL, then the potentials are,

$$\text{LXI.} \ldots P = \text{OU} . \text{TV}, \quad \text{and} \quad P' = \text{OU} . \text{T}'\text{V}';$$

and it is easy to prove, geometrically, that the *segment* U′L is the *harmonic mean* between what may be called the *ordinates*, TV, T′V′, to the *hodographic axis* HL.

(30.) If we suppose the point U to take any new but near position U, in the plane, the polar chord TT′, and (in general) the length u of the tangent UT, will change; and we shall have the differential relations:

$$\text{LXII.} \ldots \mathrm{d}\tau = (\tau - \upsilon)^{-1}\mathrm{S}(\tau - \kappa)\,\mathrm{d}\upsilon;$$

$$\text{LXII}'. \ldots \mathrm{d}\tau' = (\tau' - \upsilon)^{-1}\mathrm{S}(\tau' - \kappa)\,\mathrm{d}\upsilon;$$

and

$$\text{LXIII.} \ldots \mathrm{d}u = u^{-1}\mathrm{S}(\kappa - \upsilon)\,\mathrm{d}\upsilon.$$

(31.) Conceiving next that U moves along the line OU, or LU, so that we may write,

$$\text{LXIV.} \ldots \upsilon = (x - e')(\mu - \lambda), \quad \text{if} \quad x = \frac{\text{LU}}{\text{LM}} = \frac{\text{LM}}{\text{LU}'}, \quad \text{and} \quad e' = \frac{\text{LO}}{\text{LM}},$$

we shall have,

$$\text{LXV.} \ldots \mathrm{d}\upsilon = (\mu - \lambda)\,\mathrm{d}x = \upsilon\,(x - e')^{-1}\mathrm{d}x, \text{ with } x > 1 > e',$$

if U be on LM prolonged, and if O be on the concave side of the arc TMT′; and thus, by LIII., the differential expressions (30.) become,

$$\text{LXVI.} \ldots \mathrm{d}\tau = (\upsilon - \tau)^{-1}P(x - e')^{-1}\mathrm{d}x; \quad \mathrm{d}\tau' = (\upsilon - \tau')^{-1}P'(x - e')^{-1}\mathrm{d}x;$$

and $$\text{LXVII.} \ldots \mathrm{d}u = u^{-1}\mathrm{S}q.(x - e')^{-1}\mathrm{d}x, \quad \text{with } \mathrm{S}q = \upsilon(\lambda - \upsilon);$$

so that $$\text{LXVIII.} \ldots \mathrm{Td}\tau = \frac{P\mathrm{d}x}{u(x - e')}, \quad \mathrm{Td}\tau' = \frac{P'\mathrm{d}x}{u(x - e')}, \quad \text{if} \quad \mathrm{d}x > 0.$$

Such then are the *lengths* of the two *elementary arcs* TT, and T′T,′ of the hodograph, intercepted between two near secants NTT′ and NT,T,′ drawn from the pole N of the chord MM′, and having U and U, for their own poles; and we see that these *arcs* are proportional to the *potentials*, P and P', or by LXI. to the *ordinates*, TV, T′V′, or finally to the lines NT, NT′: and accordingly we have the *inverse similarity* (comp. 118), of the two small triangles with N for vertex,

$$\text{LXIX.} \ldots \Delta\,\mathrm{NTT}, \propto' \mathrm{NT}'_{,}\mathrm{T}',$$

as appears on inspection of the annexed Figure 86.

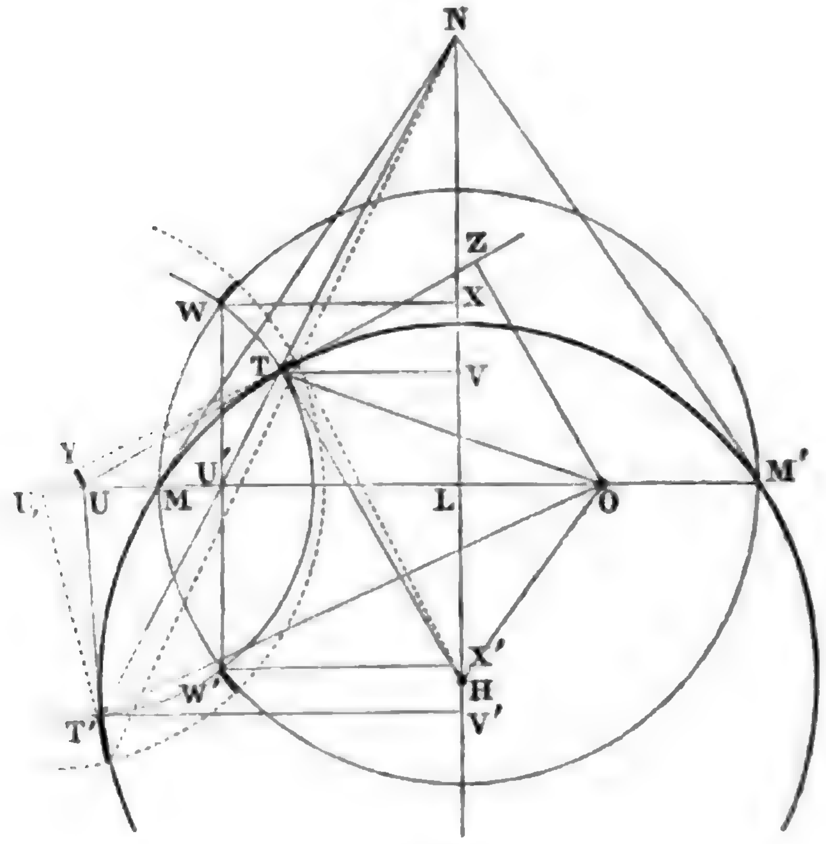

Fig. 86.

(32.) For *any motion of a point*, however complex, the *element* dt of *time* which corresponds to a given element dDα of the *hodograph*, is found by *dividing* the latter element by the *vector* D$^2\alpha$ of *accelerating force*; if then we denote by dt and dt' the times corresponding to the elements dτ and dτ' (31.), we have the expressions,

$$\text{LXX.} \ldots \mathrm{d}t = M.\,P^{-2}.\,\mathrm{Td}\tau = \frac{M\mathrm{d}x}{Pu(x - e')} = \frac{r\mathrm{d}x}{u\,(x - e')},$$

$$\text{LXX}'. \ldots \mathrm{d}t' = M.\,P'^{-2}.\,\mathrm{Td}\tau' = \frac{M\mathrm{d}x}{P'u\,(x - e')} = \frac{r'\mathrm{d}x}{u\,(x - e')},$$

because, for the motion here considered, the measure or quantity of the force is, by I. and LIII.,

$$\text{LXXI.}\ldots \text{T}D^2\alpha = Mr^{-2} = M^{-1}P^2.$$

(33.) The *times of hodographically describing* the *two small circular arcs*, T,T and T′T,′, are therefore *inversely* proportional to the *potentials*, or *directly* proportional to the *distances* in the *orbit;* and their *sum* is,

$$\text{LXXII.}\ldots \mathrm{d}t + \mathrm{d}t' = \left(\frac{M}{P} + \frac{M}{P'}\right)\frac{u^{-1}\mathrm{d}x}{x-e'} = \frac{(r+r')\,\mathrm{d}x}{u\,(x-e')};$$

that is, by LX. and LXIV.,

$$\text{LXXIII.}\ldots \mathrm{d}t + \mathrm{d}t' = \frac{2Mx\mathrm{d}x}{u\,(x-e')^2 g^2}, \quad \text{if} \quad \text{LXXIV.}\ldots g = \text{T}(\mu - \lambda) = \overline{\text{LM}}.$$

(34.) We have also the relations,

$$\text{LXXV.}\ldots u = (x^2-1)^{\frac{1}{2}} g, \quad \text{and} \quad \text{LXXVI.}\ldots \frac{M}{a} = (1-e'^2)g^2;$$

so that the sum of the two small times may be thus expressed,

$$\text{LXXVII.}\ldots \mathrm{d}t + \mathrm{d}t' = \frac{2(a(1-e'^2))^{\frac{3}{2}}}{M^{\frac{1}{2}}}\cdot\frac{(1-e'x^{-1})^{-2}\,\mathrm{d}x}{x\,(x^2-1)^{\frac{1}{2}}},$$

or finally,

$$\text{LXXVIII.}\ldots \mathrm{d}t + \mathrm{d}t' = 2\left(\frac{a^3(1-e'^2)^3}{M}\right)^{\frac{1}{2}}\cdot\frac{\mathrm{d}w}{(1-e'\cos w)^2},$$

if $\text{LXXIX.}\ldots x = \sec w, \quad \text{or} \quad w = \angle\,\text{MLW}$ in Fig. 86,

in which Figure U′W is an ordinate of a semicircle, with the chord MM′ of the hodograph for its diameter.

(35.) The two near secants (31.), from the pole N of that chord, have been here supposed to cut the half chord LM *itself*, as in the cited Figure 86; but if they were to cut the *other* half chord LM′, it is easy to prove that the formulæ LXXVIII. LXXIX. would still hold good, the only difference being that the angle w, or MLW, would be now *obtuse*, and its secant $x < -1$.

(36.) A *circle*, with U for centre, and u for radius, *cuts* the hodograph *orthogonally* in the points T and T′; and in like manner a *near circle*, with U, for centre, and $u + \mathrm{d}u$ for radius, is *another orthogonal*, cutting the same hodograph in the near points T, and T,′ (31.). And by conceiving a *series* of such orthogonals, and observing that the differential expression LXXVIII. depends only on the *four scalars*, $M^{-1}a^3$, e', w, and $\mathrm{d}w$, which are all known when the *mass* M and the *five points* O, L, M, U, U, are given, so that they do not change when we retain that mass and those points, but alter the radius h of the hodograph, or the perpendicular HL let fall from its centre H on the fixed chord MM′, we see that the *sum of the times* (comp. (33.), of *hodographically describing any two circular arcs*, such as T,T and T′T,′, even if they be *not small*, but intercepted between *any two secants* from the *pole* N of the *fixed chord*, is *independent of the radius* (h), or of the *height* HL of the *centre* H of the hodograph.

(37.) *If then two circular hodographs*, such as the two in Fig. 86, having a *common chord* MM′, which passes through, or tends towards, a *common centre of force* O, with a *common mass* M there situated, be *cut by any two common orthogonals*, the *sum of the two times* of *hodographically describing* (33.) the *two intercepted arcs* (small or large) will be the *same* for those *two* hodographs.

(38.) And as a *case* of this *general* result, we have the following *Theorem*[*] of *Hodographic Isochronism* (or *Synchronism*):

"*If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.*"

For example, in Fig. 86, we have the equation,

LXXX. . . *Time of* TMT′ = *time of* WMW′.

(39.) The time of thus describing the arc TMT′ (Fig. 86), if this arc be throughout *concave*[†] towards O (so that $x>1>e'$, as in LXV.), is expressed (comp. LXXVIII.) by the definite integral,

$$\text{LXXXI.}\ldots \textit{Time of } \mathrm{TMT'} = 2\left(\frac{a^3(1-e'^2)^3}{M}\right)^{\frac{1}{2}}\int_0^w \frac{\mathrm{d}w}{(1-e'\cos w)^2};$$

and the time of describing the *remainder* of the hodographic circle, if *this* remaining arc T′M′T be throughout concave towards the centre O of force, is expressed by this other integral,

$$\text{LXXXII.}\ldots \textit{Time of } \mathrm{T'M'T} = 2\left(\frac{a^3(1-e'^2)^3}{M}\right)^{\frac{1}{2}}\int_w^\pi \frac{\mathrm{d}w}{(1-e'\cos w)^2}.$$

(40.) Hence, for the *case* of a *closed orbit* ($e'^2<1$, $e<1$, $a>0$), if n denote the *mean angular velocity*, we have the formula,

$$\text{LXXXIII.}\ldots \textit{Periodic Time} = \frac{2\pi}{n} = 2\left(\frac{a^3}{M}\right)^{\frac{1}{2}}(1-e'^2)^{\frac{3}{2}}\int_0^\pi \frac{\mathrm{d}w}{(1-e'\cos w)^2}$$
$$= 2\pi\left(\frac{a^3}{M}\right)^{\frac{1}{2}};$$

or

$$\text{LXXXIV.}\ldots M = a^3n^2, \text{ as usual.}$$

The same result, for the same case of *elliptic motion*, may be more rapidly obtained, by conceiving the chord MM′ through O to be perpendicular to OH; for, in this position of that chord, its middle point L coincides with O, and $e'=0$ by LXIV.

(41.) In general, by LXXVI., we are at liberty to make the substitution,

$$\text{LXXXV.}\ldots \left(\frac{a^3(1-e'^2)^3}{M}\right)^{\frac{1}{2}} = \frac{M}{g^3}, \text{ with } g = \textit{half chord of the hodograph};$$

supposing then that $e'=-1$, or placing O at the extremity M′ of the chord, we have by LXXXI.,

$$\text{LXXXVI.}\ldots \textit{Parabolic time of } \mathrm{TMT'} = \frac{2M}{g^3}\int_0^w \frac{\mathrm{d}w}{(1+\cos w)^2};$$

for, when the *centre of force* is thus situated *on the circumference* of the hodographic circle, we have by (8.) the *excentricity* $e=1$, and the *orbit* becomes by XV. a *para-*

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[*] This Theorem, in which it is understood that the *common centre* of force (O) is occupied by a *common mass* (M), was communicated to the Royal Irish Academy on the 16th of March, 1847. (See the *Proceedings* of that date, Vol. III., page 417.) It has since been treated as a subject of investigation by several able writers, to whom the author cannot hope to do justice on this subject, within the very short space which now remains at his disposal.

[†] Compare the Note to page 721.

bola. For *hyperbolic motion* ($e'^2 > 1$, $e > 1$, $a < 0$), the formula LXXXI. (with or without the substitution LXXXV.) is to be employed if $e' < -1$, that is, if O be on LM' prolonged; and the formula LXXXII., if $e' > 1$, $e' < \sec w$, that is, if O be situated between M and U.

(42.) For *any law* of *central force*, if P, P' be the points of the *orbit* which correspond to the points T, T' of the *hodograph*, and if Q be the point of meeting of the tangents to the orbit at P, P', as in the annexed Figure 87, while the tangents to the hodograph at T, T' meet as before in U, we shall have the parallelisms,

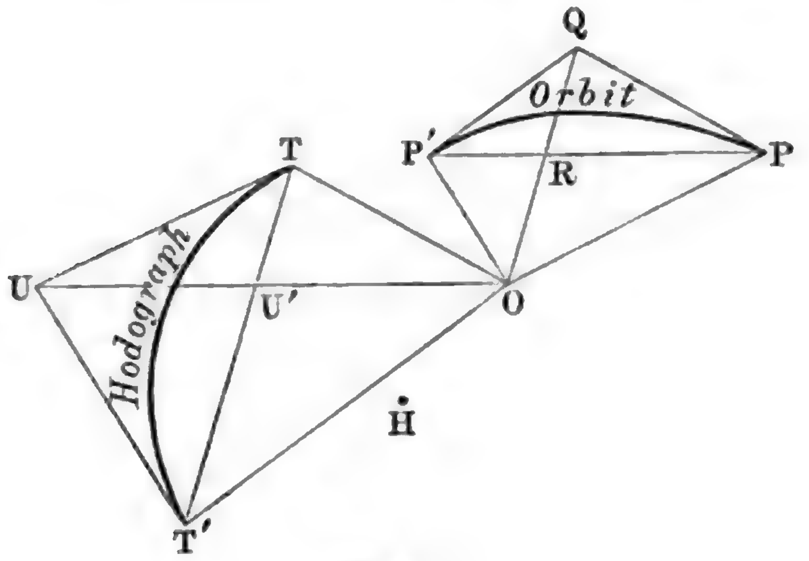

Fig. 87.

$$\text{LXXXVII.} \ldots \text{OP} \parallel \text{UT}, \quad \text{OP}' \parallel \text{T}'\text{U}, \quad \text{PQ} \parallel \text{OT}, \quad \text{QP}' \parallel \text{OT}';$$

writing then,

$$\text{LXXXVIII.} \ldots \text{OP} = \alpha, \ \text{OP}' = \alpha', \ \text{OT} = \text{D}\alpha = \tau, \ \text{OT}' = \text{D}\alpha' = \tau', \ \text{OU} = \upsilon, \ \text{OQ} = \omega,$$

most of which notations have occurred before, we have the equations,

$$\text{LXXXIX.} \ldots 0 = \text{V}\alpha(\tau - \upsilon) = \text{V}\alpha'(\upsilon - \tau') = \text{V}\tau(\omega - \alpha) = \text{V}\tau'(\alpha' - \omega);$$

thus $\quad \text{XC.} \ldots \text{V}\alpha\upsilon = \text{V}\alpha\tau = \beta = \text{V}\alpha'\tau' = \text{V}\alpha'\upsilon, \quad \alpha' - \alpha \parallel \upsilon, \quad \text{PP}' \parallel \text{OU},$

and $\quad \text{XCI.} \ldots \text{V}\tau\omega = \text{V}\tau\alpha = -\beta = \text{V}\tau'\alpha' = \text{V}\tau'\omega', \quad \tau - \tau' \parallel \omega, \quad \text{T}'\text{T} \parallel \text{OQ}.$

Geometrically, the *constant parallelogram* (16.) under OP, OT, or under OP', OT', is equal, by LXXXVII., to each of the four following parallelograms: I. under OP, OU; II. under OP', OU; III. under OQ, OT; and IV. under OQ, OT'; whence PP' ∥ OU, and T'T ∥ OQ, as before.

(43.) The *parallelism* XC. may be otherwise deduced for the law of the *inverse square*, with recent notations, from the quaternion formulæ,

$$\text{XCII.} \ldots \frac{\alpha' - \alpha}{r + r'} = \frac{u}{\lambda - \upsilon} = \frac{\upsilon - \upsilon'}{u}, \quad \text{in which,} \quad \text{XCII}'. \ldots \upsilon' = \frac{r\tau + r'\tau'}{r + r'},$$

and which may be obtained in various ways; whence it may also be inferred, that if s denote the *length* $\text{T}(\alpha' - \alpha)$ of the *chord* PP' of the *orbit*, then (comp. Fig. 86),

$$\text{XCIII.} \ldots \frac{s}{r + r'} = \frac{u}{\text{T}(\lambda - \upsilon)} = \overline{\text{UT}} : \overline{\text{UL}} = \&\text{c.} = \sin w;$$

w being the same auxiliary angle as in (34.), &c.

(44.) It is easy to prove that

$$\text{XCIV.} \ldots \lambda - \tau = \left(1 + \frac{u}{\gamma}\right)\frac{P}{v}, \quad \lambda - \tau' = \left(1 - \frac{u}{\gamma}\right)\frac{P}{v},$$

whence

$$\text{XCV.} \ldots \mathrm{T}\,\frac{\tau' - \lambda}{\tau - \lambda} = \frac{P'}{P} = \frac{r}{r'}, \quad \text{and} \quad \text{XCVI.} \ldots P'^{-1}(\tau' - \lambda)\,v = \mathrm{K}.P^{-1}(\tau - \lambda)\,v;$$

the lines LT, LT′ are therefore in *length* proportional to the *potentials*, P, P'; and their *directions* are *equally inclined* to that of OU, but at *opposite sides* of it, so that the line LU *bisects the angle* TLT′. Accordingly (see Fig. 86), the three points T, L, T′ are on the *circle* (not drawn in the Figure) which has HU for diameter; so that the angles ULT′, TLU are equal to each other, as being respectively equal to the angles UTT′, TT′U, which the chord TT′ of the hodograph makes with the tangents at its extremities: the triangles TLV, T′LV′ are therefore similar, and $\overline{\mathrm{LT}}$ is to $\overline{\mathrm{LT}}'$ as TV to T′V′, that is, by LXI., as P to P', or as r' to r.

(45.) Again, calculation with quaternions gives,

$$\text{XCVII.} \ldots \frac{(v - \tau)(\lambda - \tau)}{v' - \tau} = \frac{(v - \tau')(\lambda - \tau')}{v' - \tau'} = (v - \kappa)(v - \lambda)(v - \kappa)^{-1},$$

whence

$$\text{XCVIII.} \ldots \mathrm{T}\,\frac{v' - \tau}{\lambda - \tau} = \mathrm{T}\,\frac{v' - \tau'}{\lambda - \tau'} = \mathrm{T}\,\frac{\tau - v}{\lambda - v} = \overline{\mathrm{UT}} : \overline{\mathrm{UL}} = \sin w;$$

such then is the *common ratio*, of the *segments* $\overline{\mathrm{TU}'}$, $\overline{\mathrm{U'T}'}$ of the *base* TT′ of the triangle TLT′, to the adjacent *sides* $\overline{\mathrm{LT}}$, $\overline{\mathrm{LT}}'$, which are to *each other* as r' to r (44.); and because this ratio is also that of s to $r + r'$, by (43.), we have the proportion,

$$\text{XCIX.} \ldots \overline{\mathrm{OP}} : \overline{\mathrm{OP}}' : \overline{\mathrm{PP}}' = r : r' : s = \overline{\mathrm{LT}}' : \overline{\mathrm{LT}} : \overline{\mathrm{TT}}',$$

and the formula of inverse similarity (118),

$$\text{C.} \ldots \Delta\,\mathrm{LT'T} \propto' \mathrm{OPP'}.$$

Accordingly (comp. the two last Figures), the base angles OPP′, OP′P of the second triangle are respectively equal, by the parallelisms (42.), to the angles TUL, T′UL, and therefore, by the circle (44.), to the base angles TT′L, T′TL, of the first triangle but the two rotations, round O from P to P′, and round L from T′ to T, are oppositely directed.

(46.) The investigations of the three last subarticles have not assumed any knowledge of the *form* of the *orbit* (as *elliptic*, &c.), but only the *law* of *attraction* according to the *inverse square*, or by (6.) the *Law of the Circular Hodograph.* And the same general principles give not only the expression LXXVI. for the constant Ma^{-1}, but also (by LX. LXIV. LXXIV. LXXIX.) this other expression,

$$\text{CI.} \ldots \frac{2M}{r + r'} = (1 - e' \cos w)\,g^2; \quad \text{whence} \quad \text{CII.} \ldots \frac{r + r'}{2a} = \frac{1 - e'^2}{1 - e' \cos w},$$

which last may be considered as a quadratic in e', assigning two values (real or imagiuary) for that scalar, when the first member of CII. and the angle w are given; the sine of this latter angle being already expressed by XCIII.

(47.) Abstracting, then, from any *ambiguity*[*] of solution, we see, by the definite integrals in (39.), that the *time of describing an arc* PP' *of an orbit*, with the law of the *inverse square*, is a *function* (comp. (36.)) *of the three ratios*,

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[*] That there *ought* to be some such ambiguity is evident from the consideration that when a *focus* O, and *two points* P, P′ of an *elliptic orbit* are *given*, it is still

$$\text{CIII.} \ldots \frac{a^3}{M}, \quad \frac{r+r'}{a}, \quad \frac{s}{r+r'};$$

which is a form *of Lambert's Theorem*, but presents itself *here* as *deduced* from the recently stated *Theorem of Hodographic Isochronism* (38.), *without the employment of any property of conic sections.*

(48.) The differential equation I. of the present relative motion may be thus written (comp. 418, I., and generally the preceding Series 418):

$$\text{CIV.} \ldots \mathrm{S}.\mathrm{D}^2\alpha\delta\alpha + \delta P = 0, \quad \text{whence} \quad \text{CV.} \ldots T = P + H,$$

as in 418, X., if we now write,

$$\text{CVI.} \ldots T = -\tfrac{1}{2}\mathrm{D}\alpha^2 = -\tfrac{1}{2}\tau^2, \quad \text{and} \quad \text{CVII.} \ldots H = \frac{-M}{2a};$$

in fact (by LIII., comp. (20.) (21.)),

$$\text{CVIII.} \ldots -2H = 2(P - T) = 2P + \tau^2 = \kappa^2 - \gamma^2 = \frac{M}{a}.$$

(49.) Integrating CIV. by parts, &c., and writing (as in 418, XII. XXII.),

$$\text{CIX.} \ldots F = \int_0^t (T + P)\,\mathrm{d}t, \quad \text{and} \quad \text{CX.} \ldots V = \int_0^t 2T\mathrm{d}t,$$

so that *F* may again be called the *Principal Function* and *V* the *Characteristic Function* of the motion, we have the variations,

$$\text{CXI.} \ldots \delta F = \mathrm{S}\tau\delta\alpha - \mathrm{S}\tau'\delta\alpha' - H\delta t; \qquad \text{CXII.} \ldots \delta V = \mathrm{S}\tau\delta\alpha - \mathrm{S}\tau'\delta\alpha' + t\delta H;$$

in which α, α' (instead of α_0, α) denote now what may be called the *initial* and *final vectors* (OP, OP') of the *orbit;* whence follow the *partial derivatives*,

$$\text{CXIII.} \ldots \mathrm{D}_\alpha F = \mathrm{D}_\alpha V = \tau; \qquad \text{CXIII}'. \ldots \mathrm{D}_{\alpha'}F = \mathrm{D}_{\alpha'}V = -\tau';$$
$$\text{CXIV.} \ldots (\mathrm{D}_t F) = -H; \quad \text{and} \quad \text{CXV.} \ldots \mathrm{D}_H V = t;$$

F being here a scalar function of α, α', t, while *V* is a scalar function of α, α', H, if *M* be treated as given.

(50.) The two *vectors* α, α' can enter into these two *scalar functions*, only through their dependent scalars r, r', s (comp. 418, (17.)); but

$$\text{CXVI.} \ldots \delta r = -r^{-1}\mathrm{S}\alpha\delta\alpha, \quad \delta r' = -r'^{-1}\mathrm{S}\alpha'\delta\alpha', \quad \delta s = -s^{-1}\mathrm{S}(\alpha' - \alpha)(\delta\alpha' - \delta\alpha);$$

confining ourselves then, for the moment, to the function *V*, and observing that we have by CXII. the formula,

$$\text{CXVII.} \ldots \mathrm{S}(\tau\delta\alpha - \tau'\delta\alpha') = \mathrm{D}_r V.\delta r + \mathrm{D}_{r'} V.\delta r' + \mathrm{D}_s V.\delta s,$$

in which the variations $\delta\alpha$, $\delta\alpha'$ are arbitrary*, we find the expressions,

$$\text{CXVIII.} \ldots \tau = -\alpha r^{-1}\mathrm{D}_r V + (\alpha' - \alpha)s^{-1}\mathrm{D}_s V;$$
$$\text{CXVIII}'. \ldots \tau' = +\alpha' r'^{-1}\mathrm{D}_r' V + (\alpha' - \alpha)s^{-1}\mathrm{D}_s V;$$

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permitted to conceive the motion to be performed along *either* of the *two elliptic arcs*, PP', P'P, which together make up the whole periphery. But into details of this kind we cannot enter here.

which give these others,

$$\text{CXIX.} \ldots \mathrm{D}_r V = r\mathrm{V}(\alpha' - \alpha)\,\tau : \mathrm{V}\alpha\alpha' ;$$

$$\text{CXIX}'. \ldots \mathrm{D}_{r'} V = r'\mathrm{V}(\alpha - \alpha')\,\tau' : \mathrm{V}\alpha\alpha' ;$$

and

$$\text{CXX.} \ldots \mathrm{D}_s V = s\beta : \mathrm{V}\alpha\alpha',$$

because

$$\mathrm{V}\alpha\tau = \mathrm{V}\alpha'\tau' = \beta.$$

(51.) But, by XCII'.,

$$\text{CXXI.} \ldots r\tau + r'\tau' = (r + r')\,\upsilon' \parallel \upsilon \parallel \alpha' - \alpha,$$

the chord TT' of the hodograph, in Figures 86, 87, being divided at U' into segments TU', U'T', which are inversely as the distances r, r', or as the lines OP, OP' in the orbit; we have therefore the partial differential equation,

$$\text{CXXII.} \ldots \mathrm{D}_r V = \mathrm{D}_{r'} V, \quad \text{and similarly,} \quad \text{CXXIII.} \ldots \mathrm{D}_r F = \mathrm{D}_{r'} F;$$

so that *each* of the two functions, F and V, depends on the ***distances*** r, r', only by depending on their *sum*, $r + r'$.

(52.) Hence, if for greater generality we now treat M as ***variable***, the *Principal Function* F, and therefore by CXIV. its partial derivative $H = -(\mathrm{D}_t F)$, are functions of the *four* scalars,

$$\text{CXXIV.} \ldots r + r', \quad s, \quad t, \quad \text{and} \quad M.$$

(53.) And in like manner, the *Characteristic Function* (or *Action-Function*) V, and *its* partial derivative (by CXV.) the *Time*, $t = \mathrm{D}_H V$, may be considered as functions of this *other system* of four scalars (comp. (47.)),

$$\text{CXXV.} \ldots r + r', \quad s, \quad H, \quad \text{and} \quad M;$$

no knowledge whatever being here assumed, of the form or properties of the *orbit*, but only of the *law* of attraction.

(54.) But this dependence of the *time*, t, on the four scalars CXXV., is a new form of *Lambert's Theorem* (47.); which celebrated theorem is thus obtained in a new way, by the foregoing *quaternion analysis*.

(55.) Squaring the equations CXVIII. CXVIII'., attending to the relation CXXII., and changing signs, we get these new partial differential equations,

$$\text{CXXVI.} \ldots 2P + 2H = (\mathrm{D}_r V)^2 + (\mathrm{D}_s V)^2 + \frac{r^2 - r'^2 + s^2}{rs}\,\mathrm{D}_r V . \mathrm{D}_s V;$$

$$\text{CXXVI}'. \ldots 2P' + 2H = (\mathrm{D}_r V)^2 + (\mathrm{D}_s V)^2 + \frac{r'^2 - r^2 + s^2}{r's}\,\mathrm{D}_r V . \mathrm{D}_s V;$$

because

$$\text{CXXVII.} \ldots \alpha^2 = -r^2, \quad \alpha'^2 = -r'^2, \quad (\alpha' - \alpha)^2 = -s^2.$$

Hence, by merely algebraical combinations (because $P = Mr^{-1}$, and $P' = Mr'^{-1}$), we find:

$$\text{CXXVIII.} \ldots \tfrac{1}{2}\left((\mathrm{D}_r V)^2 + (\mathrm{D}_s V)^2\right) = H + \frac{M}{r + r' + s} + \frac{M}{r + r' - s};$$

$$\text{CXXIX.} \ldots \mathrm{D}_r V . \mathrm{D}_s V = \frac{M}{r + r' + s} - \frac{M}{r + r' - s};$$

$$\text{CXXX.} \ldots (\mathrm{D}_r V + \mathrm{D}_s V)^2 = 2H + \frac{4M}{r + r' + s} = M\left(\frac{4}{r + r' + s} - \frac{1}{a}\right);$$

$$\text{CXXX}'. \ldots (\mathrm{D}_r V - \mathrm{D}_s V)^2 = 2H + \frac{4M}{r + r' - s} = M\left(\frac{4}{r + r' - s} - \frac{1}{a}\right).$$

(56.) But, by CXII. CXVII. CXXII., we have the variation,

$$\text{CXXXI.} \ldots \delta V - t\delta H = \tfrac{1}{2}(\mathrm{D}_r V + \mathrm{D}_s V)\,\delta(r + r' + s) + \tfrac{1}{2}(\mathrm{D}_r V - \mathrm{D}_s V)\,\delta(r + r' - s);$$

and the function V vanishes with t, and therefore with s, at least at the commencement of the motion; whence it is easy to infer the expressions,*

$$\text{CXXXII.} \ldots V = \int_{-s}^{s} \left(\frac{M}{r + r' + s} + \frac{H}{2} \right)^{\frac{1}{2}} \mathrm{d}s = \int_{-s}^{s} \left(\frac{M}{r + r' + s} - \frac{M}{4a} \right)^{\frac{1}{2}} \mathrm{d}s;$$

$$\text{CXXXIII.} \ldots t = \tfrac{1}{4} \int_{-s}^{s} \left(\frac{M}{r + r' + s} + \frac{H}{2} \right)^{-\frac{1}{2}} \mathrm{d}s = \tfrac{1}{2} \int_{-s}^{s} \left(\frac{4M}{r + r' + s} - \frac{M}{a} \right)^{-\frac{1}{2}} \mathrm{d}s.$$

As a verification,† when t and s are small, and therefore r' nearly $= r$, we have thus the approximate values,

$$\text{CXXXIV.} \ldots V = (2P + 2H)^{\frac{1}{2}} s = (2T)^{\frac{1}{2}} s = 2Tt;$$
$$\text{CXXXV.} \ldots t = (2P + 2H)^{-\frac{1}{2}} s = (2T)^{-\frac{1}{2}} s;$$

in which s may be considered to be a *small arc* of the *orbit*, and $(2T)^{\frac{1}{2}}$ the *velocity* with which that arc is described.

(57.) Some not inelegant constructions, deduced from the theory of the hodograph, might be assigned for the case of a *closed orbit*, to represent the *excentric* and *mean anomalies*; but whether the orbit be closed or *not*, the *arc* TMT′ of the *hodographic circle*, in Fig. 86, represents the arc of *true anomaly* described: for it subtends at the hodographic *centre* H an *angle* THT′, which is equal to the *angular motion* POP′ in the orbit.

(58.) We may add that, whatever the *special form* of the orbit may be, the equations CXVIII. CXVIII′. give, by CXXII.,

$$\text{CXXXVI.} \ldots \tau' - \tau = (\mathrm{U}\alpha' + \mathrm{U}\alpha)\,\mathrm{D}_r V;$$

from which it follows that the *chord* TT′ of the hodograph is *parallel* to the *bisector* of the *angle* POP′ in the orbit: and therefore, by XCI., that this angle is bisected by OQ in Fig. 87, so that the *segments* PR, RP′, in that Figure, of the *chord* PP′ *of the orbit*, are *inversely proportional* to the segments TU′, U′T′ of the *chord* TT′ *of the hodograph*.

(59.) We arrive then thus, in a new way, and as a new verification, at this known theorem: that *if two tangents* (QP, QP′) *to a conic section be drawn from*

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* Expressions by definite integrals equivalent to these, for the *action* and *time* in the relative motion of a binary system, were deduced by the present writer, but by an entirely different analysis, in the *First Essay*, &c., already cited, and will be found in the *Phil. Trans.* for 1834, Part II., pages 285, 286. It is supposed that the radical in CXXXIII. does not become infinite within the extent of the integration; if it did so become, transformations would be required, on which we cannot enter here.

† An analogous verification may be applied to the definite integral LXXXI.; in which however it is to be observed that *both* $r + r'$ and s *vary*, along with the variable w: whereas, in the recent integrals CXXXII. CXXXIII., $r + r'$ is treated as *constant*.

any common point (Q), *they subtend equal angles at a focus* (O), whatever the special form of the conic may be.

(60.) And although, in several of the preceding sub-articles, *geometrical constructions* have been used only to *illustrate* (and so to *confirm*, if confirmation were needed) *results* derived through *calculation with quaternions*; yet the eminently *suggestive* nature of the present Calculus enables us, in this as in many other questions, to *dispense with its own processes*, when once they have indicated a definite train of *geometrical investigation*, to serve as their substitute.

(61.) Thus, after having in any manner been led to perceive that, for the motion above considered, the *hodograph* is a *circle** (5.), of which the *radius* HT is equal (7.) to the attracting *mass* M, divided by the constant *parallelogram* (16.) under the vectors OP, OT of position and velocity, in the recent Figures 86 and 87, which parallelogram is equal to the *rectangle* under the distance OP in the orbit, and the *perpendicular* OZ let fall from the centre O of force on the tangent UT to the hodograph, we see *geometrically* that the *potential* P, or the mass divided by the distance, for the point P of the orbit corresponding to the point T of the hodograph, is equal (as in (27.)) to the rectangle under HT and OZ, and therefore, by the similar triangles HTV, UOZ, to the rectangle under OU and TV (as in (29.)).

(62.) In like manner, the three potentials corresponding to the second point T′ of the first hodograph, and to the points W and W′ of the second hodograph, in Fig. 86, are respectively equal to the rectangles under the same line OU, and the three other perpendiculars T′V′, WX, W′X′, on what we have called (29.) the *hodographic axis*, HL; so that, for these *two pairs of points*, in which the *two circular hodographs*, with a *common chord* MM′, are cut by a *common orthogonal* with U for centre, the *four potentials* are directly proportional to the *four hodographic ordinates* (29.).

(63.) And because the force (Mr^{-2}) is equal to the *square* of the *potential* (Mr^{-1}), divided by the *mass* (M), the *four forces* are directly as the *squares* of the *four ordinates* corresponding; each force, when divided by the square of the corresponding hodographic ordinate, giving the constant or *common quotient*,

$$\text{CXXXVII.} \;.\;.\;.\; \overline{\text{OU}}^2 : M.$$

(64.) It has been already seen (31.) to be a geometrical consequence of the two pairs of similar triangles, NTT$_{,}$, NT′$_{,}$T′, and NTV, NT′V′, that the *two small arcs* of the *first hodograph*, near T and T′, intercepted between two near secants from the pole N of the *fixed chord* MM′, or between two near orthogonal circles, with U and U$_{,}$ for centres, are proportional to the *two ordinates*, TV, T′V′.

(65.) Accordingly, if we draw, as in Fig. 86, the *near radius* (represented by a dotted line from H) of the first hodograph, and also the *small perpendicular* UY, erected at the centre U of the first orthogonal to the tangent UT, and terminated in Y by the tangent from the near centre $U_,$, the two new pairs of similar triangles, $THT_,$, UTY, and THV, $UU_,Y$, give the proportion,

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* This follows, among other ways, from the general value XXVI. for the *radius of curvature* of the hodograph, with *any law* of *central force*; which value was *geometrically deduced*, as stated in the Note to page 720, compare the Note to page 719, by the present writer, in a Paper read before the Royal Irish Academy in 1846, and published in their *Proceedings*. In fact, that *general expression* for the radius of hodographic curvature may be obtained with great facility, by dividing the element fdt of the hodograph (in which f denotes the force), by the corresponding element $cr^{-2}dt$ of angular motion in the orbit.

$$\text{CXXXVIII.} \ldots \overline{TT_,} : \overline{TV} = \overline{UU_,} : \overline{UT};$$

which not merely confirms what has just been stated (64.), for the case of the *first* hodograph, but proves that the *four small arcs*, of the *two circular hodographs* in Fig. 86, intercepted between the two near orthogonals, are directly proportional to the *four ordinates* already mentioned.

(66.) But the *time* of describing any small hodographic arc is the quotient (32.) of that *arc* divided by the *force;* and therefore, by (63.), (65.), the *four small times* are *inversely proportional* to the *four ordinates.* And the *harmonic mean* U'L between the *two* ordinates TV, T'V' of the *first* hodograph, does not vary when we pass to the *second*, or to *any other hodograph*, with the *same fixed chord* MM', and the *same orthogonal circles;* it follows then, *geometrically*, that the *sum* (33.) of the *two small times* is the *same*, in any *one* hodograph as in any *other*, under the conditions above supposed: and that this sum is equal to the expression,

$$\text{CXXXIX.} \ldots \frac{2M.\overline{UU'}}{\overline{OU}^2.\overline{UT}.\overline{U'L}} = \frac{2M.\overline{UU'}.\overline{UL}}{\overline{OU}^2.\overline{LM}^2.\overline{UT}},$$

which agrees with the formula LXXIII.

(67.) On the whole, then, it is found that the *Theorem of Hodographic Isochronism* (38.) admits of being *geometrically*[*] *proved*, although by processes *suggested* (60.) by quaternions: and sufficient *hints* have been already given, in connexion with Figure 87, as regards the *geometrical passage* from *that* theorem to the well-known *Theorem of Lambert*, without necessarily employing any property of *conic sections.*

420. As a *fifth specimen*, we shall deduce by quaternions an equation, which is adapted to assist in the determination of the *distance* of a *comet*, or *new planet*, from the *earth.*

(1.) Let M be the mass of the sun, or (somewhat more exactly) the sum of the masses of sun and earth; and let α and ω be the heliocentric vectors of earth and comet. Write also,

$$\text{I.} \ldots \text{T}\alpha = r, \quad \text{T}\omega = w, \quad \text{T}(\omega - \alpha) = z, \quad \text{U}(\omega - \alpha) = \rho,$$

so that r and w are the distances of earth and comet from the sun, while z is their distance from each other, and ρ is the unit-vector, directed from earth to comet. Then (comp. 419, I.),

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[*] It appears from an unprinted memorandum, to have been nearly thus that the author orally deduced the theorem, in his communication of March, 1847, to the Royal Irish Academy; although, as usually happens in cases of invention, his own previous processes of investigation had involved principles and methods, of a much less simple character.

$$\text{II.} \ldots \ \mathrm{D}^2\alpha = -Mr^{-3}\alpha, \quad \mathrm{D}^2\omega = -Mw^{-3}\omega,$$

and

$$\text{III.} \ldots \ \mathrm{D}^2 . z\rho = \mathrm{D}^2(\omega - \alpha) = M(r^{-3} - w^{-3})\,\alpha - Mzw^{-3}\rho,$$

with

$$\text{IV.} \ldots \ w^2 = -(\alpha + z\rho)^2 = r^2 + z^2 - 2z\mathrm{S}\alpha\rho.$$

(2.) The vector α, with its tensor r, and the mass M, are given by the theory of the earth (or sun); and ρ, $\mathrm{D}\rho$, $\mathrm{D}^2\rho$ are deduced from three (or more) near observations of the comet; operating then on III. with $\mathrm{S}.\rho\mathrm{D}\rho$, we arrive at the formula,

$$\text{V.} \ldots \ \frac{\mathrm{S}\rho\mathrm{D}\rho\mathrm{D}^2\rho}{\mathrm{S}\rho\mathrm{D}\rho\mathrm{U}\alpha} = \frac{r}{z}\left(\frac{M}{r^3} - \frac{M}{w^3}\right);$$

which becomes by IV., when cleared of fractions and radicals, and divided by z, an algebraical equation of the seventh degree, whereof one root is the sought distance[*] z, of the comet (or planet) from the earth.

421. As a *sixth specimen*, we shall indicate a method, suggested by quaternions, of developing and geometrically decomposing the disturbing force of the sun on the moon, or of a relatively superior on a relatively inferior planet.

(1.) Let α, σ be the geocentric vectors of moon and sun; r, s their geocentric distances ($r = \mathrm{T}\alpha$, $s = \mathrm{T}\sigma$); M the sum of the masses of earth and moon; and S the mass of the sun; then the differential equation of motion of the moon about the earth may be thus written (comp. 418, 419),

$$\text{I.} \ldots \ \mathrm{D}^2\alpha = M.\phi\alpha + S.(\phi\sigma - \phi(\sigma - \alpha)),$$

if D be still the mark of derivation relatively to the time, and

$$\text{II.} \ldots \ \phi\alpha = \phi(\alpha) = \alpha^{-1}\mathrm{T}\alpha^{-1};$$

so that $\phi\alpha$ is here a *vector-function* of α, but *not* a *linear* one.

(2.) If we confine ourselves to the *term* $M\phi\alpha$, in the second member of I., we fall back on the equation 419, I., and so are conducted anew to the laws of *undisturbed relative elliptic motion*.

(3.) If we denote the *remainder* of that second member by η, then η may be called the *Vector of Disturbing Force;* and we propose now to *develope* this vector, according to *descending powers* of $\mathrm{T}(\sigma : \alpha)$, or according to *ascending* powers of the *quotient* $r : s$, of the *distances* of moon and sun from the earth.

(4.) The expression for that vector may be thus transformed:

$$\begin{aligned} \text{III.} \ldots \ \textit{Vector of Disturbing Force} &= \eta = \mathrm{D}^2\alpha - M\phi\alpha \\ &= Ss^{-1}\sigma^{-1}\{1 - (1 - \alpha\sigma^{-1})^{-1}\mathrm{T}(1 - \alpha\sigma^{-1})^{-1}\} \\ &= Ss^{-1}\sigma^{-1}\{1 - (1 - \alpha\sigma^{-1})^{-\frac{3}{2}}(1 - \sigma^{-1}\alpha)^{-\frac{1}{2}}\} \end{aligned}$$

$$= Ss^{-1}\sigma^{-1}\left\{1 - \left(1 + \frac{3}{2}\alpha\sigma^{-1} + \frac{3.5}{2.4}(\alpha\sigma^{-1})^2 + \ldots\right)\left(1 + \tfrac{1}{2}\sigma^{-1}\alpha + \frac{1.3}{2.4}(\sigma^{-1}\alpha)^2 + \ldots\right)\right\};$$

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[*] Compare the equation in the *Mécanique Céleste* (Tom. I., p. 241, new edition, Paris, 1843). Laplace's rule for determining, by inspection of a globe, which of the two bodies is the nearer to the sun, results at once from the formula V.

that is, $$\text{IV.}\ldots\ \eta = \eta_1 + \eta_2 + \eta_3 + \&\text{c.},$$

if $$\text{V.}\ldots\ \eta_1 = -Ss^{-1}\sigma^{-1}(\tfrac{1}{2}\sigma^{-1}\alpha + \tfrac{3}{2}\alpha\sigma^{-1}) = \frac{S}{2s^3}(\alpha + 3\sigma\alpha\sigma^{-1}) = \eta_{1,1} + \eta_{1,2};$$

$$\text{VI.}\ldots\ \eta_2 = \frac{3Sr^2}{8s^5}(\alpha\sigma\alpha^{-1} + 2\sigma + 5\sigma\alpha\sigma\alpha^{-1}\sigma^{-1}) = \eta_{2,1} + \eta_{2,2} + \eta_{2,3};\ \&\text{c.}$$

the *general term** of this development being easily assigned.

(5.) We have thus a *first group* of *two* component and disturbing forces, which are of the same order as $\frac{Sr}{s^3}$; a *second group* of *three* such forces, of the same order as $\frac{Sr^2}{s^4}$; a *third* group of *four* forces, and so on.

(6.) The *first component* of the *first group* has the following tensor and versor,

$$\text{VII.}\ldots\ \mathrm{T}\eta_{1,1} = \frac{Sr}{2s^3},$$

$$\mathrm{U}\eta_{1,1} = \mathrm{U}\alpha;$$

it is therefore a purely *ablatitious force* MN, acting along the moon's geocentric vector EM prolonged, as in the annexed Figure 88.

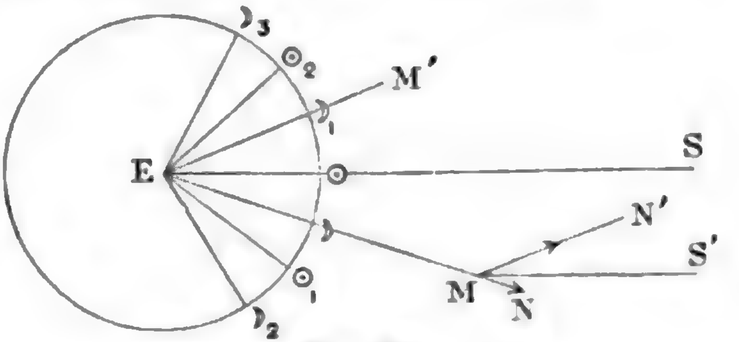

Fig. 88.

(7.) The *second component* MN′, of the same first group, has an exactly *triple intensity*, $\overline{\text{MN}}' = 3\overline{\text{MN}}$; and its *direction* is such that the *angle* NMN′, between these two forces of the first group, is *bisected* by a line MS′ from the moon, which is *parallel to the sun's geocentric vector* ES.

(8.) If then we conceive a line EM′ from the earth, having the same direction as the last force MN′, this new line will meet the heavens in what may be called for the moment a *fictitious moon* ☽$_1$, such that the *arc* ☽☽$_1$ of a *great circle*, connecting it with the *true moon* ☽ in the heavens, shall be *bisected by the sun* $\odot$, as represented in Fig. 88.

(9.) Proceeding to the *second group* (5.), we have by VI. for the *first component* of this group,

$$\text{VIII.}\ldots\ \mathrm{T}\eta_{2,1} = \frac{3Sr^2}{8s^4},\quad \mathrm{U}\eta_{2,1} = \mathrm{U}\alpha\sigma\alpha^{-1} = \frac{\alpha\mathrm{U}\sigma}{\alpha};$$

a line from the earth, parallel to this new force, meets therefore the heavens in what may be called a *first fictitious sun*, $\odot_1$, such that the arc of a great circle, $\odot\odot_1$, connecting it with the true sun, is *bisected by the moon* ☽, as in the same Fig. 88.

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* Such a general term was in fact assigned and interpreted in a communication of June 14, 1847, to the Royal Irish Academy (*Proceedings*, Vol. III., p. 514); and in the *Lectures*, page 616. The development may also be obtained, although less easily, by *Taylor's Series* adapted to quaternions. Compare pp. 427, 428, 430, 431 of the present work; and see page 332, &c., for the interpretation of such symbols as $\sigma\alpha\sigma^{-1}$, $\alpha\sigma\alpha^{-1}$,

(10.) The *second* component force, of the same second group, has an *intensity* exactly *double* that of the *first* ($T\eta_{2,2} = 2T\eta_{2,1}$); and in *direction* it is parallel to the sun's geocentric vector ES, so that a line drawn in its direction from the earth would meet the heavens in the place of the sun $\odot$.

(11.) The *third* component of the present group has an intensity which is precisely *five-fold* that of the *first* component ($T\eta_{2,3} = 5T\eta_{2,1}$); and a line drawn in its direction from the earth meets the heavens in a *second fictitious sun* $\odot_2$, such that the arc $\odot_1 \odot_2$, connecting these *two* fictitious suns, is *bisected by the true sun* $\odot$.

(12.) There is no difficulty in extending this analysis, and this interpretation, to *subsequent groups* of component disturbing forces, which forces *increase in number*, and *diminish in intensity*, in passing from any one group to the next; their *intensities*, for each *separate* group, bearing *numerical ratios* to each other, and their *directions* being connected by simple *angular relations*.

(13.) For example, the *third group* consists (5.) of *four small forces*, $\eta_{3,1} \ldots \eta_{3,4}$, of which the *intensities* are represented by $\frac{Sr^3}{16s^6}$, multiplied respectively by the four whole numbers, 5, 9, 15, and 35; and which have *directions* respectively parallel to lines drawn from the earth, towards a second fictitious moon $☽_2$, the true moon, the first fictitious moon $☽_1$ (8.), and a third fictitious moon $☽_3$; these *three* fictitious *moons*, like the *two* fictitious *suns* lately considered, being all situated in the *momentary plane* of the *three bodies* E, M, S: and the *three* celestial *arcs*, $☽_2☽$, $☽☽_1$, $☽_1☽_3$, being each equal to double the arc $☽\odot$ of apparent *elongation* of sun from moon in the heavens, as indicated in the above cited Fig. 88.

(14.) An exactly similar method may be employed to develope or decompose the disturbing force of one *planet* on another, which is nearer than it to the sun; and it is important to observe that no supposition is here made, respecting any *smallness* of *excentricities* or *inclinations*.

422. As a *seventh specimen* of the physical application of quaternions, we shall investigate briefly the construction and some of the properties of *Fresnel's Wave Surface*, as deductions from his principles or hypotheses* respecting light.

(1.) Let ρ be a *Vector of Ray-Velocity*, and μ the corresponding *Vector of Wave-Slowness* (or *Index-Vector*), for propagation of light from an origin O, within a biaxal crystal; so that

$$\text{I.} \ldots S\mu\rho = -1; \qquad \text{II.} \ldots S\mu\delta\rho = 0; \quad \text{and therefore} \quad \text{III.} \ldots S\rho\delta\mu = 0,$$

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* The present writer desires to be understood as not expressing any opinion of his own, respecting these or any rival hypotheses. In the next Series (423), as an *eighth specimen* of application, he proposes to deduce, from a *quite different set of physical principles respecting light*, expressed however still in the language of the present Calculus, Mac Cullagh's Theorem of the *Polar Plane;* intending then, as a *ninth* and *final specimen*, to give briefly a quaternion transformation of a celebrated equation in partial differential coefficients, of the first order and second degree, which occurs in the theory of *heat*, and in that of the *attraction of spheroids*.

if $\delta\rho$ and $\delta\mu$ be any infinitesimal variations of the vectors ρ and μ, consistent with the scalar equations (supposed to be as yet unknown), of the *Wave-Surface* and its *Reciprocal* (with respect to the unit-sphere round O), namely the *Surface of Wave-Slowness*, or (as it has been otherwise called) the *Index*[*]*-Surface:* the velocity of light in a vacuum being here represented by unity.

(2.) The variation $\delta\rho$ being next conceived to represent a small *displacement*, tangential to the wave, of a particle of ether in the crystal, it was supposed by Fresnel that such a displacement $\delta\rho$ gave rise to an *elastic force*, say $\delta\epsilon$, *not generally* in a direction exactly *opposite* to that displacement, but still a *function* thereof, which function is of the kind called by us (in the Sections III. ii. 6, and III. iii. 7) *linear*, *vector*, and *self-conjugate;* and which there will be a convenience (on account of its connexion with certain *optical constants*, a, b, c) in denoting here by $\phi^{-1}\delta\rho$ (instead of $\phi\delta\rho$): so that we shall have the two converse formulæ,

$$\text{IV.} \ldots \delta\rho = \phi\delta\epsilon; \qquad \text{V.} \ldots \delta\epsilon = \phi^{-1}\delta\rho.$$

(3.) The *ether* being treated as *incompressible*, in the theory here considered, so that the *normal component* $\mu^{-1}S\mu\delta\epsilon$ of the elastic force may be neglected, or rather suppressed, there remains only the *tangential component*,

$$\text{VI.} \ldots \mu^{-1}V\mu\delta\epsilon = \delta\epsilon - \mu^{-1}S\mu\delta\epsilon,$$

as regulating the *motion*, tangential to the wave, of a disturbed and *vibrating particle*.

(4.) If then it be admitted that, for the propagation of a *rectilinear vibration*, tangential to a wave of which the velocity is $T\mu^{-1}$, the *tangential force* (3.) must be *exactly opposite* in direction to the *displacement* $\delta\rho$, and *equal* in quantity to that displacement multiplied by the *square* ($T\mu^{-2}$) of the *wave-velocity*, we have, by V. and VI., the equation,

$$\text{VII.} \ldots \phi^{-1}\delta\rho - \mu^{-1}S\mu\delta\epsilon = \mu^{-2}\delta\rho, \quad \text{or} \quad \text{VIII.} \ldots \delta\rho = (\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}S\mu\delta\epsilon;$$

combining which with II., we obtain at once this *Symbolical Form* of the scalar equation of the *Index Surface*,

$$\text{IX.} \ldots 0 = S\mu^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1};$$

or by an easy transformation,

$$\text{X.} \ldots 1 = S\mu\phi^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1};$$

or finally,

$$\text{XI.} \ldots 1 = S\mu(\mu^2 - \phi)^{-1}\mu;$$

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[*] This brief and expressive name was proposed by the late Prof. Mac Cullagh (Trans. R. I. A., Vol. XVIII., Part I., page 38), for that *reciprocal* of the wave-surface which the present writer had previously called the *Surface of Components of Wave-Slowness*, and had employed for various purposes: for instance, to pass from the *conical cusps* to the *circular ridges* of the *Wave*, and so to establish a geometrical connexion between the theories of the *two conical refractions*, *internal* and *external*, to which his own methods had conducted him (Trans. R. I. A., Vol. XVII., Part I., pages 125–144). He afterwards found that the same Surface had been otherwise employed by M. Cauchy (*Exercises de Mathématiques*, 1830 p. 36), who did not seem however to have perceived its *reciprocal relation* to the Wave.

while the *direction* of the vibration $\delta\rho$, for any given tangent plane to the wave, is determined *generally* by the formula VIII.

(5.) That formula for the displacement, combined with the expression V. for the elastic force resulting, gives

$$\text{XII.}\ldots\ \delta\rho = -\phi\nu S\mu\delta\epsilon, \quad \text{and} \quad \text{XIII.}\ldots\ \delta\epsilon = -\nu S\mu\delta\epsilon,$$

if

$$\text{XIV.}\ldots\ (\phi - \mu^2)\nu = \mu, \quad \text{or} \quad \text{XV.}\ldots\ \nu = (\phi - \mu^2)^{-1}\mu,$$

ν being thus an auxiliary vector; and because the equation XI. of the index surface gives,

$$\text{XVI.}\ldots\ S\mu\nu = -1, \quad \text{while} \quad \text{XVII.}\ldots\ V\nu\delta\epsilon = 0, \qquad \text{by XIII.,}$$

it follows that the vector ν, if drawn like ρ and μ from o, *terminates on the tangent plane to the wave*, and is *parallel to the direction of the elastic force.*

(6.) The equations XIV. XVI. give,

$$\text{XVIII.}\ldots\ \mu^2\nu^2 - S\nu\phi\nu = 1, \quad \text{whence} \quad \text{XIX.}\ldots\ \nu^2 S\mu\delta\mu = S\mu\delta\nu = -S\nu\delta\mu,$$

because $\delta S\mu\nu = 0$, by XVI., and $\delta S\nu\phi\nu = 2S(\phi\nu . \delta\nu)$, by the self-conjugate property of ϕ; comparing then XIX. with III., we see that $\pm\rho$ (as being $\perp$ every $\delta\mu$) has the direction of $\mu + \nu^{-1}$, and therefore, by I. and XVI., that we may write,

$$\text{XX.}\ldots\ \rho^{-1} = -\mu - \nu^{-1}; \qquad \text{XXI.}\ldots\ \rho^{-2} = \mu^2 - \nu^{-2}; \qquad \text{XXII.}\ldots\ S\rho\nu = 0;$$

which last equation shows, by (5.), that *the ray is perpendicular* (on Fresnel's principles) ***to the elastic force*** $\delta\epsilon$, produced by the displacement $\delta\rho$.

(7.) The equations XX. and XXI. show by XIV. that

$$\text{XXIII.}\ldots\ (\rho^{-2} - \phi)\nu = \rho^{-1}, \quad \text{whence} \quad \text{XXIV.}\ldots\ \nu = (\rho^{-2} - \phi)^{-1}\rho^{-1};$$

we have therefore, by XXII., the following *Symbolical Form* (comp. (4.)) *of the Equation of the Wave Surface*,

$$\text{XXV.}\ldots\ 0 = S\rho^{-1}(\phi - \rho^{-2})^{-1}\rho^{-1};$$

or, by transformations analogous to X. and XI.,

$$\text{XXVI.}\ldots\ 1 = S\rho\phi(\phi - \rho^{-2})^{-1}\rho^{-1}; \qquad \text{XXVII.}\ldots\ 1 = S\rho(\rho^2 - \phi^{-1})^{-1}\rho;$$

and we see that we can *return* from each *equation of the wave*, to the corresponding *equation of the index surface*, by merely changing ρ to μ, and ϕ to ϕ^{-1}: but this result will soon be seen to be included in one more general, which may be called the *Rule of the Interchanges.*

(8.) The equation XXV. may also be thus written,

$$\text{XXVIII.}\ldots\ S\rho(\phi - \rho^{-2})^{-1}\rho = 0;$$

but under this last form it coincides with the equation 412, XLI.; hence, by 412, (19.), the *Wave Surface* may be *derived* from the *auxiliary* or *Generating Ellipsoid*,

$$\text{XXIX.}\ldots\ S\rho\phi\rho = 1,$$

by the following *Construction*, which was in fact assigned by Fresnel[*] himself, but as the result of far more complex calculations:— *Cut the ellipsoid (abc) by an arbitrary plane through its centre, and at that centre erect perpendiculars to that plane, which shall have the lengths of the semiaxes of the section ; the locus of the extremities of the perpendiculars so erected will be the sought wave surface.*

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[*] See Sir John F. W. Herschel's Treatise on Light, in the *Encyclopædia Metropolitana*, page 545, Art. 1017.

(9.) And we see, by IX., that the *Index Surface* may be *derived*, by an exactly similar construction, from that *Reciprocal Ellipsoid*, of which the equation is, on the same plan,

$$\text{XXX.} \ldots \mathrm{S}\rho\phi^{-1}\rho = 1.$$

(10.) If the scalar equations, XXVII. and XI., of the wave and index surface, be denoted by the abridged forms,

$$\text{XXXI.} \ldots \mathrm{f}\rho = 1, \quad \text{and} \quad \text{XXXII.} \ldots \mathrm{F}\mu = 1,$$

then the relations I. II. III. enable us to infer the expressions (comp. the notation in 418, (2.)),

$$\text{XXXIII.} \ldots \mu = \frac{-\mathrm{D}_\rho \mathrm{f}\rho}{\mathrm{S}\rho \mathrm{D}_\rho \mathrm{f}\rho}; \qquad \text{XXXIV.} \ldots \rho = \frac{-\mathrm{D}_\mu \mathrm{F}\mu}{\mathrm{S}\mu \mathrm{D}_\mu \mathrm{F}\mu};$$

in which (comp. 412, (36.), and the Note to that sub-article),

$$\text{XXXV.} \ldots \tfrac{1}{2}\mathrm{D}_\rho \mathrm{f}\rho = (\rho^2 - \phi^{-1})^{-1}\rho - \rho \mathrm{S}\rho\,(\rho^2 - \phi^{-1})^{-2}\rho = -\omega - \omega^2\rho,$$

and $$\text{XXXVI.} \ldots \tfrac{1}{2}\mathrm{D}_\mu \mathrm{F}\mu = (\mu^2 - \phi)^{-1}\mu - \mu \mathrm{S}\mu\,(\mu^2 - \phi)^{-2}\mu = -\upsilon - \upsilon^2\mu;$$

υ being the same auxiliary vector XV. as before, and ω being a new auxiliary vector, such that (by XXIV. XXVII. and IX. XV.),

$$\text{XXXVII.} \ldots \omega = (\phi^{-1} - \rho^2)^{-1}\rho = \phi\upsilon; \qquad \text{XXXVIII.} \ldots \mathrm{S}\rho\omega = -1;$$
$$\text{XXXIX.} \ldots \mathrm{S}\mu\omega = 0;$$

whence also $\omega \parallel \delta\rho$ by XII., so that (comp. (5.)) if ω be drawn (like ρ, μ, and υ) from the point O, *this new vector terminates on the tangent plane to the index surface*, and is *parallel to the displacement on the wave;* also $\delta\rho : \delta\epsilon = \omega : \upsilon$.

(11.) Hence, by XXXIII. XXXV. XXXVIII.,

$$\text{XL.} \ldots \mu = \frac{\omega + \omega^2\rho}{1 - \omega^2\rho^2} = \frac{\omega^{-1} + \rho}{\omega^{-2} - \rho^2} = -(\omega^{-1} + \rho)^{-1}, \quad \text{or} \quad \text{XLI.} \ldots -\mu^{-1} = \rho + \omega^{-1};$$

and similarly, by XXXIV. XXXVI. and XVI.,

$$\text{XLII.} \ldots \rho = \frac{\upsilon + \upsilon^2\mu}{1 - \upsilon^2\mu^2} = \frac{\upsilon^{-1} + \mu}{\upsilon^{-2} - \mu^2} = -(\upsilon^{-1} + \mu)^{-1}, \text{ or } -\rho^{-1} = \mu + \upsilon^{-1}, \text{ as in XX.};$$

so that, with the help of the expressions XV. and XXXVII. for υ and ω, the *ray-vector* ρ and the *index-vector* μ are expressed as *functions of each other:* which functions are *generally definite*, although we shall soon see *cases*, in which one or other becomes partially *indeterminate*.

(12.) It is easy now to enunciate the *rule of the interchanges*, alluded to in (7.), as follows:—*In any formula involving the vectors* ρ, μ, υ, ω, and the *functional symbol* ϕ, or some of them, it is *permitted to exchange* ρ *with* μ, υ *with* ω, *and* ϕ *with* ϕ^{-1}; *provided* that we at the same time interchange $\delta\rho$ with $\delta\epsilon$ (but *not** generally with $\delta\mu$), when either $\delta\rho$ or $\delta\epsilon$ occurs.

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* It is true that, in passing from II. to III. (instead of passing to XLIII.), we may be said to have exchanged not only ρ with μ, but *also* $\delta\rho$ with $\delta\mu$. But *usually*, in the present investigation, $\delta\rho$ represents a small *displacement* (2.), which is conceived to have a *definite direction*, tangential to the wave; whereas $\delta\mu$

For example, we pass thus from XX. to XLI., and conversely from the latter to the former; from II. we pass by the same rule, to the formula,

XLIII. . . $S\rho\delta\iota = 0$, which agrees by XVII. with XXII.;

and, as other verifications, the following equations may be noticed,

XLIV. . . $\delta\rho = \mu V\mu\delta\iota$; XLV. . . $\delta\iota = \rho V\rho\delta\rho$; XLVI. . . $S\mu\delta\iota = S\rho\delta\rho$.

(13.) The relations between the vectors may be illustrated by the annexed Figure 89; in which,

XLVII. . . $OP = \rho$, $OQ = \mu$, $OU = \upsilon$, $OW = \omega$,

and XLVIII. . . $OP' = -\rho^{-1}$, $OQ' = -\mu^{-1}$, $OU' = -\upsilon^{-1}$, $OW' = -\omega^{-1}$;

in fact it is evident on inspection, that

XLIX. . . $OP \,.\, OP' = OQ \,.\, OQ' = OU \,.\, OU' = OW \,.\, OW'$;

and the common value of these four scalar products is here taken as negative unity.

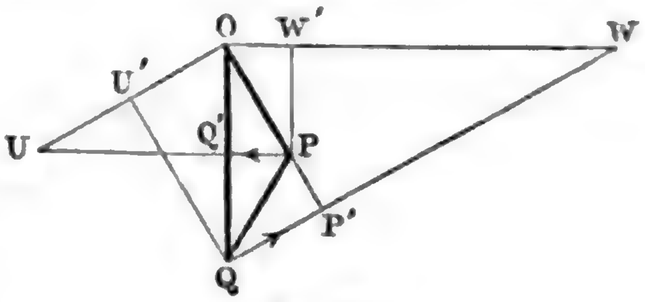

Fig. 89.

(14.) As examples of such illustration, the equation XX. becomes $P'O = QU'$; XLI. becomes $OQ' = W'P$; XXIII. may be written as $\omega + \rho^{-1} = \rho^{-2}\upsilon$, or as $P'W : OU = P'O : OP$; &c. And because the lines $PQ'U$ and $QP'W$ are sections of the tangent planes, to the wave at the extremity P of the ray, and to the index surface at the extremity Q of the index vector, made by the plane of those two vectors ρ and μ, while $\delta\rho$ and $\delta\iota$ (as being parallel to ω and υ) have the directions of PQ' and QP'; we see that the *displacement* (or vibration) has *generally*, in Fresnel's theory, the direction of the *projection of the ray on the tangent plane to the wave;* and that the *elastic force* resulting has the direction of the *projection of the index vector on the tangent plane to the index surface:* results which might however have been otherwise deduced, from the formulæ alone.

(15.) It may be added, as regards the reciprocal deduction of the two vectors μ and ρ from each other, that (by XLI. XXXVIII., and XX. XVI.) we have the expressions,

L. . . $-\mu^{-1} = \omega^{-1}V\omega\rho$, and LI. . . $-\rho^{-1} = \upsilon^{-1}V\upsilon\mu$;

which answer in Fig. 89 to the relations, that OQ' is the part (or component) of OP, perpendicular to OW; and that OP' is, in like manner, the part of $OQ \perp OU$.

(16.) We have also the expressions,

LII. . . $-\mu^{-1} = \omega^{-1}V\omega\upsilon$, and LIII. . . $-\rho^{-1} = \upsilon^{-1}V\upsilon\omega$,

which may be similarly interpreted; and which conduct to the relations,

LIV. . . $-(V\upsilon\omega)^2 = \upsilon^2\rho^{-2} = \omega^2\mu^{-2} = S\upsilon\omega$.

Hence, *the Locus of each of the two Auxiliary Points* U and W, in Fig. 89, *is a Surface of the Fourth Degree;* the scalar equations of these two loci being,

LV. . . $(V\upsilon\phi\upsilon)^2 + S\upsilon\phi\upsilon = 0$, and LVI. . . $(V\omega\phi^{-1}\omega)^2 + S\omega\phi^{-1}\omega = 0$;

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continues, as in (1.) to represent *any infinitesimal tangent* to the *index surface*, while $\delta\iota$ still denotes the *elastic-force* (2.), resulting from the displacement $\delta\rho$.

from which it would be easy to deduce *constructions* for those surfaces, with the help of the two *reciprocal ellipsoids*, XXIX. and XXX.

(17.) The equations XII. XXII., combined with the self-conjugate property of ϕ, give

$$\text{LVII.} \ldots 0 = S(\phi^{-1}\rho \,.\, \delta\rho), \quad \text{or} \quad \text{LVIII.} \ldots 0 = \delta S\rho\phi^{-1}\rho;$$

hence (between suitable limits of the constant), *every ellipsoid* of the form,

$$\text{LIX.} \ldots S\rho\phi^{-1}\rho = h^4 = \text{const.},$$

which is thus *concentric and coaxal with the reciprocal ellipsoid* XXX., being also *similar* to it, and *similarly placed*, *contains* upon its surface what may be called a *Line of Vibration** *on the Wave;* the *intersection* of this *new* ellipsoid LIX. with the wave surface being generally such, that the *tangent* at each point of that line (or curve) has the *direction* of Fresnel's vibration.

(18.) The fundamental connexion (2.) of the *function* ϕ with the *optical constants*, a, b, c, of the crystal, is expressed by the *symbolical cubic* (comp. 350, I., and 417, XXV.),

$$\text{LX.} \ldots (\phi + a^{-2})(\phi + b^{-2})(\phi + c^{-2}) = 0;$$

from which it is easy to infer, by methods already explained, that if e be *any scalar*, and if we write,

$$\text{LXI.} \ldots E = (e - a^{-2})(e - b^{-2})(e - c^{-2}),$$

we have then this *formula of inversion*,

$$\text{LXII.} \ldots E(\phi + e)^{-1} = e^2 - e(\phi + a^{-2} + b^{-2} + c^{-2}) - a^{-2}b^{-2}c^{-2}\phi^{-1}.$$

(19.) Changing then e to $-\rho^{-2}$, the equation XXVIII. of the wave becomes,

$$\text{LXIII.} \ldots 0 = \rho^{-2} + a^{-2} + b^{-2} + c^{-2} + S\rho^{-1}\phi\rho - a^{-2}b^{-2}c^{-2}S\rho\phi^{-1}\rho:$$

the *Wave* is therefore (as is otherwise known) a *Surface of the Fourth Degree:* and (as is likewise well known), the *Index Surface* is of the *same* degree, its equation (found by changing ρ, ϕ, a, b, c to μ, ϕ^{-1}, a^{-1}, b^{-1}, c^{-1}) being, on the same plan,

$$\text{LXIV.} \ldots 0 = \mu^{-2} + a^2 + b^2 + c^2 + S\mu^{-1}\phi^{-1}\mu - a^2b^2c^2S\mu\phi\mu.$$

(20.) These equations may be variously transformed, with the help of the cubic LX. in ϕ, which gives the analogous cubic in ϕ^{-1},

$$\text{LXV.} \ldots (\phi^{-1} + a^2)(\phi^{-1} + b^2)(\phi^{-1} + c^2) = 0;$$

for instance, another form of the equation of the wave is,

$$\text{LXVI.} \ldots 0 = S\rho\phi^{-2}\rho + (\rho^2 + a^2 + b^2 + c^2)S\rho\phi^{-1}\rho - a^2b^2c^2;$$

in which it may be remarked that $S\rho\phi^{-2}\rho = (\phi^{-1}\rho)^2 < 0$, whereas $S\rho\phi^{-1}\rho > 0$.

(21.) Substituting then, for $S\rho\phi^{-1}\rho$ in LXVI., its value h^4 from (17.), we find that this *second variable ellipsoid*, with h for an arbitrary constant or parameter,

$$\text{LXVII.} \ldots 0 = (\phi^{-1}\rho)^2 + h^4(\rho^2 + a^2 + b^2 + c^2) - a^2b^2c^2,$$

contains upon its surface the *same line of vibration* as the first variable ellipsoid LIX., which involves the same arbitrary constant h; and therefore that the *line* in question is a *quartic curve*, or *Curve of the Fourth Degree*, as being the intersection of these *two* variable but connected *ellipsoids*: and that the *wave* itself is the *locus* of all such quartic curves.

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* Such *lines of vibration* were discussed by the present writer, but by means of a quite different analysis, in his Memoir of 1832 (*Third Supplement on Systems of Rays*), which was published in the following year, in the Transactions of the Royal Irish Academy. See reference in the Note to page 737.

(22.) The *Generating Ellipsoid* ($S\rho\phi\rho = 1$) has a, b, c for its semiaxes ($a > b > c > 0$); and for *any vector* ρ, in the *plane* of bc, we have the *symbolical quadratic* (comp. 353, (9.)),

$$\text{LXVIII.} \ldots (\phi + b^{-2})(\phi + c^{-2}) = 0, \quad \text{or} \quad \text{LXIX.} \ldots -b^{-2}c^{-2}\phi^{-1} = \phi + b^{-2} + c^{-2};$$

making then this last substitution for $\phi + b^{-2} + c^{-2}$ in LXIII., we find, for the *section* of the wave by this principal plane of the ellipsoid XXIX., an equation which breaks up into the *two factors*,

$$\text{LXX.} \ldots \rho^{-2} + a^{-2} = 0, \quad \text{and} \quad \text{LXXI.} \ldots 1 - b^{-2}c^{-2}S\rho\phi^{-1}\rho = 0;$$

whereof the *first* represents (the *plane* being understood) a *circle*, with *radius* = a, which we may call briefly *the circle* (a); while the *second* represents (with the same understanding) an *ellipse*, which may by analogy be called here *the ellipse* (a): its two semiaxes having the *lengths* of c and b, but in the *directions* of b and c, for which directions $\phi + b^{-2} = 0$ and $\phi + c^{-2} = 0$, respectively, so that *this* ellipse (a) is merely the *elliptic section* (bc) of the *ellipsoid* (abc), *turned through a right angle* in its own plane, as by the *construction* (8.) it evidently ought to be. And an exactly similar analysis shows, what indeed is otherwise known, that the plane of ca cuts the wave in the system of a circle (b), and an ellipse (b); and that the plane of ab cuts the same wave surface, in a *circle* (c), and an *ellipse* (c).

(23.) The circle (a) is entirely *exterior* to the ellipse (a); and the circle (c) is wholly *interior* to the ellipse (c); but the circle (b) *cuts* the ellipse (b), in *four real points*, which are therefore (in a sense to be soon more fully examined) *cusps* (or *nodal points*) on the wave surface, or briefly *Wave-Cusps*: and the *vectors* ρ, say $\pm\rho_0$ and $\pm\rho_1$, which are drawn from the centre O to these *four cusps*, may be called *Lines of Single Ray-Velocity*, or briefly *Cusp-Rays*.

(24.) It is clear, from the construction (8.), that these lines or rays must have the directions of the *cyclic normals* of the ellipsoid (abc); which suggests our using here the *cyclic forms*,

$$\text{LXXII.} \ldots \phi\rho = g\rho + V\lambda\rho\lambda', \quad \text{and} \quad \text{LXXIII.} \ldots S\rho\phi\rho = g\rho^2 + S\lambda\rho\lambda'\rho = 1,$$

for the function ϕ, and the generating ellipsoid (8.); λ' being written, to avoid confusion, instead of the μ of 357, &c., to represent the second cyclic normal.

(25.) Changing then μ to λ', ν to ρ, and g to $g - \rho^{-2}$, in the expression 361, XXVII. for $F\nu$ or $S\nu\phi^{-1}\nu$; equating the result to zero, and resolving the equation so obtained, as a quadratic in g; we find this *new form* of the *Equation* XXVIII. *of the Wave*,

$$\text{LXXIV.} \ldots g\rho^2 = 1 + S\lambda\rho S\lambda'\rho \pm TV\lambda\rho TV\lambda'\rho;$$

the upper sign belonging to *one sheet*, and the lower sign to the *other sheet*, of that wave surface. The new equation may also be thus written, as an expression for the *inverse square* of the *ray-velocity* $T\rho$, or of the *radius-vector*, say r, of the wave,

$$\text{LXXV.} \ldots r^{-2} = T\rho^{-2} = \frac{a^{-2} + c^{-2}}{2} + \frac{a^{-2} - c^{-2}}{2} \cos\left(\angle\frac{\rho}{\lambda} \mp \angle\frac{\rho}{\lambda'}\right),$$

because, by 405, (2.), (6.), &c.,

$$\text{LXXVI.} \ldots a^{-2} = -g - T\lambda\lambda', \quad b^{-2} = -g + S\lambda\lambda', \quad c^{-2} = -g + T\lambda\lambda';$$

and we have the verification, for a *cusp-ray* (23.), that

$$\text{LXXVII.} \ldots r^{-2} = b^{-2}, \quad \text{or} \quad r = \mathrm{T}\rho = b, \quad \text{if} \quad \rho \parallel \lambda \text{ or } \lambda'.$$

(26.) If we write (comp. XXXI.),

$$\text{LXXVIII.} \ldots \mathfrak{f}\rho = -\rho^{-2}(1 + \mathrm{S}\rho\phi\rho) + a^{-2}b^{-2}c^{-2}\mathrm{S}\rho\phi^{-1}\rho,$$

the equation LXIII. of the wave takes the form,

$$\text{LXXIX.} \ldots \mathfrak{f}\rho = a^{-2} + b^{-2} + c^{-2} = \text{const.};$$

and we have the partial derivative (comp. XXXV.),

$$\begin{aligned}\text{LXXX.} \ldots \tfrac{1}{2}\mathrm{D}_\rho\mathfrak{f}\rho &= \rho^{-3}(1 + \mathrm{S}\rho\phi\rho) - \rho^{-2}\phi\rho + a^{-2}b^{-2}c^{-2}\phi^{-1}\rho \\ &= \rho^{-3}(1 - \mathrm{V}\rho\phi\rho) + a^{-2}b^{-2}c^{-2}\phi^{-1}\rho;\end{aligned}$$

which gives by XXXIII. the expression,

$$\text{LXXXI.} \ldots \mu = \frac{\rho^{-3}(\mathrm{V}\rho\phi\rho - 1) - a^{-2}b^{-2}c^{-2}\phi^{-1}}{\rho^{-2} + a^{-2}b^{-2}c^{-2}\mathrm{S}\rho\phi^{-1}\rho},$$

and therefore a *generally definite value* (comp. (11.)) for the *index vector* μ, when the *ray* ρ is given.

(27.) If the ray be *in the plane* of ac, then (comp. LXIX.),

$$\text{LXXXII.} \ldots \phi\rho + (a^{-2} + c^{-2})\rho + a^{-2}c^{-2}\phi^{-1}\rho = 0,$$

whence $\text{LXXXIII.} \ldots \mathrm{V}\rho\phi\rho = -a^{-2}c^{-2}\mathrm{V}\rho\phi^{-1}\rho = a^{-2}c^{-2}(\mathrm{S}\rho\phi^{-1}\rho - \rho\phi^{-1}\rho);$

and therefore by LXXXI.,

$$\text{LXXXIV.} \ldots \mu = \frac{\rho^{-3}(\mathrm{S}\rho\phi^{-1}\rho - a^2c^2) - (\rho^{-2} + b^{-2})\phi^{-1}\rho}{b^{-2}(\mathrm{S}\rho\phi^{-1}\rho - a^2c^2) + (\rho^{-2} + b^{-2})a^2c^2};$$

an expression which gives, *definitely*,

$$\text{LXXXV.} \ldots \mu = -\rho^{-1}, \quad \text{if} \quad \text{LXXXVI.} \ldots \rho^{-2} + b^{-2} = 0,$$

but *not* $\qquad \text{LXXXVII.} \ldots \mathrm{S}\rho\phi^{-1}\rho = a^2c^2,$

that is (comp. (22.)), if the ray terminate on the *circle* (b), at any point which is *not also on the ellipse* (b); and with equal *definiteness*,

$$\text{LXXXVIII.} \ldots \mu = -a^{-2}c^{-2}\phi^{-1}\rho, \quad \text{if} \quad \text{LXXXVII. but } \textit{not} \text{ LXXXVI. hold good,}$$

that is, if the ray terminate on the *ellipse* (b), at any point which is *not also* on the *circle*.

(28.) The *normal* then to the *wave*, in each of the two cases last mentioned, *coincides* with the normal to the *section*, made by the plane of ac; and if we abstract for a moment from the *cusps* (23.), we see that the *wave* is *touched*, *along the circle* (b), by the concentric *sphere* LXXXVI. with radius $= b$, which we may call *the sphere* (b); and *along the ellipse* (b) by the concentric *ellipsoid* LXXXVII. which may on the same plan be called *the ellipsoid* (b).

(29.) An exactly similar analysis shows that the wave is touched along the circles (a) and (c), by two other concentric spheres, with radii a and c, which may be briefly called *the spheres* (a) *and* (c); and along the ellipses (a) and (c) by two other concentric and similar ellipsoids, which may by analogy be called *the ellipsoids* (a) *and* (c). And by comparing the equation LXXXVII. of the ellipsoid (b) with the form LIX., we see that the *three elliptic sections* (a) (b) (c) of the wave, made by the *three principal planes* of the generating ellipsoid (abc), are *lines of vibration* (17.); the constant h^4 receiving the three values, b^2c^2, c^2a^2, a^2b^2, for these three ellipses respectively.

(30.) But *at a cusp* the *two* equations LXXXVI. and LXXXVII. *coexist*, and the expression LXXXIV. for μ takes the *indeterminate form* $\frac{0}{0}$; in fact, there is in this case no reason for preferring *either* to the *other* of the *two values*, *within the plane* of *ac*,

$$\text{LXXXIX.} \ldots \mu = -\rho_0^{-1}, \qquad \text{XC.} \ldots \mu = \mu_0, \quad \text{if} \quad \text{XCI.} \ldots \mu_0 = -a^{-2}c^{-2}\phi^{-1}\rho_0;$$

in which ρ_0 is the *cusp-ray* (23.), and the *first* value of μ corresponds to the *circle*, but the *second* to the *ellipse* (*b*).

(31.) The *indetermination* of μ, at a *wave-cusp*, is however even *greater* than this. For, if we observe that the equations LXXIX. and LXXX. give, for this case, by LXXXIII. LXXXVI. LXXXVII.,

$$\text{XCII.} \ldots \mathrm{f}\rho_0 = a^{-2} + b^{-2} + c^{-2}, \quad \text{and} \quad \text{XCIII.} \ldots \mathrm{D}_\rho \mathrm{f}\rho = 0, \quad \text{for} \quad \rho = \rho_0,$$

we shall see that if ρ be changed to $\rho_0 + \rho'$ in the expression LXXVIII. for $\mathrm{f}\rho$, and only terms which are of the *second dimension* in ρ' retained, the result equated to zero will represent a *cone of tangents* ρ', or a *Tangent Cone to the Wave at the Cusp:* which cone is of the *second degree*, and *every normal* μ to which, if limited by the condition I., is here to be considered as *one value of the vector* μ, corresponding to the value ρ_0 of ρ.

(32.) And it is evident, by the law (12.) of *transition* from the wave to the index surface, that if $\pm\nu_0$, $\pm\nu_1$ be the *Lines of Single Normal Slowness*, or the four values of μ which are *analogous*[*] to the *four cusp-rays* $\pm\rho_0$, $\pm\rho_1$ (23.), then, at the end of each such new line, there must be a *Conical Cusp on the Index Surface*, analogous to the *Conical Cusp* (31.) *on the Wave*, which is in like manner one of *four* such cusps.

(33.) In forming and applying the equation above indicated (31.), of the tangent cone to the wave at a cusp, the following transformations are useful:

$$\begin{aligned}\text{XCIV.} \ldots -(\rho + \rho')^{-2} &= -\rho^{-2}(1 + \rho^{-1}\rho')^{-1}\,(1 + \rho'\rho^{-1})^{-1} \\ &= -\rho^{-2} + 2\rho^{-2}\mathrm{S}\rho'\rho^{-1} + \rho^{-4}\rho'^2 - 4\rho^{-6}(\mathrm{S}\rho\rho')^2 + \&\text{c.},\end{aligned}$$

the terms not written being of the third and higher dimensions in ρ', and ρ, ρ' being *any two vectors* such that $\mathrm{T}\rho' < \mathrm{T}\rho$ (comp. 421, (4.)); also, without neglecting *any terms*, the self-conjugate property of ϕ gives (comp. 362),

$$\text{XCV.} \ldots \mathrm{S}(\rho + \rho')\,\phi(\rho + \rho') = \mathrm{S}\rho\phi\rho + 2\mathrm{S}\rho'\phi\rho + \mathrm{S}\rho'\phi\rho',$$

with an analogous transformation for the corresponding expression in ϕ^{-1}; while the cubic LX. in ϕ, or LXV. in ϕ^{-1}, gives for an *arbitrary* ρ,

$$\text{XCVI.} \ldots \phi(\phi + a^{-2})\,(\phi + c^{-2})\rho = -b^{-2}(\phi + a^{-2})\,(\phi + c^{-2})\rho,$$
$$\text{XCVII.} \ldots \phi^{-1}(\phi + a^{-2})\,(\phi + c^{-2})\rho = -b^{2}(\phi + a^{-2})\,(\phi + c^{-2})\rho;$$

and therefore, among other transformations of the same kind,

$$\text{XCVIII.} \ldots (\phi + a^{-2})^2\,(\phi + c^{-2})^2\rho = (a^{-2} - b^{-2})\,(c^{-2} - b^{-2})\,(\phi + a^{-2})\,(\phi + b^{-2})\rho.$$

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[*] This word "analogous" is here more proper than "corresponding"; in fact, the *cusps* on *each* of the two surfaces will soon be seen to *correspond* to *circles* on the *other*, in virtue of the *law of reciprocity*.

We have also for a *cusp*, the values,

$$\text{XCIX.} \ldots \phi\rho_0 = \mu_0 - (a^{-2} + c^{-2})\rho_0; \qquad \text{XCIX}'. \ldots 1 + S\rho_0\phi\rho_0 = (a^{-2} + c^{-2})b^2$$

$$\text{C.} \ldots \mu_0^2 = a^{-4}c^{-4}S\rho_0\phi^{-2}\rho_0 = a^{-2}b^2c^{-2} - (a^{-2} + c^{-2}).$$

(34.) In this way the *equation of the tangent cone* is easily found to take the form,

$$\text{CI.} \ldots 0 = b^4 S\rho'(\phi + a^{-2})(\phi + c^{-2})\rho' - 4S\rho'\rho_0 S\rho'\mu_0$$

and to give, by operating with $D_{\rho'}$ (comp. (10.) (26.) (31.)),

$$\text{CII.} \ldots x\mu = b^4(\phi + a^{-2})(\phi + c^{-2})\rho' - 2\rho_0 S\rho'\mu_0 - 2\mu_0 S\rho'\rho_0,$$

the scalar coefficient x being determined, for each direction of the tangent ρ' to the wave at the cusp, by the condition I., which here becomes (31.),

$$\text{CIII.} \ldots S\mu\rho_0 = S\mu_0\rho_0 = -1;$$

also, by CII., &c., we have after some slight reductions,

$$\text{CIV.} \ldots xS\mu\rho_0 = 2(b^2 S\rho'\mu_0 + S\rho'\rho_0);$$

$$\text{CV.} \ldots xS\mu\mu_0 = 2(S\rho'\mu_0 - \mu_0^2 S\rho'\rho_0);$$

$$\text{CVI.} \ldots x^2\mu^2 = 4(b^2\mu_0^2 + 1)S\rho'\rho_0 S\rho'\mu_0 + 4(\rho_0 S\rho'\mu_0 + \mu_0 S\rho'\rho_0)^2$$
$$= -4b^2(S\rho'\mu_0)^2 + 4(b^2\mu_0^2 - 1)S\rho'\rho_0 S\rho'\mu_0 + 4\mu_0^2(S\rho'\rho_0)^2;$$

but this last expression is equal, by CIV. CV., to $-x^2 S\mu\rho_0 S\mu\mu_0$; the equation of the *cone of perpendiculars*, let fall from the *wave-centre* O on the *tangent planes at the cusp*, takes then this very simple form,

$$\text{CVII.} \ldots \mu^2 + S\mu\rho_0 S\mu\mu_0 = 0;$$

so that *this cone* of the second degree has the two vectors ρ_0 and μ_0 at once for *sides* and *cyclic normals* (comp. 406, (7.)); and it is *cut*, by the *plane* CIII., in a *circle*, of which the *diameter* is,

$$\text{CVIII.} \ldots T(\mu_0 + \rho_0^{-1}) = (T\mu_0^2 - b^{-4})^{\frac{1}{2}} = b(b^{-2} - a^{-2})^{\frac{1}{2}}(c^{-2} - b^{-2})^{\frac{1}{2}};$$

and therefore *subtends*, at the centre O, and in the plane of *ac*, the *angle*,

$$\text{CIX.} \ldots \angle\frac{\mu_0}{\rho_0} = \tan^{-1}.b^2(b^{-2} - a^{-2})^{\frac{1}{2}}(c^{-2} - b^{-2})^{\frac{1}{2}}.$$

(35.) And by combining the equations CIII. CVII., we see that this circle (34.) is a *small circle of the sphere*,

$$\text{CX.} \ldots \mu^2 = S\mu\mu_0, \quad \text{or} \quad \text{CX}'. \ldots S\mu^{-1}\mu_0 = 1$$

which passes through the wave-centre, and has the vector μ_0 for a diameter, passing also through the extremity of the vector $-\rho_0^{-1}$.

(36.) This circle is, by III., a *curve of contact* of the *plane* CIII. with the *surface* of which μ is the vector, because *every vector μ of the curve corresponds*, by (31.), to the *one vector* ρ_0 of the *wave;* it is therefore one of *Four Circular Ridges on the Index Surface*, the three others having *equal diameters*, and corresponding to the three remaining *cusp-rays*, $-\rho_0$, ρ_1, $-\rho_1$ (28.); and there are, in like manner, *Four Circular Ridges on the Wave*, along which it is *touched by the four planes*,

$$\text{CXI.} \ldots S\rho\nu_0 = -1, \quad S\rho\nu_0 = +1, \quad S\rho\nu_1 = -1, \quad S\rho\nu_1 = +1,$$

$\pm\nu_0$, $\pm\nu_1$ being the four lines introduced in (32.); also the *common length* of the *diameters*, of these four *circles on the wave*, is (comp. CVIII.),

$$\text{CXII.} \ldots T(\sigma_0 + \nu_0^{-1}) = (T\sigma_0^2 - b^2)^{\frac{1}{2}} = b^{-1}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}},$$

where $\text{CXIII.} \ldots \sigma_0 = -a^2c^2\phi\nu_0$, $\text{CXIV.} \ldots T\nu_0 = b^{-1}$, and $\text{CXV.} \ldots S\nu_0\sigma_0 = -1$;

finally, $-\nu_0^{-1}$ and σ_0 are the two values* of ρ, in the plane of ac, for the first of the four new circles: and the *angle* between these two vectors, or the angle which the diameter of the circle, in the same plane, *subtends at the wave-centre*, is (comp. CIX.),

$$\text{CXVI.} \ldots \angle \frac{\sigma_0}{\nu_0} = \tan^{-1}. b^{-2}(a^2 - b^2)^{\frac{1}{2}} (b^2 - c^2)^{\frac{1}{2}}.$$

(37.) In the recent calculations (33.) (34.), the *circle of contact* (36.) on the index surface was deduced *from the tangent cone* at a wave-cusp, as a *section* of a certain *cone of normals* CVII. to that tangent cone CI., made by the plane CIII.; but the following is a simpler, and perhaps more elegant method, of deducing and representing the *same circle* by means of its *vector equation* (comp. 392, IX. &c.), and *without* assuming *any previous knowledge* of the character, or even the *existence*, of that *conical wave-cusp*.

(38.) In general, by eliminating the auxiliary vector υ between XX. and XXIII., we arrive at the following equation,

$$\text{CXVII.} \ldots (\phi - \rho^{-2})(\mu + \rho^{-1})^{-1} = \rho^{-1};$$

which holds good for *every pair of corresponding vectors* ρ and μ, of the wave and index surface. And *in general*, this relation is *sufficient*, to determine the index-vector μ, when the ray-vector ρ is given: because $(\phi + e)^{-1}0$ is generally $= 0$.

(39.) But when e is a root of the equation $E = 0$, with the signification LXI. of E, then, by the formula of inversion LXII., the symbol $(\phi + e)^{-1}0$ takes the indeterminate form $\frac{0}{0}$; and therefore, for *every point* of each of the *three circles* (a) (b) (c) of the wave, the *formula* CXVII. *fails to determine* μ: although it is *only* at a *cusp* (23.), that the *value* of μ becomes in fact *indeterminate* (comp. (27.) (28.) (29.) (30.) (31.)).

(40.) At such a cusp ($\rho = \rho_0$), the equation CXVII. takes the symbolical form,

$$\text{CXVIII.} \ldots (\mu + \rho_0^{-1})^{-1} = (\phi + b^{-2})^{-1}\rho_0^{-1} = (\mu_0 + \rho_0^{-1})^{-1} + (\phi + b^{-2})^{-1}0;$$

μ_0 retaining its recent signification XCI., and the symbol $(\phi + b^{-2})^{-1}0$ denoting *any vector* of the *form* $y\beta$, if β be the *mean vector semiaxis* of the *generating ellipsoid* XXIX., so that

$$\text{CXIX.} \ldots S\beta\phi\beta = 1, \quad (\phi + b^{-2})\beta = 0, \quad T\beta = b.$$

(41.) Writing then for abridgment (comp. XX.),

$$\text{CXX.} \ldots \upsilon_0 = -(\mu_0 + \rho_0^{-1})^{-1},$$

the *Vector Equation* of the *Index Ridge* (36.) is obtained under the sufficiently simple form,

$$\text{CXXI.} \ldots V\beta(\mu + \rho_0^{-1})^{-1} + V\beta\upsilon_0 = 0;$$

and this equation does in fact represent a *Circle* (comp. 296, (7.)), which is easily

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[*] It is not difficult to show that these are the vectors of two points, in which the circle and ellipse (b), wherein the wave is cut by the plane of ac, are touched by a *common tangent*.

proved to be the *same* as the *circular section* (34.), of the *cone* CVII. by the *plane* CIII.; its *diameter* CVIII. being thus found anew under the form,

$$\text{CXXII.} \ldots \mathrm{T}\nu_0^{-1} = b\mathrm{TV}\lambda\lambda' = b(b^{-2} - a^{-2})^{\frac{1}{2}}(c^{-2} - b^{-2})^{\frac{1}{2}},$$

with the significations (24.) (25.) of λ, λ'; in fact we have now the expressions,

$$\text{CXXIII.} \ldots \rho_0 = b\mathrm{U}\lambda, \quad \nu_0 = \rho_0^{-1}(\mathrm{V}\lambda\lambda')^{-1},$$

with the verification, that

$$\text{CXXIV.} \ldots (\phi + b^{-2})\nu_0 = \lambda\mathrm{S}\lambda'\nu_0 + \lambda'\mathrm{S}\lambda\nu_0 = b^{-1}\mathrm{U}\lambda = -\rho_0^{-1}.$$

(42.) And by a precisely similar analysis, we have first the ***new general relation*** (comp. CXVII.), for ***any two corresponding vectors***, ρ and μ,

$$\text{CXXV.} \ldots (\phi^{-1} - \mu^{-2})(\rho + \mu^{-1})^{-1} = \mu^{-1};$$

and then in particular (comp. CXVIII.), for $\mu = \nu_0$,

$$\text{CXXVI.} \ldots (\rho + \nu_0^{-1})^{-1} = (\phi^{-1} + b^2)^{-1}\nu_0^{-1} = (\sigma_0 + \nu_0^{-1})^{-1} + (\phi^{-1} + b^2)^{-1}0;$$

so that finally, if we write for abridgment (comp. XLI. CXX.),

$$\text{CXXVII.} \ldots \omega_0 = -(\sigma_0 + \nu_0^{-1})^{-1},$$

the *Vector Equation of a Wave-Ridge* is found (comp. CXXI.) to be,

$$\text{CXXVIII.} \ldots \mathrm{V}\beta(\rho + \nu_0^{-1})^{-1} + \mathrm{V}\beta\omega_0 = 0,$$

β being still (as in CXIX.) the ***mean vector semiaxis*** of the ***generating ellipsoid*** ($\mathrm{S}\rho\phi\rho = 1$): and the *diameter* CXII., of this ***circle of contact*** of the ***wave*** with the first plane CXI., is thus found ***anew*** (comp. CXXII.), ***without any reference to cusps*** (37.), as the value of $\mathrm{T}\omega_0^{-1}$.

(43.) Several of the foregoing results may be illustrated, by a new use of the last diagram (13.). Thus if we suppose, in that Fig. 89, that we have the values,

$$\text{CXXIX.} \ldots \mathrm{OP} = \rho_0, \quad \mathrm{OQ} = \mu_0, \quad \mathrm{OU} = \nu_0, \quad \text{whence} \quad \text{CXXX.} \ldots \mathrm{OP}' = -\rho_0^{-1}, \ \&\text{c.},$$

then the ***index-ridge*** (36.), corresponding to the ***wave-cusp*** P (23.), will be the ***circle*** which has P′Q for diameter, in a ***plane*** perpendicular to the plane of the Figure, which is here the plane of *ac*; the ***cone of normals*** μ (34.), to the ***tangent cone*** to the ***wave*** at P, has the ***wave-centre*** O for its ***vertex***, and ***rests*** on the last-mentioned circle, having also for a subcontrary ***section*** that ***second circle*** which has PQ′ for diameter, and has its plane in like manner at right angles to the plane of POQ; also if R and S be any two points on the second and first circles, such that ORS is a right line, namely a ***side*** μ of the cone here considered, then the ***chord*** PR of the second circle is ***perpendicular*** to this last line, and has the ***direction*** of the ***vibration*** $\delta\rho$, which answers here to the ***two*** vectors $\rho\,(=\rho_0)$ and μ: because (comp. (14.)) this chord is perpendicular to μ, but complanar with ρ and μ.

(44.) Again, to illustrate the theory of the ***wave-ridge*** (36.), which ***corresponds*** to a ***cusp*** (32.) on the ***index-surface***, we may suppose that ***this*** cusp is at the point Q in Fig. 89, writing ***now*** (instead of CXXIX. CXXX.),

$$\text{CXXXI.} \ldots \mathrm{OQ} = \nu_0, \quad \mathrm{OP} = \sigma_0, \quad \mathrm{OW} = \omega_0, \quad \mathrm{OQ}' = -\nu_0^{-1}, \ \&\text{c.};$$

for then the *ridge* (or *circle of contact*) on the *wave* will coincide with the ***second circle*** (43.), and the ***cone of rays*** ρ from O, which rests upon ***this*** circle, will have the ***first*** circle (43.) for a sub-contrary section: also the ***vibration***, at any point R of the wave-ridge, will have the direction of the *chord* RQ′, for reasons of the same kind as before.

(45.) Let K and K′ denote the *bisecting points* of the lines PQ′ and QP′, in the same Fig. 89; then K′ is the *centre of the index ridge*, in the case (43.); while, in the case (44.), K is the *centre of the wave-ridge.*

(46.) In the *first* of these two cases, the point K is *not* the *centre* of *any ridge*, on *either* wave or index-surface; but it *is* the centre of a certain *subcontrary* and *circular section* (43.), of the *cone* with O for vertex which *rests* upon an index-ridge; and *each* of its *chords* PR has the *direction* (43.) of a *vibration* $\delta\rho_0$, at the *wave-cusp* P corresponding: so that this *cusp-vibration revolves*, in the *plane* of the *circle* last mentioned, with exactly *half the angular velocity* of the *revolving radius* KR.

(47.) And *every one* of those *cusp-vibrations* $\delta\rho_0$, which (as we have seen) are all situated *in one plane*, namely in the *tangent plane* at the *cusp* P to the *ellipsoid* (*b*) of (28.), has (as by (14.) it ought to have) the direction of the *projection* of the *cusp-ray* ρ_0, on *some tangent plane* to the *tangent cone* to the *wave*, at that point P: to the determination of which *last cone*, by some new methods, we purpose shortly to return.

(48.) In the *second* of the two cases (45.), namely in the case (44.), PQ′ is a *diameter* of a *wave-ridge*, with K for the *centre* of that circle, and with a *plane* (perpendicular to that of the Figure) which *touches* the *wave* at *every point* of the same circular ridge; and the *vibration*, at any such point R, has been seen to have the direction of the *chord* RQ′, which is in fact the *projection* (14.) of the *ray* OR upon the *tangent plane* at R to the wave.

(49.) And we see that, in passing from one point to another of this *wave-ridge*, the *vibration* RQ′ *revolves* (comp. (46.)) round the *fixed point* Q′ of that circle, namely round the *foot* of the *perpendicular from* O on the *ridge-plane*, with (again) *half the angular velocity* of the *revolving radius* KR.

(50.) These *laws* of the *two sets of vibrations*, at a *cusp* and at a *ridge* upon the *wave*, are intimately connected with the *two conical polarizations*, which accompany the *two conical refractions*,* *external* and *internal*, in a *biaxal crystal;* because, on the one hand, the *theoretical deduction* of those two *refractions* is associated with, and was in fact accomplished by, the consideration of those *cusps* and *ridges:* while, on the other hand, in the theory of Fresnel, the *vibration* is always *perpendicular* to the *plane of polarization*. But into the details of such investigations, we cannot enter here.

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* The writer's anticipation, from theory, of the two Conical Refractions, was announced at a general meeting of the Royal Irish Academy, on the 22nd of October, 1832, in the course of a final reading of that *Third Supplement on Systems of Rays*, which has been cited in a former Note (p. 737). The very elegant experiments, by which his friend, the Rev. Humphrey Lloyd, succeeded shortly afterwards in exhibiting the expected results, are detailed in a Paper *On the Phenomena presented by Light, in its passage along the Axes of Biaxal Crystals*, which was read before the same Academy on the 28th of January, 1833, and is published in the same First Part of Volume XVII. of their Transactions. Dr. Lloyd has also given an account of the same phenomena, in a separate work since published, under the title of an *Elementary Treatise on the Wave Theory of Light* (London, Longman and Co., 1857, Chapter XI.).

(51.) It is not difficult to show, by decomposing ρ' into two other vectors, ρ_1' and ρ_2', perpendicular and parallel to the plane of *ac*, that we have the *general transformation*, for *any* vector ρ',

$$\text{CXXXII.} \ldots b^4 S\rho' (\phi + a^{-2}) (\phi + c^{-2}) \rho' = (S\mu_0\rho_0\rho')^2;$$

the equation CI. of the *tangent cone* at a *wave-cusp* may therefore be thus more briefly written,

$$\text{CXXXIII.} \ldots (S\mu_0\rho_0\rho')^2 = 4S\rho_0\rho' S\mu_0\rho';$$

and under this form, the *cone* in question is easily proved to be the *locus* of the *normals* from the *cusp*, to that *other cone* CVII., which has μ for a side, and the *wave-centre* o for its vertex: while the same cone CVII. is now seen, more easily than in (34.), to be reciprocally the locus of the perpendiculars from o on the tangent planes to the wave at the cusp, in virtue of the new equation CXXXIII., of the tangent cone at that point.

(52.) Another form of the equation of the cusp-cone may be obtained as follows. The equation LXXIV. of the *wave* may be thus modified (comp. LXXVI.), by the introduction of the two non-opposite cusp-rays, $\rho_0 = bU\lambda$ (CXXIII.), and $\rho_1 = bU\lambda'$:

$$\text{CXXXIV.} \ldots 2a^2b^2c^2 + (a^2 + c^2)\, b^2\rho^2 + (a^2 - c^2)\, S\rho_0\rho \,.\, S\rho_1\rho$$
$$= \mp (a^2 - c^2)\, TV\rho_0\rho \,.\, TV\rho_1\rho;$$

where it will be found that the first member vanishes, as well as the second, at the cusp for which $\rho = \rho_0$.

(53.) Changing then ρ to $\rho_0 + \rho'$, and retaining only terms of *first* dimension in ρ' (comp. (31.)), we find an equation of *unifocal form* (comp. 359, &c.),

$$\text{CXXXV.} \ldots S\beta_0\rho' = \mp TV\alpha_0\rho', \quad \text{or} \quad \text{CXXXV}'. \ldots (V\alpha_0\rho')^2 + (S\beta_0\rho')^2 = 0;$$

with the two constant vectors,

$$\text{CXXXVI.} \ldots \alpha_0 = (b^{-2} - a^{-2})^{\frac{1}{2}} (c^{-2} - b^{-2})^{\frac{1}{2}} \rho_0; \quad \text{CXXXVI}'. \ldots \beta_0 = \mu_0 - \rho_0^{-1};$$

and this equation CXXXV. or CXXXV'. represents the *tangent cone*, with ρ' for side, $S\beta_0\rho'$ being positive for one sheet, but negative for the other.

(54.) As regards the calculations which conduct to the recent expressions for α_0, β_0, it may be sufficient here to observe that those expressions are found to give the equations,

$$\text{CXXXVII.} \ldots 2a^2b^2c^2\alpha_0 = (a^2 - c^2)\, \rho_0 TV\rho_0\rho_1;$$
$$\text{CXXXVII}'. \ldots 2a^2b^2c^2\beta_0 = 2\,(a^2 + c^2)\, b^2\rho_0 + (a^2 - c^2)\, (\rho_0 S\rho_0\rho_1 - b^2\rho_1);$$

and that, in deducing these, we employ the values,

$$\text{CXXXVIII.} \ldots S\rho_0\rho_1 = \frac{b^2 S\lambda\lambda'}{T\lambda\lambda'}, \quad TV\rho_0\rho_1 = \frac{b^2 TV\lambda\lambda'}{T\lambda\lambda'};$$

together with the formula XCIX., and the following,

$$\text{CXXXIX.} \ldots \phi\,(\rho_0 - \rho_1) = -a^{-2}\,(\rho_0 - \rho_1); \quad \phi\,(\rho_0 + \rho_1) = -c^{-2}\,(\rho_0 + \rho_1).$$

(55.) It is not difficult to show that the equation CXXXV. or CXXXV'., of the tangent cone at a cusp, can be transformed into the equation CXXXIII.; but it may be more interesting to assign here a *geometrical interpretation*, or *construction*, of the *unifocal form* last found (53.).

(56.) Retaining then, for a moment, the use made in (48.) of Fig. 89, as serving to illustrate the case of a wave-cusp at P, with the signification (45.) of the new point K′ as bisecting the line P′Q, or as being the centre of the index-ridge; and conceiving a *parallel cone*, with O instead of P for *vertex*, and with a variable *side* OT $= \rho'$; then the *cusp-ray* OP $(= \rho_0 \parallel \alpha_0)$ is a *focal line* of the new cone, and the line OK′ $(= \frac{1}{2}(\mu_0 - \rho_0^{-1}) = \frac{1}{2}\beta_0)$ is the *directive normal*, or the normal to the *director plane* corresponding; and the formula CXXXV. is found to conduct to the following,

$$\text{CXL.} \ldots \cos \text{K}'\text{OT} = \sin \text{POK}' \sin \text{POT},$$

which may be called a *Geometrical Equation of the Cusp-Cone:* or (more immediately) of that *Parallel Cone*, which has (as above) its vertex removed to the wave-centre O.

(57.) Verifications of CXL. may be obtained, by supposing the side OT to be one of the two right lines, ρ_1', ρ_2', in which the cone is *cut* by the *plane* of the figure (or of *ac*); that is, by assuming either

$$\text{CXLI.} \ldots \text{OT} = \rho_1' = \mu_0 + \rho_0^{-1} \parallel \text{OU}, \quad \text{or} \quad \text{CXLI}'. \ldots \text{OT} = \rho_2' = \rho_0 + \mu_0^{-1} \parallel \text{OW};$$

and it is easy to show, not only that these *two sides*, OU, OW, make (as in Fig. 89) an *obtuse angle* with *each other*, but also that they belong to one *common sheet*, of the cone here considered, because *each* makes an *acute angle* with the directive normal OK′.

(58.) Another way of arriving at this result, is to observe that the equation CXXXIII. takes easily the *rectangular form*,

$$\text{CXLII.} \ldots (\text{S}\rho'(\text{U}\mu_0 + \text{U}\rho_0))^2 = (\text{S}\rho'(\text{U}\mu_0 - \text{U}\rho_0))^2 + \text{T}\mu_0\rho_0 (\text{S}\rho'\text{U}\mu_0\rho_0)^2;$$

the *internal axis* of the *cusp-cone* has therefore the direction of $\text{U}\mu_0 + \text{U}\rho_0$, that is, of the *internal bisector* of the angle POQ, while the *external bisector* of the same angle is *one* of the two *external axes*, and the *third* axis is perpendicular to the *plane* of ρ_0, μ_0; but $\text{S}\rho'(\text{U}\mu_0 + \text{U}\rho_0) < 0$, whether $\rho' = \rho_1'$, or $= \rho_2'$: and therefore these *two sides*, ρ_1' and ρ_2', belong (as above) to *one sheet*, because *each* is inclined at an acute angle to the internal *axis* $\text{U}\mu_0 + \text{U}\rho_0$.

(59.) It is easy to see that the *second focal line* of the *parallel cone* (56.) is μ_0, or OQ; and that the *second directive normal* corresponding is the line OK (45.), in the same Fig. 89; whence may be derived (comp. CXL.) this *second geometrical equation* of the cone at O,

$$\text{CXLIII.} \ldots \cos \text{KOT} = \sin \text{KOQ} \sin \text{QOT}; \quad \text{with} \quad \text{KOQ} = \text{POK}'.$$

(60.) And finally, as a *bifocal* but still *geometrical form* of the equation of the *cusp-cone*, with its vertex thus *transferred* to O, we may write,

$$\text{CXLIV.} \ldots \angle\, \text{POT} + \angle\, \text{QOT} = \text{const.} = \angle\, \text{WOU}.$$

(61.) *Any legitimate form* of any *one* of the *four functions*, $\phi\rho$, $\phi^{-1}\rho$, $\text{S}\rho\phi\rho$, $\text{S}\rho\phi^{-1}\rho$, when treated by rules of the present Calculus which have been already stated and exemplified, not only conducts to the connected forms of the *three other* functions of the group, but also gives the corresponding forms of equation, of the *Wave* and the *Index-Surface.*

(62.) For instance, with the significations (32.) of ν_0 and ν_1, the scalar function $S\rho\phi^{-1}\rho$, which is $= 1$ in the equation XXX. of the *Reciprocal Ellipsoid* (9.), may be expressed by the following *cyclic form*, with ν_0, ν_1 for the *cyclic normals* of that ellipsoid,

$$\text{CXLV.}\ldots\ S\rho\phi^{-1}\rho = -b^2\rho^2 + (a^2 - c^2)b^2 S\nu_0\rho S\nu_1\rho\,;$$

reciprocating which (comp. 361), we are led to a *bifocal form* of the function $S\rho\phi\rho$, which function was made $= 1$ in the equation XXIX. of the *Generating Ellipsoid* (8.), and is now expressed by this other equation (comp. 360, 407),

$$\text{CXLVI.}\ldots\ \frac{4a^2c^2}{(a^2-c^2)^2}(S\rho\phi\rho + b^{-2}\rho^2) = (S\nu_0\rho)^2 + (S\nu_1\rho)^2 - 2\,\frac{a^2+c^2}{a^2-c^2}\,S\nu_0\rho S\nu_1\rho\,;$$

ν_0, ν_1 being here the two (real) *focal lines* of the same ellipsoid (8.), or of its (imaginary) asymptotic cone.

(63.) Substituting then these forms (62.), of $S\rho\phi\rho$ and $S\rho\phi^{-1}\rho$, in the equation LXIII., we find (after a few reductions) this *new form* of the *Equation of the Wave:*

$$\text{CXLVII.}\ldots\ (2\rho^2 - (a^2 - c^2)S\nu_0\rho S\nu_1\rho + a^2 + c^2)^2 = (a^2 - c^2)^2\{1 - (S\nu_0\rho)^2\}\{1 - (S\nu_1\rho)^2\}\,;$$

whence it follows at once, that *each of the four planes* CXI. *touches the wave, along the circle in which it cuts the quadric*, with ν_0, ν_1 for cyclic normals, which is found by equating to zero the expression squared in the first member of CXLVII. For example, the *first* plane CXI. touches the wave along that *circle*, or *wave-ridge*, of which on this plan the equations are,

$$\text{CXLVIII.}\ldots\ S\nu_0\rho + 1 = 0, \quad 2\rho^2 + (a^2 - c^2)S\nu_1\rho - (a^2 + c^2)S\nu_0\rho = 0\,;$$

and because

$$\text{CXLIX.}\ldots\ \phi(\nu_0 + \nu_1) = -a^{-2}(\nu_0 + \nu_1), \quad \phi(\nu_0 - \nu_1) = -c^{-2}(\nu_0 - \nu_1),$$

and therefore, with the value CXIII. of σ_0,

$$\text{CL.}\ldots\ \sigma_0 = -a^2c^2\phi\nu_0 = \tfrac{1}{2}((a^2 + c^2)\nu_0 - (a^2 - c^2)\nu_1),$$

the second equation CXLVIII. represents (comp. CX.) the *diacentric sphere*,

$$\text{CLI.}\ldots\ \rho^2 = S\sigma_0\rho, \quad \text{or} \quad \text{CLI}'.\ldots\ S\sigma_0\rho^{-1} = 1,$$

which passes through the *wave-centre* O, and of which the *ridge* here considered is a *section*. The *diameter* of that ridge may thus be shown again to have the value CXII.; and it may be observed that the circle is a section also of the *cone*,

$$\text{CLII.}\ldots\ S\nu_0\rho S\sigma_0\rho = -\rho^2, \quad \text{or} \quad \text{CLII}'.\ldots\ S\nu_0\rho S\sigma_0\rho^{-1} = -1.$$

(64.) It was shown in (17.) that the *vibration* $\delta\rho$, at *any point* of the wave-surface, or at the end of *any ray* ρ, is perpendicular to $\phi^{-1}\rho$, as well as to μ by II.; and is therefore *tangential* to the variable *ellipsoid* LIX., as well as to the *wave itself*. Hence it is easy to infer, that this vibration must have generally the direction of the auxiliary vector ω, because not only $S\mu\omega = 0$, by XXXIX., but also $S\omega\phi^{-1}\rho = S\rho\phi^{-1}\omega = S\rho\upsilon = 0$, by XXII. and XXXVII. Indeed, this parallelism of $\delta\rho$ to ω results at once by XXXVII. from XII.

(65.) If then we denote by $\delta'\rho$ an infinitesimal vector, such as $\mu\delta\rho$, which is *tangential to the wave*, but *perpendicular to the vibration* $\delta\rho$, the parallelism $\delta\rho \parallel \omega$ will give,

$$\text{CLIII.} \ldots \delta'\rho = \mu\delta\rho \parallel \mu\omega \perp \rho, \quad \text{because} \quad \text{CLIII}'. \ldots \mathrm{S}\rho\mu\omega = 0;$$

whence $\text{CLIV.} \ldots \mathrm{S}\rho\delta'\rho = 0, \quad \delta'\mathrm{T}\rho = 0, \quad \text{or} \quad \text{CLV.} \ldots \mathrm{T}\rho = r = \text{const.},$

for this *new direction* $\delta'\rho$ of motion upon the wave.

(66.) And thus (or otherwise) it may be shown, that the *Orthogonal Trajectories to the Lines of Vibration* (17.) are the curves in which the *Wave* is *cut* by *Concentric Spheres*, such as CLV.; that is, by the spheres $\rho^2 + r^2 = 0$, in which the radius r is *constant* for any *one*, but *varies* in passing from one to *another*.

(67.) The *spherical curves* (r), which are thus *orthogonal* to what we have called the *lines* (h) of vibration, are *sphero-conics* on the wave; either because each such curve (r) is, by XXVIII., situated on a *concentric and quadric cone*, namely,

$$\text{CLVI.} \ldots 0 = \mathrm{S}\rho(\phi + r^{-2})^{-1}\rho;$$

or because, by XXVII., it is on this *other concentric quadric*,

$$\text{CLVII.} \ldots -1 = \mathrm{S}\rho(\phi^{-1} + r^2)^{-1}\rho.$$

(68.) It is easy to prove (comp. LXXV.)) that, for any *real* point of the wave, r^2 cannot be less than c^2, nor greater than a^2; and that the squares of the scalar semiaxes of the new quadric CLVII. are, in algebraically ascending order, $r^2 - a^2$, $r^2 - b^2$, $r^2 - c^2$; so that this surface is generally an *hyperboloid*, with *one sheet* or with *two*, according as $r >$ or $< b$.

(69.) And we see, at the same time, that the *conjugate hyperboloid*,

$$\text{CLVIII.} \ldots +1 = \mathrm{S}\rho(\phi^{-1} + r^2)^{-1}\rho,$$

which has *two sheets* or *one*, in the same two cases, $r > b$, $r < b$, and has (in descending order) the values,

$$\text{CLIX.} \ldots a^2 - r^2, \quad b^2 - r^2, \quad c^2 - r^2,$$

for the squares of *its* scalar semiaxes, is *confocal* with the *generating ellipsoid* XXIX.: so that the quadric CLVII. itself is the *conjugate of such a confocal*.

(70). To form a distinct conception (comp. (67.)) of the *course* of a curve (r) upon the wave, it may be convenient to distinguish the *five* following *cases*:

$$\text{CLX.} \ldots (\alpha) \,.\,.\, r = a; \quad (\beta) \,.\,.\, r < a, > b; \quad (\gamma) \,.\,.\, r = b; \quad (\delta) \,.\,.\, r < b, > c; \quad (\varepsilon) \,.\,.\, r = c.$$

(71.) In each of the *three* cases (α) (γ) (ε), the *conic* (r) becomes a *circle*, in one or other of the three principal planes: namely the circle (a), for the case (α); (b) for (γ); and (c) for (ε).

(72.) In the case (β), the *curve* (r) is one of *double curvature*, and consists of *two closed ovals*, *opposite* to each other on the *wave*, and *separated* by the *plane* (a), which plane is *not* (really) *met*, in *any point*, by the complete *sphero-conic* (r); and *each* separate oval *crosses the plane* (b) *perpendicularly*, in *two* (real) *points* of the *ellipse* (b), which are *external* to the *circle* (b): while the *same oval* crosses *also* the *plane* (c) at *right angles*, in some *two* real points of the *ellipse* (c).

(73.) Finally, in the remaining case (δ), the ovals are separated by the plane (c), and each crosses the plane (b) at right angles, in *two* points of the ellipse (b), which are *interior* to the circle (b); crossing also perpendicularly the plane (a), in two points of the ellipse (a).

(74.) Analogous remarks apply to the *lines of vibration* (h); which are either the *ellipses* (a) (b) (c), or else *orthogonals* to the *circles* (a) (b) (c), and generally to the *sphero-conics* (r), as appears easily from foregoing results.

(75.) It may be here observed, that when we only know the *direction* ($U\mu$), but not the *length* ($T\mu$), of an *index-vector* μ, so that we have *two parallel tangent planes* to the *wave*, at one *common side* of the *centre*, the *directions* of the *vibrations* $\delta\rho$ *differ* generally for these *two planes*, according to a *law* which it is easy to assign as follows.

(76.) The *second values* of μ and $\delta\rho$ being denoted by $\mu_{,}$ and $\delta\rho_{,}$, we have, by the equation IX. of the index-surface, these two other equations:

$$\text{CLXI.} \ldots 0 = S\mu(\phi^{-1} - \mu^{-2})^{-1}\mu; \qquad \text{CLXI}'. \ldots 0 = S\mu(\phi^{-1} - \mu_{,}^{-2})^{-1}\mu;$$

of which the difference gives, suppressing the factor $\mu_{,}^{-2} - \mu^{-2}$,

$$\text{CLXII.} \ldots 0 = S\mu(\phi^{-1} - \mu_{,}^{-2})^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu;$$

or

$$\text{CLXII}'. \ldots 0 = S(\phi^{-1} - \mu^{-2})^{-1}\mu(\phi^{-1} - \mu_{,}^{-2})^{-1}\mu,$$

because $(\phi^{-1} - \mu_{,}^{-2})^{-1}$, as a functional operator, is *self-conjugate*, so that μ may be transferred from one side of it to the other; just as, if $\nu = \phi\rho$ be such a self-conjugate function of ρ, then $\nu^2 = S\nu\phi\rho = S\rho\phi\nu = S\rho\phi^2\rho$, &c.

(77.) But, by VIII., we have the parallelisms,

$$\text{CLXIII.} \ldots \delta\rho \parallel (\phi^{-1} - \mu^{-2})^{-1}\mu; \qquad \text{CLXIII}'. \ldots \delta\rho_{,} \parallel (\phi^{-1} - \mu_{,}^{-2})^{-1}\mu;$$

hence, by CLXII'., we have the very simple relation,

$$\text{CLXIV.} \ldots S\,\delta\rho\,\delta\rho_{,} = 0;$$

that is, *the two vibrations*, in the *two parallel planes*, are *mutually rectangular*.

(78.) The following quite different method has however the advantage of not only proving anew this *known relation* of *rectangularity*, but also of assigning *quaternion expressions* for the *two directions separately*: and, at the same time, that of leading easily to what appears to be a *new* and elegant *Geometrical Construction*, simpler in some respects than the *known* one, which can indeed be deduced from it.

(79.) By the first principles of Fresnel's theory (comp. (3.)), the *vibration* ($\delta\rho$), on any *one* tangent plane to the *wave*, is situated in the *normal plane* (through μ), which contains the direction ($\delta\epsilon$) of the *elastic force*; that is to say, we have the *Equation of Complanarity*,

$$\text{CLXV.} \ldots S\mu\,\delta\rho\,\delta\epsilon = 0.$$

(80.) We have then, by II. and V., the system of the two equations,

$$\text{CLXVI.} \ldots S\mu\delta\rho = 0, \quad S\mu\delta\rho\phi^{-1}\delta\rho = 0;$$

comparing which with the equations of the same form,

$$S\nu\tau = 0, \quad S\nu\tau\phi\tau = 0, \qquad 410, \text{V. VI.}$$

we derive at once the following *Construction, which may* also be expressed as a *Theorem*:—

"*At either of the two points* Q *of the Reciprocal Ellipsoid* XXX., *the tangent plane at which is parallel to that at the given point* P *of the Wave, the tangents to the Lines of Curvature on the Ellipsoid are parallel to the tangents to the Lines of Vibration on the Wave*;" namely, to *one* at that given point P *itself*, and to *another* at the *other point* P', on the *same side* of the *centre*, at which the tangent plane is parallel to each of the two others above mentioned.

(81.) Thus for *each* of the two points P, P′ the line of *vibration* is parallel to *one* of the lines of *curvature* at Q; and it is evident, from what precedes, that the *other* of these last lines has the direction of the corresponding *Orthogonal* (66.) at P or P′: nor is there any danger of confusion.

(82.) As regards *quaternion expressions*, for the *two vibrations* on a *given wave-front*, the sub-article, 410, (8.), with notations suitably modified, shows by its formulæ XIX. XXII. that we have here the equations,

$$\text{CLXVII.} \ldots 0 = \mathrm{S}\mu\,\delta\rho\,\nu_0\,\delta\rho\,\nu_1$$

$$= \mathrm{S}\mu\,\delta\rho\,\nu_0\,\mathrm{S}\nu_1\,\delta\rho + \mathrm{S}\mu\,\delta\rho\,\nu_1\,\mathrm{S}\nu_0\,\delta\rho,$$

and

$$\text{CXVIII.} \ldots \delta\rho \parallel \mathrm{UV}\mu\nu_0 \pm \mathrm{UV}\mu\nu_1,$$

if ν_0, ν_1 be, as in earlier formulæ of the present Series 422, the *cyclic normals* of the *reciprocal ellipsoid*, which are often called the *Optic Axes* of the *Crystal*.

(83.) And hence may be deduced the *known* construction, namely, that "for any *given direction of wave-front*, the *two planes of polarization*, perpendicular respectively to the two vibrations in Fresnel's theory, *bisect* the *two supplementary* and *diedral angles*, which the *two optic axes subtend at the normal* to the front:" or that these planes of polarization *bisect*, *internally and externally*, the angle between the *two planes*, $\mu\nu_0$ and $\mu\nu_1$.

(84.) It may not be irrelevant here to remark, that if μ and $\mu_,$ be any two index-vectors, which have (as in (76.)) the *same direction*, but *not* the *same length*, the equation LXIV. enables us to establish the two converse relations:

$$\text{CLXIX.} \ldots abc\mathrm{T}\mu_, = (\mathrm{S}\mu\phi\mu)^{-\frac{1}{2}}; \qquad \text{CLXIX}'. \ldots abc\mathrm{T}\mu = (\mathrm{S}\mu_,\phi\mu_,)^{-\frac{1}{2}}.$$

(85.) Either by changing a, b, c, ϕ, μ to a^{-2}, b^{-2}, c^{-2}, ϕ^{-1}, ρ, or by treating the *form* LXIII., in (19.), of the *Equation of the Wave*, as we have just treated the form LXIV., of the equation of *Index Surface*, in the same sub-article (19.), we see that if ρ and $\rho_,$ be *any two condirectional rays* $(\mathrm{U}\rho_, = \mathrm{U}\rho)$, then,

$$\text{CLXX.} \ldots (abc)^{-1}\mathrm{T}\rho_, = (\mathrm{S}\rho\phi^{-1}\rho)^{-\frac{1}{2}}, \quad \text{or,} \quad abc\mathrm{T}\rho_,^{-1} = (\mathrm{S}\rho\phi^{-1}\rho)^{\frac{1}{2}};$$

and

$$\text{CLXX}'. \ldots (abc)^{-1}\mathrm{T}\rho = \mathrm{S}\,(\rho_,\phi^{-1}\rho_,)^{-\frac{1}{2}}, \quad \text{or,} \quad abc\mathrm{T}\rho^{-1} = (\mathrm{S}\rho_,\phi^{-1}\rho_,)^{\frac{1}{2}}.$$

(86.) A somewhat interesting geometrical consequence may be deduced from these last formulæ, when combined with the equation LIX. of that *variable ellipsoid*, $\mathrm{S}\rho\phi^{-1}\rho = h^4$, which *cuts the wave* in a *line of vibration* (h). For if we introduce this symbol h^4 for $\mathrm{S}\rho\phi^{-1}\rho$, and write $r_,$ instead of $\mathrm{T}\rho_,$ to denote the *length* of the *second ray* $\rho_,$, the first equation CLXX. will take this simple form,

$$\text{CLXXI.} \ldots r_, = abch^{-2},$$

which shows at once that $r_,$ and h are *together constant*, or *together variable*; and therefore, that "*a Line of Vibration on one Sheet of the Wave is projected into an Orthogonal Trajectory to all such Lines on the other Sheet, and conversely the latter into the former, by the Vectors* ρ *of the Wave*:" so that *one* of these *two curves* would appear to be *superposed* upon the *other*, to an eye placed at the *Wave-Centre* O.

(87.) The *visual cone*, here conceived, is represented by the equation CLVI., with some constant value of r; and as being a surface of the *second* degree, it ought to *cut the wave*, which is one of the *fourth*, in some *curve* of the *eighth degree*; or in some *system* of curves, which have the *product* of their dimensions equal to *eight*.

Accordingly we now see that the *complete intersection*, of the *cone* CLVI. with the *wave*, consists of *two curves*, *each* of the *fourth degree ; one* of these being, as in (67.), a *complete sphero-conic* (r), and the *other* a complete *line of vibration* (h): a new geometrical *connexion* being thus established between these *two quartic curves*.

(88.) As additional verifications, we may regard the *three principal planes*, as *limits of the cutting cones ;* for then, in the *plane* (a) for instance, the *circle* (a) and the *ellipse* (a), which are (in a sense) *projections* of each other, and of which the *latter* has been seen to be a *line of vibration*, are represented respectively by the two equations,

$$\text{CLXXII.}\ldots r = a, \quad \text{and} \quad \text{CLXXII}'.\ldots bc = h^2,$$

in agreement with CLXXI. ; and similarly for the two other planes.

(89.) It was an early result of the quaternions, that an ellipsoid with its centre at the origin might be adequately represented by the equation (comp. 281, XXIX., or 282, XIX.),

$$\text{CLXXIII.}\ldots \mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \quad \text{if} \quad \mathrm{T}\iota > \mathrm{T}\kappa;$$

or, without *any* restriction on the *two* vector constants, ι, κ, by this *other* equation,*

$$\text{CLXXIII}'.\ldots \mathrm{T}(\iota\rho + \rho\kappa)^2 = (\kappa^2 - \iota^2)^2.$$

(90.) Comparing this with $\mathrm{S}\rho\phi\rho = 1$, as the equation XXIX. of the *Generating Ellipsoid*, we see that we are to satisfy, *independently of* ρ, or as an *identity*, the relation (comp. 336):

$$\begin{aligned}\text{CLXXIV.}\ldots (\kappa^2 - \iota^2)^2\,\mathrm{S}\rho\phi\rho &= (\iota\rho + \rho\kappa)\ (\rho\iota + \kappa\rho \\ &= (\iota^2 + \kappa^2)\,\rho + 2\mathrm{S}\iota\rho\kappa\rho;\end{aligned}$$

which is done by assuming (comp. again 336) this *cyclic form* for ϕ,

$$\begin{aligned}\text{CLXXV.}\ldots (\kappa^2 = \iota^2)^2\,\phi\rho &= (\iota^2 + \kappa^2)\,\rho + 2\mathrm{V}\kappa\rho\iota \\ &= (\iota - \kappa)^2\rho + 2\iota\mathrm{S}\kappa\rho + 2\kappa\mathrm{S}\iota\rho;\end{aligned}$$

or as in (24.) comp. 359, III. IV.,

$$\phi\rho = g\rho + \mathrm{V}\lambda\rho\lambda', \quad \mathrm{S}\rho\phi\rho = g\rho^2 + \mathrm{S}\lambda\rho\lambda'\rho = 1; \qquad \text{LXXII. LXXIII.}$$

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* This equation, CLXXIII'. or CLXXII., which had been assigned by the author as a form of the equation of an ellipsoid, has been selected by his friend Professor Peter Guthrie Tait, now of Edinburgh, as the basis of an admirable Paper, entitled: "Quaternion Investigations connected with Fresnel's Wave-Surface," which appeared in the May number for 1865, of the *Quarterly Journal of Pure and Applied Mathematics ;* and which the present writer can strongly recommend to the careful perusal of all quaternion students. Indeed, Professor Tait, who has already published tracts on *other* applications of Quaternions, mathematical and physical, including some on Electro-Dynamics, appears to the writer eminently fitted to carry on, happily and usefully, this new branch of mathematical science: and likely to become in it, if the expression may be allowed, one of the chief successors to its inventor.

with expressions for the constants g, λ, λ', which give, by LXXVI., the following values for the scalar semiaxes,*

$$\text{CLXXVI.}\ldots\ a = \mathrm{T}\iota + \mathrm{T}\kappa\,; \quad b = \frac{\kappa^2 - \iota^2}{\mathrm{T}(\iota - \kappa)}\,; \quad c = \mathrm{T}\iota - \mathrm{T}\kappa\,;$$

whence conversely,

$$\text{CLXXVII.}\ldots\ \mathrm{T}\iota = \frac{a+c}{2} \quad \mathrm{T}\kappa = \frac{a-c}{2}\,; \quad \mathrm{T}(\iota - \kappa) = \frac{ac}{b}\,;\ \&\text{c.}$$

(91.) Knowing thus the form CLXXV. of the function ϕ, which answers in the present case to the given equation CLXXIII. of the generating ellipsoid, there would be no difficulty in carrying on the calculations, so as to reproduce, in connexion with the *two* constants ι, κ, all the preceding theorems and formulæ of the present Series, respecting the Wave and the Index-Surface. But it may be more useful to show briefly, before we conclude the Series, how we can *pass* from *Quaternions* to *Cartesian Co-ordinates*, in any question or formula, of the kind lately considered.

(92.) The *three italic letters*, *ijk*, conceived to be connected by the *four fundamental relations*,

$$i^2 = j^2 = k^2 = ijk = -1, \qquad \text{(A), 183,}$$

were *originally* the *only peculiar symbols* of the present Calculus; and although they are not *now* so much used, as in the *early practice* of quaternions, because certain general *signs of operation*, such as S, V, T, U, K, have since been introduced, yet they (the symbols *ijk*) may be supposed to be *still familiar* to a student, *as links* between *quaternions* and *co-ordinates*.

(93.) We shall therefore merely write down here some leading expressions, of which the meaning and utility seem likely to be at once perceived, especially after the Calculations above performed in this Series.

(94.) The vector semiaxes of the generating ellipsoid being called α, β, γ (comp. (40.) (42.)), we may write,

$$\text{CLXXVIII.}\ldots\ \alpha = ia, \quad \beta = jb, \quad \gamma = kc\,;$$

$$\text{CLXXIX.}\ldots\ \phi\rho = \alpha^{-1}\mathrm{S}\alpha^{-1}\rho + \beta^{-1}\mathrm{S}\beta^{-1}\rho + \gamma^{-1}\mathrm{S}\gamma^{-1}\rho = \Sigma\alpha^{-1}\mathrm{S}\alpha^{-1}\rho = -\Sigma i a^{-2} x\,;$$

$$\text{CLXXX.}\ldots\ \mathrm{S}\rho\phi\rho = \Sigma(\mathrm{S}\alpha^{-1}\rho)^2 = \Sigma a^{-2}x^2\,; \qquad \text{CLXXXI.}\ldots\ \mathrm{S}\rho\phi^{-1}\rho = \Sigma a^2 x^2\,;$$

$$\text{CLXXXII.}\ldots\ (\phi + e)\rho = \Sigma\alpha\,(a^{-2} + e)\,\mathrm{S}\alpha^{-1}\rho\,;$$

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* The reader, at this stage, might perhaps usefully turn back to that *Construction of the Ellipsoid*, illustrated by Fig. 53 (p. 226), with the Remarks thereon, which were given in the few last Series of the Section II. i. 13, pages 223–233. It will be seen there that the *three vectors*, ι, κ, $\iota - \kappa$, of which the lengths are expressed by CLXXVII., are the *three sides*, CB, CA, AB, of what may be called the *Generating Triangle* ABC in the Figure; and that the deduction CLXXVI., of the *three semiaxes*, *abc*, from the *two vector constants*, ι, κ, with many connected results, can be very simply exhibited by *Geometry*. The whole subject, of the equation $\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2$ of the ellipsoid, was very fully treated in the *Lectures*; and the calculations may be made more general, by the transformations assigned in the long but important Section III. ii. 6 of the present *Elements*, so that it seems unnecessary to dwell more on it in this place.

$$\text{CLXXXIII.} \ldots (\phi + e)^{-1}\rho = \Sigma\alpha\,(\alpha^{-2} + e)^{-1}\,\mathrm{S}\alpha^{-1}\rho\,;$$

$$\text{CLXXXIV.} \ldots \text{if } r^2 = \mathrm{T}\rho^2 = \Sigma x^2, \text{ then } v = r^{-2}(\phi + r^{-2})^{-1}\rho$$

$$= r^{-2}\Sigma\,\frac{\alpha\mathrm{S}\alpha^{-1}\rho}{r^{-2} - a^{-2}} = -\,\Sigma\,\frac{ia^2x}{r^2 - a^2}\,;$$

$$\text{CLXXXV.} \ldots \text{for } \textit{Wave},\ 0 = \mathrm{S}\rho v = \Sigma\,\frac{a^2x^2}{r^2 - a^2} = \frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2}\,;$$

$$\text{or CLXXXVI.} \ldots 1 = -\mathrm{S}\rho\omega = -\mathrm{S}\rho\phi v = -\,\mathrm{S}v\phi\rho$$

$$= \Sigma\,\frac{x^2}{r^2 - a^2} = \frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2}\,;$$

and the *Index-Surface* may be treated similarly, or obtained from the *Wave* by changing *abc* to their reciprocals.

423. As an *eighth specimen* of physical application we shall investigate, by quaternions, MacCullagh's *Theorem of the Polar Plane*,[*] and some things therewith connected, for an important case of incidence of polarized light on a biaxal crystal: namely, for what was called by him the case of *uniradial vibrations*.

(1.) Let homogeneous light in air (or in a vacuum), with a velocity[†] taken for unity, fall on a plane face of a doubly refracting crystal, with such a polarization that only *one* refracted ray shall result; let ρ, ρ', ρ'' denote the *vectors of ray-velocity* of the incident, refracted, and reflected lights respectively, ρ having the direction of the *incident ray*, prolonged *within the crystal*, but ρ'' that of the *reflected ray outside*; and let μ' be the *vector of wave-slowness*, or the *index-vector* (comp. 422, (1.)), for the refracted light: these *four* vectors being all drawn from a given point of incidence o, and μ', like ρ', being *within* the crystal.

(2.) Then, by *all*[‡] *wave theories of light*, translated into the present notation, we have the equations,

$$\text{I.} \ldots \rho^2 = \mathrm{S}\mu'\rho' = \rho'^2 = -1\,;$$

$$\text{II.} \ldots \rho'' = -\,\nu\rho\nu^{-1}, \quad \text{with} \quad \text{II}'. \ldots \nu = \mu' - \rho,$$

where ν is a *normal to the face*; whence also,

$$\text{III.} \ldots \rho'' = \rho\mathrm{S}\,\frac{\mu' + \rho}{\mu' - \rho} - 2\mu'\mathrm{S}\,\frac{\rho}{\mu' - \rho}\,;$$

$$\text{IV.} \ldots \rho'' + \rho = 2\iota, \quad \text{if} \quad \text{IV}'. \ldots \iota = \nu^{-1}\mathrm{V}\mu'\rho = \nu^{-1}\mathrm{V}\nu\rho\,;$$

and

$$\text{V.} \ldots \rho'' - \rho = -\,2\nu\mathrm{S}\rho\nu^{-1} = -\,2\nu^{-1}\mathrm{S}\rho\nu\,;$$

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[*] See pp. 39, 40 of the Paper by that great mathematical and physical philosopher, "*On the Laws of Crystalline Reflexion and Refraction*," already referred to in the Note to page 737 (Trans. R. I. A., Vol. XVIII., Part I.).

[†] Of course, by a suitable choice of the *units* of time and space, the *velocities* and *slownesses*, here spoken of, may be represented by *lines* as *short* as may be thought convenient.

[‡] These equations may be deduced, for example, from the principles of Huyghens, as stated in his *Tractatus de Lumine* (Opera reliqua, Amst., 1728).

so that the *three vectors*, ρ, μ', ρ'', terminate *on one right line*, which is *perpendicular to the face of the crystal:* and the *bisector* of the *angle* between the *first* and *third* of them, or between the incident and reflected rays, is the *intersection* ι of the *plane of incidence* with the same plane face.

(3.) Let τ, τ', τ'' be the *vectors of vibration* for the *three rays* ρ, ρ', ρ'', conceived to be drawn from their respective extremities; then, by *all*[*] theories of *tangential* vibration, we have the equations,

$$\text{VI.} \ldots \mathrm{S}\rho\tau = 0; \qquad \text{VII.} \ldots \mathrm{S}\mu'\tau' = 0; \qquad \text{VIII.} \ldots \mathrm{S}\rho''\tau'' = 0;$$

to which Mac Cullagh *adds* the supposition (a), that the vibration *in the crystal* is *perpendicular to the refracted ray:* or, with the present symbols, that

$$\text{IX.} \ldots \mathrm{S}\rho'\tau' = 0; \quad \text{whence} \quad \text{X.} \ldots \tau' \parallel \mathrm{V}\mu'\rho',$$

the *direction* of the *refracted vibration* τ' being thus in general *determined*, when those of the vectors ρ' and μ' are given.

(4.) To deduce from τ' the *two other* vibrations, τ and τ'', Mac Cullagh assumes, (b), the *Principle of Equivalent Vibrations*, expressed here by the formula,

$$\text{XI.} \ldots \tau - \tau' + \tau'' = 0,$$

in virtue of which the *three* vibrations are *parallel* to one *common plane*, and the refracted vibration is the *vector sum* (or *resultant*) of the other two; (c), the *Principle of the Vis Viva*, by which the *reflected* and *refracted lights* are *together equal* to the *incident light*, which is conceived to have *caused* them; and (d), the *Principle of Constant Density of the Ether*, whereby the *masses of ether*, disturbed by the *three lights*, are simply *proportional* to their *volumes:* the *two last* hypotheses[†] being here jointly expressed by the equation,

$$\text{XII.} \ldots \mathrm{S}\nu(\rho\tau^2 - \rho'\tau'^2 + \rho''\tau''^2) = 0.$$

(5.) Eliminating ρ'' and τ'' from XII. by V. and XI., τ^2 goes off; and we find, with the help of I. and II'., the following *linear equation* in τ,

$$\text{XIII.} \ldots 2\mathrm{S}\frac{\tau}{\tau'} = 1 + \frac{\mathrm{S}\nu\rho'}{\mathrm{S}\nu\rho} = \frac{\mathrm{S}\rho\nu'}{\mathrm{S}\rho\nu}, \quad \text{if} \quad \text{XIII}'. \ldots \nu' = \mu' - \rho';$$

a *second* such equation is obtained by eliminating ρ'' and τ'' by III. and XI. from VIII., and attending to I. VI. VII., namely,

$$\text{XIV.} \ldots 2\mathrm{S}\rho\nu\mathrm{S}\mu'\tau = (\rho^2 - \mu'^2)\mathrm{S}\rho\tau' = -\mathrm{S}\mu'\nu'\mathrm{S}\rho\tau';$$

and a *third* linear equation in τ is given immediately by VI.

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[*] The equations VI. VII. VIII. hold good, for instance, on Fresnel's principles; but Fresnel's tangential vibration in the crystal has a direction *perpendicular* to that adopted by Mac Cullagh.

[†] In the concluding Note (p. 74) to this Paper, Professor Mac Cullagh refers to an elaborate Memoir by Professor Neumann, published in 1837 (in the Berlin Transactions for 1835), as containing precisely the *same system* of hypothetical principles respecting Light. But there was evidently a complete mutual independence, in the researches of those two eminent men. Some remarks on this subject will be found in the *Proceedings* of the R. I. A., Vol. I., pp. 232, 374, and Vol. II., p. 96.

(6.) Solving then for τ, by the rules of the present Calculus, this system of the three linear and scalar equations VI. XIII. XIV., we find for the *incident vibration* the following *vector expression*,*

$$\text{XV.} \ldots \tau = \frac{\mathrm{V}\rho\nu'\tau'}{2\mathrm{S}\rho\nu}; \quad \text{or} \quad \text{XV}'. \ldots 2\tau\mathrm{S}\rho\nu = \tau'\mathrm{S}\rho\nu' - \nu'\mathrm{S}\rho\tau';$$

and accordingly it may be verified by mere inspection, with the help of VII. and IX., that this vector value of τ satisfies the three scalar equations (5.). And when the *incident* vibration has been thus deduced from the *refracted* vibration τ', the *reflected* vibration τ'' is at once given by the formula XI., or by the expression,

$$\text{XVI.} \ldots \tau'' = \tau' - \tau;$$

(7.) The relation XV′. gives at once the *equation of complanarity*,

$$\text{XVII.} \ldots \mathrm{S}\nu'\tau\tau' = 0, \quad \text{or the formula} \quad \text{XVIII.} \ldots \mu' - \rho' \mathbin{|||} \tau, \tau';$$

if then a *plane* be anywhere so drawn, as to be *parallel* (4.) to the *three vibrations* τ, τ', τ'', it will be parallel *also* to the line $\mu' - \rho'$, which *connects two corresponding points*, on the *wave* and *index surface* in the crystal: but this is one form of enunciation of Professor Mac Cullagh's *Theorem of the Polar Plane*, which theorem is thus deduced with great simplicity by quaternions, from the principles above supposed.

(8.) For example, if we suppose that OP and OQ, in Fig. 89, represent the *refracted ray* ρ', and the *index vector* μ' corresponding, and if we draw through the line PQ a plane perpendicular to the plane of the Figure, then the plane so drawn will *contain* (on the principles here considered) the *refracted vibration* τ', and will be *parallel* to *both* the *incident vibration* τ and the *reflected vibration* τ''; whence the *directions* of the two latter vibrations may be in general determined, as being also *perpendicular* respectively to the *incident* and *reflected rays*, ρ and ρ'': and then the *relative intensities* ($\mathrm{T}\tau^2$, $\mathrm{T}\tau'^2$, $\mathrm{T}\tau''^2$) of the *three lights* may be deduced from the *relative amplitudes* ($\mathrm{T}\tau$, $\mathrm{T}\tau'$, $\mathrm{T}\tau''$) of the *three vibrations*, which may themselves be found from the *three complanar directions*, by a simple *resolution* of one line τ' into two others, of which it is the *vector sum*, *as if* the vibrations were *forces*.

(9.) The equations II′. IV′. V. and XIII′. enable us to express the four vectors, $\mu' (= \rho + \nu)$, $\iota (= \rho - \nu^{-1}\mathrm{S}\nu\rho)$, $\rho'' (= \rho - 2\nu^{-1}\mathrm{S}\nu\rho)$, and $\rho' (= \rho + \tau - \nu')$, in terms of the three vectors ρ, ν, ν', which are connected with each other by the relation,

$$\text{XIX.} \ldots \iota (= \rho - \nu^{-1}\mathrm{S}\nu\rho), \quad \rho'' (= \rho - 2\nu^{-1}\mathrm{S}\nu\rho), \quad \text{and} \quad \rho' (= \rho + \nu - \nu').$$

$$\text{XIX.} \ldots \nu^2 + 2\mathrm{S}\nu\rho = \mathrm{S}\nu'(\rho + \nu), \quad \text{because} \quad \text{XIX}'. \ldots \mathrm{S}\nu\rho' = \mathrm{S}(\nu' - \nu)\rho,$$

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* The expressions XV. XVI. enable us to determine, not only the *directions* $\mathrm{U}\tau$, $\mathrm{U}\tau''$ of the *incident* and *reflected vibrations*, but also their *amplitudes* $\mathrm{T}\tau$, $\mathrm{T}\tau''$, or the *intensities* $\mathrm{T}\tau^2$, $\mathrm{T}\tau''^2$ of the incident and reflected *lights*, for any given or assumed amplitude $\mathrm{T}\tau'$ of the *refracted vibration*, or intensity $\mathrm{T}\tau'^2$ of the *refracted light*, after having determined the *direction* $\mathrm{U}\tau'$ of the refracted vibration by means of the formula X.

as in XIII., or because $\mu'^2 - \rho^2 = \mathrm{S}\mu'\nu'$) by I. and XIII'.; and with which τ' is connected (VII. and IX.), by the two equations,

$$\text{XX.} \ldots \mathrm{S}(\rho+\nu)\tau' = 0, \quad \text{and} \quad \text{XXI.} \ldots \mathrm{S}\nu'\tau' = 0;$$

while τ and τ'' are connected with the same three vectors, and with τ', by the relations VI. VIII. XI. XIII., which conduct, by elimination of τ'', to the following system (comp. (5.)) of three linear and scalar equations in τ,

$$\text{XXII.} \ldots \mathrm{S}\rho\tau = 0; \quad 2\mathrm{S}\nu\rho\mathrm{S}\nu\tau = \mathrm{S}\nu'(\rho+\nu)\mathrm{S}\nu\tau'; \quad 2\mathrm{S}\nu\rho\mathrm{S}\tau'^{-1}\tau = \mathrm{S}\nu'\rho;$$

and therefore to the vector expression,

$$2\tau\mathrm{S}\nu\rho = \mathrm{V}\rho\nu'\tau', \text{ as in XV.}$$

(10.) By these or other transfomations, there is no difficulty in deducing this new equation, in which ω may be any vector,

$$\text{XXIII.} \ldots \mathrm{V}\nu\mathrm{V}\{(\rho-\omega)\tau - (\rho'-\omega)\tau' + (\rho''-\omega)\tau_1''\}\tau' = 0;$$

and conversely, when ω is thus treated as *arbitrary*, the formula XXIII., with the relations (9.) between the vectors ρ, ρ', ρ'', ν, ν', μ', but *without* any restriction (*except itself*) on τ, τ', τ'', is *sufficient* to give the *two* vector equations,

$$\text{XI.} \ldots \tau - \tau' + \tau'' = 0, \quad \text{and} \quad \text{XXIV.} \ldots \rho\tau - \rho'\tau' + \rho''\tau'' = x\nu^{-1} + y,$$

in which

$$\text{XXV.} \ldots x = \mathrm{S}\nu(\rho\tau - \rho'\tau' + \rho''\tau'') = \mathrm{S}\nu\nu'\tau', \text{ and XXI.} \ldots y = \mathrm{S}(\rho\tau - \rho'\tau'\nu + \rho''\tau'');$$

and which conduct to the *two* scalar equations (among others),

$$\text{XXVII.} \ldots \mathrm{S}\kappa(\rho\tau - \rho'\tau' + \rho''\tau'') = 0, \quad \text{if} \quad \text{XXVII'.} \ldots \mathrm{S}\kappa\nu = 0,$$

and

$$\text{XXVIII.} \ldots \mathrm{S}\nu\rho(\mathrm{S}\rho\tau - \mathrm{S}\rho''\tau'') = \mathrm{S}\nu\rho'\mathrm{S}\mu'\tau';$$

so that if we *now* suppose the equations VI. VIII. IX. to be *given*, the equation VII. will *follow*, by XXVIII.; while, as a *case* of XXVII., and with the signification IV. or IV'. of ι, we have the equation,

$$\text{XXIX.} \ldots \mathrm{S}\iota(\rho\tau - \rho'\tau' + \rho''\tau'') = 0.$$

(11.) And thus (or otherwise) it may be shown, that the *three* scalar equations VI. VIII. IX., combined with the *one* vector formula XXIII., which (on account of the arbitrary ω) is equivalent to *five* scalar equations, are sufficient to give the *same direction* of τ', and the *same dependencies* of τ and τ'' thereon, as those expressed by the equations X. XV. XVI.; and therefore (among other consequences), to the formulæ XII. and XVII.

(12.) But the equations VI. VIII. IX. contain what may be called the *Principle of Rectangular Vibrations* (or of *vibrations rectangular to rays*); and the formula XXIII. is easily interpreted (416.), as expressing what may be termed the *Principle of the Resultant Couple:* namely the theorem, that *if the three vibrations* (or displacements), τ, τ', τ'', *be regarded as three forces*, RT, R'T', R''T'', *acting at the ends of the three rays*, ρ, ρ', ρ'', or OR, OR', OR'' (drawn in the directions (1.) from the point of incidence O), *then this other system of three forces*, RT, − R'T', R''T'' (conceived as applied to a solid body), *is equivalent to a single couple, of which the plane is parallel* (or the axis perpendicular) *to the face of the crystal.*

(13.) It follows then, by (10.) and (11.), that from these two principles,[*] (I.) and (II.), we can infer *all* the following:

(III.) the *Principle of Tangential Vibrations* (or of vibrations *tangential to the waves*);

(IV.) the *Principle of Equivalent Vibrations* (4.);

(V.) the *Principle of the Vis Viva*, as expressed (in conjunction with that of the *Constant Density of the Ether*) by the equation XII.;

(VI.) the *Principle* (or *Theorem*) *of the Polar Plane*;

And (VII.) what may be called the *Principle of Equivalent Moments*,[†] namely

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[*] The word "Principle" is here employed with the usual latitude, as representing *either* an *hypothesis assumed*, or a *theorem deduced*, but made a *ground* of *subsequent* deduction. The *principle* (I.) of *rectangular vibrations coincides*, for the case of an *ordinary medium*, with the principle (III.) of *tangential* vibrations; but, for an *extraordinary* medium, except for the case (not here considered) of *ordinary rays* in an *uniaxal crystal*, these two principles are *distinct*, although *both* were assumed by Mac Cullagh and Neumann. The present writer has already disclaimed (in the Note to page 736) any responsibility for the *physical hypotheses*; so that the results given above are offered merely as instances of *mathematical deduction* and *generalization* attained through the Calculus of Quaternions.

[†] In a very clear and able Memoir, by Arthur Cayley, Esq. (now Professor Cayley), "On Professor Mac Cullagh's Theorem of the Polar Plane," which was read before the Royal Irish Academy on the 23rd of February, 1857, and has been printed in Vol. VI. of the *Proceedings* of that Academy (pages 481-491), this *name* "principle of equivalent moments," is given to a statement (p. 489), that "the moment of $R't'$ round the axis AH, is equal to the sum of the moments of Rt and $R''t''$ round the same axis"; the line AH being (p. 487) the intersection of the plane of incidence with the plane of separation of the two media, that is, with the *face* of the crystal: while Rt, $R't'$, $R''t''$ are lines representing (p. 488) the *three vibrations* (incident, refracted, and reflected), at the ends of the *three rays* AR, AR' AR'', which are drawn from the point of incidence A, so as to lie, *all three* (p. 487), *within the crystal*. And in fact, if this statement be modified, either by *changing the sign* of the moment of $R''t''$ (p. 491), or by drawing the *reflected ray* AR'', like the line OR'' of the present investigation *in the air* (or *in vacuo*), *instead of prolonging it backwards within* the biaxal crystal, it *agrees* with the *case* XXIX. of the more general formula XXVII., which is itself *included* in what has been called above the *Principle of the Resultant Couple*. In venturing thus to point out, as the subject obliged him to do, what seemed to him to be a slight inadvertence in a Paper of such interest and value, the present writer hopes that he will not be supposed to be deficient in the admiration (long since publicly expressed by him), which is due to the vast attainments of a mathematician so eminent as Professor Cayley.

Since the preceding Series 423, including its Notes (so far), was copied and sent to the printers, the writer's attention has been drawn to a *later* Paper by Mac Cullagh (read December 9th, 1839, and published in Vol. XXI., Part I., of the Transactions of the Royal Irish Academy, pp. 17-50), entitled "*An Essay towards a Dynamical Theory of crystalline Reflexion and Refraction*;" in which there is given at p. 43) a *theorem* essentially equivalent to the above-stated "Principle of the theorem that *the Moment of the Refracted Vibration* ($\mathrm{R'T'}$) *is equal to the Sum of the Moments of the Incident and Reflected Vibrations* (RT and $\mathrm{R''T''}$), *with respect to any line, which is on, or parallel to, the Face of the Crystal.*

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[It appears by the Table of Initial Pages (see p. lix.), that the Author had intended to complete the work by the addition of Seven Articles.]

Resultant Couple," but expressed so as to include the case where the vibrations are *not uniradial*, so that the *double* refraction of the crystal is allowed to manifest itself. Mac Cullagh speaks, in his enunciation of the theorem, of measuring each ray, *in the direction of propagation:* which agrees with, but of course anticipates, the *direction of the reflected ray*, adopted in the preceding investigation. The writer believes that subsequent experiments, by Jamin and others, are considered to diminish much the *physical value* of the theory above discussed.

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